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Homotopy exponents of Harper's spaces

By

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Abstract

For an odd prime p, we show that the p-primary homotopy exponent of Harper's rank 2 finite mod-p H-space K_p is p^{p^2+p}. We then use this to show that the 3-primary homotopy exponent of each of the exceptional Lie groups F_4 and E_6 is 3^{12}.

1. Introduction

Localize spaces and maps at an odd prime p. The *homotopy exponent* of a space X is the least power of p which annihilates the p-torsion in $\pi_*(X)$. We write this as $\exp(X) = p^r$, or if the prime deserves extra emphasis, we instead write $\exp_p(X) = p^r$. Harper [H] constructed a rank 2 finite mod-p H-space K_p which is analogous to the Lie group G_2 at the prime 2, as

$$H^*(K_p; \mathbf{Z}/p\mathbf{Z}) = \Lambda(x_3, y_{2p+1}) \otimes \mathbf{Z}/p\mathbf{Z}[z_{2p+2}]/(z_{2p+2}^p)$$

with $\mathcal{P}^1(x) = y$ and $\beta(y) = z$. We show:

Theorem 1.1. *For any odd prime p, $\exp(K_p) = p^{p^2+p}$.*

Theorem 1.1 is proven by showing that upper and lower bounds for the homotopy exponent coincide. Davis [D1] has shown that K_p has v_1-periodic homotopy groups of order p^{p^2+p}. As the v_1-periodic homotopy groups of any space X represent actual summands in $\pi_*(X)$, these calculations give lower bounds for the homotopy exponents. We approach the problem from the other side and find upper bounds for the homotopy exponents of matching order.

One interesting consequence of Theorem 1.1 concerns the exceptional Lie groups F_4 and E_6 at the prime 3. These are both examples of a *torsion Lie group*, that is, a Lie group which has torsion in its mod-p cohomology. For the compact simple Lie groups, the torsion cases are F_4, E_6, E_7, and E_8 at the prime 3, and E_8 at the prime 5. Harper [H] has shown that there is a 3-local equivalence $F_4 \simeq K_3 \times B(11, 15)$, where $B(11, 15)$ is a spherically resolved

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space. Harris [Hs] has shown there is a 3-local equivalence $E_6 \simeq F_4 \times (E_6/F_4)$, while Bendersky and Davis [BD] have shown that E_6/F_4 is spherically resolved. Since $\pi_m(X) = \pi_m(Y) \times \pi_m(Z)$ if $X \simeq Y \times Z$, these decompositions reduce the calculation of an upper bound for the homotopy exponents of F_4 and E_6 to finding upper bounds for the homotopy exponents of each of their factors. Theorem 1.1 gives the upper bound on the homotopy exponent of the factor K_3. In Section 2 we discuss a general method for obtaining an upper bound on the homotopy exponent of the total space in a fibration over a sphere. Lower bounds on the homotopy exponents are given by Bendersky and Davis [BD]. They show F_4 and E_6 each have v_1-periodic homotopy groups of order 3^{12}. Note that this is the value of $\exp_3(K_3)$. We show:

Theorem 1.2. $\exp_3(F_4) = \exp_3(E_6) = 3^{12}$.

Another space related to K_p is $J_{p-1}(S^{2n})$, the $(p-1)^{st}$ stage of the James construction on S^{2n} (equivalently, the $2n(p-1)$-skeleton of ΩS^{2n+1}). As suggested by the cohomology of K_p, Davis [D1], giving an unpublished proof of Harper, has shown the existence of a map $K_p \longrightarrow J_{p-1}(S^{2p+2})$ which is a cohomological inclusion. We show that, in general:

Proposition 1.1. *For any odd prime p,* $\exp(J_{p-1}(S^{2n})) \leq p^{np}$.

One application of Proposition 1.1 concerns the Cayley projective plane W. In the context of the exceptional Lie groups, there is a fibration $Spin(9) \longrightarrow F_4 \longrightarrow W$. Davis and Mahowald [DM] showed there is an integral homotopy fibration $S^7 \longrightarrow \Omega W \longrightarrow \Omega S^{23}$ which splits if $p \geq 5$. Bendersky and Davis [BD] showed that when $p = 3$, there is a homotopy equivalence $\Omega W \simeq \Omega J_2(S^8)$, and further showed that there exist elements of order 3^{12} in $\pi_*(W)$. Combining this with the upper bound on $\exp_3(J_2(S^8))$ from Proposition 1.1 proves the following.

Corollary 1.1. $\exp_3(W) = 3^{12}$.

The three remaining cases of torsion Lie groups, E_7 and E_8 at the prime 3, and E_8 at the prime 5, have been well studied. Davis [D2] has shown that $\exp_3(E_7) \geq 3^{19}$, $\exp_3(E_8) \geq 3^{30}$, and $\exp_5(E_8) \geq 5^{30}$. Wilkerson [W] has shown that there is a 5-local equivalence $E_8 \simeq X \times Y$ where $H^*(X; \mathbf{Z}/5\mathbf{Z}) \cong \Lambda(x_{15}, x_{23}, x_{39}, x_{47})$ and $H^*(Y; \mathbf{Z}/5\mathbf{Z}) \cong \mathbf{Z}/5\mathbf{Z}[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{27}, x_{35})$. Gonçalves [Go] has shown that X is indecomposable. Homologically, it looks as though Y may have K_5 as a factor. At one point Harper claimed this was the case but later agreed with an objection by Kono. The issue was finally resolved by Davis [D2] when he showed that Y was in fact indecomposable. However, there should exist fibrations $B(27, 35) \longrightarrow Y \longrightarrow K_5$ and $S^{27} \longrightarrow B(27, 35) \longrightarrow S^{35}$, in which case the methods of this paper would apply and one should be able to obtain a good, perhaps optimal, upper bound for $\exp_5(E_8)$. The cases of E_7 and E_8 at the prime 3 appear more difficult. Kono and Mimura [KM] have shown that both E_7 and E_8 are indecomposable at 3, which makes the computation of a best possible upper bound on their homotopy exponents more difficult. In the case of E_7, however, there is a fibration $F_4 \longrightarrow E_7 \longrightarrow E_7/F_4$ and it is conjectured that E_7/F_4 is spherically resolved. If so then again our methods apply. On the other hand, the author knows of no such advantageous fibration for E_8 so this case remains problematic.

2. A method for computing upper bounds on exponents

Typically, an upper bound for the exponent of a space Y is estimated by identifying homotopy fibrations $X \longrightarrow Y \longrightarrow Z$ in which the exponents of both X and Z are known. Then $\exp(Y) \leq \exp(X) \cdot \exp(Z)$. Often, though, this is a poor estimate. This section shows that a better estimate can be obtained in certain cases, in particular for spherically resolved spaces. We then consider some examples.

We begin with the following Lemma, which is a sort of Mayer-Vietoris sequence, and is trivial to prove.

Lemma 2.1. *Suppose there is a homotopy pullback diagram*

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "h"] & P \arrow[d, "i"] \\ M \arrow[r, "g"] & N, \end{tikzcd}$$

where N is an H-space. Then there is a homotopy fibration

$$Q \xrightarrow{f\times h} P \times M \xrightarrow{i\cdot(-g)} N.$$

Let $p^r : S^{2n+1} \longrightarrow S^{2n+1}$ be the map of degree p^r. Let $S^{2n+1}\{p^r\}$ be its homotopy fiber. By [N], $\exp(S^{2n+1}\{p^r\}) = p^r$.

Lemma 2.2. *Suppose there is a homotopy fibration*

$$X \xrightarrow{f} Y \xrightarrow{q} S^{2n+1}$$

where Y is an H-space and there is a map $S^{2n+1} \xrightarrow{i} Y$ such that $q \circ i \simeq p^r$. Then there is a homotopy fibration

$$\Omega X \times \Omega S^{2n+1} \xrightarrow{\Omega f\cdot(-\Omega i)} \Omega Y \longrightarrow S^{2n+1}\{p^r\}.$$

Consequently, $\exp(Y) \leq p^r \cdot \max(\exp(X), \exp(S^{2n+1}))$.

Proof. The homotopy $q \circ i \simeq p^r$ results in a homotopy pullback diagram

$$\begin{tikzcd} S^{2n+1}\{p^r\} \arrow[r] \arrow[d] & S^{2n+1} \arrow[r, "p^r"] \arrow[d, "i"] & S^{2n+1} \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "q"] & S^{2n+1}. \end{tikzcd}$$

Apply Lemma 2.1 to get a homotopy fibration $S^{2n+1}\{p^r\} \longrightarrow X \times S^{2n+1} \xrightarrow{f\cdot(-i)} Y$. Continuing the fibration sequence to the left two steps gives the desired fibration. The exponent consequence follows. $\square$

Two slight modifications of Lemma 2.2 are useful. Both concern altered hypotheses on the initial fibration $X \xrightarrow{f} Y \xrightarrow{q} S^{2n+1}$. First, if Y is not an H-space then we can instead consider the fibration $\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\Omega q} \Omega S^{2n+1}$. Second, the initial fibration may have the form $Z \longrightarrow \Omega Y \xrightarrow{q} \Omega S^{2n-1}$, where q is not a loop map. The following Lemma covers both cases.

Lemma 2.3. *Suppose there is a homotopy fibration*

$$Z \xrightarrow{f} \Omega Y \xrightarrow{q} \Omega S^{2n+1}$$

and there is a map $\Omega S^{2n+1} \xrightarrow{i} \Omega Y$ *such that* $q \circ i \simeq p^r$. *Then there is a homotopy fibration*

$$\Omega Z \times \Omega^2 S^{2n+1} \xrightarrow{\Omega f\cdot(-\Omega i)} \Omega^2 Y \longrightarrow \Omega S^{2n+1}\{p^r\}.$$

Consequently, $\exp(Y) \leq p^r \cdot \max(\exp(X), \exp(S^{2n+1}))$.

Proof. As in Lemma 2.2. $\square$

We now consider some examples of Lemmas 2.2 and 2.3 which play a role in our exponent calculations. Suppose there is a homotopy fibration

$$S^{2m+1} \longrightarrow B \longrightarrow S^{2n+1},$$

where $n > m$. The $(2n+1)$-skeleton of B is the cofiber C of a map $f : S^{2n} \longrightarrow S^{2m+1}$. Suppose f has order p^r. Recall from [CMN] that $\exp(S^{2n+1}) = p^n$.

Lemma 2.4. $\exp(B) \leq p^{n+r}$.

Proof. Since f has order p^r there is a homotopy cofibration diagram

$$\begin{tikzcd}
S^{2n} \arrow[r] \arrow[d, "p^r"] & * \arrow[r] \arrow[d] & S^{2n+1} \arrow[r, equal] \arrow[d] & S^{2n+1} \arrow[d, "p^r"] \\
S^{2n} \arrow[r, "f"] & S^{2m+1} \arrow[r] & C \arrow[r] & S^{2n+1}.
\end{tikzcd}$$

Since the map $B \longrightarrow S^{2n+1}$ is an extension of the map $C \longrightarrow S^{2n+1}$ we have a composition $S^{2n+1} \longrightarrow C \longrightarrow B \longrightarrow S^{2n+1}$ which is degree p^r. If B is an H-space apply Lemma 2.2 to get a homotopy fibration $\Omega S^{2m+1} \times \Omega S^{2n+1} \longrightarrow \Omega B \longrightarrow S^{2n+1}\{p^r\}$. If B is not an H-space apply Lemma 2.3 to ΩB to get a homotopy fibration $\Omega^2 S^{2m+1} \times \Omega^2 S^{2n+1} \longrightarrow \Omega^2 B \longrightarrow \Omega S^{2n+1}\{p^r\}$. The exponent conclusion then follows. $\square$

Example 2.1. Let $q = 2(p-1)$. Let $\alpha_1 \in \pi^S_{q-1}(S^0)$ be a generator of the stable stem. Following Mimura and Toda [MT], for $m \geq 1$ define a space $B(2m+1, 2m+q+1)$ as the homotopy pullback

$$\begin{tikzcd} S^{2m+1} \arrow[r] \arrow[d, equal] & B(2m+1,2m+q+1) \arrow[r] \arrow[d] & S^{2m+q+1} \arrow[d, "\alpha_1"] \\ S^{2m+1} \arrow[r] & S^{4m+3} \arrow[r, "w"] & S^{2m+2}, \end{tikzcd}$$

where w is the Whitehead product of the identity map on S^{2m} with itself. Since α_1 has order p we have $\exp(B(2m+1, 2m+q+1) \leq p^{m+p}$.

Example 2.2. Replacing α_1 in Example 2.1 with $\alpha_2 \in \pi^S_{2q-1}(S^0)$, we obtain a homotopy fibration $S^{2m+1} \longrightarrow B_2(2m+1, 2m+2q+1) \longrightarrow S^{2m+2q+1}$. Again, since α_2 has order p we have $\exp(B_2(2m+1, 2m+2q+1) \leq p^{m+2p-1}$.

Another example involves the filtration of the James construction on spheres. The James construction on a connected space X is a model for $\Omega\Sigma X$. Let $J_k(X) = (\prod_{i=1}^k X)/\sim$ where

$$(x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots x_k) \sim (x_1, \ldots, x_i, *, x_{i+2}, \ldots x_k)$$

for $1 \leq i \leq k-1$. By adding $*$'s on the right we have an inclusion $J_k(X) \longrightarrow J_{k+1}(X)$. Let $J(X) = \varinjlim J_k(X)$. Then $J(X) \simeq \Omega\Sigma X$.

Of particular interest is the case when X is an even sphere S^{2n}. Then $J(S^{2n}) \simeq \Omega S^{2n+1}$ and, localized at a prime p, there is an EHP fibration

$$J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1}.$$

If $p = 2$ then a similar fibration holds for $X = S^{2n+1}$ but at odd primes the second EHP fibration is

$$S^{2n-1} \longrightarrow \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1},$$

where T is the Toda map. In [Gr] it was shown that T can be chosen to be an H-map.

Let $s : S^{2np-1} \longrightarrow J_{p-1}(S^{2n})$ be the attaching map whose cofiber is $J_p(S^{2n})$.

Lemma 2.5. *There is a homotopy commutative diagram*

$$\begin{tikzcd} \Omega S^{2np-1} \arrow[r, "p"] \arrow[d, "\Omega s"] & \Omega S^{2np-1} \arrow[d, equal] \\ \Omega J_{p-1}(S^{2n}) \arrow[r, "T"] & \Omega S^{2np-1}. \end{tikzcd}$$

Proof. The proof is standard. One property of the James construction is that any H-map $\Omega\Sigma X \longrightarrow Y$ into a homotopy associative H-space Y is uniquely determined by its restriction to X. Let $E : S^{2np-2} \longrightarrow \Omega S^{2np-1}$ be the inclusion. A homology calculation shows that $T \circ \Omega s \circ E$ has degree p. Note that $T \circ \Omega s$ is a composite of H-maps and so is an H-map. Thus $T \circ \Omega s$ is homotopic to $\Omega S^{2np-1} \xrightarrow{\Omega p} \Omega S^{2np-1}$, which in turn is homotopic to the p^{th}-power map. □

We now prove Proposition 1.1, which states that $\exp(J_{p-1}(S^{2n})) \leq p^{np}$.

Proof. Start with the homotopy fibration $S^{2n-1} \longrightarrow \Omega J_{p-1}(S^{2n}) \xrightarrow{T} \Omega S^{2np-1}$. By Lemmas 2.5 and 2.3 there is a homotopy fibration

$$\Omega S^{2n-1} \times \Omega^2 S^{2np-1} \longrightarrow \Omega^2 J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2np-1}\{p\}.$$

The conclusion follows. □

Remark 1. The factorization of the p^{th}-power map on ΩS^{2np-1} through $\Omega J_{p-1}(S^{2n})$ also implies $\exp(J_{p-1}(S^{2n})) \geq \frac{1}{p}\exp(S^{2np-1}) = p^{np-2}$. In fact, Davis [D1] has shown that $\exp(J_{p-1}(S^{2p+2}) \geq p^{p^2+p}$, showing Proposition 1.1 is sharp in this case. It is likely to always be sharp.

3. Harper's finite H-spaces

In [H] Harper constructs finite H-spaces K_p, one for each odd prime p, satisfying

$$H^*(K_p; \mathbf{Z}/p\mathbf{Z}) = \Lambda(x_3, y_{2p+1}) \otimes \mathbf{Z}/p\mathbf{Z}[z_{2p+2}]/(z_{2p+2}^p)$$

with $\mathcal{P}^1(x) = y$ and $\beta(y) = z$. We wish to find an upper bound for the homotopy exponent of K_p. To do this we want to wrap around K_p a suitable homotopy fibration which allows us to use Lemma 2.3.

We begin by recording some cohomological information. The three-connected cover $K_p\langle 3\rangle$ of K_p is defined by the homotopy fibration

$$K_p\langle 3\rangle \longrightarrow K_p \longrightarrow K(\mathbf{Z}, 3).$$

In [K] Kono shows:

Lemma 3.1. *We have*

$$H^*(K_p\langle 3\rangle; \mathbf{Z}/p\mathbf{Z}) \cong \Lambda(b_{2p^2+1}, c_{2p^2+2p-1}) \otimes \mathbf{Z}/p\mathbf{Z}[a_{2p^2}]$$

where $\beta(a_{2p^2}) = b_{2p^2+1}$ *and* $\mathcal{P}^1(b_{2p^2+1}) = c_{2p^2+2p-1}$. □

We now set up the homotopy fibration that will allow us to compute $\exp(K_p)$. Davis [D1], proving an unpublished result of Harper, showed that there is a homotopy fibration

$$B(3, 2p+1) \longrightarrow K_p \xrightarrow{\pi} J_{p-1}(S^{2p+1}),$$

where $B(3, 2p+1)$ is one case of the spaces considered in Example 2.1. Taking three-connected covers, looping, and composing with the Toda map gives a homotopy pullback

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & S^{2p+1} \arrow[r] \arrow[d] & B(3,2p+1)\langle 3\rangle \arrow[d, equal] \\
\Omega K_p\langle 3\rangle \arrow[r, "\Omega\pi"] \arrow[d, "\overline{T}"] & \Omega J_{p-1}(S^{2p+2}) \arrow[r] \arrow[d, "T"] & B(3,2p+1)\langle 3\rangle \\
\Omega S^{p^2+2p-1} \arrow[r, equal] & \Omega S^{2p^2+2p-1},
\end{tikzcd}$$

where $\overline{T}$ is defined as the composite $T \circ \Omega\pi$, and X is simply a name for the pullback. In particular, note that $\overline{T}$ is an H-map since both T and $\Omega\pi$ are. Note also that the map $S^{2p+1} \longrightarrow B(3, 2p+1)\langle 3\rangle$ along the top row of the diagram is the inclusion of the bottom cell. Toda [T1] first studied this map and calculated $H^*(X; \mathbf{Z}/p\mathbf{Z}) \cong \Lambda(x_{2p^2-1}) \otimes \mathbf{Z}/p\mathbf{Z}[y_{2p^2}]$ with $\beta(x) = y$.

The following Lemma is the analogue of Lemma 2.5.

Lemma 3.2. *There is a homotopy commutative square*

$$\begin{tikzcd}
\Omega S^{2p^2+2p-1} \arrow[r, "\Omega t"] \arrow[d, "p"] & \Omega K_p\langle 3\rangle \arrow[d, "\overline{T}"] \\
\Omega S^{2p^2+2p-1} \arrow[r, equal] & \Omega S^{2p^2+2p-1},
\end{tikzcd}$$

for some map map t.

Proof. Consider the homotopy fibration $X \xrightarrow{f} \Omega K_p\langle 3\rangle \xrightarrow{\overline{T}} \Omega S^{2p^2+2p-1}$. By Lemma 3.1 the $(2p^2 + 2p - 2)$-skeleton of $\Omega K_p\langle 3\rangle$ is a space $\overline{C}$ whose mod-p cohomology has vector space basis a_{2p^2-1}, b_{2p^2}, and c_{2p^2+2p-2} with $\beta(a_{2p^2-1}) = b_{2p^2}$ and $\mathcal{P}^1(b_{2p^2}) = c_{2p^2+2p-2}$. In particular, there is a cofibration $S^{2p^2+2p-3} \xrightarrow{\overline{\alpha}_1} P^{2p^2}(p) \longrightarrow \overline{C}$, where $P^{2p^2}(p)$ is the Moore space of dimension $2p^2$ and order p, and $\overline{\alpha}_1$ is a lift of $S^{2p^2+2p-3} \xrightarrow{\alpha_1} S^{2p^2}$ through the pinch map $P^{2p^2}(p) \longrightarrow S^{2p^2}$.

Observe that the cohomology of X implies that its $(4p^2 - 2)$-skeleton is $P^{2p^2}(p)$. Since f is $(2p^2+2p-3)$-connected we must therefore have a homotopy commutative diagram

$$\begin{tikzcd}
P^{2p^2}(p) \arrow[r] \arrow[d] & \overline{C} \arrow[r] \arrow[d] & S^{2p^2+2p-2} \arrow[d, "E"] \\
X \arrow[r, "f"] & \Omega K_p\langle 3\rangle \arrow[r, "\overline{T}"] & \Omega S^{2p^2+2p-1},
\end{tikzcd}$$

where the top row is a cofibration and all the vertical maps are inclusions.

Since $\overline{\alpha}_1$ has order p, there is a homotopy cofibration diagram

$$\begin{tikzcd}
S^{2p^2+2p-3} \arrow[r] \arrow[d, "p"] & * \arrow[r] \arrow[d] & S^{2p^2+2p-2} \arrow[r, equal] \arrow[d] & S^{2p^2+2p-2} \arrow[d, "p"] \\
S^{2p^2+2p-3} \arrow[r, "\overline{\alpha}_1"] & P^{2p^2}(p) \arrow[r] & \overline{C} \arrow[r] & S^{2p^2+2p-2}.
\end{tikzcd}$$

Define $\overline{\lambda}$ as the composite $\overline{\lambda} : S^{2p^2+2p-2} \longrightarrow \overline{C} \longrightarrow \Omega K_p\langle 3\rangle$. Then the two diagrams above imply $\overline{T} \circ \overline{\lambda} \simeq E \circ p$. The James construction lets us extend $\overline{\lambda}$ to an H-map $\lambda : \Omega S^{2p^2+2p-1} \longrightarrow \Omega K_p\langle 3\rangle$. A standard argument with the James construction shows that an H-map $\Omega\Sigma X \longrightarrow \Omega Z$ is homotopic to a loop map. Thus $\lambda \simeq \Omega t$ for some map $t : S^{2p^2+2p-1} \longrightarrow K_p$.

Now consider the H-map $\Omega S^{2p^2+2p-1} \xrightarrow{\overline{T}\circ\Omega t} \Omega S^{2p^2+2p-1}$. One property of the James construction is that any H-map $\Omega\Sigma X \longrightarrow Y$ into a homotopy associative H-space Y is uniquely determined by its restriction to X. Thus $\overline{T} \circ \overline{\lambda} \simeq E \circ p$ implies $\overline{T} \circ \Omega t \simeq p$. □

Lemma 3.3. $\exp(X) = p$.

Proof. The composition $S^{2p+1} \xrightarrow{i} B(3, 2p+1)\langle 3\rangle \longrightarrow B(3, 2p+1) \longrightarrow S^{2p+1}$ is degree p. Thus there is a homotopy pullback diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & S^{2p+1}\{p\} \arrow[r] \arrow[d] & S^3\langle 3\rangle \arrow[d] \\
X \arrow[r] & S^{2p+1} \arrow[r, "i"] \arrow[d, "p"] & B(3, 2p+1)\langle 3\rangle \arrow[d] \\
& S^{2p+1} \arrow[r, equal] & S^{2p+1}.
\end{tikzcd}$$

By [S], the fibration along the top row splits when looped twice, $\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3\langle 3\rangle \times \Omega^2 X$. Since $\exp(S^{2p+1}\{p\}) = p$, we therefore have $\exp(X) \leq p$. The inclusion of the bottom Moore space into X shows $\exp(X) \geq p$. □

We now prove Theorem 1.1, which states that $\exp(K_p) = p^{p^2+p}$.

Proof. Lemmas 3.2 and 2.3 imply there is a homotopy fibration

$$\Omega X \times \Omega^2 S^{2p^2+2p-1} \longrightarrow \Omega^2 K_p \longrightarrow \Omega S^{2p^2+2p-1}\{p\}.$$

Thus $\exp(K_p) \leq p \cdot \max(\exp(X), \exp(S^{2p^2+2p-1}))$. By Lemma 3.3, $\exp(T) = p$ while $\exp(S^{2p^2+2p-1}) = p^{p^2+p-1}$. Thus $\exp(K_p) \leq p^{p^2+p}$. On the other hand, Davis [D1] showed that $\pi_*(K_p)$ has elements of order p^{p^2+p} so $\exp(K_p) \geq p^{p^2+p}$. □

4. The exponents of F_4 and E_6 at 3

We first record lower bounds for the homotopy exponents. As mentioned in the Introduction, Bendersky and Davis [BD] showed that $\exp_3(F_4) \geq 3^{12}$ and $\exp_3(E_6) \geq 3^{12}$. We now prove Theorem 1.2 by showing that the upper bounds on the 3-primary homotopy exponents of F_4 and E_6 match the lower bounds.

Proof. First, Harper [H] showed there is a 3-local equivalence

$$F_4 \simeq K_3 \times B(11, 15).$$

Theorem 1.1 shows that $\exp_3(K_3) \leq 3^{12}$. On the other hand, Example 2.1 shows that $\exp_3(B(11, 15)) \leq 3^8$. Thus $\exp_3(F_4) \leq 3^{12}$. Next, Harris [Hs] showed there is a 3-local equivalence

$$E_6 \simeq F_4 \times (E_6/F_4).$$

Bendersky and Davis [BD] showed there is a 3-local equivalence $E_6/F_4 \simeq B_2(9, 17)$. Example 2.2 shows that $\exp_3(B_2(9, 17)) \leq 3^9$. Thus the upper bound for the homotopy exponent of E_6 equals that of F_4, and so $\exp_3(E_6) \leq 3^{12}$. $\square$

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