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On invariants of curves in centro-affine geometry

By

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Abstract

Let $GL(n, R)$ be the general linear group of $n \times n$ real matrices. Definitions of $GL(n, R)$-equivalence and the centro-affine type of curves are introduced. All possible centro-affine types are founded. For every centro affine type all invariant parametrizations of a curve are described. The problem of $GL(n, R)$-equivalence of curves is reduced to that of paths. A generating system of the differential field of invariant differential rational functions of a path is described. They can be viewed as centro-affine curvatures of a path. Global conditions of $GL(n, R)$-equivalence of curves are given in terms of the centro-affine type and the generating differential invariants. Independence of elements of the generating differential invariants is proved.

1. Introduction

The fundamental theorem of curves in n-dimensional centro-affine geometry is obtained by Gardner and Wilkens [5]. In the paper they used Cartan's method [4] of moving frames in order to find the formulation of the local rigidity theorem for curves that is amenable to direct application to problems in control theory. They provide a method for constructing the centro-affine curvatures $\kappa_1(s), \ldots, \kappa_n(s)$. They have obtained explicit formulae for computing a centro-affine arclength in terms of an arbitrary parameter and the first curvature $\kappa_1(s)$. In this paper there are no explicit formulae for centro-affine curvatures $\kappa_2(s), \ldots, \kappa_n(s)$, but their orders are defined. A discussion of centro-affine plane curves, as well as a very brief discussion of centro-affine space curves, can be found in ([15], [13], [17]). A very detailed discussion of the centro-affine theory of plane curves can be found in Laugwitz [11]. The first comprehensive treatment of affine geometry is given in the seminal work of Blaschke [3]. For further developments of the subject, we refer the reader to [14], and the more modern texts [20], [12], commentaries [16], [17], and survey papers [19], [2],

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[18]. Equi-affine invariants of 3-dimensional space curves are investigated by Izumiya and Sano [7]. For curvatures of curves in n-dimensional equi-affine geometry see ([6, pp. 170–172], [13]). But in all these works, equivalence of curves is investigated locally. The global $SL(n)$-equivalence of paths in R^n and C^n is considered by Khadjiev [8] and Suhtaeva [21]. Complete systems of global equi-affine invariants for plane and space paths are obtained by Angelis, Moons, Van Gool and Verstraelen [1]. The complete system of global differential and integral invariants for curves in n-dimensional equi-affine geometry is obtained by Khadjiev and Pekşen [9].

Our paper is concerned with the problem of the global equivalence of centro-affine curves. We introduce a centro-affine type of a curve. The centro-affine type of a curve coincides with the centro-affine arclength if it is finite. But curves with infinite centro-affine arclength have three different centro-affine types. We describe all possible invariant parametrizations of a curve for every centro-affine type. We obtained a generating system of the differential field of all centro-affine invariant differential rational functions of a path. We can consider elements of the generating system as curvatures of a path. Our curvatures coincide with $c_1(s), \ldots, c_n(s)$ functions of Gardner and Wilkens ([5, p. 398]). We give the explicit formulae of curvatures in terms of the centro-affine invariant parameter, the conditions of the global centro-affine equivalence of curves in terms of the centro-affine type and curvatures of a curve. We prove an independence of curvatures.

2. The centro-affine type of a curve and the theorem on reduction

Let R be the field of real numbers and $I = (a, b)$ be an open interval of R.

Definition 2.1. A C^∞-map $x : I \to R^n$ will be called an I-path (shortly, a path) in R^n.

Definition 2.2. An I_1-path $x(t)$ and an I_2-path $y(r)$ in R^n will be called D-equivalent if there exists a C^∞-diffeomorphism $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in I_2$. A class of D-equivalent paths in R^n will be called a curve in R^n, ([10, p. 9]). A path $x \in \alpha$ will be called a parametrization of a curve α.

Remark 1. There exist different definitions of a curve ([6, p. 2], [8]).

Let $G = GL(n, R)$ be the general linear group of $n \times n$ regular matrices. G acts by $(g, x) \to gx$ on R^n, where gx is the multiplication of a matrix g and a column vector $x \in R^n$.

If $x(t)$ is an I-path in R^n then $gx(t)$ is an I-path in R^n for any $g \in G$.

Definition 2.3. Two I-paths x and y in R^n will be called G-equivalent and written $x \overset{G}{\sim} y$ if there exists $g \in G$ such that $y(t) = gx(t)$.

Let α be a curve in R^n, that is, $\alpha = \{h_\tau, \tau \in Q\}$, where h_τ is a parametrization of α. Then $g\alpha = \{gh_\tau, \tau \in Q\}$ is a curve in R^n for any $g \in G$.

Definition 2.4. Two curves α and β in R^n will be called G-equivalent (or G-congruent) and written $\alpha \overset{G}{\sim} \beta$ if $\beta = g\alpha$ for some $g \in G$.

Remark 2. Our definition is essentially different from the definition ([6, p. 21]) of a congruence of curves for the group of euclidean motions. By the definition ([6, p. 21]), two curves with different lengths may be congruent.

Let x be an I-path in R^n and $x'(t)$ be the derivative of $x(t)$. Put $x^{(0)} = x$, $x^{(n)} = (x^{(n-1)})'$. For $a_k \in R^n$, $k = 1, \dots, n$, the determinant $\det(a_{ij})$ (where a_{ki} are coordinates of a_k) will be denoted by $[a_1 a_2 \dots a_n]$. So $\big[x(t)x'(t)\dots x^{(n-1)}(t)\big]$ is the determinant of the vectors $x(t)$, $x'(t), \dots$, $x^{(n-1)}(t)$.

Definition 2.5. An I-path $x(t)$ in R^n will be called substantial if both

$$\Big[x(t)x'(t)\dots x^{(n-1)}(t)\Big] \neq 0 \text{ and } \Big[x^{(n)}(t)x'(t)\dots x^{(n-1)}(t)\Big] \neq 0$$

for all $t \in I$. A curve will be called substantial if it contains a substantial path [5].

For $I = (a, b)$, $q, p \in I$ and a substantional I-path $x(t)$ put

$$l_x(q,p) = \int_q^p \left| \frac{\big[x^{(n)}(t)x'(t)\dots x^{(n-1)}(t)\big]}{\big[x(t)x'(t)\dots x^{(n-1)}(t)\big]} \right|^{\frac{1}{n}} dt$$

and $l_x(a,p) = \lim\limits_{q \to a} l_x(q,p)$, $l_x(q,b) = \lim\limits_{p \to b} l_x(q,p)$. There are only four possible cases:

(i) $l_x(a,p) < +\infty$, $l_x(q,b) < +\infty$; (ii) $l_x(a,p) < +\infty$, $l_x(q,b) = +\infty$;
(iii) $l_x(a,p) = +\infty$, $l_x(q,b) < +\infty$; (iv) $l_x(a,p) = +\infty$, $l_x(q,b) = +\infty$.

Suppose that the case (i) or (ii) holds for some q, $p \in I$. Then the number $l = l_x(a,p) + l_x(q,b) - l_x(q,p)$, where $0 \le l \le +\infty$, does not depend on q, p. In this case, we say that x belongs to the centro-affine type of $(0, l)$. The cases (iii) and (iv) do not depend on q, p. In these cases, we say that x belongs to the centro-affine types of $(-\infty, 0)$ and $(-\infty, +\infty)$, respectively. There exist paths of all types $(0, l)$ (where $0 \le l \le +\infty$), $(-\infty, 0)$ and $(-\infty, +\infty)$. The centro-affine type of a path x will be denoted by $L(x)$.

Proposition 2.1.
(i) If $x \overset{G}{\sim} y$ then $L(x) = L(y)$;
(ii) Let α be a curve and $x, y \in \alpha$. Then $L(x) = L(y)$.

Proof. It is obvious. □

The centro-affine type of a path $x \in \alpha$ will be called the centro-affine type of the curve α and denoted by $L(\alpha)$. According to Proposition 2.1, $L(\alpha)$ is a G-invariant of a curve α.

Now we define an invariant parametrization of a substantial curve in R^n.

Let $I=(a,b)$ and $x(t)$ be a substantial I-path in R^n. We define the centro-affine arc length function $s_x(t)$ for each centro-affine type as follows. We put $s_x(t)=l_x(a,t)$ for the case $L(x)=(0,l)$, where $0<l\leq+\infty$, and $s_x(t)=-l_x(t,b)$ for the case $L(x)=(-\infty,0)$. Let $L(x)=(-\infty,+\infty)$. We choose a fixed point in every interval $I=(a,b)$ of R and denote it by a_I. Let $a_I=0$ for $I=(-\infty,+\infty)$. We set $s_x(t)=l_x(a_I,t)$.

Since $s'_x(t)>0$ for all $t\in I$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $L(x)$ and $t'_x(s)>0$ for all $s\in L(x)$.

Proposition 2.2. *Let $I=(a,b)$ and x be a substantial I-path in R^n. Then*

(i) $s_{gx}(t)=s_x(t)$ and $t_{gx}(s)=t_x(s)$ for all $g\in G$;

(ii) the equalities $s_{x(\varphi)}(r)=s_x(\varphi(r))+s_0$ and $\varphi(t_{x(\varphi)}(s+s_0))=t_x(s)$ hold for any C^∞-diffeomorphism $\varphi: J=(c,d)\to I$ such that $\varphi'(r)>0$ for all $r\in J$, where $s_0=0$ for $L(x)\neq(-\infty,+\infty)$ and $s_0=l_x(\varphi(a_J),a_I)$ for $L(x)=(-\infty,+\infty)$.

Proof. (i). Let $L(x)=(0,l)$, where $0<l\leq+\infty$. Then we have

$$\begin{aligned} s_{gx}(t) &= \lim_{t_0\to a^+}\int_{t_0}^{t}\left|\frac{\big[(gx)^{(n)}(t)(gx)'(t)\dots(gx)^{(n-1)}(t)\big]}{\big[(gx)(t)(gx)'(t)\dots(gx)^{(n-1)}(t)\big]}\right|^{\frac{1}{n}}dt \\ &= \lim_{t_0\to a^+}\int_{t_0}^{t}\left|\frac{\big[x^{(n)}(t)x'(t)\dots x^{(n-1)}(t)\big]}{\big[x(t)x'(t)\dots x^{(n-1)}(t)\big]}\right|^{\frac{1}{n}}dt \\ &= s_x(t). \end{aligned}$$

For the second part, we obtain $s_{gx}(t_{gx}(s))=s$, $t_{gx}(s_{gx}(t))=t$, $s_{gx}(t_{gx}(s))=s_x(t_{gx}(s))=s$, $t_{gx}(s_{gx}(t))=t_{gx}(s_x(t))=t$. Therefore $t_{gx}(s)=t_x(s)$. Proofs of (i) for centro-affine types $(-\infty,0)$ and $(-\infty,+\infty)$ are similar.

For (ii) let $L(x)=(-\infty,+\infty)$. Then we have

$$\begin{aligned} s_{x(\varphi)}(r) &= \int_{a_J}^{r}\left|\frac{\Big[\frac{d^n}{dr^n}x(\varphi(r))\frac{d}{dr}(x(\varphi(r)))\dots\frac{d^{n-1}}{dr^{n-1}}(x(\varphi(r)))\Big]}{\Big[x(\varphi(r))\frac{d}{dr}(x(\varphi(r)))\dots\frac{d^{n-1}}{dr^{n-1}}(x(\varphi(r)))\Big]}\right|^{\frac{1}{n}}dr \\ &= \int_{a_J}^{r}\frac{d\varphi}{dr}\left|\frac{\Big[\frac{d^n}{d\varphi^n}x(\varphi(r))\frac{d}{d\varphi}(x(\varphi(r)))\dots\frac{d^{n-1}}{d\varphi^{n-1}}(x(\varphi(r)))\Big]}{\Big[x(\varphi(r))\frac{d}{d\varphi}(x(\varphi(r)))\dots\frac{d^{n-1}}{d\varphi^{n-1}}(x(\varphi(r)))\Big]}\right|^{\frac{1}{n}}dr \\ &= l_x\left(\varphi\left(a_J\right),\varphi(r)\right)=l_x\left(a_I,\varphi(r)\right)+l_x\left(\varphi\left(a_J\right),a_I\right). \end{aligned}$$

So $s_{x(\varphi)}(r)=s_x(\varphi(r))+s_0$, where $s_0=l_x\left(\varphi\left(a_J\right),a_I\right)$. This implies that $\varphi(t_{x(\varphi)}(s+s_0))=t_x(s)$. For $L(x)\neq(-\infty,+\infty)$, it is easy to see that $s_0=0$. $\square$

Let α be a substantial curve and $x \in \alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 2.6. The parametrization $x(t_x(s))$ of a substantial curve α will be called an invariant parametrization of α.

We denote the set of all invariant parametrizations of α by ϕ_α. Every $y \in \phi_\alpha$ is an I-path, where $I = L(\alpha)$.

Proposition 2.3. *Let α be a substantial curve, $x \in \alpha$ and x be an I-path, where $I = L(\alpha)$. Then the following conditions are equivalent*:

(i) x is an invariant parametrization of α;

(ii) $\left|\frac{[x^{(n)}(s)x'(t)...x^{(n-1)}(s)]}{[x(s)x'(s)...x^{(n-1)}(s)]}\right| = 1$ *for all* $s \in L(\alpha)$;

(iii) $s_x(s) = s$ *for all* $s \in L(\alpha)$.

Proof. $(i) \Rightarrow (ii)$. Let $x \in \phi_\alpha$. Then there exists $y \in \alpha$ such that $x(s) = y\,(t_y(s))$. By Proposition 2.2, $s_x(s) = s_{y(t_y)}(s) = s_y(t_y(s)) + s_0 = s + s_{0,}$ where s_0 is as in Proposition 2.2. Since s_0 does not depend on s, $\frac{ds_x(s)}{ds} = \left|\frac{[x^{(n)}(s)x'(t)...x^{(n-1)}(s)]}{[x(s)x'(s)...x^{(n-1)}(s)]}\right|^{\frac{1}{n}} = 1$. Hence $\left|\frac{[x^{(n)}(s)x'(t)...x^{(n-1)}(s)]}{[x(s)x'(s)...x^{(n-1)}(s)]}\right| = 1$ for all $s \in L(\alpha)$.

$(ii) \Rightarrow (iii)$. Let $\left|\frac{[x^{(n)}(s)x'(t)...x^{(n-1)}(s)]}{[x(s)x'(s)...x^{(n-1)}(s)]}\right| = 1$ for all $s \in L(\alpha)$. By the definition of $s_x(t)$, we have $\frac{ds_x(s)}{ds} = \left|\frac{[x^{(n)}(s)x'(t)...x^{(n-1)}(s)]}{[x(s)x'(s)...x^{(n-1)}(s)]}\right|^{\frac{1}{n}} = 1$. Therefore $s_x(s) = s + c$ for some $c \in R$. In the case $L(x) \neq (-\infty, +\infty)$, $s_x(s) = s + c$ and $s_x(s) \in L(\alpha)$ for all $s \in L(\alpha)$ implies $c = 0$, that is, $s_x(s) = s$. In the case $L(x) = (-\infty, +\infty)$, $s_x(s) = l_x(a_I, s) = l_x(0, s) = s + c$ implies $0 = l_x(0,0) = c$, that is, $s_x(s) = s$.

$(iii) \Rightarrow (i)$. The equality $s_x(s) = s$ implies $t_x(s) = s$. Therefore $x(s) = x(t_x(s)) \in \phi_\alpha$. $\square$

Proposition 2.4. *Let α be a substantial curve and $L(\alpha) \neq (-\infty, +\infty)$. Then there exists the unique invariant parametrization of α.*

Proof. Let $x, y \in \alpha$, x be an I_1-path and y be an I_2-path. Then there exists a C^∞-diffeomorphism $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $y(r) = x(\varphi(r))$ for all $r \in I_2$. By Proposition 2.2 and $L(\alpha) \neq (-\infty, +\infty)$, we obtain $y(t_y(s)) = x(\varphi(t_y(s)) = x(\varphi(t_{x(\varphi)}(s))) = x(t_x(s))$. $\square$

Let α be a substantial curve and $L(\alpha) = (-\infty, +\infty)$. Then it is easy to see that the set ϕ_α is not countable.

Proposition 2.5. *Let α be a substantial curve, $L(\alpha) = (-\infty, +\infty)$ and $x \in \phi_\alpha$. Then $\phi_\alpha = \{y : y\,(s) = x\,(s + s')\,, s' \in (-\infty, +\infty)\}$.*

Proof. Let $x, y \in \phi_\alpha$. Then there exist $h, k \in \alpha$ such that $x(s) = h(t_h(s))$, $y(s) = k(t_k(s))$, where h is an I_1-path and k is an I_2-path. Since $h, k \in \alpha$ there exists $\varphi : I_2 \to I_1$ such that $\varphi'(r) > 0$ and $k(r) = h(\varphi(r))$ for all $r \in I_2$. By Proposition 2.2, $y(s) = k(t_k(s)) = h(\varphi(t_k(s)) = h(\varphi(t_{h(\varphi)}(s))) = h(t_h(s - s_0)) = x(s - s_0)$.

Let $x \in \phi_\alpha$ and $s' \in (-\infty, +\infty)$. We prove $x(\psi) \in \phi_\alpha$, where $\psi(s) = s + s'$. By Proposition 2.3, $\left|\frac{[x^{(n)}(s)x'(t)\ldots x^{(n-1)}(s)]}{[x(s)x'(s)\ldots x^{(n-1)}(s)]}\right| = 1$ and $s_x(s) = s$. Put $z(s) = x(\psi(s))$. Since ψ is a C^∞-diffeomorphism of $(-\infty, +\infty)$ onto $(-\infty, +\infty)$, then $z = x(\psi) \in \alpha$. Using Proposition 2.2 and $s_x(s) = s$, we get $s_z(s) = s_{x(\psi)}(s) = s_x(\psi(s)) + s_1 = (s + s') + s_1$, where

$$s_1 = \int_{\psi(0)}^{0} \left|\frac{[x^{(n)}(s)x'(t)\ldots x^{(n-1)}(s)]}{[x(s)x'(s)\ldots x^{(n-1)}(s)]}\right|^{\frac{1}{n}} ds.$$

This, in view of $\left|\frac{[x^{(n)}(s)x'(t)\ldots x^{(n-1)}(s)]}{[x(s)x'(s)\ldots x^{(n-1)}(s)]}\right| = 1$, implies $s_1 = -\psi(0) = -s'$. Then $s_z(s) = (s + s') - s' = s$. By Proposition 2.3, $z \in \phi_\alpha$. □

Theorem 2.1. *Let α, β be substantial curves and $x \in \phi_\alpha$, $y \in \phi_\beta$. Then,*

(i) for $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{G}{\sim} \beta$ if and only if $x(s) \overset{G}{\sim} y(s)$;

(ii) for $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{G}{\sim} \beta$ if and only if $x(s) \overset{G}{\sim} y(s + s')$ for some $s' \in (-\infty, +\infty)$.

Proof. (*i*) Let $\alpha \overset{G}{\sim} \beta$ and $h \in \alpha$. Then there exists $g \in G$ such that $\beta = g\alpha$. This implies $gh \in \beta$. Using Propositions 2.2 and 2.4, we get $x(s) = h(t_h(s))$, $y(s) = (gh)(t_{gh}(s))$ and $gx(s) = g(h(t_h(s))) = (gh)(t_h(s)) = (gh)(t_{gh}(s)) = y(s)$. Thus $x \overset{G}{\sim} y$. Conversely, let $x \overset{G}{\sim} y$, that is, there exists $g \in G$ such that $gx = y$. Then $\alpha \overset{G}{\sim} \beta$.

(*ii*) Let $\alpha \overset{G}{\sim} \beta$. Then there exist I-paths $h \in \alpha$, $k \in \beta$ and $g \in G$ such that $k(t) = gh(t)$. We have $k(t_k(s)) = k(t_{gh}(s)) = k(t_h(s)) = (gh)(t_h(s))$. By Proposition 2.5, $x(s) = k(t_k(s + s_1))$, $y(s) = h(t_h(s + s_2))$ for some s_1, $s_2 \in (-\infty, +\infty)$. Therefore $x(s - s_1) = gy(s - s_2)$. This implies that $x(s) \overset{G}{\sim} y(s + s')$, where $s' = s_1 - s_2$. Conversely, let $x(s) \overset{G}{\sim} y(s + s')$ for some $s' \in (-\infty, +\infty)$. Then there exists $g \in G$ such that $y(s + s') = gx(s)$. Since $y(s + s') \in \beta$, then $\alpha \overset{G}{\sim} \beta$. □

Definition 2.7. Two I-paths x and y in R^n, where $I = R$, will be called (G, R)-equivalent if there exists $g \in G$ and $s' \in R = (-\infty, +\infty)$ such that $y(s) = gx(s + s')$ for all $s \in (-\infty, +\infty)$.

Theorem 2.1 reduces the problem of the G-equivalence of substantial curves to that of paths for the case $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$. But for the case

$L(\alpha) = L(\beta) = (-\infty, +\infty)$, Theorem 2.1 reduces the problem of G-equivalence of substantional curves to the (G, R)-equivalence of paths.

3. The generating system of the differential field of invariant differential rational functions of a path and the problem of equivalence of curves

Let $x(t)$ be an I-path in R^n.

Definition 3.1. A polynomial $p(x, x', \ldots, x^{(k)})$ of x and a finite number of derivatives $x', \ldots, x^{(k)}$ of x with coefficients from R will be called a differential polynomial of x. It will be denoted by $p\{x\}$ [9].

We denote the set of all differential polynomials of x by $R\{x\}$. It is a differential R-algebra. It is also an integral domain. Therefore there exists a quotient field $R\langle x\rangle$ of $R\{x\}$ and every element of $R\langle x\rangle$ is of the form $f\langle x\rangle = \frac{p\{x\}}{q\{x\}}$, where $p\{x\}$ and $q\{x\} \neq 0$ are differential polynomials of x. Any element of $R\langle x\rangle$ will be called a differential rational function of x. The derivative operator of $R\{x\}$ can be extended uniquely to $R\langle x\rangle$ as $f'\langle x\rangle = \left(\frac{p\{x\}}{q\{x\}}\right)' = \frac{p'\{x\}q\{x\} - p\{x\}q'\{x\}}{(q\{x\})^2}$. Let G be a subgroup of $GL(n, R)$.

Definition 3.2. A differential rational function $f\langle x\rangle$ will be called G-invariant if $f\langle gx\rangle = f\langle x\rangle$ for all $g \in G$.

The set of all G-invariant differential rational functions of x will be denoted by $R\langle x\rangle^G$. It is a differential subfield of $R\langle x\rangle$.

Definition 3.3. A subset S of $R\langle x\rangle^G$ will be called a generating system of $R\langle x\rangle^G$ if the smallest differential subfield containing S is $R\langle x\rangle^G$.

Theorem 3.1. *The system*

$$\frac{\left[x^{(n)}x'\ldots x^{(n-1)}\right]}{\left[xx'\ldots x^{(n-1)}\right]}, \ \frac{\left[xx'\ldots x^{(i-1)}x^{(n)}x^{(i+1)}\ldots x^{(n-1)}\right]}{\left[xx'\ldots x^{(n-1)}\right]}, \ i = 1, \ldots, n-1,$$

is a generating system of $R\langle x\rangle^G$.

Proof. For the proof, see ([8, p. 79]) □

Theorem 3.2. *Let α, β be substantial curves in R^n and $x \in \phi_\alpha$, $y \in \phi_\beta$. Then,*

(i) for $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{G}{\sim} \beta$ if and only if

$$sgn\frac{[x^{(n)}(s)x'(s)\ldots x^{(n-1)}(s)]}{[x(s)x'(s)\ldots x^{(n-1)}(s)]} = sgn\frac{[1y^{(n)}(s)y'(s)\ldots y^{(n-1)}(s)]}{[y(s)y'(s)\ldots y^{(n-1)}(s)]},$$

$$(3.1) \qquad \frac{[x(s)x'(s)\ldots x^{(i-1)}(s)x^{(n)}(s)x^{(i+1)}(s)\ldots x^{(n-1)}(s)]}{[x(s)x'(s)\ldots x^{(n-1)}(s)]} = \frac{[y(s)y'(s)\ldots y^{(i-1)}(s)y^{(n)}(s)y^{(i+1)}(s)\ldots y^{(n-1)}(s)]}{[y(s)y'(s)\ldots y^{(n-1)}(s)]}$$

for all $s \in L(\alpha) = L(\beta)$ *and* $i = 1, \ldots, n-1$.

(*ii*) *for* $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{G}{\sim} \beta$ *if and only if there exists* $b \in (-\infty, +\infty)$ *such that*

$$sgn\frac{\left[x^{(n)}(s)x'(s)\ldots x^{(n-1)}(s)\right]}{\left[x(s)x'(s)\ldots x^{(n-1)}(s)\right]} = sgn\frac{\left[y^{(n)}(s+b)y'(s+b)\ldots y^{(n-1)}(s+b)\right]}{\left[y(s+b)y'(s+b)\ldots y^{(n-1)}(s+b)\right]},$$

$$\frac{\left[x(s)x'(s)\ldots x^{(i-1)}(s)x^{(n)}(s)x^{(i+1)}(s)\ldots x^{(n-1)}(s)\right]}{\left[x(s)x'(s)\ldots x^{(n-1)}(s)\right]}$$
$$= \frac{\left[y(s+b)y'(s+b)\ldots y^{(i-1)}(s+b)y^{(n)}(s+b)y^{(i+1)}(s+b)\ldots y^{(n-1)}(s+b)\right]}{\left[y(s+b)y'(s+b)\ldots y^{(n-1)}(s+b)\right]}$$

for all $s \in (-\infty, +\infty)$ *and* $i = 1, \ldots, n-1$.

Proof. (*i*) Let $\alpha \overset{G}{\sim} \beta$. By claim (*i*) of Theorem 2.1, $x \overset{G}{\sim} y$. By Proposition 2.3, $\left|\frac{\left[x^{(n)}x'\ldots x^{(n-1)}\right]}{\left[xx'\ldots x^{(n-1)}\right]}\right| = \left|\frac{\left[y^{(n)}y'\ldots y^{(n-1)}\right]}{\left[yy'\ldots y^{(n-1)}\right]}\right| = 1$. This, in view of $x \overset{G}{\sim} y$, yields the formulae (3.1). Now suppose that (3.1) holds. By Proposition 2.3 , we have $\left|\frac{\left[x^{(n)}x'\ldots x^{(n-1)}\right]}{\left[xx'\ldots x^{(n-1)}\right]}\right| = \left|\frac{\left[y^{(n)}y'\ldots y^{(n-1)}\right]}{\left[yy'\ldots y^{(n-1)}\right]}\right| = 1$. Using (3.1), we obtain

$$\frac{\left[x^{(n)}(s)x'(s)\ldots x^{(n-1)}(s)\right]}{\left[x(s)x'(s)\ldots x^{(n-1)}(s)\right]} = \frac{\left[y^{(n)}(s)y'(s)\ldots y^{(n-1)}(s)\right]}{\left[y(s)y'(s)\ldots y^{(n-1)}(s)\right]},$$
$$\frac{\left[x\,(s)\,x'(s)\ldots x^{(i-1)}(s)x^{(n)}(s)x^{(i+1)}(s)\ldots x^{(n-1)}(s)\right]}{\left[x(s)x'(s)\ldots x^{(n-1)}(s)\right]}$$
$$= \frac{\left[y(s)y'(s)\ldots y^{(i-1)}(s)y^{(n)}(s)y^{(i+1)}(s)\ldots y^{(n-1)}(s)\right]}{\left[y(s)y'(s)\ldots y^{(n-1)}(s)\right]}.$$

Let us consider the matrix

$$A_x(t) = \left\|x(t)x'(t)\ldots x^{(n-1)}(t)\right\|.$$

By the substantiality of x $\det A_x(t) = \left[x(t)x'(t)\ldots x^{(n-1)}(t)\right] \neq 0$ for all t in I. Therefore there exists the matrix $A_x^{-1}(t)$. We consider the matrices

$$A'_x(t) = \left\|x'(t)x^{(2)}(t)\ldots x^{(n)}(t)\right\| \text{ and } A_x^{-1}(t)A'_x(t) = \|c_{ij}(t)\|.$$

It is easily obtained that

1. $c_{j+1j}(t) = 1$ for all t in I, $j : 1 \leq j \leq n-1$;
2. $c_{ij}(t) = 0$ for all t in I, $j \neq n$, $i \neq j+1$, $1 \leq i \leq n$;
3. $c_{i+1n}(t) = \frac{\left[x(t)x'(t)\ldots x^{(i-1)}(t)x^{(n)}(t)x^{(i+1)}(t)\ldots x^{(n-1)}(t)\right]}{\left[x(t)x'(t)\ldots x^{(n-1)}(t)\right]}$ for all t in I, $0 \leq i \leq n-1$.

Similarly, for $A_y^{-1}(t)A'_y(t) = \|d_{ij}(t)\|$ we have

1. $d_{j+1j}(t) = 1$ for all t in I, $j : 1 \leq j \leq n-1$;

2. $d_{ij}(t) = 0$ for all t in I, $j \neq n$, $i \neq j+1$, $1 \leq i \leq n$;

3. $d_{i+1n}(t) = \frac{\left[y(t)y'(t)\ldots y^{(i-1)}(t)y^{(n)}(t)y^{(i+1)}(t)\ldots y^{(n-1)}(t)\right]}{\left[y(t)y'(t)\ldots y^{(n-1)}(t)\right]}$ for all t in I, $0 \leq i \leq n-1$.

We obtain from (3.1) that $c_{ij}(t) = d_{ij}(t)$ for all t in I, $i, j = 1, \ldots, n$. Then

$$A_x^{-1}(t)A'_x(t) = A_y^{-1}(t)A'_y(t).$$

We have

$$\begin{aligned}(A_yA_x^{-1})' &= A'_yA_x^{-1} + A_y(A_x^{-1})' = A'_yA_x^{-1} + A_y(-A_x^{-1}A'_xA_x^{-1}) \\ &= A_y(A_y^{-1}A'_y - A_x^{-1}A'_x)A_x^{-1} = 0.\end{aligned}$$

Therefore $A_y(t)A_x^{-1}(t)$ does not depend on t. Put $g = A_y(t)A_x^{-1}(t)$. As $\det A_x(t) \neq 0$ for all t in I, and $A_y(t) \neq 0$ for all t in I, then $\det g \neq 0$. We have $A_y(t) = gA_x(t)$. Therefore $y(t) = gx(t)$ for all t in I. Thus $x \overset{G}{\sim} y$. The proof of (ii) follows similarly from claim (ii) of Theorem 2.1. □

Let T be one of the sets $(0, l)$ (where $l \leq +\infty$), $(-\infty, 0)$, $(-\infty, +\infty)$.

Theorem 3.3. *Let $h_1(s), \ldots, h_n(s)$ be C^∞-functions on T, where $|h_1(s)| = 1$ for all $s \in T$. Then there exists an invariant parametrization x of a substantial curve such that*

$$\begin{aligned}&\frac{[x^{(n)}(s)x(s)x'(s)\ldots x^{(n-1)}(s)]}{[x(s)x'(s)\ldots x^{(n-1)}(s)]} = h_1(s),\\ &\frac{[x(s)x'(s)\ldots x^{(i-1)}(s)x^{(n)}(s)x^{(i+1)}(s)\ldots x^{(n-1)}(s)]}{[x(s)x'(s)\ldots x^{(n-1)}(s)]} = h_{i+1}(s)\end{aligned} \tag{3.2}$$

for all $s \in T$ and $i = 1, \ldots, n-1$.

Proof. Let $C(s)$ be the matrix $\|c_{ij}(s)\|$, where $c_{j+1j}(s) = 1$ for all $s \in T$, $1 \leq j \leq n-1$; $c_{ij}(s) = 0$ for all $s \in T$, $j \neq n$, $i \neq j+1$, $1 \leq i \leq n$; $c_{in}(s) = h_i(s)$, $i = 1, \ldots, n$. It is known from the theory of differential equations that there exists a solution of the differential equation

$$A'_x(s) = A_x(s)C(s) \tag{3.3}$$

such that $\det A_x(s) \neq 0$ for all $s \in T$, where $A_x(s) = \left\|x(s)x'(s)\ldots x^{(n-1)}(s)\right\|$ is the matrix of column vectors $x(s)$, $x'(s), \ldots, x^{(n-1)}(s)$ and $A'_x(s) = \left\|x'(s)x''(s)\ldots x^{(n)}(s)\right\|$ is the matrix of column vectors $x'(s)$, $x''(s), \ldots, x^{(n)}(s)$. Let $A_x(s)$ be such solution of the differential equation (3.3). From (3.3) we have $A_x^{-1}(s)A'_x(s) = C(s)$. From this equation we get the equalities (3.2) □

Remark 3. The functions

$$\frac{\left[x(s)x'(s)\ldots x^{(i-1)}(s)x^{(n+1)}(s)x^{(i+1)}(s)\ldots x^{(n)}(s)\right]}{\left[x(s)x'(s)\ldots x^{(n)}(s)\right]},$$

where $i = 1, \ldots, n$, equal to the functions $c_1(s), \ldots, c_n(s)$ of Gardner and Wilkens ([5, p. 398]).

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