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Homotopy genus of BU and the Bott map

By

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1. Introduction

The homotopy genus of a nilpotent finite CW-complex X is defined as follows ([5], [7]):

$$\{\,[Y]\,|\,Y \simeq_p X \text{ for each prime } p\,\}.$$

The homotopy genus of certain spaces are computed, for example, the order of the homotopy genus of a classifying space of a compact connected Lie group is uncountable infinite. But the homotopy genus of $BU = BU(\infty)$ is not known yet. The purpose of this paper is to determine the homotopy genus of the pair of BU and the Bott map of BU. The main theorem below says that it is unique.

Theorem. *Let X be a pointed of finite type simply connected CW-complex equipped with a map $\lambda : S^2 \wedge X \to X$ and a homotopy equivalences $h_p : X_{(p)} \to BU_{(p)}$ for each prime p such that they satisfy the following homotopy commutative diagram*

$$\begin{array}{ccc}
(S^2 \wedge X)_{(p)} & \xrightarrow{1\wedge h_p} & (S^2 \wedge BU)_{(p)} \\
\lambda_{(p)}\big\downarrow & & \big\downarrow\beta_{(p)} \\
X_{(p)} & \xrightarrow[h_p]{} & BU_{(p)},
\end{array}$$

where $\beta : S^2 \wedge BU \to BU$ is the Bott map. Then we have a homotopy equivalence $h : X \xrightarrow{\sim} BU$ which satisfies the following homotopy commutative diagram.

$$\begin{array}{ccc}
S^2 \wedge X & \xrightarrow{1\wedge h} & S^2 \wedge BU \\
\lambda\big\downarrow & & \big\downarrow\beta \\
X & \xrightarrow[h]{} & BU
\end{array}$$

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2. The Bott map of BU

Let us recall the Bott map $\beta : S^2 \wedge BU \to BU$. Let η and ξ_n be the Hopf bundle of S^2 and the universal bundle of $BU(n)$. The Bott map

$$\beta : S^2 \wedge BU \to BU$$

is defined as the classifying map of the virtual complex vector bundle $(\eta - \mathbf{1}) \wedge \lim(\xi_n - \mathbf{n})$ on $S^2 \wedge BU$, where $\mathbf{1}$ and $\mathbf{n}$ is of rank 1 and n trivial complex vector bundle. It is well known that the Bott map gives the Bott periodicity of BU which is

$$\widetilde{\mathrm{ad}\beta} : BU \xrightarrow{\simeq} \Omega^2 BSU,$$

where $\widetilde{\mathrm{ad}\beta}$ is the lift of $\mathrm{ad}\beta : BU \to \Omega^2 BU$ ([2]). We have the following as a consequence of the Bott periodicity.

Proposition 2.1. *Let $g_1 : S^2 \to BU$ represent a generator of $\pi_2(BU) \cong \mathbf{Z}$. Then a generator of $\pi_{2n}(BU) \cong \mathbf{Z}$ $(n > 1)$ is represented by*:

$$\begin{aligned} g_n = {} & \beta(1 \wedge \beta)(1 \wedge 1 \wedge \beta) \cdots \\ & \cdots (1 \wedge \cdots \wedge 1 \wedge \beta)(1 \wedge \cdots \wedge 1 \wedge g_1) : S^2 \wedge \cdots \wedge S^2 = S^{2n} \to BU. \end{aligned}$$

Corollary 2.1. *Let X, λ be as in Theorem, $\imath : S^2 \to S^2_{(0)}$ be the rationalization and $g'_1 : S^2 \to X_{(0)}$ represent a generator of $\pi_2(X_{(0)}) \cong \mathbf{Q}$. Then we have that a generator of $\pi_{2n}(X_{(0)}) \cong \mathbf{Q}$ $(n > 1)$ is represented by*:

$$\begin{aligned} g'_n = {} & \lambda_{(0)} \circ (\imath \wedge \lambda_{(0)}) \circ (1 \wedge \imath \wedge \lambda_{(0)}) \circ \cdots \\ & \cdots \circ (1 \wedge \cdots \wedge 1 \wedge \imath \wedge \lambda_{(0)}) \circ (1 \wedge \cdots \wedge 1 \wedge \imath \wedge g'_1) : S^2 \wedge \cdots \wedge S^2 = S^{2n} \to X_{(0)}. \end{aligned}$$

3. Proof of Theorem

To prove Theorem we need to construct a homotopy equivalence by patching together the homotopy equivalences between localized spaces. The following is the well-known pull-back theorem ([4]).

Lemma 3.1. *Let X and Y be finite nilpotent spaces with a homotopy equivalence $h_p : X_{(p)} \to Y_{(p)}$ such that $h_{p(0)} \simeq h_{q(0)}$, for each prime p, q. Then we have a homotopy equivalence $h : X \to Y$ such that $h_{(p)} \simeq h_p$ for each prime p.*

Proposition 3.1. *Let X be of finite type pointed CW-complex with a homotopy equivalence $h^n : X^n \to BU^n$ for each n such that $h^{n+1}|_{X^n} \simeq h^n$, where X^n and BU^n are n-skeleta of X and* BU. *Then we have a homotopy equivalence $h : X \to BU$ such that $h|_{X^n} \simeq h^n$ for any n.*

Proof. By Milnor's short exact sequence ([6])

$$0 \to \varprojlim{}^1 \widetilde{K}^{-1}(X^n) \to \widetilde{K}(X) \to \varprojlim \widetilde{K}(X^n) \to 0,$$

we have a map $h : X \to BU$ such that $h|_{X^n} \simeq h^n$. Since $h_* = \varprojlim h^n_* : \pi_*(X) \to \pi_*(BU)$ is an isomorphism, h is a homotopy equivalence by J.H.C. Whitehead theorem. $\square$

It is easily seen that

$$H^*(BU_{(p)};\mathbf{Q}) \cong H^*(BU;\mathbf{Q}) \cong \mathbf{Q}[c_1, c_2, c_3, \ldots],$$

where c_n is the n-th Chern class.

Lemma 3.2. *Let $g_n : S^{2n} \to BU, g'_n : S^{2n} \to X$ be as in Proposition* 2.1 *and Corollary* 2.1. *Then we have $\bar{g}_n : K(2n,\mathbf{Q}) \to BU_{(0)}$, $\bar{g'}_n : K(2n,\mathbf{Q}) \to X_{(0)}$ such that $\bar{g}_n i \simeq g_{n(0)}$ and $\bar{g'}_n i \simeq g'_{n(0)}$, where $i : S^{2n}_{(0)} \to K(2n,\mathbf{Q})$ is the rationalization of a generator of $\pi_{2n}(K(2n,\mathbf{Z})) \cong \mathbf{Z}$.*

Proof. It is well know that

$$\pi_k(S^{2n}_{(0)}) \cong \begin{cases} \mathbf{Q}, & k = 2n, 4n-1, \\ 0, & \text{otherwise.} \end{cases}$$

We construct a space K_1 which is the rationalization of the adjunction space $S^{2n}_{(0)} \cup e^{4n}$, where the attaching map is a generator of $\pi_{4n-1}(S^{2n}_{(0)})$. Then we have the following by [3, Proposition 13.12].

$$\pi_k(K_1) \cong \begin{cases} \mathbf{Q}, & k = 2n, 6n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\pi_{4n-1}(BU_{(0)}) = \pi_{4n-1}(X_{(0)}) = 0$, we see that $g_{n(0)}, g'_{n(0)}$ can be extended to maps $g^1_n : K_1 \to BU_{(0)}$, ${g^1_n}' : K_1 \to X_{(0)}$ such that $g^1_n i \simeq g_{n(0)}$, ${g^1_n}' i \simeq g'_{n(0)}$. Inductively we construct a space K_r such that

$$\pi_k(K_r) \cong \begin{cases} \mathbf{Q}, & k = 2n, 2(r+2)n-1, \\ 0, & \text{otherwise} \end{cases}$$

and the maps $g^r_n : K_r \to BU_{(0)}$, ${g^r_n}' : K_r \to X_{(0)}$ such that $g^r_n i \simeq g_{n(0)}$, ${g^r_n}' i \simeq g'_{n(0)}$ by the same way as the above. Let $\bar{g_n} = \varinjlim g^r_n$, $\bar{g_n}' = \varinjlim {g^r_n}'$. Since $\varinjlim K_r = K(\mathbf{Q}, 2n)$ and $(g^r_n i)^* = {g_n}^*_{(0)} : H^{2n}(BU_{(0)};\mathbf{Q}) \to H^*(K(\mathbf{Q},2n);\mathbf{Q})$, $({g^r_n}' i)^* = {g'_n}^*_{(0)} : H^{2n}(X_{(0)};\mathbf{Q}) \to H^*(K(\mathbf{Q},2n);\mathbf{Q})$, the proof is completed. $\square$

Proof of Theorem. Since $h_p : X_{(p)} \to BU_{(p)}$ is a homotopy equivalence, we have

$${g'_{1(p)}}^* {h_p}^*(c_1) = k_p {g_{1(p)}}^*(c_1) \text{ for } k_p \in \mathbf{Z}^\times_{(p)}.$$

It is well known that

$${\beta_{(0)}}^*(s_k) = ke \otimes s_{k-1},$$

where s_k is the k-th power sum in $\{c_n\}$ and $e = {g_{1(0)}}^*(c_1)$. Then we see that

$$ {g_{n(0)}}^*(s_n) = (k_p)^n {g'_{n(0)}}^* {h_{p(0)}}^*(s_n) \in H^{2n}(S^{2n}_{(0)}; \mathbf{Q}). $$

It is well known that there exists the inverse of the localized Adams operator $\psi^m_{(p)} : \widetilde{K}(\,\cdot\,)_{(p)} \to \widetilde{K}(\,\cdot\,)_{(p)}$, when $p \nmid m$. Denote $k_p = \pm a/b$ such that $a > 0$, $b > 0$ and $p \nmid a$. Let $h'_p : X_{(p)} \to BU_{(p)}$ be $(\psi^a_{(p)})^{-1} \psi^b_{(p)} \psi^{\operatorname{sgn} k_p}_{(p)}(h_p)$, then we have h'_p is a homotopy equivalence and ${h'_{p(0)}}^*(s_n) = (k_p)^n {h_{p(0)}}^*(s_n)$ ([1]). Then we have

$$ {g_{n(0)}}^*(s_n) = {g'_{n(0)}}^* {h'_{p(0)}}^*(s_n) \in H^{2n}(S^{2n}_{(0)}; \mathbf{Q}). $$

Since $\prod s_n : BU_{(0)} \to \prod K(\mathbf{Q}; 2n)$ is a homotopy equivalence, we have

$$ g_{n(0)} \simeq h'_{p(0)} g'_{n(0)}. $$

By Lemma 3.2 we have $\bar{g_n} i \simeq h'_{p(0)} \bar{g'_n} i$. Therefore we have $\bar{g_n} \simeq h'_{p(0)} \bar{g'_n}$. Since $\prod \bar{g'_n} : K(2n, \mathbf{Q}) \simeq X_{(0)}$, we obtain for each prime p and q,

$$ h'_{p(0)} \simeq \left(\prod \bar{g_n} \right) \left(\prod \bar{g'_n} \right)^{-1} \simeq h'_{q(0)}. $$

By Lemma 3.1 and Proposition 3.1 we obtain a homotopy equivalence $h : X \xrightarrow{\simeq} BU$. Since $[S^2 \wedge X, BU]_* \cong [S^2 \wedge BU, BU]_* \cong \widetilde{K}^{-2}(BU)$ is a free abelian group, we see that $h\lambda \simeq \beta(1 \wedge h)$ by $(h\lambda)_{(0)} \simeq (\beta(1 \wedge h))_{(0)}$. □

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