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Nodal curves and Riccati solutions of Painlevé equations

By

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Abstract

In this paper, we study Riccati solutions of Painlevé equations from a view point of geometry of Okamoto-Painlevé pairs (S, Y). After establishing the correspondence between (rational) nodal curves on $S - Y$ and Riccati solutions, we give the complete classification of the configurations of nodal curves on $S - Y$ for each Okamoto-Painlevé pair (S, Y). As an application of the classification, we prove the non-existence of Riccati solutions of Painlevé equations of types $P_I, P_{III}^{\tilde{D}_8}$ and $P_{III}^{\tilde{D}_7}$. We will also give a partial answer to the conjecture in [STT] and [T1] that the dimension of the local cohomology $H^1_{Y_{red}}(S, \Theta_S(-\log Y_{red}))$ is one.

1. Introduction

A pair (S, Y) of a projective smooth surface S and an effective anti-canonical divisor Y on S is called an Okamoto-Painlevé pair if it satisfies a suitable condition (see (2.1) in Section 2). In [STT], we established the theory of Okamoto-Painlevé pairs (S, Y) and characterize the Painlevé equations by means of the special deformation of Okamoto-Painlevé pairs. There exist 8 types of rational Okamoto-Painlevé pairs which correspond to the Painlevé equations. The types are classified by the types of the dual graphs of the configurations of Y, which are the affine Dynkin diagram of types $R = \tilde{D}_k, 4 \leq k \leq 8, \tilde{E}_l, 6 \leq l \leq 8$. For each R, we obtain the global family of Okamoto-Painlevé pairs

$$(1.1)\qquad \begin{array}{ccc} \mathcal{S}_R & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & \end{array},$$

where $\mathcal{B}_R$ is an affine open subset of the t-affine line $\operatorname{Spec} \mathbf{C}[t]$. In [STT], the deformation with respect to the t-direction can be characterized by the local

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cohomology group $H^1_D(S, \Theta_S(-\log D))$ where $D = Y_{red}$. Furthermore we can show that the vector field $\frac{\partial}{\partial t}$ has a unique lifting to a rational global vector field

$$\tilde{v} \in H^0(\mathcal{S}_R, \Theta_{\mathcal{S}_R}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{D})) \tag{1.2}$$

which induces the Painlevé differential equations on $\mathcal{S}_R - \mathcal{D}$.

In the theory of Painlevé equations, it is important to determine all classical solutions, like algebraic solutions and Riccati solutions. (For the definition of classical solutions of Painlevé equations, see §1 in [U-W1]). In this direction, there are a considerable number of works by many authors. Here, we list up only a part of the references: (e.g., [DM], [Grm1], [Grm2], [Grm3], [Grm4], [Grm5], [Grm6], [Gr-Lu], [Gr-Ts], [Luk1], [Luk2], [Maz], [Mu1], [Mu2], [Ni], [Ohy], [O3], [U1], [U2], [U-W1], [U-W2], [V], [W1], [W2], [Y]). For example, in order to prove the irreducibility of the Painlevé equations, one has to determine the cases when the given Painlevé equations admit the Riccati solutions (cf. [U1], [U2], [NO], [U-W1], [U-W2]).

In this paper, a smooth rational curve $C \simeq \mathbf{P}^1$ on a surface S with $C^2 = -2$ is called a *nodal curve or a* (-2)*-curve*[*1]. One of the main purpose of this paper is to characterize the Riccati solutions of Painlevé equations by means of geometry of nodal curves on $S - Y_{red}$ for the corresponding rational Okamoto-Painlevé pairs (S, Y).

Since our characterization of Painlevé vector field $\tilde{v}$ (1.2) in [STT] is intrinsic, that is, coordinate free, so is the characterization of Riccati solutions.

Moreover we shall give the complete classification theorem (Theorem 3.1) of configurations of nodal curves on $S - Y_{red}$ for all rational Okamoto-Painlevé pairs (S, Y) of non-fibered type and of additive type. As a corollary to Theorem (3.1), we can show that Painlevé equations $P_I, P_{III}^{\tilde{D_7}}, P_{III}^{\tilde{D_8}}$ have no Riccati solutions for any parameters in the equations.

The following is a rough outline of this paper.

In Section 2, we characterize the Riccati solutions of the Painlevé equations by means of (-2)-curve (or nodal curve) C on $S - Y_{red}$. If for a given $(\boldsymbol{\alpha}_0, t_0) \in \mathcal{M}_R \times \mathcal{B}_R$, the fiber S of π in (1.1) over $(\boldsymbol{\alpha}_0, t_0)$ contains a nodal curve $C \subset S - Y_{red}$, we can extend the nodal curve C in the t-direction and obtain a family of nodal curves $\mathcal{C} \longrightarrow \{\boldsymbol{\alpha}_0\} \times U$ where U is an (analytic or étale) open neighborhood of t_0 in $\mathcal{B}_R$. Then the restriction $\tilde{v}_{|\mathcal{C}}$ is tangent to $\mathcal{C}$ which induces the Riccati equation on $\mathcal{C}$. It seems that this approach is essentially equivalent to Umemura's theory of invariant divisors for the Painlevé equations (cf. e.g., [U-W1]). However we believe that our approach gives a clearer geometric viewpoint of Riccati solutions of Painlevé equations.

In Section 3, we shall give the complete classification of configurations of nodal curves on $S - Y$ for all rational Okamoto-Painlevé pairs (S, Y) of non-fibered type and of additive type. The classification is based on the structure theorem of the lattice induced by the intersection form on $H^2(S, \mathbf{Z})$. We

[*1] We can always contract a nodal curve $C \subset S$ to a singular point which is called the nodal surface singularity or the A_1-singularity.

can show that the sub-lattice generated by the nodal curves C on $S - Y$ is a sub-lattice of E_8^-, the unique even unimodular negative-definite lattice of rank 8. Then taking account into the sub-lattice generated by the irreducible components of Y, we can obtain the list of the possible configurations. For the existence of the possible configurations, we quote the Oguiso-Shioda's classification theorem of singular fiber or Mordell-Weil group for rational elliptic surfaces. Note that a rational elliptic surface with a fixed fiber is a rational Okamoto-Painlevé pairs of fibered type in our terminology. Using the Oguiso-Shioda's existence theorem and the deformation theory of Okamoto-Painlevé pairs, we shall show the existence of all possible configurations for some rational Okamoto-Painlevé pairs.

In Section 4, as a corollary to the classification theorem, we shall prove the non-existence of Riccati solutions of the Painlevé equations of type $R = P_I, P_{III}^{\tilde{D}_7}, P_{III}^{\tilde{D}_8}$. Though there are other proofs for this result for P_I and $P_{III}^{\tilde{D}_7}$ (e.g., [U1], [U2] and [Ohy]), our proof clarify the point that the obstruction to the existence of Riccati solutions lies in the topological conditions.

In Section 5, we give explicit examples of nodal curves and Riccati solutions of Painlevé equations associated to the nodal curves.

In Section 6, we shall give an example of the confluence of the Riccati solutions for $R = \tilde{E}_6, (P_{IV})$ and also the confluence of nodal curves. Moreover we give a remark on rational solutions coming from the intersection of two different Riccati solutions.

In Appendix A, as a corollary to Theorem 3.1, we shall give a partial answer to the Conjecture A.1 presented in [STT] and [T1] about the dimension of the local cohomology group.

2. (-2)-curves (Nodal curves) and Riccati solutions

In this section, we shall review the theory of Okamoto-Painlevé pairs and their relations to the Painlevé equations which were introduced in [STT].

2.1. Okamoto-Painlevé pairs

Definition 2.1. Let (S, Y) be a pair of a complex projective surface S and an effective anti-canonical divisor $Y \in |-K_S|$ of S. Let $Y = \sum_{i=1}^{r} m_i Y_i$ be the irreducible decomposition of Y. We call a pair (S, Y) an *Okamoto-Painlevé pair* if for all $i, 1 \leq i \leq r$,

$$Y \cdot Y_i = \deg[Y]_{|Y_i} = 0. \tag{2.1}$$

An Okamoto-Painlevé pair (S, Y) is called *rational* if S is a rational surface.

Remark 1. An Okamoto-Painlevé pair (S, Y) in Definition 2.1 is called a generalized Okamoto-Painlevé pair in [STT]. However, in this paper, we shall use this terminology. Note that in the original definition of an Okamoto-Painlevé pair (S, Y) in [STT] we assume that $S - Y_{red}$ contains $\mathbf{C}^2$ as a Zariski open set and Y_{red} is a normal crossing divisor. (See also [Sa-Ta]).

2.2. Okamoto-Painlevé pairs and Painlevé equations

Let (S,Y) be a rational Okamoto-Painlevé pair with the irreducible decomposition $Y=\sum_{i=1}^{r} m_i Y_i$ and set $D=Y_{red}=\sum_{i=1}^{r} Y_i$. Denote by $M(Y)$ the sub-lattice of $\mathrm{Pic}(S)\simeq H^2(S,\mathbf{Z})$ generated by the irreducible components $\{Y_i\}_{i=1}^{r}$. With the bilinear form on $M(Y)$ which is (-1) times the intersection pairing on S, $M(Y)$ becomes a root lattice of affine type (cf. [STT, Section 1], [Sakai]). Let $R(Y)$ denote the type of the root lattice. One can classify rational Okamoto-Painlevé pairs (S,Y) in terms of the type $R(Y)$. (See [STT, Section 1], [Sakai]).

Not all types of rational Okamoto-Painlevé pairs correspond to the Painlevé equations. The Table 1 is the list of the types of Okamoto-Painlevé pairs which correspond to the Painlevé equations. We shall explain the meaning of the correspondence in Theorem 2.1. Note that classically, Painlevé equations were classified into 6 types, however now we should classify them into 8 types. Actually, the third Painlevé equations P_{III} can be classified further into 3 types $P_{III}^{\tilde{D}_6}, P_{III}^{\tilde{D}_7}$ and $P_{III}^{\tilde{D}_8}$ corresponding to the types of $R=R(Y)$. The classical third Painlevé equations correspond to $P_{III}^{\tilde{D}_6}$, which form a two parameter family of equations. The equations $P_{III}^{\tilde{D}_7}$ and $P_{III}^{\tilde{D}_8}$ can be obtained by specializations of these parameters.

Okamoto-Painlevé pairs and Painlevé equations

$R=R(Y)$	$\tilde{E}_8$	$\tilde{E}_7$	$\tilde{D}_8$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$
Painlevé equation	P_I	P_{II}	$P_{III}^{\tilde{D}_8}$	$P_{III}^{\tilde{D}_7}$	$P_{III}^{\tilde{D}_6}$	P_{IV}	P_V	P_{VI}

Table 1.

Here we shall recall one more important definition (cf. [STT, Section 1]).

Definition 2.2. A rational Okamoto-Painlevé pair (S,Y) will be called *of fibered type* if there exists an elliptic fibration $f: S \longrightarrow \mathbf{P}^1$ such that $f^*(\infty)=Y$ as divisors. We say that a rational Okamoto-Painlevé pair is *of non-fibered type* if (S,Y) is not of fibered type.

The following theorem (cf. [STT, Proposition 5.1 and Theorem 6.1]) explains how one can give correspondences between rational Okamoto-Painlevé pairs and Painlevé equations in Table 1.

Theorem 2.1 ([STT, Proposition 5.1]). *Let $R=R(Y)$ be one of types of the root systems in Table 1 (i.e., $R=\tilde{D}_i, 4\le i\le 8$ or $\tilde{E}_j, 6\le j\le 8$) and let r be the number of irreducible components of $D=Y_{red}$ and set $s=s(R)=9-r$.*

Then there exist affine open subschemes $\mathcal{M}_R \subset \mathbf{C}^s = \operatorname{Spec}\mathbf{C}[\alpha_1, \ldots, \alpha_s]$, $\mathcal{B}_R \subset \mathbf{C} = \operatorname{Spec}\mathbf{C}[t]$, *and the following commutative diagram satisfying the conditions below*:

$$
\begin{array}{ccc}
\mathcal{S} & \hookleftarrow & \mathcal{D} \\
\pi \downarrow & \swarrow & \varphi \\
\mathcal{M}_R \times \mathcal{B}_R & & .
\end{array}
\tag{2.2}
$$

(1) $\mathcal{S}$ *is a smooth quasi-projective manifold and* $\mathcal{D}$ *is a divisor with normal crossing of* $\mathcal{S}$. *Moreover* π *is a smooth and projective morphism and* φ *is a flat morphism such that the above diagram is a deformation of non-singular pairs of projective surfaces and normal crossing divisors in the sense of Kawamata* [Kaw].

(2) *There is a rational relative* 2*-form*

$$\omega_{\mathcal{S}} \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{M}_R \times \mathcal{B}_R}(*\mathcal{D})) \tag{2.3}$$

which has poles only along $\mathcal{D}$. *If we denote by* $\mathcal{Y}$ *the pole divisor of* $\omega_{\mathcal{S}}$, *then for each point* $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$, $(\mathcal{S}_{\boldsymbol{\alpha},t}, \mathcal{Y}_{\boldsymbol{\alpha},t})$ *is a rational Okamoto-Painlevé pair of type* $R = R(Y)$ *and* $\mathcal{Y}_{red} = \mathcal{D}$.

(3) *There is a unique global rational vector field*

$$\tilde{v} \in \Gamma(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{D})) \tag{2.4}$$

on $\mathcal{S}$ *which is a lift of* $\frac{\partial}{\partial t}$, *that is,* $\pi_*(\tilde{v}) = \frac{\partial}{\partial t}$. *Moreover the restriction of* $\tilde{v}$ *to* $\mathcal{S} - \mathcal{D}$ *gives a regular algebraic vector field which corresponds to the Painlevé equation of type* R. *We call the systems of differential equations determined by the vector field* $\tilde{v}$ the Painlevé system of type R. (*See* (2.10) *below*).

We can state more about the family in (2.2) as follows.

(1) The family is semi-universal at each point $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$, that is, the Kodaira-Spencer map

$$\rho : T_{\boldsymbol{\alpha},t}(\mathcal{M}_R \times \mathcal{B}_R) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \tag{2.5}$$

is an isomorphism. For a point $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$ at which the corresponding Okamoto-Painlevé pair is of non-fibered type, one can obtain the following commutative diagram:

(2.6)

$$
\begin{array}{ccc}
 & & 0 \\
 & & \uparrow \\
0 \to H^1_{\mathcal{D}_{\boldsymbol{\alpha},t}}(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) & \hookrightarrow & H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \\
\uparrow & & || \\
H^0(\mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}) \otimes N_{\mathcal{D}_{\boldsymbol{\alpha},t}}) \simeq \mathbf{C} \cdot \rho\left(\dfrac{\partial}{\partial t}\right) & \hookrightarrow & H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \\
\wr \uparrow & & \wr \uparrow \rho \\
T_{\boldsymbol{\alpha},t}(\mathcal{B}_R) \simeq \mathbf{C} \cdot \dfrac{\partial}{\partial t} & \hookrightarrow & T_{\boldsymbol{\alpha},t}(\mathcal{M}_R \times \mathcal{B}_R) \\
\uparrow & & \uparrow \\
0 & & 0.
\end{array}
$$

(2) Let M_R and B_R denote the affine coordinate rings of $\mathcal{M}_R$ and $\mathcal{B}_R$ respectively so that $\mathcal{M}_R = \operatorname{Spec} M_R$ and $\mathcal{B}_R = \operatorname{Spec} B_R$. (Note that M_R and B_R are obtained by some localizations of $\mathbf{C}[\alpha_1, \ldots, \alpha_s]$ and $\mathbf{C}[t]$ respectively.) There exists an affine open covering $\{\tilde{U}_i\}_{i=1}^{l+k}$ of $\mathcal{S}$ such that for each i

$$\begin{aligned} \tilde{U}_i &\simeq \operatorname{Spec}\left((M_R \otimes B_R)\left[x_i, y_i, \frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)}\right]\right) \subset \operatorname{Spec} \mathbf{C}[\boldsymbol{\alpha}, t, x_i, y_i] \\ &\simeq \mathbf{C}^{s+3} \simeq \mathbf{C}^{12-r}. \end{aligned} \tag{2.7}$$

Here $f_i(x_i, y_i, \boldsymbol{\alpha}, t)$ is a polynomial in $(M_R \otimes B_R)[x_i, y_i]$. Moreover, we may assume that $\mathcal{S} - \mathcal{D}$ can be covered by $\{\tilde{U}_i\}_{i=1}^{l}$, and for each i, the restriction of the rational 2-form $\omega_{\mathcal{S}}$ can be written as

$$\omega_{\mathcal{S}|\tilde{U}_i} = \frac{dx_i \wedge dy_i}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)^{m_i}}. \tag{2.8}$$

(3) By using the local coordinates of $\mathcal{S} - \mathcal{D}$, the global rational vector field $\tilde{v}$ on $\mathcal{S}$ obtained in (2.4) can be written on each open set $\tilde{U}_i$ for $1 \leq i \leq l$ (corresponding to the open covering of $\mathcal{S} - \mathcal{D}$) as

$$\tilde{v}_{|\tilde{U}_i} = \frac{\partial}{\partial t} - \theta_i = \frac{\partial}{\partial t} - \eta_i \frac{\partial}{\partial x_i} - \zeta_i \frac{\partial}{\partial y_i}, \tag{2.9}$$

where $\theta_i = \eta_i \frac{\partial}{\partial x_i} + \zeta_i \frac{\partial}{\partial y_i}$ is a regular algebraic vector field on $\tilde{U}_i$.
This explicit expression of $\tilde{v}_{|\tilde{U}_i}$ gives a system of differential equations

$$\begin{cases} \dfrac{dx_i}{dt} = -\eta_i(x_i, y_i, \boldsymbol{\alpha}, t), \\ \dfrac{dy_i}{dt} = -\zeta_i(x_i, y_i, \boldsymbol{\alpha}, t), \end{cases} \tag{2.10}$$

which is equivalent to the Painlevé equation of type R.

Remark 2. One can show that the deformation corresponding to $\rho(\frac{\partial}{\partial t})$ preserves the relative rational 2-form $\omega_{\mathcal{S}}$ in (2.3). This fact explains the reason why the systems of differential equations in (2.10) can be written in Hamiltonian systems. For more details, see [STT, Section 6].

2.3. Riccati equations

Let $U \subset \mathbf{C}$ be an open complex domain (in analytic topology) with a local analytic coordinate t and $a(t), b(t), c(t)$ holomorphic functions defined in U.

Consider a Riccati equation

$$x' = a(t)x^2 + b(t)x + c(t). \tag{2.11}$$

By the change of unknown

$$x = -\frac{1}{a(t)} \frac{d}{dt} \log(u) = -\frac{1}{a(t)} \frac{u'}{u}, \tag{2.12}$$

the equation (2.11) is transformed into the linear equation

$$u'' - \left[\frac{a'(t)}{a(t)} + b(t) \right] u' + a(t)c(t)u = 0. \tag{2.13}$$

Therefore the movable singularities of the solution $x(t) = -\frac{1}{a(t)} \frac{u'}{u}$ of (2.11) are only poles. This condition is called *the Painlevé property* for an algebraic ordinary differential equation (cf. [IKSY, Ch. 3, 3.1]).

Remark 3. Riccati equations above are defined in the space $\mathbf{P}^1 \times U$ with the coordinates (x, t). The equation (2.11) is equivalent to a rational global vector field on $\mathbf{P}^1 \times U$ as

$$\tilde{v} = \frac{\partial}{\partial t} + [a(t)x^2 + b(t)x + c(t)] \frac{\partial}{\partial x}. \tag{2.14}$$

By the coordinate change $u = \frac{1}{x}$, $\tilde{v}$ can be transformed into the form

$$\tilde{v} = \frac{\partial}{\partial t} - [a(t) + b(t)u + c(t)u^2] \frac{\partial}{\partial u}.$$

This shows that the vector field $\tilde{v}$ is holomorphic even at $x = \infty$, hence $\tilde{v}$ is a global holomorphic vector field on $\mathbf{P}^1 \times U$. (Conversely, one can show that any holomorphic vector field on $\mathbf{P}^1 \times U$ which is a lift of $\frac{\partial}{\partial t}$ can be written as in (2.14).) Therefore the space $\mathbf{P}^1 \times U$ can be considered as the space of initial conditions for the Riccati equation above.

2.4. Nodal curves on Okamoto-Painlevé pairs and Riccati equations

Let (S, Y) be a rational Okamoto-Painlevé pair of type $R = R(Y)$ corresponding to Painlevé equations of type R. Then, as we see in Theorem 2.1, one can construct a global rational vector field $\tilde{v}$ on the semi-universal deformation family of (S, Y) which gives the Painlevé equation of type R.

In what follows, we will show that Painlevé equations can be reduced to the Riccati equations if and only if the corresponding rational Okamoto-Painlevé pair (S, Y) contains $\mathbf{P}^1$ on $S - Y_{red}$. Roughly speaking, we have the following correspondences.

$$\begin{array}{ccc} \boxed{\text{Painlevé equations}} & \Leftrightarrow & \boxed{\begin{array}{l}\text{Special deformations of} \\ \text{Okamoto-Painlevé pairs } (S, Y)\end{array}} \\ \cup & & \cup \\ \boxed{\text{Riccati equations}} & \Leftrightarrow & \boxed{\text{Nodal curves } C \simeq \mathbf{P}^1 \subset S - Y_{red}} \end{array} \tag{2.15}$$

In order to explain this scheme more explicitly, let us consider the Hamiltonian systems of the Painlevé equation of type $\tilde{E}_6$ $(= P_{IV})$ with two auxiliary parameters κ_0, κ_∞;

$$\begin{cases} \dfrac{dx_0}{dt} = 4x_0y_0 - x_0^2 - 2tx_0 - 2\kappa_0, \\ \dfrac{dy_0}{dt} = -2y_0^2 + 2(x_0 + t)y_0 - \kappa_\infty. \end{cases} \tag{2.16}$$

When $\kappa_0 = 0$, if we set $x_0 \equiv 0$, the first equation of the system (2.16) is automatically satisfied, and the second equation can be reduced to the equation

$$\frac{dy_0}{dt} = -2y_0^2 + 2ty_0 - \kappa_\infty, \tag{2.17}$$

which is nothing but a Riccati equation. One can easily check that $\{x_0 = 0\}$ defines a smooth $\mathbf{P}^1$ on $S - Y_{red}$ (see Section 4).

Note that if $C \subset S - Y_{red}$ is a smooth irreducible rational curve in $S - Y_{red}$, we see that $K_S \cdot C = -Y \cdot C = 0$, hence, by the adjunction formula, we have

$$C^2 = K_S \cdot C + C^2 = -2.$$

Therefore a smooth irreducible rational curve $C \subset S - Y_{red}$ is always *a nodal curve or a* (-2)*-curve.*

The following proposition gives a characterization of Riccati equations obtained from the Painlevé equations in terms of rational nodal curves on Okamoto-Painlevé pair (S, Y) (see Figure 1).

Proposition 2.1. *Under the same notation as in Theorem* 2.1*, let us consider the family* $\pi : \mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ *of the Okamoto-Painlevé pairs of type* R *in* (2.2)*.*

(1) *Assume that for a point* $t_0' = (\boldsymbol{\alpha}_0, t_0) \in \mathcal{M}_R \times \mathcal{B}_R$*, there exists a smooth rational curve* $C \subset \mathcal{S}_{(\boldsymbol{\alpha}_0, t_0)} - \mathcal{D}_{(\boldsymbol{\alpha}_0, t_0)}$*. Then there exists an* (*analytic or étale*) *open neighborhood* U *of* t_0 *of* $\mathcal{B}_R$ *satisfying the following conditions.*

(a) *There exist a flat family of rational curves* $\varphi : \mathcal{C} \longrightarrow \{\boldsymbol{\alpha}_0\} \times U$ *and an inclusion* $\iota : \mathcal{C} \hookrightarrow \mathcal{S} - \mathcal{D}|_{\{\boldsymbol{\alpha}_0\} \times U}$ *such that the following diagram is commutative*:

$$\begin{array}{ccccc} C & \hookrightarrow & \mathcal{C} & \stackrel{\iota}{\hookrightarrow} & \mathcal{S} - \mathcal{D}|_{\{\boldsymbol{\alpha}_0\}\times U} \\ \downarrow & & \varphi \downarrow & \swarrow \pi & \\ (\boldsymbol{\alpha}_0, t_0) & \in & \{\boldsymbol{\alpha}_0\} \times U & & . \end{array} \tag{2.18}$$

(b) *The restriction of the vector field* $\tilde{v} \in \Gamma(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{D}))$ *in* (2.4) *to* $\mathcal{C}$ *is tangent to* $\mathcal{C}$*, that is,*

$$\tilde{v}|_{\mathcal{C}} \in H^0(\mathcal{C}, \Theta_{\mathcal{C}}). \tag{2.19}$$

Moreover $\tilde{v}_{|\mathcal{C}}$ *defines a Riccati equation.*

(2) *Conversely, assume that the restriction of Painlevé equation* $\tilde{v}_{|\mathcal{S}'}$ *to the family* $\pi' : \mathcal{S}' := \mathcal{S}_{|\{\boldsymbol{\alpha}_0\}\times\mathcal{B}_R} \longrightarrow \{\boldsymbol{\alpha}_0\} \times \mathcal{B}_R$ *can be reduced to a Riccati equation on an open neighborhood* $\{\boldsymbol{\alpha}_0\} \times U$ *of a point* $(\boldsymbol{\alpha}_0, t_0) \in \{\boldsymbol{\alpha}_0\} \times \mathcal{B}_R$*. Then there exist a family of rational nodal curves* $\mathcal{C} \longrightarrow \{\boldsymbol{\alpha}_0\} \times U$ *on* $\pi' : \mathcal{S}' - \mathcal{D}' \longrightarrow \{\boldsymbol{\alpha}_0\} \times U$*.*

Proof. Let us set $\mathcal{B}_{\boldsymbol{\alpha}_0} = \{\boldsymbol{\alpha}_0\} \times \mathcal{B}_R \hookrightarrow \mathcal{M}_R \times \mathcal{B}_R$, $t_0' = (\boldsymbol{\alpha}_0, t_0)$. Restricting the family $\mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ to $\mathcal{B}_{\boldsymbol{\alpha}_0}$, we obtain a smooth projective family of surfaces:

$$\pi' : \mathcal{S}' := \mathcal{S}_{|\mathcal{B}_{\boldsymbol{\alpha}_0}} \longrightarrow \mathcal{B}_{\boldsymbol{\alpha}_0}.$$

Moreover, we set $S_{t'_0} = {\pi'}^{-1}(t'_0)$. Fix a relatively ample line bundle H for $\pi' : \mathcal{S}' \longrightarrow \mathcal{B}_{\boldsymbol{\alpha}_0}$. Consider the connected component T of the Hilbert scheme $\mathrm{Hilb}(\mathcal{S}'/\mathcal{B}_{\boldsymbol{\alpha}_0})$ which contains a point $[C]$ and let $\mathcal{C} \longrightarrow T$ denote the corresponding universal family. (Since π' is projective and smooth, the universal family $\tau : \mathcal{C} \longrightarrow T$ exists (cf. [Kol, Ch. 1, Theorem 1.4])).

Moreover we have a natural morphism $\phi : T \to \mathcal{B}_{\boldsymbol{\alpha}_0}$ and a natural inclusion $\iota : \mathcal{C} \hookrightarrow T \times_{\mathcal{B}_{\boldsymbol{\alpha}_0}} \mathcal{S}'$, so that $\tau : \mathcal{C} \longrightarrow T$ can be factorized into $\tau = p_1 \circ \iota$ where p_1 denotes the first projection.

Let (Q, m_Q) be the local ring of T at $[C]$. Then from [Kol, Ch. 1, Theorem 2.10], one can see the following:

(1) The $\mathcal{O}_{\mathcal{B}_{\boldsymbol{\alpha}_0}, t'_0}$ -algebra Q can be written as the quotient of a local $\mathcal{O}_{\mathcal{B}_{\boldsymbol{\alpha}_0}, t'_0}$ -algebra P, where

$$\operatorname{Spec} P \longrightarrow \mathcal{B}_{\boldsymbol{\alpha}_0}$$

is smooth of relative dimension $d = \dim H^0(C, N_{C/S_{t'_0}})$.

(2) The kernel $K = \ker[P \to Q]$ is generated by $\dim Obs(C)$ elements where $Obs(C)$ denotes the space of obstructions.

Since $C \subset S_{t'_0}$ is a (-2)-curve, we see that $N_{C/S_{t'_0}} \simeq \mathcal{O}_C(-2)$, and hence we have $H^0(C, N_{C/S_{t'_0}}) = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)) = \{0\}$. Therefore $\operatorname{Spec} P \longrightarrow \mathcal{B}_{\boldsymbol{\alpha}_0}$ is smooth of relative dimension 0. Now we claim that:

$$\boxed{\textbf{Claim: } Obs(C) = \{0\}.} \tag{2.20}$$

Assuming the claim, we see that

$$P \simeq Q \simeq \mathcal{O}_{\mathcal{B}_{\boldsymbol{\alpha}_0}, t'_0},$$

hence this implies that T is a smooth variety of dimension 1 at the point $[C]$ and the morphism $\phi : T \longrightarrow \mathcal{B}_{\boldsymbol{\alpha}_0}$ is also an isomorphism near $[C]$ (étale or analytic) locally. Hence we obtain an open neighborhood U' of $[C]$ in T on which the morphism ϕ induces the isomorphism $\phi_{|U'} : U' \xrightarrow{\simeq} \phi(U') \subset \mathcal{B}_{\boldsymbol{\alpha}_0}$. It is clear that $U' = \boldsymbol{\alpha}_0 \times U$ for some open neighborhood of t_0 in $\mathcal{B}_R$ and the restriction of the family $\mathcal{C} \longrightarrow T$ to U' gives a family of rational curves $\mathcal{C} \longrightarrow \{\boldsymbol{\alpha}_0\} \times U$ which is a deformation of the rational curve C in $S_{t'_0}$.

Now we show the claim (2.20).

From [Kol, Ch. 1, Proposition 2.14], one see that the space of the obstructions $Obs(C)$ lies in $H^1(C, N_{C/S_{t'_0}})$. Consider the natural homomorphisms of cohomology groups

$$H^1(S_{t'_0}, \Theta_{S_{t'_0}}) \xrightarrow{\nu} H^1(C, \Theta_{S_{t'_0}|C}) \xrightarrow{\mu} H^1(C, N_{C/S_{t'_0}}).$$

Combining the Kodaira-Spencer homomorphism $\rho : T_{\mathcal{B}_{\boldsymbol{\alpha}_0}, t'_0} \longrightarrow H^1(S_{t'_0}, \Theta_{S_{t'_0}})$, it is easy to see that

$$Obs(C) = \mu \circ \nu \circ \rho(T_{\mathcal{B}_{\boldsymbol{\alpha}_0}, t'_0}). \tag{2.21}$$

For simplicity, we set $S = S_{t'_0}, Y = \mathcal{Y}_{t'_0}, D = Y_{red} = \sum_{i=1}^{r} Y_i$.

Since $C \subset S - D$, we see that

$$\Theta_S(-\log(D+C))_{|D} \simeq \Theta_S(-\log D)_{|D}.$$

Therefore we have the following exact sequence

$$\begin{aligned}0 \longrightarrow \Theta_S(-\log(D+C)) \longrightarrow \Theta_S(-\log(D+C))(D) \\ \longrightarrow \Theta_S(-\log D) \otimes N_{D/S} \longrightarrow 0.\end{aligned}$$

For an Okamoto-Painlevé pair (S, Y) of non-fibered type, we have (cf. [STT, Proposition 2.1]) $H^0(S, \Theta_S(-\log(D+C))(D)) = \{0\}$. Hence, this gives an injective homomorphism

$$0 \to H^0(D, \Theta_S(-\log D) \otimes N_{D/S}) \hookrightarrow H^1(S, \Theta_S(-\log(D+C)). \tag{2.22}$$

We also have the following commutative diagram of sheaves (cf. [STT, Lemma 2.1]):

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & & \Theta_S(-\log(D+C)) & \longrightarrow & \Theta_S(-\log D) & \longrightarrow & N_{C/S} & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & & \Theta_S(-\log C) & \longrightarrow & \Theta_S & \longrightarrow & N_{C/S} & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & & \oplus_{i=1}^r N_{Y_i/S} & \longrightarrow & \oplus_{i=1}^r N_{Y_i/S} & \longrightarrow & 0 & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ & & 0 & & 0 & & 0 & . \end{array}$$

Since $N_{Y_i/S} = \mathcal{O}_{Y_i}(-2)$ and $N_{C/S} = \mathcal{O}_C(-2)$, we have the inclusions

$$H^1(S, \Theta_S(-\log(D+C))) \hookrightarrow H^1(S, \Theta_S(-\log C)) \hookrightarrow H^1(S, \Theta_S).$$

Combining this and (2.22), we see that

(2.23)
$$\begin{array}{ccc} H^0(D, \Theta_S(-\log D) \otimes N_{D/S}) & \hookrightarrow & H^1(S, \Theta_S(-\log(D+C))) \\ & & \cap \\ & & \ker[\mu \circ \nu : H^1(S, \Theta_S) \longrightarrow H^1(C, N_{C/S})]. \end{array}$$

From (2.6), we have

$$\rho(T_{\mathcal{B}_{\boldsymbol{\alpha}_0}, t_0'}) \simeq H^0(D, \Theta_S(-\log D) \otimes N_{D/S}),$$

and hence

$$\mu \circ \nu \circ \rho(T_{\mathcal{B}_{\boldsymbol{\alpha}_0}, t_0'}) = \{0\}.$$

Together with (2.21), this shows the claim (2.20).

Next, let us consider the family

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{S}'_{|U} \\ & \searrow & \downarrow \pi \\ & & U. \end{array} \tag{2.24}$$

Since $\mathcal{D} \cap \mathcal{C} = \emptyset$, we have $\Theta_{\mathcal{S}'|\mathcal{C}} = \Theta_{\mathcal{S}'}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}'}(\mathcal{D})_{|\mathcal{C}}$, and hence we obtain the following exact sequence:

$$0 \longrightarrow \Theta_{\mathcal{C}} \longrightarrow \Theta_{\mathcal{S}'}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}'}(\mathcal{D})_{|\mathcal{C}} \longrightarrow N_{\mathcal{C}/\mathcal{S}'} \longrightarrow 0.$$

Since $N_{\mathcal{C}/\mathcal{S}'|\mathcal{S}'_t} = N_{\mathcal{C}_t/\mathcal{S}'_t} = \mathcal{O}_{\mathcal{C}_t}(-2)$, we can show that $\pi_*(N_{\mathcal{C}/\mathcal{S}'}) = \{0\}$. Then we have $\Gamma(\mathcal{C}, N_{\mathcal{C}/\mathcal{S}'}) = \{0\}$. This implies that

$$H^0(\mathcal{C}, \Theta_{\mathcal{C}}) \simeq H^0(\mathcal{S}', \Theta_{\mathcal{S}'}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}'}(\mathcal{D})_{|\mathcal{C}}).$$

Hence $\tilde{v}_{|\mathcal{C}} \in H^0(\mathcal{C}, \Theta_{\mathcal{C}})$.

Moreover, we may assume that $\mathcal{C} \longrightarrow U$ is a trivial $\mathbf{P}^1$-bundle, that is, $\mathcal{C} \simeq \mathbf{P}^1 \times U$ analytically. Since $\tilde{v}_{|\mathcal{C}}$ defines a holomorphic vector field on $\mathbf{P}^1 \times U$, it is easy to see that $\tilde{v}_{|\mathcal{C}}$ is equivalent to a Riccati equation (cf. Remark 3).

The second assertion is now obvious, because the space of initial conditions of a Riccati equation must be a family of $\mathbf{P}^1$ (cf. Remark 3). □

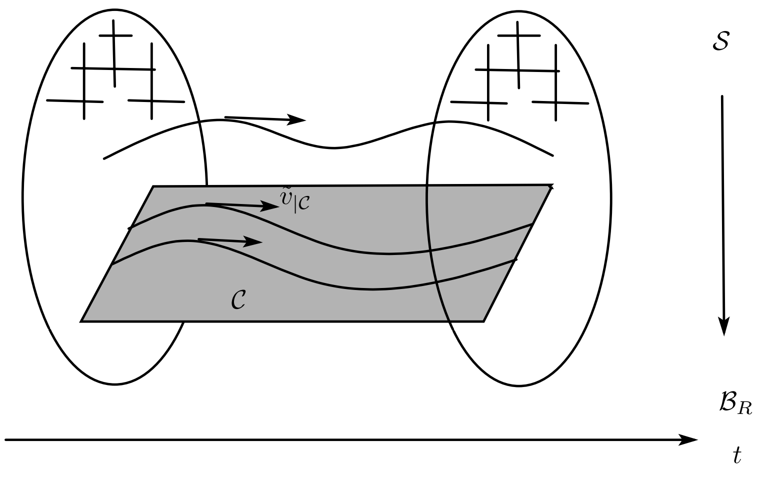

Figure 1. Nodal curves and Riccati equations for $\tilde{E}_6$ (P_{IV})

Remark 4 (Global deformation of a (-2)-curve C)**.** Let us consider the connected component T of the Hilbert scheme $\mathrm{Hilb}(\mathcal{S}'/\mathcal{B}_{\boldsymbol{\alpha}_0})$ which contains a point $[C]$ and the corresponding universal family $\tau : \mathcal{C} \longrightarrow T$ in the proof of Proposition 2.1. The argument in the proof shows that $\dim T = 1$ and the natural morphism

$$\phi : T \longrightarrow \mathcal{B}_{\boldsymbol{\alpha}_0}$$

is projective, and hence surjective. We see that ϕ is a finite morphism of degree $d \geq 1$. Assume that ϕ is an isomorphism, i.e., $d = 1$. Then we have the global family of rational curves $\mathcal{C} \subset \mathcal{S}'$ over the affine curve $\mathcal{B}_{\boldsymbol{\alpha}_0}$:

$$\begin{array}{ccccc} C & \subset & \mathcal{C} & \hookrightarrow & \mathcal{S}' \\ \downarrow & & \downarrow & & \downarrow \\ t'_{\mathbf{0}} & \in & T & = & \mathcal{B}_{\boldsymbol{\alpha}_0} \end{array} . \tag{2.25}$$

Then the vector field $\tilde{v}_{|\mathcal{C}}$ becomes an algebraic regular vector field on $\mathcal{C}$ and defines a Riccati equation over the affine algebraic curve $\mathcal{B}_{\boldsymbol{\alpha}_0}$. In this case, we call the differential equation defined by $\tilde{v}_{|\mathcal{C}}$ the *Riccati equation* associated to the rational curve $C(\subset S - D)$.

We do not know whether the case with $d > 1$ occurs or does not occur. However, if $\phi : T \longrightarrow \mathcal{B}_{\boldsymbol{\alpha}_0}$ is of degree $d > 1$, we see that $\phi^{-1}(\phi([C]))$ consists of d rational curves of $\mathcal{S}_{t'_0}$ $C_1 := C, C_2, \ldots, C_d$ which are in the flat family of rational curves in $\mathcal{S}'$ parametrized by a connected variety T.

In Section 2, we see that there exists an Okamoto-Painlevé pair (S, Y) which contains more than one rational curves $C_i \subset S - D$, $i \geq 2$.

Definition 2.3. Under the same notation and assumptions in Proposition 2.1, we call the differential equations determined by the vector field $\tilde{v}|_{\mathcal{C}}$ in (2.14) *Riccati equation associated with the rational curve* $C \subset S - Y_{red}$. Moreover we call a solution of the Riccati equation $\tilde{v}|_{\mathcal{C}}$ *a Riccati solution of the Painlevé system* (*associated with* $C \subset S - Y_{red}$). (Note that all solutions of $\tilde{v}|_{\mathcal{C}}$ remain in the family of rational curves in (2.18).)

3. Classification of (-2)-rational curves (nodal rational curves) on $S - D$

Let (S, Y) be a rational Okamoto-Painlevé pair of non-fibered type which corresponds to a Painlevé equation (cf. Table 1).

In this section, we will classify all configurations of (-2)-curves on $S - D$ for a rational Okamoto-Painlevé pair (S, Y) of non-fibered type. The classification of the configurations are based on the similar classification for rational Okamoto-Painlevé pairs (S, Y) of fibered type with the elliptic fibration $f : S \longrightarrow \mathbf{P}^1$ and some deformation arguments.

3.1. Notations and the Result

Let S be a projective smooth surface over $\mathbf{C}$. We denote by $Div(S)$ the free abelian group generated by all irreducible curves on S. Let $\sim_a$ and $\sim$ denote the algebraic equivalence and the linear equivalence of divisors respectively. We define the Néron-Severi group and the Picard group of S by

$$\mathrm{NS}(S) = Div(S)/\sim_a, \tag{3.1}$$

$$\mathrm{Pic}(S) = Div(S)/\sim . \tag{3.2}$$

In what follows, we assume that S is a rational surface. Then we have the natural isomorphisms

$$\mathrm{Pic}(S) \simeq \mathrm{NS}(S) \simeq H^2(S, \mathbf{Z}), \tag{3.3}$$

and these groups are free $\mathbf{Z}$-modules of rank $b_2(S)$. For any divisor C, we also denote by the same letter C the class of the divisor in $\mathrm{NS}(S) \simeq H^2(S, \mathbf{Z})$. Moreover $C = D$ means that the two divisors are linear equivalent to each other. We can consider the lattice structure on these free $\mathbf{Z}$-modules by the intersection form $<\ ,\ >$ on $\mathrm{NS}(S)$ or equivalently by the cup product on $H^2(S, \mathbf{Z})$. Let E_8 be the unique even unimodular positive-definite lattice of rank 8. For a lattice $L = (L, <\ ,\ >)$, we denote by $L^- = (L, (-1)\times <\ ,\ >)$, the opposite lattice of L. Note that the opposite lattice E_8^- of E_8 is negative-definite.

Let (S, Y) be a rational Okamoto-Painlevé pair and let

$$Y = \sum_{i=1}^{r} m_i Y_i \tag{3.4}$$

be the irreducible decomposition of Y. Since S is a rational surface with $b_2(S) = \operatorname{rank} H^2(S, \mathbf{Z}) = 10$, by the Hodge index theorem, the bilinear form $<\ ,\ >$ on $H^2(S, \mathbf{Z})$ can be written as the diagonal matrix $(1, \underbrace{-1, \ldots, -1}_{9})$. The sub-lattice $M(Y)$ generated by $\{Y_i\}_{i=1}^r$ in $H^2(S, \mathbf{Z})$ is a root lattice of an affine type, say $R = R(Y)$. Since S is not relatively minimal, S contains a (-1)-rational curve O on S. Then by the adjunction formula, one has $Y \cdot O = -K_S \cdot O = 1$. Hence, there exists a $i_0, 1 \leq i_0 \leq r$ such that $m_{i_0} = 1$ and $Y_{i_0} \cdot O = 1$. By renumbering i, we may assume that $i_0 = 1$. Define the sub-lattice by

$$M'(Y) = \langle Y_2, \ldots, Y_r \rangle_{\mathbf{Z}} \subset M(Y), \tag{3.5}$$

which is a root lattice of classical type R'. For example, if $R = \tilde{D}_4$, then $R' = D_4$. Let $M(S - Y_{red})$ be the sub-lattice $H^2(S, \mathbf{Z})$ generated by all (-2)-curves C on $S - Y$. Note that we have the orthogonal sum

$$M'(Y) \oplus M(S - Y_{red}) \subset H^2(S, \mathbf{Z}). \tag{3.6}$$

Lemma 3.1. *Assume that (S, Y) is of non-fibered type. Then $M'(Y) \oplus M(S - Y_{red})$ is a root sub-lattice of E_8^-.*

Proof. The sub-lattice $\langle Y, O \rangle_{\mathbf{Z}}$ generated by Y and O has the intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

Then the orthogonal complement $\langle Y, O \rangle^{\perp}$ in $H^2(S, \mathbf{Z})$ is an even, negative-definite unimodular lattice of rank 8, which is isomorphic to the root lattice E_8^-. (Since $K_S = -Y$, the adjunction formula implies that $\langle Y, O \rangle^{\perp}$ is even.) Since $Y \cdot O = 1$, we see that the orthogonal complement $\langle Y \rangle^{\perp}$ is given by

$$\langle Y \rangle^{\perp} \simeq \langle Y, O \rangle^{\perp} \oplus \mathbf{Z}Y \simeq \tilde{E}_8^-.$$

Since $M(S-Y_{red})$ is generated by (-2)-curves on $S-Y_{red}$, we see that $M(S-Y_{red}) \subset \langle Y \rangle^{\perp}$. Moreover by definition of Okamoto-Painlevé pair (cf. (2.1)), $M'(Y) \subset \langle Y \rangle^{\perp}$. (In fact, we have $M'(Y) \subset \langle Y, O \rangle^{\perp}$.) Set

$$N(Y) := M'(Y) \oplus M(S-Y_{red}). \tag{3.7}$$

Then $N(Y) \subset \langle Y \rangle^{\perp}$.

Let us consider the natural projection map

$$\pi : \langle Y \rangle^{\perp} \simeq \langle Y, O \rangle^{\perp} \oplus \mathbf{Z}Y \longrightarrow \langle Y, O \rangle^{\perp}.$$

We claim that:

$$\textbf{Claim}: \quad \pi_{|N(Y)} \quad \text{is injective.} \tag{3.8}$$

If the claim is true, we see that $N(Y) \simeq \pi(N(Y)) \subset \langle Y, O \rangle^{\perp} \simeq E_8^-$. This implies that $N(Y)$ is a negative-definite lattice generated by (-2)-elements. Hence one can see that $N(Y)$ is a root lattice which is a direct sum of root lattices of type A_i, D_j, E_k. (This also implies that $M'(Y)$ and $M(S-Y_{red})$ are direct sums of root lattices of type A_i, D_j, E_k.) To show the claim (3.8), it suffices to show that $\operatorname{Ker} \pi_{|N(Y)} = \operatorname{Ker} \pi \cap N(Y) = \{0\}$. Since $\operatorname{Ker} \pi = \mathbf{Z}[Y]$ with $Y^2 = 0$ and $M'(Y)$ is negative-definite, we have

$$\begin{aligned} \operatorname{Ker} \pi \cap N(Y) &= \mathbf{Z}[Y] \cap N(Y) = \mathbf{Z}[Y] \cap (M'(Y) \oplus M(S-Y_{red})) \\ &= \mathbf{Z}[Y] \cap M(S-Y_{red}). \end{aligned}$$

Hence we have to show that $\mathbf{Z}[Y] \cap M(S-Y_{red}) = \{0\}$. Take $\gamma \in \operatorname{Ker} \pi_{|M(S-Y_{red})}$ and assume that $\gamma \neq 0$. Since $\operatorname{Ker} \pi = \mathbf{Z} \cdot Y$, we can write γ as $\gamma = b \cdot Y$ with $b \neq 0$. We may assume that $b > 0$. On the other hand, since $\gamma \in M(S-Y_{red})$, we can write γ as

$$\gamma = C - D$$

with

$$C = \sum_{i=1}^{l} a_i C_i, \quad D = \sum_{j=1}^{t} b_j D_j,$$

where C_i $(1 \leq i \leq l)$ and D_j $(1 \leq j \leq t)$ are different (-2)-curves in $S-Y_{red}$ and $a_i \geq 0, b_j \geq 0$. Assume that $D = 0$. Then we see that bY and C are linear equivalent to each other. Since bY and C are different effective divisors, we see that $\dim H^0(S, \mathcal{O}_S(bY)) \geq 2$. This contradicts to the fact that (S, Y) is of non-fibered type (cf. [STT, Proposition 1.3]). Therefore we may assume that both of C and D are non-zero effective divisors. Recall that the lattice $\langle Y \rangle^{\perp}$ is negative semi-definite. Hence one has

$$0 \geq C^2 = (D + bY)^2 = D^2 = D \cdot C \geq 0.$$

(Here we used the fact that $D \cdot Y = C \cdot Y = 0$.) This implies that

$$C^2 = D^2 = C \cdot D = 0.$$

An element $G \in \langle Y \rangle^{\perp}$ with $G^2 = 0$ must be proportional to Y, that is, $G = cY$. Therefore we see that $C = b'Y$ with $b' > 0$, which again contradicts to the fact that (S, Y) is of non-fibered type. We have proved that $\mathrm{Ker}\, \pi_{|M(S-Y_{red})} = \{0\}$ and hence $\mathrm{Ker}\, \pi_{|N(Y)} = \{0\}$ as in (3.8). $\square$

By Lemma 3.1, there are only finitely many (-2) curves $\{C_i\}_{i=1}^{l}$ on $S - Y_{red}$. The dual graph of configurations of (-2)-curves on S can be classified by the Dynkin diagram of ADE types. The following theorem is the main theorem in this section.

Theorem 3.1. *Let (S, Y) be a rational Okamoto-Painlevé pair of non-fibered type which corresponds to a Painlevé equation (cf. Table 1). The type of the root lattice $M(S - Y_{red})$, or equivalently, the dual graph of the configuration of (-2)-curves on $S - Y$ are classified in Table 2.*

Painlevé equations	$R(Y)$	the type of the dual graph of configuration of (-2)-curves on $S - Y$
P_{VI}	$\tilde{D}_4$	D_4 , (A_1, A_1, A_1, A_1), A_3 (A_1, A_1, A_1), A_2, (A_1, A_1), A_1
P_V	$\tilde{D}_5$	A_3, A_2, (A_1, A_1), A_1
$P_{III}^{\tilde{D}_6}$	$\tilde{D}_6$	(A_1, A_1), A_1
$P_{III}^{\tilde{D}_7}$	$\tilde{D}_7$	none
$P_{III}^{\tilde{D}_6}$	$\tilde{D}_8$	none
P_{IV}	$\tilde{E}_6$	A_2, A_1
P_{II}	$\tilde{E}_7$	A_1
P_I	$\tilde{E}_8$	none

Table 2. Configuration of (-2)-curves on $S - Y$ for a rational Okamoto-Painlevé pair (S, Y) of non-fibered type

3.2. The case of fibered type

Oguiso and Shioda [O-S] give the complete structure theorem of the Mordell-Weil group of rational elliptic surfaces $f : S \longrightarrow \mathbf{P}^1$ with a section. Let (S, Y) be a rational Okamoto-Painlevé pair of fibered type, i.e. there exists an elliptic fibration $f : S \to \mathbf{P}^1$ such that $f^*(\infty) = Y$. Since $K_S = f^*(-\infty) = -Y$, by the adjunction formula, it is easy to check that an irreducible curve C is a (-2)-curve if and only if it is one of the irreducible components of the reducible singular fibers. Hence, to give the complete structure of (-2)-curves on $S - Y$, we quote the structure of the reducible singular fibers which is a part of the structure theorem of the Mordell-Weil group of $f : S \longrightarrow \mathbf{P}^1$.

We will introduce some notations. Let (S, Y) be a rational Okamoto-Painlevé pairs of fibered type with an elliptic fibration $f : S \longrightarrow \mathbf{P}^1$ such that $f^*(\infty) = Y$. (Here, we do not assume that the type of Y is in Table 1.) We also assume that there exists a section $O \subset S$ and we denote by F the class of a general fiber of f so that Y and F are linearly equivalent to each other, or equivalently, have the same class in $H^2(S, \mathbf{Z})$. For a lattice L, let us denote by L^- the opposite lattice of L, i.e.,

$$L^- = \text{the module } L \text{ with the pairing } (-1) \times < \, , \, > .$$

Let $F_v := f^{-1}(v)$ denote the fiber over the closed point $v \in \mathbf{P}^1$, and set

$$Sing(f) := \{v \in \mathbf{P}^1 | F_v = f^{-1}(v) \text{ is singular } \},$$
$$\mathcal{R} = Red(f) := \{v \in \mathbf{P}^1 | F_v = f^{-1}(v) \text{ is reducible } \}.$$

For each $v \in \mathcal{R}$, let

$$F_v = f^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} \mu_{v,i} \Theta_{v,i} \quad (\mu_{v,i} \geq 1, \mu_{v,0} = 1)$$

be the irreducible decomposition of F_v where $\Theta_{v,0}$ is the unique component of F_v meeting the zero section O and m_v is the number of irreducible components. We set

$$T_v := \langle \Theta_{v,i} | 1 \leq i \leq m_v - 1 \rangle_{\mathbf{Z}} \subset \mathrm{NS}(S), \tag{3.9}$$

and

$$T := \bigoplus_{v \in \mathcal{R}} T_v. \tag{3.10}$$

Note that the notation T is used for another lattice in [Shi].

By the classification of singular fibers (cf. [Kod]), (and using the intersection matrix $(\Theta_{v,i} \cdot \Theta_{v,j})_{1 \leq i,j \leq m_v - 1}$), we have the following

Lemma 3.2 ([Shi, Lemma 7.2]). *The opposite lattice T_v^- is a root lattice of rank $m_v - 1$, determined by the type of the singular fiber F_v as follows*:

Type of F_v	I_m	I_m^*	II^*	III^*	IV^*	IV	III
T_v^-	A_{m-1}	D_{m+4}	E_8	E_7	E_6	A_2	A_1

Furthermore, we have (cf. [Shi, (7.2)])

$$\langle O, F, \Theta_{v,i} \ (0 \leq i \leq m_v - 1, v \in \mathcal{R}) \rangle_{\mathbf{Z}} = \langle O, F \rangle_{\mathbf{Z}} \oplus T \subset \mathrm{NS}(S)$$
$$\text{(orthogonal direct sum)}$$

where F is the class of a fiber of f. As we see in the previous subsection, we see that $\langle O, F \rangle^{\perp} \simeq E_8^-$.

Hence we have an embedding

$$T^- = \bigoplus_{v \in \mathcal{R}} T_v^- \hookrightarrow E_8. \tag{3.11}$$

Now we recall Dynkin's results on the classification of root lattices contained in E_8, which is equivalent to the classification of regular semisimple subalgebras of the exceptional Lie algebra of type E_8.

Theorem 3.2 ([D, Ch. II, Table 11]). *Let L be a root lattice of* rank s *which is embedded as a sub-lattice of E_8, other than $\{0\}$ and E_8. Then L is isomorphic to one in Table 3.*

s	L
8	$A_8,\ D_8,\ A_7 \oplus A_1,\ A_5 \oplus A_2 \oplus A_1,\ A_4^{\oplus 2},\ A_2^{\oplus 4},\ E_6 \oplus A_2,\ E_7 \oplus A_1$ $D_6 \oplus A_1^{\oplus 2},\ D_5 \oplus A_3,\ D_4^{\oplus 2},\ D_4 \oplus A_1^{\oplus 4},\ A_3^{\oplus 2} \oplus A_1^{\oplus 2},\ A_1^{\oplus 8}$
7	$A_6 \oplus A_1,\ A_4 \oplus A_2 \oplus A_1,\ A_5 \oplus A_2,\ A_2^{\oplus 3} \oplus A_1,\ E_6 \oplus A_1,\ E_7,\ D_7,$ $D_5 \oplus A_1^{\oplus 2},\ D_4 \oplus A_1^{\oplus 3},\ A_3^{\oplus 2} \oplus A_1,\ A_1^{\oplus 7},\ D_6 \oplus A_1, D_5 \oplus A_2,$ $A_3 \oplus A_2 \oplus A_1^{\oplus 2},\ D_4 \oplus A_3,\ A_3 \oplus A_1^{\oplus 4}, A_4 \oplus A_3,\ A_5 \oplus A_1^{\oplus 2},\ A_7$
6	$A_2^{\oplus 3},\ E_6,\ D_6,\ D_4 \oplus A_1^{\oplus 2},\ A_3^{\oplus 2},\ D_5 \oplus A_1,\ A_3 \oplus A_1^{\oplus 3},\ D_4 \oplus A_2,$ $A_1^{\oplus 6}, A_2 \oplus A_1^{\oplus 4},\ A_4 \oplus A_1^{\oplus 2},\ A_6,\ A_3 \oplus A_2 \oplus A_1,\ A_5 \oplus A_1,\ A_4 \oplus A_2$ $A_2^{\oplus 2} \oplus A_1^{\oplus 2}$
5	$D_5,\ A_3 \oplus A_1^{\oplus 2},\ A_3 \oplus A_2,\ A_5\ , A_1^{\oplus 5},\ A_4 \oplus A_1,\ D_4 \oplus A_1$ $A_2 \oplus A_1^{\oplus 3},\ A_2^{\oplus 2} \oplus A_1$
4	$D_4,\ , A_1^{\oplus 4},\ A_2 \oplus A_1^{\oplus 2},\ A_2^{\oplus 2},\ A_3 \oplus A_1,\ A_4$
3	$A_3,\ A_2 \oplus A_1,\ A_1^{\oplus 3}$
2	$A_2,\ A_1^{\oplus 2}$
1	A_1

Table 3. Root sub-lattices of E_8

From Theorem 3.2, one can classify the root sub-lattice of E_8, hence T must be one of the root lattices in the Table 3.

However, as for the existence, we quote the following

Theorem 3.3 (cf. [O-S, Remark 2.7]). *For every type given in Table 3 except for the type*

$$D_4 \oplus A_1^{\oplus 4},\ A_1^{\oplus 8} \text{ and } A_1^{\oplus 7},$$

there exists a rational elliptic surface whose T^- is of given type.

Remark 5 (cf. [O-S, Remark 3.4]). The sum of the local Euler number of the reducible singular fibers cannot exceed 12, the Euler number of a rational elliptic surface. Therefore, the types $D_4 \oplus A_1^{\oplus 4},\ A_1^{\oplus 8}$ and $A_1^{\oplus 7}$ do not appear.

In the case of a rational Okamoto-Painlevé pair (S, Y) of fibered type in Table 1, the type of root lattice T_∞ is determined by the type of Y. Thus, we obtain the classification theorem as follows.

Proposition 3.1. *Let (S, Y) be a rational Okamoto-Painlevé pair of fibered type in Table 1. The type of root lattice $\bigoplus_{v \in \mathcal{R} - \infty} T_v^-$ are classified by Table 4.*

Type of Y	$\bigoplus_{v\in\mathcal{R}-\infty} T_v^-$
$\tilde{D}_4 = I_0^*$	$D_4, \quad A_3, \quad A_1^{\oplus 3}, \quad A_2, \quad A_1^{\oplus 2}, \quad A_1$
$\tilde{D}_5 = I_1^*$	$A_3, \quad A_2, \quad A_1^{\oplus 2}, \quad A_1$
$\tilde{D}_6 = I_2^*$	$A_1^{\oplus 2}, \quad A_1$
$\tilde{D}_7 = I_3^*$	none
$\tilde{D}_8 = I_4^*$	none
$\tilde{E}_6 = IV^*$	$A_2, \quad A_1$
$\tilde{E}_7 = III^*$	A_1
$\tilde{E}_8 = II^*$	none

Table 4. The list of root lattices $\bigoplus_{v\in\mathcal{R}-\infty} T_v^-$ (fibered type)

Remark 6. By Lemma 3.2 and Proposition 3.1, the structure of configuration of (-2)-curves (i.e. type of F_v's) is 'almost' determined. For $T_v = A_1$, the type of F_v cannot be distinguished between I_2 and III. Similarly, for $T_v = A_2$, the type of F_v cannot be distinguished between I_2 and IV. (For other types, we can determine the type of F_v.) In the case of $Y = \tilde{D}_4 = I_0^*$, one has the root lattice $T_v^- \simeq D_4$. In this case the corresponding fiber F_v is of type $\tilde{D}_4 = I_0^*$. Note that there is a sub-lattice $A_1^{\oplus 4}$ (or the configuration of (-2)-curves (A_1, A_1, A_1, A_1)) in $\tilde{D}_4$. This sub-lattice gives the configuration of type (A_1, A_1, A_1, A_1) after the deformation from fibered type to non-fibered type (see Figure 2).

3.3. Proof of Theorem 3.1

Now we prove Theorem 3.1. Let (S, Y) be a rational Okamoto-Painlevé pair of non-fibered type with a given type $R = R(Y)$ of Y in the Table 1. Let $M'(Y)$ and $M(S - Y_{red})$ be the sub-lattices defined in (3.6). By Lemma 3.1, the orthogonal sum $M'(Y)^- \oplus M(S - Y_{red})^-$ is a root sub-lattice of E_8. Then since the type $R'(Y)$ of $M'(Y)^-$ is $D_k, 4 \leq k \leq 8$ or E_6, E_7, E_8, by the Classification Theorem 3.2, we can obtain the list of possible types for $M(S - Y_{red})^-$ as in Table 2. Therefore, it suffices to show for each type R'' of root lattices listed in Table 2, there exsits a rational Okamoto-Painlevé pair (S, Y) of non-fibered types with the root sub-lattice $M(S - Y_{red})$ of type R''.

First, let (S, Y) be a rational Okamoto-Painlevé pair (S, Y) of fibered type with a given type of Y in the Table 1 and let $f : S \longrightarrow \mathbf{P}^1$ be the elliptic fibration with $f^*(\infty) = Y$. From Proposition 3.1, we can determine the possible configuration of (-2)-curves on $S - Y_{red} = S - f^{-1}(\infty)$ by the classification of the other reducible singular fibers. (Note that Proposition 3.1 says the existence of such a fibration.) Let $Y = \sum_{i=1}^r m_i Y_i$ be the irreducible decomposition of Y. Set $D = Y_{red} = \sum_{i=1}^r Y_i$, and take all (-2) curves $\{C_1, \dots, C_l\}$ on $S - Y_{red}$. Note that each C_i is an irreducible component of reducible singular fibers of f.

Now we will use the following deformation argument.

Lemma 3.3. *Let (S, Y) be a rational Okamoto-Painlevé pair of fibered type with the irreducible decomposition $Y = \sum_{i=1}^{r} m_i Y_i$ such that $D = Y_{red}$ is a normal crossing divisor, and let $C = \sum_{j=1}^{s} C_j$ be a normal crossing divisor of S satisfying the following conditions*:

(1) $C \subset S - D$,

(2) $C_j \simeq \mathbf{P}^1$,

(3) *The classes of curves $\{Y_i, C_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ are linearly independent in $H^2(S, \mathbf{C}) \simeq \mathrm{Pic}(S) \otimes_{\mathbf{Z}} \mathbf{C}$.*

Then there exists a rational Okamoto-Painlevé pair (S', Y') such that

(1) *(S', Y') is of non-fibered type,*

(2) *the type of Y' is same as the type of Y,*

(3) *$S' - Y'_{red}$ contains (-2) curves $\{C'_j\}_{j=1}^{s}$ with the same configurations as $\{C_j\}_{j=1}^{s}$, and*

(4) *S' is a deformation of S.*

Proof. Let F be an arbitrary fiber at $\mathbf{P}^1 - \{\infty\} - Sing(f)$, which is an elliptic curve and $F \subset S - (D + C)$. In Lemma 3.4, we will show

$$H^2(\Theta_S(-\log(D + C + F))) = \{0\}. \tag{3.12}$$

By (3.12), the exact sequence of sheaves

$$0 \to \Theta_S(-\log(D + C + F)) \to \Theta_S(-\log(D + C)) \to N_F \to 0,$$

yields the exact sequence

$$H^1(\Theta_S(-\log(D+C+F))) \to H^1(\Theta_S(-\log(D+C))) \xrightarrow{\phi} H^1(N_F) \to 0. \tag{3.13}$$

Since F and Y are linearly equivalent, we get $N_F = [F]_{|F} = [Y]_{|F} = \mathcal{O}_F$, and hence $H^1(N_F) = H^1(\mathcal{O}_F) = \mathbf{C}$. From (3.13) together with (3.12), we see that there exists an element $\theta \in H^1(\Theta_S(-\log(D+C)))$ such that $\phi(\theta) \neq 0$. Such an element θ induces an infinitesimal deformation of the pair $(S, D+C)$ which does not preserve the elliptic curve F. Since we see that $H^2(S, \Theta_S(-\log(D+C))) = \{0\}$, such an infinitesimal deformation θ induces a one parameter deformation

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} + \mathcal{C} \\ \varphi \downarrow & \swarrow & \\ \Delta & & \end{array}$$

of $(S, D + C)$ where $\Delta = \{z \in \mathbf{C} \mid |z| < \epsilon\}$ is a small neighborhood of the origin. Note that we also have the relative divisor $\mathcal{Y}_i$ for φ which gives the deformation of Y_i. Hence we have the relative divisor $\mathcal{Y} = \sum_{i=1}^{r} m_i \mathcal{Y}_i$. For $z \in \Delta$, denote by $\mathcal{S}_z, \mathcal{Y}_{i,z}, \mathcal{D}_z, \mathcal{C}_z, \mathcal{Y}_z$ the corresponding fibers of $\mathcal{S}, \mathcal{Y}_i, \mathcal{D}, \mathcal{C}$ and $\mathcal{Y}$ over z respectively. It is obvious that for every $z \in \Delta$ each $\mathcal{Y}_{i,z}$ is a (-2)-curve on $\mathcal{S}_z$ and $\mathcal{Y}_z$ satisfies the numerical condition (2.1) that $\mathcal{Y}_z \cdot \mathcal{Y}_{i,z} = 0$ for all i.

Consider the divisor $K_{\mathcal{S}} + \mathcal{Y}$ on $\mathcal{S}$ and set $\mathcal{L} = \mathcal{O}_{\mathcal{S}}(K_{\mathcal{S}} + \mathcal{Y})$. We know the following two facts:

(1) $\mathcal{L}_{|\mathcal{S}_0} \sim \mathcal{O}_{\mathcal{S}_0}$.

(2) Since $\mathcal{S}_z$ is a projective smooth rational surface for every $z \in \Delta$, we see that $H^i(\mathcal{S}_z, \mathcal{O}_{\mathcal{S}_z}) = 0$ for $i \geq 1$ and every $z \in \Delta$. In particular, $R^i\pi_*\mathcal{O}_{\mathcal{S}} = 0$ for $i \geq 1$.

Then by the upper-semicontinuity theorem, we see that $\dim H^i(\mathcal{S}_z, \mathcal{L}_{|\mathcal{S}_z}) = 0$ for every $i \geq 1$ and $z \in \Delta$. Noting that $\pi_*\mathcal{L} \simeq \mathcal{O}_\Delta$, we see that there is a non-trivial homomorphism $s : \pi^*(\pi_*\mathcal{L}) = \mathcal{O}_{\mathcal{S}} \to \mathcal{L}$. Applying the same argument for the dual sheaf $\mathcal{L}^\vee$, we also have a non-trivial homomorphism $s' : \mathcal{O}_{\mathcal{S}} \to \mathcal{L}^\vee$. Then we conclude that $\mathcal{O}_{\mathcal{S}}(K_{\mathcal{S}}+\mathcal{Y}) = \mathcal{L} \simeq \mathcal{O}_{\mathcal{S}}$. Therefore we see that $K_{\mathcal{S}} = -\mathcal{Y}$ and hence $K_{\mathcal{S}_z} \sim -\mathcal{Y}_z$ for every $z \in \Delta$. This implies that $(\mathcal{S}_z, \mathcal{Y}_z)$ is a rational Okamoto-Painlevé pair for $z \in \Delta$. Next we claim that there exists $z \in \Delta - \{0\}$ such that $\dim H^0(\mathcal{S}_z, \mathcal{Y}_z) = 1$. This also implies that $(\mathcal{S}_z, \mathcal{Y}_z)$ is of non-fibered type. If $\dim H^0(\mathcal{S}_z, \mathcal{Y}_z) \geq 2$ for every $z \in \Delta$, we can show that there exists an elliptic fibration $f_z : \mathcal{S}_z \longrightarrow \mathbf{P}^1$ with $f_z^*(\infty) = \mathcal{Y}_z$ which is a deformation of the original elliptic fibration $f : \mathcal{S}_0 \longrightarrow \mathbf{P}^1$. Since the general fiber F of f does not extend over $z \in \Delta - \{0\}$, this deduces the contradiction. Note that the type of $(\mathcal{S}_z, \mathcal{Y}_z)$ is same as the type of $(\mathcal{S}_0, \mathcal{Y}_0) = (S, Y)$ and $\mathcal{S}_z - (\mathcal{Y}_z)_{red}$ contains (-2)-curves $\mathcal{C}_z$ whose configuration is same as the configuration of $\mathcal{C}_0 = C$. $\square$

Now we shall prove the claim (3.12).

Lemma 3.4. *Under the same assumption of Lemma* 3.3*, we have*

$$H^2(S, \Theta_S(-\log(D + C + F))) = \{0\},$$

where F is a smooth fiber of the elliptic fibration $f : S \longrightarrow \mathbf{P}^1$.

Proof. By the Serre duality, it suffices to show that

(3.14)
$$H^0(S, \Omega^1_S(\log(D + C + F)) \otimes K_S) \simeq H^0(S, \Omega^1_S(\log(D + C + F))(-F)) = \{0\}.$$

(Note that $K_S \sim -F$.) Set $\tilde{D} = \prod_{i=1}^r Y_i, \tilde{C} = \prod_{j=1}^s C_j$. Then we have the following commutative diagram of sheaves:

(3.15)
$$\begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & & \downarrow & \\
 & 0 & & 0 & \to & \mathcal{O}_F(-F) & \xrightarrow{\delta} \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \Omega^1_S(-F) & \to & \Omega^1_S & \to & (\Omega^1_S)_{|F} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow \mu & \\
0 \to & \Omega^1_S(\log(D + C + F))(-F) & \to & \Omega^1_S(\log(D + C + F)) & \to & \Omega^1_S(\log(F))_{|F} & \to 0 \\
 & \downarrow P.R. & & \downarrow P.R. & & \downarrow & \\
\xrightarrow{\delta} & \oplus\mathcal{O}_{Y_i} \oplus \mathcal{O}_{C_j} \oplus \mathcal{O}_F(-F) & \to & \oplus\mathcal{O}_{Y_i} \oplus \mathcal{O}_{C_j} \oplus \mathcal{O}_F & \to & \mathcal{O}_F & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & .
\end{array}$$

Here the map $P.R. : \Omega^1_S(\log(D+C+F)) \longrightarrow \oplus_{i=1}^r \mathcal{O}_{Y_i} \oplus_{j=1}^s \mathcal{O}_{C_j}$ is the Poincaré residue map and the image of $\mu : (\Omega^1_S)_{|F} \longrightarrow \Omega^1_S(\log F)_{|F}$ coincides with Ω^1_F

so that the following sequences are exact.

$$0 \longrightarrow \mathcal{O}_F(-F) \longrightarrow (\Omega^1_S)_{|F} \longrightarrow \Omega^1_F \longrightarrow 0, \tag{3.16}$$

$$0 \longrightarrow \Omega^1_F \longrightarrow (\Omega^1_S(\log F))_{|F} \longrightarrow \mathcal{O}_F \longrightarrow 0. \tag{3.17}$$

Noting that $N_F^\vee \simeq \mathcal{O}_F(-F) \simeq \mathcal{O}_F$ and $H^0(\Omega^1_S) = 0$, from the first and second rows of (3.15), we obtain the exact sequence of cohomology

$$\begin{array}{cccc} & 0 & & \\ & \downarrow & & \\ & H^0(\mathcal{O}_F(-F)) \simeq \mathbf{C} & & \\ & \downarrow H_1 & \searrow & \\ 0 \longrightarrow & H^0((\Omega^1_S)_{|F}) & \to & H^1(\Omega^1_S(-F)). \end{array} \tag{3.18}$$

From the first column of (3.15), $H^0(\Omega^1_S(\log(D + C + F))(-F))$ is isomorphic to the kernel of Gysin map

$$\oplus_{i=1}^r H^0(\mathcal{O}_{Y_i}) \oplus_{j=1}^s H^0(\mathcal{O}_{C_j}) \oplus H^0(\mathcal{O}_F(-F)) \xrightarrow{G_1} H^1(\Omega^1_S(-F)). \tag{3.19}$$

We will show that the Gysin map G_1 is injective, which implies the assertion (3.14).

By (3.15) and (3.18), we can decompose the map G_1 as follows:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_F(-F)) \simeq \mathbf{C} & \xrightarrow{H_1} & H^0((\Omega^1_S)_{|F}) \\ \downarrow \tau & & \downarrow \\ \left.\begin{array}{c} H^0(\mathcal{O}_F(-F)) \simeq \mathbf{C} \\ \oplus \\ \oplus_{i=1}^r H^0(\mathcal{O}_{Y_i}) \oplus_{j=1}^s H^0(\mathcal{O}_{C_j}) \end{array}\right\} & \xrightarrow{G_1} & H^1(\Omega^1_S(-F)) \\ \downarrow \mu_1 & & \downarrow \nu \\ \oplus_{i=1}^r H^0(\mathcal{O}_{Y_i}) \oplus_{j=1}^s H^0(\mathcal{O}_{C_j}) & \xrightarrow{G_2} & H^1(\Omega^1_S). \end{array} \tag{3.20}$$

Here μ_1 is just the projection and G_2 is the natural Gysin map. Since H_1 is injective (cf. (3.18)), a diagram chasing shows that G_1 is injective if G_2 is injective. The image of 1_{Y_i} and 1_{C_j} by G_2 are the class of the divisors of Y_i and C_j in $H^1(\Omega^1_S) \simeq H^2(S, \mathbf{C})$. Since $\{Y_i, C_j, 1 \leq i \leq r, \quad 1 \leq j \leq s\}$ are linearly independent in $H^2(S, \mathbf{C}) \simeq H^1(\Omega^1_S)$ by assumption of Lemma 3.3, G_2 is injective, hence we have proved the assertion. □

Now together with Lemma 3.3 and Proposition 3.1 the following lemma shows the existence part of Theorem 3.1 and hence completes the proof of Theorem 3.1 (see Example 3.1).

Lemma 3.5. *Let R be a type of affine root lattice in Table 1, that is $R = \tilde{E}_k, (k = 8, 7, 6)$ or $R = \tilde{D}_l, (l = 8, 7, 6, 5, 4)$. Let (S, Y) be a rational Okamoto-Painlevé pair of fibered-type and let $f : S \longrightarrow \mathbf{P}^1$ be the elliptic fibration with*

$f^*(\infty) = Y = \sum_{i=1}^{r} m_i Y_i$. Let $\{C_j\}_{j=1}^{s}$ be a set of different irreducible (-2) curves on $S - Y_{red}$ such that no linear combination of $\{C_j\}_{j=1}^{s}$ has the same class of general fiber (= the class of Y). Then $\{ Y_i, C_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ are linearly independent in $H^2(S, \mathbf{Q})$.

Proof. From the condition of the set $\{C_j\}_{j=1}^{s}$, we see that the sub-lattice $\langle C_j\rangle_{j=1}^{s} \subset H^2(S, \mathbf{Z})$ generated by $\{C_j\}_{j=1}^{s}$ is negative-definite. Then we have an orthogonal decomposition

$$\langle C_j\rangle_{j=1}^{s} \oplus \langle Y_i\rangle_{i=1}^{r} \subset H^2(S, \mathbf{Z}),$$

which shows the assertion. □

Example 3.1. From Proposition 3.1, we have an Okamoto-Painlevé pair (S, Y) of fibered type with another singular fiber F_1 where the pair (Y, F_1) has the type $(\tilde{D}_4, \tilde{D}_4)$. Take a proper subset $\{C_j\}_{j=1}^{s}$ of all of irreducible components of F_1. Then the type of M_1 coincides with the proper subgraph of the Dynkin diagram $\tilde{D}_4$ of F_1, that is, one of the types; D_4, (A_1, A_1, A_1, A_1) A_3, (A_1, A_1, A_1), A_2, (A_1, A_1) and A_1. It is easy to see that the set of classes $\{Y_i, C_j, 1 \leq i \leq 5, 1 \leq j \leq s\}$ are linearly independent in $H^2(S, \mathbf{Q})$. Therefore from Lemma 3.3 and Lemma 3.5, we see that there exists a rational Okamoto-Painlevé pair (S', Y') of non-fibered type, such that:

(1) the type of Y' is $\tilde{D}_4$,

(2) there exist (-2)-curves $\{C'_j\}_{j=1}^{s}$ on $S' - Y'_{red}$ with the same Dynkin type of $\{C_j\}_{j=1}^{s}$.

Therefore, we can obtain the assertion of Theorem 3.1 for $\tilde{D}_4$. We can treat the other cases similarly.

4. Non-existence of Riccati solutions for P_I, $P_{III}^{\tilde{D}_8}$, $P_{III}^{\tilde{D}_7}$

As a corollary to Theorem 3.1, we obtain the following

Corollary 4.1. *Let (S, Y) be a rational Okamoto-Painlevé pair of non-fibered type, with the type $R = R(Y) = \tilde{E}_8, \tilde{D}_8$ or $\tilde{D}_7$. Then $S - Y_{red}$ does not contain a rational nodal curve C. Therefore all the Painlevé equations of types P_I, $P_{III}^{\tilde{D}_8}$, $P_{III}^{\tilde{D}_7}$ do not admit Riccati solutions.*

Proof. The first assertion directly follows from Theorem 3.1 and the last assertion follows from the first and Proposition 2.1. □

Remark 7.

(1) Umemura proved that the Painlevé equation of type P_I has no classical solution and hence in particular no Riccati solution (cf. [Ni], [U1], [U2]).

(2) Ohyama [Ohy] showed that all the Painlevé equations of type $P_{III}^{\tilde{D}_7}$ has no Riccati solutions by proving that they have no invariant divisor with respect to the vector field (2.4).

(3) It is worth while remarking that the obstruction to the existence of nodal curves in $S - Y_{red}$ is a topological one and hence so is the obstruction to the existence of Riccati solutions. In fact, the sub-lattice $M(S - Y_{red})$ is classified only by the intersection theory of the surface S and the structure of the sub-lattice does not depend on the complex structure of S.

For other types R, by the similar argument in the proof of Lemma (A.2), we can show the following proposition. This proposition shows that for a general parameter $\boldsymbol{\alpha} \in \mathcal{M}_R$, the corresponding Painlevé equations do not admit any Riccati solution.

Proposition 4.1. *Let (S, Y) be a rational Okamoto-Painlevé pair of non-fibered type and of type R which corresponds to a Painlevé equation and assume that $S - Y_{red}$ contains a nodal curve C. Then there exists a one parameter deformation of Okamoto-Painlevé pairs of non-fibered type and of the given type R, $\mathcal{Y} \hookrightarrow \mathcal{S} \longrightarrow \Delta = \{z \in \mathbf{C} | |z| < \epsilon\}$ of (S, Y) such that $\mathcal{S}_z - \mathcal{Y}_z$ does not contains any nodal curve for $z \in \Delta - \{0\}$. Hence for $z \in \Delta - \{0\}$ the Painlevé equation corresponding to $(\mathcal{S}_z, \mathcal{Y}_z)$ does not admit any Riccati solutions.*

5. Examples of (-2)-curves on $S - D$

In this section, we will give examples of (-2)-curves C on $S - D$ for some rational Okamoto-Painlevé pairs (S, Y) and Riccati equations associated to C.

Here we will use the explicit description of families of Okamoto-Painlevé pairs

$$(5.1)\qquad \begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & \end{array}$$

in [T2]. As we explained in Section 2, we have isomorphisms $\mathcal{M}_R = \operatorname{Spec} M_R$ and $\mathcal{B}_R = \operatorname{Spec} B_R$ such that $\operatorname{Spec} M_R$ and $\operatorname{Spec} B_R$ are affine open subschemes of $\operatorname{Spec} \mathbf{C}[\alpha_1, \ldots, \alpha_s] \simeq \mathbf{C}^s$ and $\operatorname{Spec} \mathbf{C}[t]$ respectively. Moreover $\mathcal{S}$ can be covered by affine open sets $\{\tilde{U}_i\}_{i=1}^{l+k}$ such that for each i

(5.2)

$$\begin{array}{rcl} \tilde{U}_i \simeq \operatorname{Spec}\left((M_R \otimes B_R)\left[x_i, y_i, \frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)}\right]\right) \subset \operatorname{Spec} \mathbf{C}[\boldsymbol{\alpha}, t, x_i, y_i] & \simeq & \mathbf{C}^{s+3} \\ & & || \\ & & \mathbf{C}^{12-r}, \end{array}$$

where $f_i(x_i, y_i, \alpha, t)$ is an element of $(M_R \otimes B_R)[x_i, y_i]$. (Note that in most cases $f_i(x_i, y_i, \alpha, t) \equiv 1$ and we may assume that $\mathcal{S} - \mathcal{D}$ is covered by $\{\tilde{U}_i\}_{i=1}^{l}$.)

For a given point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathcal{M}_R$, we denote the restriction of the family $\pi : \mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ to $\{\boldsymbol{\alpha}\} \times \mathcal{B}_R$ by $\mathcal{S}_{\boldsymbol{\alpha}} \longrightarrow \{\boldsymbol{\alpha}\} \times \mathcal{B}_R$. Moreover we set

(5.3)

$$U_{i\boldsymbol{\alpha}} := \tilde{U}_i \cap \mathcal{S}_{\boldsymbol{\alpha}} \subset \operatorname{Spec} B_R[x_i, y_i], \qquad U_{i(\boldsymbol{\alpha}, t)} = \tilde{U}_i \cap \mathcal{S}_{\boldsymbol{\alpha}, t} \subset \operatorname{Spec} \mathbf{C}[x_i, y_i],$$

where $\mathcal{S}_{\boldsymbol{\alpha},t} = \pi^{-1}((\boldsymbol{\alpha}, t))$.

Next let us consider the smooth variety obtained by patching affine planes $W_i = \operatorname{Spec} \mathbf{C}[x_i, y_i] \simeq \mathbf{C}^2$ ($i = 1, 2$) by the coordinate transformation

$$x_1 = \frac{1}{x_2}, \quad y_1 = {x_2}^2 y_2. \tag{5.4}$$

It is easy to see that the equations $\{y_1 = y_2 = 0\}$ define a (-2)-curve C in W.

Example 5.1 ($\tilde{E}_7$–type (P_{II}))**.** In the case of $R = \tilde{E}_7$ (P_{II}), the family is constructed as follows (cf. [MMT], [T2], [SU]). Let us set

$$\mathcal{M}_R = \operatorname{Spec} \mathbf{C}[\alpha] \simeq \mathbf{C}, \quad \mathcal{B}_R = \operatorname{Spec} \mathbf{C}[t] \simeq \mathbf{C}.$$

Here we only give the affine covering of the family $\pi : \mathcal{S} - \mathcal{D} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$. Take three affine schemes $i = 0, 1, 2$

$$\tilde{U}_i = \operatorname{Spec} \mathbf{C}[\alpha, t, x_i, y_i] \simeq \mathbf{C}^4, \tag{5.5}$$

and patch these affine schemes by the coordinate transformations:

$$\begin{aligned} x_0 &= \frac{1}{x_1} && = \frac{1}{x_2}, \\ y_0 &= x_1\left(\left(-\alpha - \frac{1}{2}\right) - x_1 y_1\right) && = 2x_2^{-2} + t + \left(\alpha - \frac{1}{2}\right) x_2 - y_2 x_2^2. \end{aligned} \tag{5.6}$$

On $\tilde{U}_0$, the Painlevé vector field $\tilde{v}$ in (2.4) is explicitly given by

$$\tilde{v} = \frac{\partial}{\partial t} + \left[y_0 - x_0^2 - \frac{t}{2}\right] \frac{\partial}{\partial x_0} + \left[2x_0 y_0 + \alpha + \frac{1}{2}\right] \frac{\partial}{\partial y_0}, \tag{5.7}$$

which is equivalent to the equation:

$$\begin{cases} \dfrac{dx_0}{dt} = y_0 - x_0^2 - \dfrac{t}{2}, \\ \dfrac{dy_0}{dt} = 2x_0 y_0 + \alpha + \dfrac{1}{2}. \end{cases} \tag{5.8}$$

Then for $\alpha = -\frac{1}{2}$, on $U_{0,-\frac{1}{2}} \cup U_{1,-\frac{1}{2}}$, we obtain a family of (-2)-curves $\mathcal{C}_{-\frac{1}{2}} \longrightarrow \{-\frac{1}{2}\} \times \mathcal{B}_{\tilde{E}_7}$ defined by

$$\mathcal{C}_{-\frac{1}{2}} = \{y_0 = y_1 = 0\} \subset U_{0,-\frac{1}{2}} \cup U_{1,-\frac{1}{2}} \subset \mathcal{S}_{-\frac{1}{2}} - \mathcal{D}_{-\frac{1}{2}}. \tag{5.9}$$

Moreover, on the family $\mathcal{C}_{-\frac{1}{2}} \longrightarrow \{-\frac{1}{2}\} \times \mathcal{B}_{\tilde{E}_7}$, the equation (5.8) can be reduced to

$$\frac{dx_0}{dt} = -x_0^2 - \frac{t}{2}. \tag{5.10}$$

It is known that Bäcklund transformations give isomorphisms between $\mathcal{S}_\alpha$ and $\mathcal{S}_{\alpha \pm 1}$. Hence for $\alpha \in -\frac{1}{2} + \mathbf{Z}$, the family $\mathcal{S}_\alpha - \mathcal{D}_\alpha$ also contains a family of (-2)-curves (cf. [SU], [U-W1]). Moreover, Noumi and Okamoto [NO] proved the following Theorem (cf. [NO, Theorem 2] and remark after it). (See also [U-W1, Theorem 2.1].)

Theorem 5.1 ([NO, Theorem 2]). *Let us denote by $P_{II}(\alpha)$ the equation in* (5.8). *Then*

(1) *For every integer $\alpha \in \mathbf{Z}$, there exists a unique rational solution of the system $P_{II}(\alpha)$.*

(2) *For every $\alpha \in \frac{1}{2} + \mathbf{Z}$, there exists a unique one parameter family of classical solutions of $P_{II}(\alpha)$, of which each solution is rationally written by a solution of the Riccati equation* (5.10).

(3) *Let (x_0, y_0) be a solution of $P_{II}(\alpha)$ different from those mentioned above. Then neither x_0 nor y_0 is classical, hence a solution of a Riccati equation.*

Note that for $\alpha = 0$, $P_{II}(0)$ in (5.8) has a rational solution $(x_0, y_0) = (0, \frac{t}{2})$. Theorem 5.1 says that this rational solution is the unique rational solution for $P_{II}(0)$.

Example 5.2 ($\tilde{D}_4$ (P_{VI})). Next let us show examples of (-2)-curves for $R = \tilde{D}_4$ (cf. [T2]). The parameter space of the semiuniversal family $\mathcal{S} - \mathcal{D} \longrightarrow \mathcal{M}_{\tilde{D}_4} \times \mathcal{B}_{\tilde{D}_4}$ are given by

$$\mathcal{M}_R = \operatorname{Spec} \mathbf{C}[\kappa_0, \kappa_1, \kappa_\infty, \kappa_t] \simeq \mathbf{C}^4, \ \mathcal{B}_R = \operatorname{Spec} \mathbf{C}[t, 1/t, 1/(t-1)] \simeq \mathbf{C} - \{0, 1\}.$$

(Here we use the parameters κ_i, $i = 0, 1, \infty, t$ for $\mathcal{M}_R$ as in [MMT] and [T2]). Take affine schemes $i = 0, 1, 2, 3, 4, 5$

$$\tilde{U}_i = \operatorname{Spec} \mathbf{C}[x_i, y_i, \kappa_0, \kappa_1, \kappa_\infty, \kappa_t, t, 1/t, 1/(t-1)] \simeq \mathbf{C}^2 \times \mathcal{M}_R \times \mathcal{B}_R \tag{5.11}$$

and patch them by the coordinate transformations:

$$\begin{aligned}
&x_0 = y_1(\kappa_0 - x_1 y_1), && y_0 = \frac{1}{y_1}, \\
&x_1 = y_0(\kappa_0 - x_0 y_0), && y_1 = \frac{1}{y_0}, \\
&x_0 = 1 + y_2(\kappa_1 - x_2 y_2), && y_0 = \frac{1}{y_2}, \\
&x_2 = y_0(\kappa_1 + y_0 - x_0 y_0), && y_2 = \frac{1}{y_0}, \\
&x_0 = t + y_3(\kappa_t - x_3 y_3), && y_0 = \frac{1}{y_3}, \\
&x_3 = y_0(\kappa_t + t y_0 - x_0 y_0), && y_3 = \frac{1}{y_0}, \\
&x_0 = \frac{1}{x_4}, && y_0 = x_4 \left(\frac{\kappa_0 + \kappa_1 + \kappa_t - 1 + \kappa_\infty}{2} - x_4 y_4 \right), \\
&x_4 = \frac{1}{x_0}, && y_4 = x_0 \left(\frac{\kappa_0 + \kappa_1 + \kappa_t - 1 + \kappa_\infty}{2} - x_0 y_0 \right), \\
&x_4 = y_5(\kappa_\infty - x_5 y_5), && y_4 = \frac{1}{y_5}, \\
&x_5 = y_4(\kappa_\infty - x_4 y_4), && y_5 = \frac{1}{y_4}.
\end{aligned} \tag{5.12}$$

On $\tilde{U_0}$, the Painlevé vector field $\tilde{v}$ in (2.4) is given by

$$\tilde{v} = \frac{\partial}{\partial t} + A(x,y,t)\frac{\partial}{\partial x_0} + B(x,y,t)\frac{\partial}{\partial y_0}, \tag{5.13}$$

where

$$\begin{aligned} A(x,y,t) &:= \frac{x_0(x_0-1)(x_0-t)}{t(t-1)}\left[2y_0 - \left(\frac{\kappa_0}{x_0} + \frac{\kappa_1}{(x_0-1)} + \frac{(\kappa_t-1)}{(x_0-t)}\right)\right], \\ B(x,y,t) &:= -\frac{1}{t(t-1)}\left[(3x_0^2 - 2(t+1)x_0 + t)y_0^2\right. \\ &\quad - (2(\kappa_0+\kappa_1+\kappa_t-1)x_0 - (\kappa_0+\kappa_1)t - \kappa_0 - \kappa_t + 1)y_0 \\ &\quad \left. + \frac{(\kappa_0+\kappa_1+\kappa_t-1)^2 - \kappa_\infty^2}{4}\right]. \end{aligned}$$

This is equivalent to the equation:

$$\begin{cases} \dfrac{dx_0}{dt} = A(x,y,t), \\ \dfrac{dy_0}{dt} = B(x,y,t). \end{cases} \tag{5.14}$$

Let us set the hyperplanes of the parameter space $\mathcal{M}_{\tilde{D}_4} \times \mathcal{B}_{\tilde{D}_4}$ as follows:

$$\begin{aligned} &H_0 = \{\kappa_0 = 0\}, \quad H_1 = \{\kappa_1 = 0\}, \quad H_t = \{\kappa_t = 0\}, \\ &H_\epsilon = \{\kappa_0+\kappa_1+\kappa_t+\kappa_\infty - 1 = 0\}, \quad H_\infty = \{\kappa_\infty = 0\}. \end{aligned} \tag{5.15}$$

Note that each hyperplane H_i is a direct product of $H_i' \subset \mathcal{M}_{\tilde{D}_4}$ and $\mathcal{B}_{\tilde{D}_4}$, i.e., $H_i = H_i' \times \mathcal{B}_{\tilde{D}_4}$. Remark also that each hyperplane is one of the reflection hyperplanes of the affine Weyl group $W(\tilde{D}_4)$ generated by Bäcklund transformations (cf. [NTY]).

We consider the deformation

$$\begin{array}{ccc} \pi^*(H_0) & \subset & \mathcal{S} - \mathcal{D} \\ \pi \downarrow & & \pi \downarrow \\ H_0 & \subset & \mathcal{M}_{\tilde{D}_4} \times \mathcal{B}_{\tilde{D}_4} \end{array}$$

which is given by restricting the parameter space $\mathcal{M}_{\tilde{D}_4} \times \mathcal{B}_{\tilde{D}_4}$ to H_0. For the subfamily $(\mathcal{S}-\mathcal{D})_{(0,\kappa_1,\kappa_t,\kappa_\infty)}$ over H_0, the coordinate transformation between $U_{0(0,\kappa_1,\kappa_t,\kappa_\infty)}$ and $U_{1(0,\kappa_1,\kappa_t,\kappa_\infty)}$ is given by

$$x_0 = -x_1{y_1}^2, \quad y_0 = \frac{1}{y_1}.$$

Therefore

$$\mathcal{C}_{0,(0,\kappa_1,\kappa_t,\kappa_\infty)} := \{x_0 = x_1 = 0\}$$

determines a family of (-2)-curves

$$\begin{array}{ccc} \mathcal{C}_{0,(0,\kappa_1,\kappa_t,\kappa_\infty)} & \hookrightarrow & (\mathcal{S}-\mathcal{D})_{(0,\kappa_1,\kappa_t,\kappa_\infty)} \\ \downarrow & \swarrow & \\ H_0 = H_0' \times \mathcal{B}_R. & & \end{array} \tag{5.16}$$

In the same way, we obtain a family of (-2)-curves over each hyperplane H_i as follows:

$$\begin{array}{ll} H_0 & : \mathcal{C}_{0,(0,\kappa_1,\kappa_t,\kappa_\infty)} := \{x_0 = x_1 = 0\} \quad \subset (\mathcal{S}-\mathcal{D})_{(0,\kappa_1,\kappa_t,\kappa_\infty)}, \\ H_1 & : \mathcal{C}_{1,(\kappa_0,0,\kappa_t,\kappa_\infty)} := \{x_0 = 1,\ x_2 = 0\} \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,0,\kappa_t,\kappa_\infty)}, \\ H_t & : \mathcal{C}_{t,(\kappa_0,\kappa_1,0,\kappa_\infty)} := \{x_0 = t,\ x_3 = 0\} \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,\kappa_1,0,\kappa_\infty)}, \\ H_\epsilon & : \mathcal{C}_{\epsilon,(\kappa_0,\kappa_1,\kappa_t,\kappa_\infty)} := \{y_0 = y_4 = 0\} \quad \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,\kappa_1,\kappa_t,\kappa_\infty)}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad (\kappa_0+\kappa_1+\kappa_t+\kappa_\infty = 1), \\ H_\infty & : \mathcal{C}_{\infty,(\kappa_0,\kappa_1,\kappa_t,0)} := \{x_4 = x_5 = 0\} \quad \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,\kappa_1,\kappa_t,0)}. \end{array} \tag{5.17}$$

By restricting the (extended) Hamiltonian system to each $\mathcal{C}_j$, we obtain the following Riccati equation.

- On $\mathcal{C}_{0,(0,\kappa_1,\kappa_t,\kappa_\infty)} \cap U_{0(0,\kappa_1,\kappa_t,\kappa_\infty)}$:

$$x_0 \equiv 0, \quad \frac{dy_0}{dt} = -\frac{1}{t(t-1)}\left(ty_0^2 + (\kappa_1 t + \kappa_t - 1)y_0 + \frac{(\kappa_1+\kappa_t-1)^2 - \kappa_\infty^2}{4}\right).$$

- On $\mathcal{C}_{1,(\kappa_0,0,\kappa_t,\kappa_\infty)} \cap U_{0(\kappa_0,0,\kappa_t,\kappa_\infty)}$:

$$\begin{cases} x_0 \equiv 1, \\ \dfrac{dy_0}{dt} = -\dfrac{1}{t(t-1)}\Bigg((1-t)y_0^2 - ((\kappa_0+\kappa_t-1) - \kappa_0 t)y_0 \\ \qquad\qquad + \dfrac{(\kappa_0+\kappa_t-1)^2 - \kappa_\infty^2}{4}\Bigg). \end{cases}$$

- On $\mathcal{C}_{t,(\kappa_0,\kappa_1,0,\kappa_\infty,t)} \cap U_{0(\kappa_0,\kappa_1,0,\kappa_\infty,t)}$:

$$\begin{cases} x_0 \equiv t, \\ \dfrac{dy_0}{dt} = -\dfrac{1}{t(t-1)}\Bigg(t(t-1)y_0^2 - ((\kappa_0+\kappa_1-2)t - \kappa_0 + 1)y_0 \\ \qquad\qquad + \dfrac{(\kappa_0+\kappa_1-1)^2 - \kappa_\infty^2}{4}\Bigg). \end{cases}$$

- On $\mathcal{C}_{\epsilon,(\kappa_0,\kappa_1,\kappa_t,1-(\kappa_0+\kappa_1+\kappa_t),t)} \cap U_{0(\kappa_0,\kappa_1,\kappa_t,1-(\kappa_0+\kappa_1+\kappa_t),t)}$:

$$\begin{cases} \dfrac{dx_0}{dt} = -\dfrac{1}{t(t-1)}(\kappa_0(x_0-1)(x_0-t) + \kappa_1 x_0(x_0-t) + (\kappa_t-1)x_0(x_0-1)), \\ y_0 \equiv 0. \end{cases}$$

fiber	(-2)-curves	configuration
$(\mathcal{S}-\mathcal{D})_{(0,0,0,1)}$	$\{\mathcal{C}_0,\mathcal{C}_1,\mathcal{C}_t,\mathcal{C}_\epsilon\}$	D_4
$(\mathcal{S}-\mathcal{D})_{(0,0,1,0)}$	$\{\mathcal{C}_0,\mathcal{C}_1,\mathcal{C}_\epsilon,\mathcal{C}_\infty\}$	D_4
$(\mathcal{S}-\mathcal{D})_{(0,1,0,0)}$	$\{\mathcal{C}_0,\mathcal{C}_t,\mathcal{C}_\epsilon,\mathcal{C}_\infty\}$	D_4
$(\mathcal{S}-\mathcal{D})_{(1,0,0,0)}$	$\{\mathcal{C}_1,\mathcal{C}_t,\mathcal{C}_\epsilon,\mathcal{C}_\infty\}$	D_4
$(\mathcal{S}-\mathcal{D})_{(0,0,0,0)}$	$\{\mathcal{C}_0,\mathcal{C}_1,\mathcal{C}_t,\mathcal{C}_\infty\}$	A_1,A_1,A_1,A_1

Table 5.

- On $\mathcal{C}_{\infty,(\kappa_0,\kappa_1,\kappa_t,0,t)}\cap U_{4(\kappa_0,\kappa_1,\kappa_t,0,t)}$:

$$x_4\equiv 0,\quad \frac{dy_4}{dt}=-\frac{1}{t(t-1)}\left(y_4^2+((\kappa_t-1)t+\kappa_1)y_4+\frac{(\kappa_1+\kappa_t-1)^2-\kappa_0^2}{4}t\right).$$

Next, choose four hyperplanes from the five hyperplanes and consider the fibers over the intersection of them. For each fiber $(\mathcal{S}-\mathcal{D})_{(0,0,0,1,t)}$ of $(0,0,0,1,t)\in H_0\cap H_1\cap H_t\cap H_\epsilon$, we can see that $\mathcal{C}_{0,(0,0,0,1,t)},\mathcal{C}_{1,(0,0,0,1,t)}$, and $\mathcal{C}_{t,(0,0,0,1,t)}$ do not intersect each other but they intersect with $\mathcal{C}_{\epsilon,(0,0,0,1,t)}$ respectively. Hence the type of the configuration of these curves is D_4. By checking the other cases, we obtain the following.

Below, we only give the tables for $\tilde{D}_k, k=5,6$. The case $\tilde{E}_6$ will be treated in Section 6. For parameters and the coordinate transformations, see [T2].

Example 5.3 ($\tilde{D}_5$ (P_V)).

$$\mathcal{M}_{\tilde{D}_5}=\operatorname{Spec}\mathbf{C}[\kappa_0,\kappa_t,\kappa_\infty]\simeq\mathbf{C}^3,\quad \mathcal{B}_{\tilde{D}_5}=\operatorname{Spec}\mathbf{C}[t,t^{-1}]\simeq\mathbf{C}^\times.$$
$$H_0=\{\kappa_0=0\},\quad H_\epsilon=\{\kappa_0+\kappa_t+\kappa_\infty=0\},\quad H_\infty=\{\kappa_\infty=0\}.$$
$$\begin{aligned}
H_0\ &:\mathcal{C}_{0,(0,\kappa_t,\kappa_\infty)}:=\{x_0=x_1=0\} &&\subset(\mathcal{S}-\mathcal{D})_{(0,\kappa_t,\kappa_\infty)},\\
H_\epsilon\ &:\mathcal{C}_{\epsilon,(\kappa_0,\kappa_t,-(\kappa_0+\kappa_t))}:=\{y_0=y_3=0\} &&\subset(\mathcal{S}-\mathcal{D})_{(\kappa_0,\kappa_t,-(\kappa_0+\kappa_t))},\\
H_\infty &:\mathcal{C}_{\infty,(\kappa_0,\kappa_t,0)}:=\{x_3=x_4=0\} &&\subset(\mathcal{S}-\mathcal{D})_{(\kappa_0,\kappa_t,0)}.
\end{aligned}$$

- On $\mathcal{C}_{0,(0,\kappa_t,\kappa_\infty)}\cap U_{0(0,\kappa_t,\kappa_\infty)}$

$$\begin{cases} x_0\equiv 0,\\ \dfrac{dy_0}{dt}=-\dfrac{1}{t}\left(y_0^2+(\kappa_t-t)y_0+\dfrac{\kappa_t^2-\kappa_\infty^2}{4}\right).\end{cases}$$

- On $\mathcal{C}_{\epsilon,(\kappa_0,\kappa_t,-(\kappa_0+\kappa_t))}\cap U_{0(\kappa_0,\kappa_t,-(\kappa_0+\kappa_t))}$

$$\frac{dx_0}{dt}=-\frac{1}{t}(\kappa_0(x_0-1)^2+\kappa_t x_0(x_0-1)+tx_0),\quad y_0\equiv 0.$$

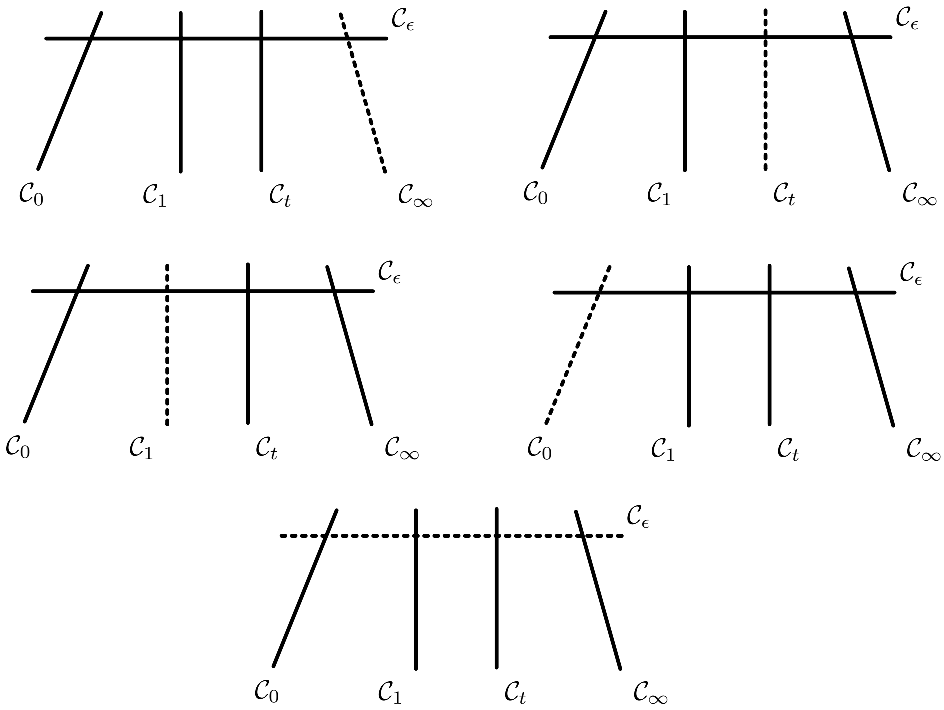

Figure 2. Maximal configurations for $R = \tilde{D}_4$.

- On $\mathcal{C}_{\infty,(\kappa_0,\kappa_t,0)} \cap U_{3(\kappa_0,\kappa_t,0)}$

$$x_3 \equiv 0, \quad \frac{dy_3}{dt} = -\frac{1}{t}\left(y_3^2 + (\kappa_t + t)y_3 + \frac{\kappa_t^2 - \kappa_0^2}{4}\right).$$

fiber	(-2)-curves	configuration
$(\mathcal{S}-\mathcal{D})_{(0,0,0)}$	$\{\mathcal{C}_0, \mathcal{C}\epsilon, \mathcal{C}_\infty\}$	A_3

Example 5.4 ($\tilde{D}_6$ (P_{III})).

$$\mathcal{M}_R = \operatorname{Spec}\mathbf{C}[\kappa_0, \kappa_\infty] \simeq \mathbf{C}^2, \quad \mathcal{B}_R = \operatorname{Spec}\mathbf{C}[t, t^{-1}] \simeq \mathbf{C}^\times.$$
$$H_1 = \{\kappa_0 + \kappa_\infty = 0\}, H_2 = \{\kappa_0 - \kappa_\infty = 0\},$$
$$H_3 = \{\kappa_0 - \kappa_\infty + 2 = 0\}, H_4 = \{\kappa_0 + \kappa_\infty + 2 = 0\}.$$

$$\begin{aligned}
&H_1 : \mathcal{C}_{1,(\kappa_0,-\kappa_0)} := \{y_0 = y_2 = 0\} && \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,-\kappa_0)},\\
&H_2 : \mathcal{C}_{2,(\kappa_0,\kappa_0)} := \{y_0 = t, y_3 = 0\} && \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,\kappa_0)},\\
&H_3 : \mathcal{C}_{3,(\kappa_0,\kappa_0+2)} := \{y_1 = 0, y_2 = t\} && \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,\kappa_0+2)},\\
&H_4 : \mathcal{C}_{4,(\kappa_0,-\kappa_0-2)} := \{y_1 = t, y_3 = t\} && \subset (\mathcal{S}-\mathcal{D})_{(\kappa_0,-\kappa_0-2)}.
\end{aligned}$$

- On $\mathcal{C}_{1,(\kappa_0,-\kappa_0)} \cap U_{0(\kappa_0,-\kappa_0)}$

$$\frac{dx_0}{dt} = \frac{1}{t}(-2tx_0^2 - (2\kappa_0+1)x_0 + 2t), \quad y_0 \equiv 0.$$

- On $\mathcal{C}_{2,(\kappa_0,\kappa_0)} \cap U_{0(\kappa_0,\kappa_0)}$

$$\frac{dx_0}{dt} = \frac{1}{t}(2tx_0^2 - (2\kappa_0+1)x_0 + 2t), \quad y_0 \equiv t.$$

- On $\mathcal{C}_{3,(\kappa_0,\kappa_0+2)} \cap U_{1(\kappa_0,\kappa_0+2)}$

$$\frac{dx_1}{dt} = \frac{1}{t}(-2tx_1^2 + (2\kappa_0+3)x_1 - 2t), \quad y_1 \equiv 0.$$

- On $\mathcal{C}_{4,(\kappa_0,-\kappa_0-2)} \cap U_{1(\kappa_0,-\kappa_0-2)}$

$$\frac{dx_1}{dt} = \frac{1}{t}(2tx_1^2 + (2\kappa_0+3)x_1 - 2t), \quad y_1 \equiv 1.$$

fiber	(-2)-curves	configuration
$(\mathcal{S}-\mathcal{D})_{(0,0)}$	$\{\mathcal{C}_1, \mathcal{C}_2\}$	A_1, A_1
$(\mathcal{S}-\mathcal{D})_{(-1,1)}$	$\{\mathcal{C}_1, \mathcal{C}_3\}$	A_1, A_1
$(\mathcal{S}-\mathcal{D})_{(-1,-1)}$	$\{\mathcal{C}_2, \mathcal{C}_4\}$	A_1, A_1
$(\mathcal{S}-\mathcal{D})_{(-2,0)}$	$\{\mathcal{C}_3, \mathcal{C}_4\}$	A_1, A_1

6. Confluences of Nodal Curves and Riccati Equations

In this section, we will discuss the confluence of nodal curves and Riccati equations for Painlevé equations. We will deal with only the case $R = \tilde{E}_6$ (P_{IV}), however one can easily extend the result to other cases like $\tilde{D}_5$ and $\tilde{D}_4$.

6.1. The confluence of nodal curves

Example 6.1 ($\tilde{E}_6$ (P_{IV}))**.**

$$\mathcal{M}_R = \operatorname{Spec} \mathbf{C}[\kappa_0, \kappa_\infty] \simeq \mathbf{C}^2, \quad \mathcal{B}_R = \operatorname{Spec} \mathbf{C}[t] \simeq \mathbf{C}.$$

An open covering of $\mathcal{S} - \mathcal{D}$ is given by

$$\mathcal{S} - \mathcal{D} = \bigcup_{i=0}^{3} \tilde{U}_i,$$

where for $i = 0, 1, 2, 3$

$$\tilde{U}_i = \operatorname{Spec} \mathbf{C}[x_i, y_i, \kappa_0, \kappa_\infty, t] \simeq \mathbf{C}^5.$$

Moreover the coordinate transformations are given by

$$
\begin{aligned}
&x_0 = y_1(\kappa_0 - x_1y_1), && y_0 = \frac{1}{y_1},\\
&x_1 = y_0(\kappa_0 - x_0y_0), && y_1 = \frac{1}{y_0},\\
&x_0 = \frac{1}{x_2}, && y_0 = x_2(\kappa_\infty - x_2y_2),\\
&x_2 = \frac{1}{x_0}, && y_2 = x_0(\kappa_\infty - x_0y_0),\\
&x_2 = x_3, && y_2 = -\frac{1/2}{x_3^3} - \frac{t}{x_3^2} + \frac{2\kappa_\infty - \kappa_0 + 1}{x_3} + y_3,\\
&x_3 = x_2, && y_3 = \frac{1/2}{x_2^3} + \frac{t}{x_2^2} - \frac{2\kappa_\infty - \kappa_0 + 1}{x_2} + y_2.
\end{aligned}
$$

Finally, on the affine open set $\tilde{U}_0$, the Painlevé system of type $\tilde{E}_6$ which is equivalent to P_{IV} is given as follows;

$$
\begin{cases}
\dfrac{dx_0}{dt} = 4x_0y_0 - x_0^2 - 2tx_0 - 2\kappa_0,\\
\dfrac{dy_0}{dt} = -2y_0^2 + 2(x_0+t)y_0 - \kappa_\infty.
\end{cases}
\tag{6.1}
$$

We have two hyperplanes H_0 and H_∞ on $\mathcal{M}_R \times \mathcal{B}_R$ and families of (-2)-curves $\mathcal{C}_0$ and $\mathcal{C}_\infty$ over H_0 and H_∞ as follows;

$$
\begin{aligned}
H_0 = \{\kappa_0 = 0\} \quad &: \mathcal{C}_{0,(0,\kappa_\infty)} := \{x_0 = x_1 = 0\} \subset (\mathcal{S} - \mathcal{D})_{(0,\kappa_\infty)},\\
H_\infty = \{\kappa_\infty = 0\} &: \mathcal{C}_{\infty,(\kappa_0,0)} := \{y_0 = y_2 = 0\} \subset (\mathcal{S} - \mathcal{D})_{(\kappa_0,0)}.
\end{aligned}
\tag{6.2}
$$

Then now it is easy to see that the Painlevé system (6.1) can be reduced to the following Riccati equations on $\mathcal{C}_0$ and $\mathcal{C}_\infty$ respectively.

- On $\mathcal{C}_{0,(0,\kappa_\infty)} \cap U_{0\,(0,\kappa_\infty)}$

$$
x_0 \equiv 0, \quad \frac{dy_0}{dt} = -2y_0^2 + 2ty_0 - \kappa_\infty.
\tag{6.3}
$$

- On $\mathcal{C}_{\infty,(\kappa_0,0)} \cap U_{0\,(\kappa_0,0)}$

$$
\frac{dx_0}{dt} = -x_0^2 - 2tx_0 - 2\kappa_0, \quad y_0 \equiv 0.
\tag{6.4}
$$

fiber	(-2)-curves	configuration
$(\mathcal{S}-\mathcal{D})_{(0,0)}$	$\{\mathcal{C}_0, \mathcal{C}_\infty\}$	A_2

Let us consider the neighborhood of $(\kappa_0, \kappa_\infty, t) = (0, 0, t) \in \mathcal{M}_R \times \mathcal{B}_R$ and the hyperplanes as in (6.2). Then, over the subvariety $H_0 \cap H_\infty = \{(0, 0, t)\}$, the family $(\mathcal{S} - \mathcal{D})_{0,0}$ contains both of families of nodal curves $\mathcal{C}_0 \cup \mathcal{C}_\infty$ (A_2-configuration), (see Figure 3). We call this phenomenon *the confluence of nodal curves* of Okamoto-Painlevé pairs.

Besides hyperplanes H_0, H_∞, we also have the hyperplane

$$H_{\kappa_0=\kappa_\infty} = \{\kappa_0 = \kappa_\infty\}.$$

Then one can easily see that over hyperplane $H_{\kappa_0=\kappa_\infty}$ there exists a family of (-2)-curves defined by

$$\mathcal{C}_{\kappa_0=\kappa_\infty} \cap \tilde{U}_{0\,\kappa_0=\kappa_\infty} = \{x_0 y_0 - \kappa_0 = 0\}.$$

Note that if κ_0 goes to 0, then the defining equation of the family becomes $x_0 y_0 = 0$. Therefore on $(\mathcal{S} - \mathcal{D})_{0,0}$ we have a homological relation:

$$\mathcal{C}_{\kappa_0=\kappa_\infty} = \mathcal{C}_0 \cup \mathcal{C}_\infty$$

(see Figure 3). On $\mathcal{C}_{\kappa_0=\kappa_\infty}$, the Painlevé system (6.1) can be reduced to

$$\frac{dx_0}{dt} = -x_0^2 - 2tx_0 + 2\kappa_0, \tag{6.5}$$

$$\frac{dy_0}{dt} = -2y_0^2 + 2ty_0 + \kappa_0, . \tag{6.6}$$

Note that if $\kappa_0 \neq 0$ the equations (6.5) and (6.6) can be transformed to each other by the coordinate change $x_0 = \kappa_0 / y_0$.

The hyperplanes are reflection hyperplanes in $\mathcal{M}_R$ with respect to the reflections of the affine Weyl group $W(\tilde{A}_2)$, which acts on both $\mathcal{M}_R$ or $\mathcal{S}$ as Bäcklund transformations (cf. [U-W1] and [NTY]). For example, by Bäcklund transformations, the Riccati equations (6.3), (6.4) and (6.5) are birational equivalent to each other. See Theorem 3.3 in [U-W1].

6.2. Rational solutions

We shall remark briefly on rational solutions of Painlevé equations. In the above example, when $(\kappa_0, \kappa_\infty) = (0, 0)$, the functions

$$(x_0, y_0) \equiv (0, 0) \tag{6.7}$$

give a solution of the system (6.1), hence gives a rational solution for the Painlevé equation P_{IV}. From the view point of the geometry of Okamoto-Painlevé pairs, it is clear that the intersection of two different families of nodal curves $\mathcal{C}_0, \mathcal{C}_\infty$ gives a solution of the Painlevé equation. In fact, Painlevé vector field $\tilde{v}$ in (2.4) is tangent to each family of rational curves by Proposition 2.1, hence tangent to their intersection (see Figure 4). It is not surprising that not all rational solutions of Painlevé equations can be obtained in this way. For example, as we explained after Theorem 5.1, the equation $S_{II}(0)$ in (5.8) has the rational solution $(x_0, y_0) = (0, \frac{t}{2})$, but no Riccati solution. It should be an interesting problem to understand the rational or algebraic solutions from the view point of the geometry of Okamoto-Painlevé pairs.

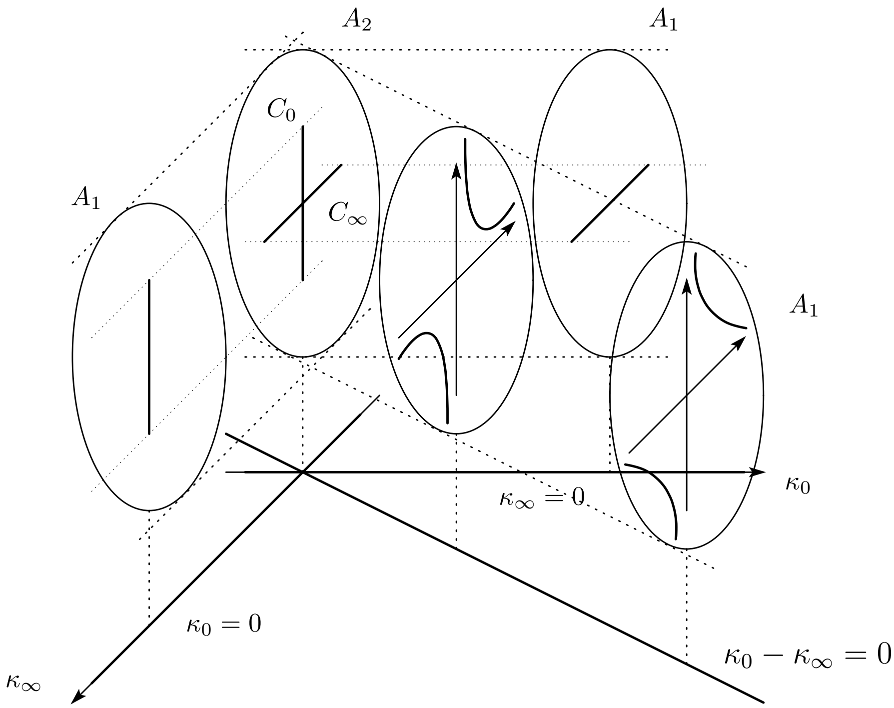

Figure 3. A Confluence of Nodal Curves in the case $\tilde{E}_6$ (P_{IV}).

Here we only remark that there are many works for the classification problems of rational and algebraic solutions (see e.g., [DM], [Maz], [Mu1], [NO], [O3], [U-W1], [U-W2]).

Appendix A. Local cohomology group $H^1_D(\Theta_S(-\log D))$

Let (S, Y) be a rational Okamoto-Painlevé pair of non-fibered type and of additive type which corresponds to Painlevé equations (i.e. of type $\tilde{D}_i (4 \le i \le 8)$ or $\tilde{E}_i (6 \le i \le 8)$), and set $D = Y_{red}$.

Applying the classification of nodal curves on $S - D$, we will investigate the local cohomology group $H^1_D(\Theta_S(-\log D))$. Note that the local cohomology group can be regarded as the space of time variables for differential equations associated to (S, D) (cf. [STT, Section 3]).

We state our conjecture for the local cohomology:

Conjecture A.1 ([STT, Conjecture 3.1], [T1]). *Let (S, Y) be a rational Okamoto-Painlevé pair (S, Y) as above. Then we have*

$$H^1_D(\Theta_S(-\log D)) \simeq \mathbf{C}. \tag{A.1}$$

For the positivity of the dimension of the cohomology group, we have the following result:

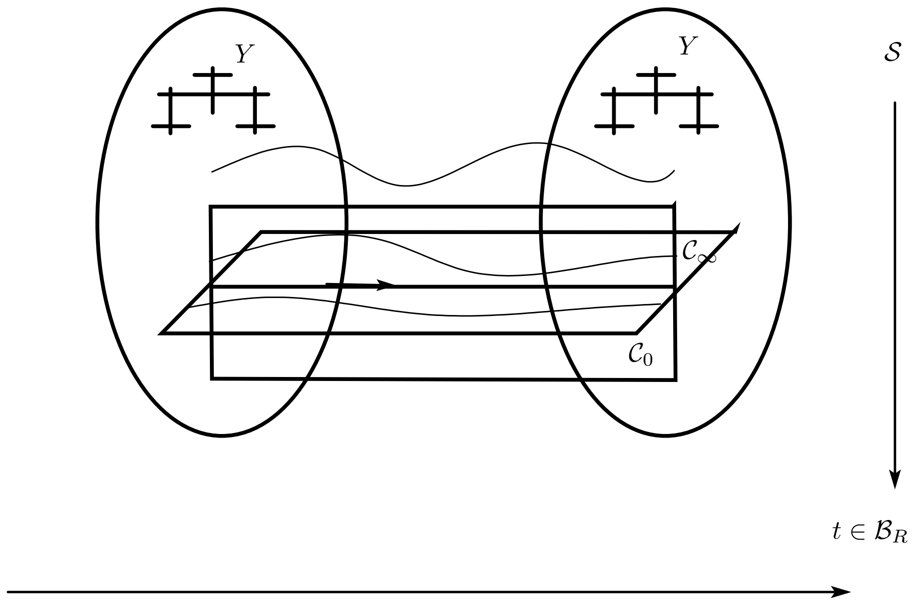

Figure 4. Rational solution coming from $\mathcal{C}_0 \cap \mathcal{C}_\infty$ for $\tilde{E}_6$ (P_{IV}).

Theorem A.1 ([T1, Theorem 2.1]).

$$\dim H^0(D, \Theta_S(-\log D) \otimes N_D) = 1. \tag{A.2}$$

Here we put $N_D = \mathcal{O}_S(D)/\mathcal{O}_S$.

In particular, a natural inclusion

$$H^0(D, \Theta_S(-\log D) \otimes N_D) \hookrightarrow H^1_D(\Theta_S(-\log D)),$$

implies

$$\dim H^1_D(\Theta_S(-\log D)) \geq 1. \tag{A.3}$$

On the other hand, in this section, we shall prove

Theorem A.2. *Let*

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & \end{array}$$

be the semi-universal deformation of rational Okamoto-Painlevé pairs (S, D) whose type is one of $\tilde{E}_8, \tilde{E}_7, \tilde{D}_8, \tilde{D}_6, \tilde{E}_6, \tilde{D}_5$ and $\tilde{D}_4$ (i.e., except for $R = \tilde{D}_7$). Then there is a Zariski open set $U \subset \mathcal{M}_R \times \mathcal{B}_R$ such that for any $(\boldsymbol{\alpha}, t) \in U$,

$$\dim H^1_{\mathcal{D}_{(\boldsymbol{\alpha},t)}}(\Theta_{\mathcal{S}_{(\boldsymbol{\alpha},t)}}(-\log \mathcal{D}_{(\boldsymbol{\alpha},t)})) = 1.$$

Remark 8. For (S,Y) of type $\tilde{D}_8$ or $\tilde{E}_8$, Theorem A.1 proves Conjecture A.1. In fact, we always have the inclusion

$$H^1_D(\Theta_S(-\log D)) \hookrightarrow H^1(S, \Theta_S(-\log D))$$

and $\dim H^1(S, \Theta_S(-\log D)) = 10 - 9 = 1$ for these cases.

From Remark 8, in order to show Theorem A.2, we will estimate the dimension of the local cohomology group for a special rational Okamoto-Painlevé pairs of other type R.

We first calculate some cohomology groups.

Lemma A.1. *Let (S,Y) be a rational Okamoto-Painlevé pair, and C a normal crossing divisor of S. Moreover, let $C = \sum_{i=1}^{s} C_i$ be an irreducible decomposition of C, and we assume that $\{C_i\}_{i=1}^{s}$ is linearly independent in $H^2(S, \mathbf{C}) \simeq \mathrm{Pic}(S) \otimes \mathbf{C}$. Then we have*

$$H^2(S, \Theta_S(-\log C)) = \{0\}.$$

Proof. We have only to replace D of [STT, Lemma 2.2 and Corollary 2.1] with C. □

Lemma A.2. *Let (S,Y) be a generalized rational Okamoto-Painlevé pair such that $D = Y_{red} = \sum_{i=1}^{r} Y_i$ is a normal crossing divisor with at least two irreducible components, say $r \geq 2$, and let $C = \sum_{i=1}^{s} C_i$ be a normal crossing divisor of S. We assume that*

(1) $C \subset S - D$,

(2) $C_i \simeq \mathbf{P}^1$,

(3) $\{Y_i, C_j | 1 \leq i \leq r, 1 \leq j \leq s\}$ is linearly independent.

Then we have

$$\dim H^1(S, \Theta_S(-\log(D + C))) = 10 - (r + s).$$

Proof. Note that assumption 1 implies $D + C$ is normal crossing and $K_S \cdot C_i = -Y \cdot C_i = 0$. We have $H^2(S, \Theta_S(-\log(D + C))) = 0$ by applying Lemma A.1 to $D + C$. Therefore by using the same argument as Proposition 2.2 in [STT], we have the assertion. □

Remark 9. We have the following exact sequence of sheaves:

$$0 \to \Theta_S(-\log(D + C)) \to \Theta_S(-\log(D + C - C_i)) \to N_{C_i/S} \to 0,$$

where $N_{C_i/S} = \mathcal{O}_S(C_i)/\mathcal{O}_S$ denotes the normal bundle of the divisor $C_i \subset S$. Note that since $N_{C_i/S} = \mathcal{O}_{C_i}(-2)$, we have $H^0(N_{C_i/S}) = \{0\}$. Then the morphism

$$H^0(\Theta_S(-\log(D + C))) \to H^0(\Theta_S(-\log(D + C - C_i)))$$

is injective. Moreover we have

$$\dim H^0(\Theta_S(-\log(D + C - C_i))) - \dim H^0(\Theta_S(-\log(D + C))) = 1$$

by Lemma A.2. This implies that there exist a deformation (S', D') of (S, D) such that only the curve C_i vanish and other nodal curves remain.

Now we obtain the following

Proposition A.1. *Let (S, Y) be a rational Okamoto-Painlevé pair "of non-fibered type " such that $D = Y_{red}$ is a normal crossing divisor with at least two irreducible components, say $r \geq 2$. We suppose the existence of a divisor $C = \sum_{i=1}^{9-r} C_i$ of S satisfying the conditions in Lemma* A.2. *Then we have*

$$\dim H^1_D(\Theta_S(-\log D)) \leq 1.$$

Proof. Let us consider the following exact sequence of local cohomology groups (cf. [Gr, Corollary 1.9])

$$H^0(S-D, \Theta_S(-\log(D+C))) \to H^1_D(\Theta_S(-\log(D+C))) \to H^1(S, \Theta_S(-\log(D+C))).$$

We have an inclusion

$$H^0(S - D, \Theta_S(-\log(D + C))) \hookrightarrow H^0(S - D, \Theta_S(-\log D)) = H^0(S - D, \Theta_S).$$

Since (S, Y) is of non-fibered type, from (2) of Proposition 2.1 in [STT], we have $H^0(S - D, \Theta_S) = \{0\}$. Therefore we have

$$H^0(S - D, \Theta_S(-\log(D + C))) = \{0\}.$$

By applying Lemma A.2, we see

$$H^1(S, \Theta_S(-\log(D + C))) \simeq \mathbf{C}.$$

Moreover, since $C \subset S - D$, we have $H^1_D(\Theta_S(-\log D)) \simeq H^1_D(\Theta_S(-\log(D + C)))$, which proves the assertion. □

Lemma A.3. *For the types $\tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{E}_7$ and $\tilde{E}_8$, there exists a rational Okamoto-Painlevé pair (S, Y) of non-fibered type satisfying the assumption of Proposition* A.1.

Proof. For each case, we only have to show the existence of nodal curves $C_j \subset S - D$ $j = 1, \ldots, 9 - r$ on a rational Okamoto-Painlevé pair (S, Y) of non-fibered type. The existence of (-2)-curves follows from Theorem 3.1. □

Remark 10. For any rational Okamoto-Painlevé pair (S, Y) of $\tilde{D}_7$, there is no (-2)-curve C on $S - D$ satisfying the condition in Lemma A.2 (cf. Table 4, 2).

Lemma A.3 and Theorem A.1 lead us the following corollary, which also implies Theorem A.2.

Corollary A.1. *For the types $\tilde{D}_4, \tilde{D}_5, \tilde{D}_6, \tilde{D}_8, \tilde{E}_7$ and $\tilde{E}_8$, there exists a rational Okamoto-Painlevé pair (S, Y) of non-fibered type such that*

$$\dim H^1_D(\Theta_S(-\log D)) = 1.$$

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