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Fundamental groups of spaces of holomorphic maps and group actions

By

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Abstract

For integers $d \geq 0$ and $1 \leq k \leq n$, let $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ denote the space consisting of all holomorphic maps $f : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ of degree d. We shall compute the fundamental group of $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ and study $\mathrm{PGL}_{n+1}(\mathbb{C})$-action on it.

1. Introduction

Let $1 \leq k \leq n$ be integers and $f : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ be a holomorphic map. Any such map can be represented by an $(n+1)$-tuple of homogeneous polynomials in $\mathbb{C}[z_0, z_1, \dots, z_k]$ of the same degree d without common roots except $\mathbf{0} = (0, \dots, 0) \in \mathbb{C}^{k+1}$ ([7]). We call the integer d as the degree of the holomorphic map f. For a fixed positive integer d, let $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ be the space consisting of all holomorphic maps $f : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ of degree d. Let $i : S^2 = \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^k$ be the natural inclusion given by $i([x_0 : x_1]) = [x_0 : x_1 : 0 : \cdots : 0]$. If $g : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ is any continuous map, the homotopy class of $g \circ i \in \pi_2(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}$ is also called the degree of g. One can show that it coincides with the polynomial degree above when g is a holomorphic map ([7]). We denote by $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ the space consisting of all continuous maps $g : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ of degree d. We also denote by $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ the subspace of $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ consisting of all base point preserving holomorphic maps $f : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ of degree d, and by $\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ the subspace of $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ consisting of base point preserving continuous maps. Let

$$\begin{cases} i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ i'_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ j'_d : \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{cases}$$

be corresponding inclusion maps.

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These spaces are of interest both from a classical and modern point of view (e.g. [1], [3]), and the author would like to study the topology of spaces $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ for the case $1 \leq k \leq n$. The principal motivation of this paper comes from the work due to G. Segal ([15]) who obtained the following remarkable theorem for the case $k = 1$.

Theorem 1.1 (G. B. Segal, [15]). *The inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences up to dimension $(2n-1)d$.

The topology of $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ have already been extensively studied for the case $k = 1$ after the work of Segal (cf. [5], [6]). So we shall mainly consider the case $2 \leq k \leq n$. As the first step to this problem, we shall compute the fundamental groups of $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$, and it is stated as follows:

Theorem A. *Let $d \geq 1$ and $1 \leq k \leq n$ be integers.*
(i) *If $k < n$, $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ are simply connected.*
(ii) *If $k = n$, there are isomorphisms*

$$\begin{cases} \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z} \\ \pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)d^n. \end{cases}$$

(iii) *The induced homomorphisms*

$$\begin{cases} i_{d*} : \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \\ j_{d*} : \pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \end{cases}$$

are isomorphisms.

Next we shall study the natural group action on $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$. Let $f \in \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ be any holomorphic map of degree one. Then it is represented by the $(n+1)$-tuple of polynomials in $\mathbb{C}[z_0, \ldots, z_n]$,

$$\begin{aligned} (f_0, f_1, \ldots, f_n) &= \left(\sum_{j=0}^{n} a_{j0} z_j, \sum_{j=0}^{n} a_{j1} z_j, \ldots, \sum_{j=0}^{n} a_{jn} z_j \right) \\ &= (z_0, z_1, \ldots, z_n) \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n0} & \cdots & \cdots & a_{nn} \end{pmatrix} \end{aligned}$$

that have no common root except $\mathbf{0} \in \mathbb{C}^{n+1}$. We note that $(f_0, \dots, f_n)$ have no common root except $\mathbf{0}$ if and only if $A = (a_{jm}) \in \mathrm{GL}_{n+1}(\mathbb{C})$. Hence, the correspondence $(f_0, \dots, f_n) \mapsto A$ induces an isomorphism of topological groups

$$\alpha_n : \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} \mathrm{PGL}_{n+1}(\mathbb{C}). \tag{1}$$

The restriction of α_n also defines an isomorphism

$$\beta_n : \mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} \mathrm{PB}_n, \tag{2}$$

where B_n and PB_n denote the subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$ consisting of all matrices of the form $\begin{pmatrix} a & \mathbf{0} \\ \mathbf{x} & A \end{pmatrix}$ $(a \in \mathbb{C}^*, \mathbf{x} \in \mathbb{C}^n, A \in \mathrm{GL}_n(\mathbb{C}))$ and the corresponding projective group, respectively. We remark that there is a homeomorphism $\mathrm{PB}_n \cong \mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}^n$. Define the right $\mathrm{PGL}_{n+1}(\mathbb{C})$ action on $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ by the usual matrix multiplication

$$\begin{array}{ccc} \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \times \mathrm{PGL}_{n+1}(\mathbb{C}) & \longrightarrow & \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ ([f_0 : \cdots : f_n], A) & \longrightarrow & [f_0 : \cdots : f_n] \cdot A \end{array} \tag{3}$$

We define the right PB_n-action on $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ in a similar way. Let $X_d^{k,n}$ be the orbit space defined by $X_d^{k,n} = \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)/\mathrm{PGL}_{n+1}(\mathbb{C})$. When $k = n$, we write $X_d^{nn} = X_d^n$. Now we recall the following result.

Theorem 1.2 (R. J. Milgram, [10]). *If $d \geq 1$ and $n = 1$, there is a homeomorphism $X_d^1 \cong \mathrm{P}(\mathcal{F}_d)$, where $\mathrm{P}(\mathcal{F}_d)$ denotes the space consisting of all projective classes of non-singular $(d \times d)$ Toeplitz matrices.*

It is very valuable to investigate the topology of spaces of finite Toeplitz matrices in the areas of applied mathematics, algebraic geometry and mathematical physics as explained in [10]. So it may be also valuable to study the topology of spaces X_d^n for $n \geq 2$, too. The second aim of this paper is to study the homotopy type of X_d^n and it is stated as follows.

Theorem B. *Let $d, n \geq 1$ be integers.*
(i) $\pi_1(X_d^n) = \mathbb{Z}/d^n$.
(ii) *There is a fibration sequence (up to homotopy)*

$$SU(n+1) \longrightarrow \tilde{X}_d^n \xrightarrow{\tilde{p}_d} \widetilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n), \tag{$*$}$$

where $\widetilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ and $\tilde{X}_d^n$ denote the universal coverings of $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ and X_d^n, respectively.

Corollary C (J. W. Havliceck, [8]). $\pi_1(\mathrm{P}(\mathcal{F}_d)) = \mathbb{Z}/d$.

Remark. (1) If $n = 1$, the fibration $(*)$ is trivial ([13]) and there is a homotopy equivalence $\tilde{X}_d^1 \simeq SU(2) \times \widetilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1)$. We do not know whether $(*)$ is trivial or not if $n \geq 2$.

(2) If $k > n$, there is no holomorphic map $\mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ except constant maps. So $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) = \emptyset$ if $k > n$ and $d \neq 0$.

This paper is organized as follows. In section 2, we shall give the proof of Theorem A, and in section 3, we shall prove Theorem B. Finally in appendix, we shall explain why the evaluation map $ev_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathbb{C}\mathrm{P}^n$ is a fibration with fiber $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. This fact might be well-known, although we cannot find any references. So we shall prove this for the completeness of this paper.

2. Fundamental groups

From now on, assume that $1 \leq k \leq n$, and let $\psi_d^{k,n} : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ denote the map given by $\psi_d^{k,n}([x_0 : \cdots : x_k]) = [x_0^d : \cdots : x_k^d : 0 : \cdots : 0]$ for $[x_0 : \cdots : x_k] \in \mathbb{C}\mathrm{P}^k$. Clearly, $\psi_d^{k,n} \in \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$, and we choose it as the base point of $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$, or of $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. Similarly, we choose the point $\mathbf{e}_m = [1 : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^m$ as the basepoint of $\mathbb{C}\mathrm{P}^m$. We denote by $ev_d : \mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathbb{C}\mathrm{P}^n$ the evaluation map defined by $ev_d(f) = f(\mathbf{e}_k)$ for $f \in \mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$.

Similarly, we also denote by $ev_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathbb{C}\mathrm{P}^n$ the restriction of ev_d to the subspace $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. There is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{i'_d} & \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev_d} & \mathbb{C}\mathrm{P}^n \\
{\scriptstyle i_d}\downarrow & & {\scriptstyle j_d}\downarrow & & =\downarrow \\
\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{j'_d} & \mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev_d} & \mathbb{C}\mathrm{P}^n
\end{array}$$

where two horizontal sequences are fibration sequences (cf. appendix).

Next define the two maps

$$\begin{cases} g_d : \mathrm{Map}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ h_d : \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{cases} \quad \text{by}$$

$$\begin{cases} g_d(g)([x_0 : \cdots : x_k]) = g([x_0^d : \cdots : x_k^d]) & g \in \mathrm{Map}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n), \\ h_d(f)([x_0 : \cdots : x_k]) = f([x_0^d : \cdots : x_k^d]) & f \in \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n). \end{cases}$$

We denoted by

$$\begin{cases} g'_d : \mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ h'_d : \mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{cases}$$

the corresponding restrictions of g_d or h_d to the subspaces $\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ or

$\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$, respectively. There are two commutative diagrams:

$$\begin{array}{ccccc}
\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{i_1'} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev_1} & \mathbb{C}\mathrm{P}^n \\
{\scriptstyle h_d'}\downarrow & & {\scriptstyle h_d}\downarrow & & {\scriptstyle =}\downarrow \\
\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{i_d'} & \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev_d} & \mathbb{C}\mathrm{P}^n
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{j_1'} & \mathrm{Map}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev_1} & \mathbb{C}\mathrm{P}^n \\
{\scriptstyle g_d'}\downarrow & & {\scriptstyle g_d}\downarrow & & {\scriptstyle =}\downarrow \\
\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{j_d'} & \mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev_d} & \mathbb{C}\mathrm{P}^n
\end{array}$$

where all horizontal sequences are fibration sequences. We recall the following two results.

Lemma 2.1 (J. M. Møller, [11])**.** *Let $d \geq 1$ and $1 \leq k \leq n$ be integers.*
(i) $\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ *is $2(n-k)$-connected and* $\pi_{2(n-k)+1}(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$.
(ii) *If $1 \leq k < n$,* $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ *is simply connected and* $\pi_2(\mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$.
(iii) *In particular, if $k = n$, then*

$$\begin{cases} \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}, \\ \pi_1(\mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)d^n. \end{cases}$$

Lemma 2.2 ([9])**.** *If $n \geq 1$ and $d = 1$, the induced homomorphisms*

$$\begin{cases} i_{1*} : \pi_1(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \\ j_{1*} : \pi_1(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \end{cases}$$

are isomorphisms.

Lemma 2.3. *The induced homomorphism*

$$\mathbb{Z} = \pi_1(\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{g'_{d*}} \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$$

is a multiplication by d^n.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/(n+1) & & \\
\| & & \| & & \| & & \\
\pi_2(\mathbb{C}\mathrm{P}^n) & \xrightarrow{\partial_1} & \pi_1(\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \xrightarrow{j'_{1*}} & \pi_1(\mathrm{Map}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle g'_{d*}}\downarrow & & {\scriptstyle g_{d*}}\downarrow & & \\
\pi_2(\mathbb{C}\mathrm{P}^n) & \xrightarrow{\partial_d} & \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \xrightarrow{j'_{d*}} & \pi_1(\mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/(n+1)d^n & &
\end{array}$$

where horizontal sequences are exact. Then it is easy to see that ∂_1 and ∂_d are identified with the multiplication maps by $(n+1)$ and by $(n+1)d^n$, respectively. Because $\partial_d = g'_{d*} \circ \partial_1$, the assertion follows. $\square$

Lemma 2.4. *The composite of induced homomorphisms*

$$\mathbb{Z} = \pi_1(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{i_{d*} \circ h'_{d*}} \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$$

is a multiplication by d^n.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) & \xrightarrow{h'_d} & \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \\
{\scriptstyle i_1}\big\downarrow & & {\scriptstyle i_d}\big\downarrow \\
\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) & \xrightarrow{g'_d} & \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)
\end{array}$$

It follows from [9] that $i_{1*} : \pi_1(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n))$ is bijective. Then the assertion follows from Lemma 2.3. $\square$

Theorem 2.1. $\pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$.

Proof. Since the case $n = 1$ was already proved in Theorem 1.1, from now on we assume $n \geq 2$. As in section 1, a map $f \in \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ can be represented as a $(n+1)$-tuple $(f_0, \dots, f_n)$ of homogeneous polynomials in $\mathbb{C}[z_0, \dots, z_n]$ of the same degree d, such that the coefficient of z_0^d of the first polynomial f_0 is 1 and the others 0, and which have no common root except $\mathbf{0}$. (This is equivalent to the condition $f(\mathbf{e}_n) = \mathbf{e}_n$.) Let $X_d \subset \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ be the subspace consisting of all $(n+1)$-tuples

$$(z_0^d, z_1^d, \dots, z_{n-2}^d, f, g) \in \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$$

such that $f, g \in \mathbb{C}[z_{n-1}, z_n]$ are homogeneous polynomials in $\mathbb{C}[z_{n-1}, z_n]$ of degree d, which satisfies the condition $[f : g] \in \mathrm{Hol}_d^*(S^2, S^2)$. Since the codimension of X_d in $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ is greater than one, the inclusion $i : X_d \to \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ induces a surjective homomorphism $i_* : \pi_1(X_d) \to \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n))$.

However, since $X_d \cong \mathrm{Hol}_d^*(S^2, S^2)$, $\pi_1(X_d) = \mathbb{Z}$ by Theorem 1.1. Hence, $\pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$ or $\mathbb{Z}/m$ for some integer $m \geq 1$. Now assume that $\pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/m$ for some integer $m \geq 1$. Consider the composite of homomorphisms

$$\begin{array}{ccccc}
\pi_1(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \xrightarrow{h'_{d*}} & \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \xrightarrow{i_{d*}} & \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \\
|| & & || & & || \\
\mathbb{Z} & & \mathbb{Z}/m & & \mathbb{Z}
\end{array}$$

Then, because i_{d*} must be trivial, $i_{d*} \circ h'_{d*}$ is trivial. However, this contradicts to Lemma 2.4, and we have $\pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$. $\square$

Theorem 2.2. *$i_{d*}:\pi_1(\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))\xrightarrow{\cong}\pi_1(\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))$ is an isomorphism for any integer $d\geq 1$.*

Proof. Since $\pi_1(\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))=\pi_1(\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))=\mathbb{Z}$, it suffices to prove that $i_{d*}:\pi_1(\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))\to\pi_1(\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))$ is surjective. Then by using the Hurewicz Theorem, it reduces to show that it induces a surjective homomorphism on $H_1(\ ,\mathbb{Z})$.

Let $i':\mathbb{C}\mathrm{P}^{n-1}\to\mathbb{C}\mathrm{P}^n$ denote the inclusion given by $i'([x_0:\cdots:x_{n-1}])=[x_0:\cdots:x_{n-1}:0]$, and define the restriction map $r:\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)\to\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^{n-1},\mathbb{C}\mathrm{P}^n)$ by $r(f)=f\circ i'$. It is a fibration with the fiber $F_d(n)$, where we take $F_d(n)=\{f\in\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n):f\circ i'=\psi^{n-1,n}_d\}$. We remark that there is a homotopy equivalence $F_d(n)\simeq\Omega^{2n}\mathbb{C}\mathrm{P}^n$ ([11]). Let $\mathrm{H}_d(n)\subset\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$ denote the subspace defined by $\mathrm{H}_d(n)=F_d(n)\cap\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$, and consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}_d(n) & \xrightarrow[\subset]{} & \mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n) & & \\
{\scriptstyle i''_d}\downarrow\cap & & {\scriptstyle i_d}\downarrow\cap & & \\
F_d(n) & \xrightarrow[\subset]{j'} & \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n) & \xrightarrow{r} & \mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^{n-1},\mathbb{C}\mathrm{P}^n).
\end{array}$$

Since $\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^{n-1},\mathbb{C}\mathrm{P}^n)$ is 2-connected (by Lemma 2.1), j' induces an isomorphism on π_1. By using the Hurewicz Theorem, it induces an isomorphism on $H_1(\ ,\mathbb{Z})$, too. Then because $i''_{d*}:H_1(\mathrm{H}_d(n),\mathbb{Z})\xrightarrow{\cong}H_1(F_d(n),\mathbb{Z})$ is an isomorphism by [12], $j'\circ i''_d$ also induces an isomorphism on $H_1(\ ,\mathbb{Z})$. Hence, i_d induces a surjective homomorphism on $H_1(\ ,\mathbb{Z})$. $\square$

Although the above proof is easier to understand, we cannot know the generator of $\pi_1(\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))=\mathbb{Z}$ explicitly. So we give the second proof.

The second proof of Theorem 2.2. As explained in the first proof, it remains to prove that $i_{d*}:\pi_1(\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))\to\pi_1(\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))$ is surjective. Let $\alpha:S^1\to\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$ denote the map defined by

$$\alpha(e^{i\theta})([\mathbf{x}])=[x_0^d:\cdots:x_{n-1}^d:e^{i\theta}x_n^d]$$

for $(e^{i\theta},[\mathbf{x}])=(e^{i\theta},[x_0:\cdots:x_n])\in S^1\times\mathbb{C}\mathrm{P}^n$.

Since $\alpha(S^1)\subset\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$, it is sufficient to show that α represents the generator of $\pi_1(\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))=\mathbb{Z}$. Because $j'_*:\pi_1(F_d(n))\xrightarrow{\cong}\pi_1(\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n))$ is bijective and $\alpha(S^1)\subset F_d(n)$, it reduces to show that α represents the generator of $\pi_1(F_d(n))=\mathbb{Z}$.

For this purpose, we recall the homotopy equivalence $\delta:\Omega^{2n}\mathbb{C}\mathrm{P}^n\xrightarrow{\simeq}F_d(n)$ defined by $\delta(h)=\nabla\circ(\psi^{n-1,n}_d\vee h)\circ\mu'$ for $h\in\Omega^{2n}\mathbb{C}\mathrm{P}^n$, where $\nabla:\mathbb{C}\mathrm{P}^n\vee\mathbb{C}\mathrm{P}^n\to\mathbb{C}\mathrm{P}^n$ and $\mu':\mathbb{C}\mathrm{P}^n\to\mathbb{C}\mathrm{P}^n\vee S^{2n}$ denote the folding map and co-action map obtained by collapsing the hemisphere of the top cell e^{2n} in $\mathbb{C}\mathrm{P}^n$, respectively.

We denote by $c:S^1\to F_d(n)$ the constant map at the base point $\psi^{n-1,n}_d$.

Then the maps c and α correspond to maps $c', \alpha' : S^1 \times \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ given by

$$\begin{cases} c'(e^{i\theta}, [x_0 : \cdots : x_n]) & = [x_0^d : \cdots : x_{n-1}^d : x_n^d], \\ \alpha'(e^{i\theta}, [x_0 : \cdots : x_n]) & = [x_0^d : \cdots : x_{n-1}^d : e^{i\theta} x_n^d]. \end{cases}$$

Two maps c' and α' agree on $(S^1 \times \mathbb{C}\mathrm{P}^{n-1}) \cup (\{1\} \times \mathbb{C}\mathrm{P}^n)$ and we wish to study the difference element between them. It will be sufficient to replace the pair $(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^{n-1})$ by the pair (D^{2n}, S^{2n-1}) using a characteristic map of the top cell e^{2n} in $\mathbb{C}\mathrm{P}^n$. A similar method given in [[14], p. 196-197] shows that the required difference element is the generator ι' of $\pi_{2n+1}(\mathbb{C}\mathrm{P}^n) = \pi_1(\Omega^{2n}\mathbb{C}\mathrm{P}^n)$. Then, by using the definition of the homotopy equivalence δ, we have $\delta_*(\iota') = \pm[\alpha] \in \pi_1(F_d(n)) = \mathbb{Z}$ and this completes the proof. □

Corollary 2.1. *If $d, n \geq 1$, the induced homomorphism*

$$\mathbb{Z} = \pi_1(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \xrightarrow{h'_{d*}} \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$$

is a multiplication by d^n.

Proof. This follows from Lemma 2.4 and Theorem 2.2. □

Theorem 2.3. *Let $d, n \geq 1$ be integers.*
(i) $\pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)d^n$.
(ii) $j_{d*} : \pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \overset{\cong}{\to} \pi_1(\mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n))$ *is an isomorphism.*

Proof. (i) Consider the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/(n+1) & & \\
\| & & \| & & \| & & \\
\pi_2(\mathbb{C}\mathrm{P}^n) & \xrightarrow{\partial_1} & \pi_1(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \xrightarrow{i'_{1*}} & \pi_1(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \longrightarrow & 0 \\
\| & & \Big\downarrow {\scriptstyle h'_{d*}} & & \Big\downarrow {\scriptstyle h_{d*}} & & \\
\pi_2(\mathbb{C}\mathrm{P}^n) & \xrightarrow{\partial_2} & \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \longrightarrow & \pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) & \longrightarrow & 0 \\
\| & & \| & & & & \\
\mathbb{Z} & & \mathbb{Z} & & & &
\end{array}$$

where horizontal sequences are exact. Because there is an isomorphism

$$\pi_1(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \underset{\cong}{\xrightarrow{\alpha_{n*}}} \pi_1(\mathrm{PGL}_{n+1}(\mathbb{C})) \cong \pi_1(\mathrm{P}SU(n+1)) = \mathbb{Z}/(n+1),$$

∂_1 is identified with the multiplication by $(n+1)$. Hence it follows from Corollary 2.1 that ∂_2 is a multiplication by $(n+1)d^n$. Hence $\pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)d^n$.

(ii) Consider the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/(n+1)d^n & & \\
\| & & \| & & \| & & \\
\pi_2(\mathbb{C}\mathrm{P}^n) & \xrightarrow{\partial_1'} & \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)) & \xrightarrow{i'_{d*}} & \pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle i_{d*}}\downarrow{\scriptstyle \cong} & & {\scriptstyle j_{d*}}\downarrow & & \\
\pi_2(\mathbb{C}\mathrm{P}^n) & \xrightarrow{\partial_2'} & \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)) & \xrightarrow{j'_{d*}} & \pi_1(\mathrm{Map}_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/(n+1)d^n & &
\end{array}$$

where horizontal sequences are exact.

Since i_{d*} is an isomorphism, j_{d*} is also an isomorphism. □

Corollary 2.2. *The induced homomorphism*

$$\mathbb{Z} = \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)) \xrightarrow{i'_{d*}} \pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)d^n$$

can be identified with the natural projection homomorphism $\mathbb{Z} \to \mathbb{Z}/(n+1)d^n$.

Proof. The assertion easily follows from the proof of Theorem 2.3. □

Proof of Theorem A. Since the assertions (ii) and (iii) easily follow from Theorems 2.1, 2.2, 2.3 and (i), it remains to prove the assertion (i).

Suppose that $1 \le k < n$ and let Y_d be the subspace of $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$ defined by the image of the map $h'_d : \mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n) \to \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$, $Y_d = h'_d(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n))$. We note that $\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$ is simply connected ([9]) and that the map h'_d induces a homeomorphism $\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n) \overset{\cong}{\to} Y_d$. Hence $\pi_1(Y_d) = 0$. Because the codimension of Y_d in $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$ is ≥ 2, the inclusion $Y_d \to \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$ induces a surjective homomorphism on π_1. Hence $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$ is simply connected.

Next, consider the exact sequence of the evaluation fibration

$$\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n) \to \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n) \xrightarrow{ev_d} \mathbb{C}\mathrm{P}^n.$$

Because $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$ and $\mathbb{C}\mathrm{P}^n$ are simply connected, $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k,\mathbb{C}\mathrm{P}^n)$ is also simply connected. □

3. $\mathrm{PGL}_{n+1}(\mathbb{C})$-action on $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n)$

In this section we shall prove Theorem B. From now on, we write

$$\mathrm{Hol}_d = \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n) \quad \text{and} \quad \mathrm{Hol}_d^* = \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n,\mathbb{C}\mathrm{P}^n).$$

Consider the right Hol_1-action on Hol_d given by the compositions of maps

$$\mathrm{Hol}_d \times \mathrm{Hol}_1 \to \mathrm{Hol}_d; \quad (f,g) \mapsto g \circ f.$$

We can consider the right Hol_1^*-action on Hol_d^* in a similar way. Using two isomorphisms α_n and β_n, we have two commutative diagrams:

$$
\begin{array}{ccc}
\mathrm{Hol}_d \times \mathrm{Hol}_1 & \xrightarrow{\circ} & \mathrm{Hol}_d \\
{\scriptstyle 1\times\alpha_n}\downarrow\cong & & =\downarrow \\
\mathrm{Hol}_d \times \mathrm{PGL}_{n+1}(\mathbb{C}) & \xrightarrow{\cdot} & \mathrm{Hol}_d
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Hol}_d^* \times \mathrm{Hol}_1^* & \xrightarrow{\circ} & \mathrm{Hol}_d^* \\
{\scriptstyle 1\times\beta_n}\downarrow\cong & & =\downarrow \\
\mathrm{Hol}_d^* \times \mathrm{PB}_n & \xrightarrow{\cdot} & \mathrm{Hol}_d^*
\end{array}
$$

Lemma 3.1.
(i) $\mathrm{PGL}_{n+1}(\mathbb{C})$ *acts on* Hol_d *freely. Similarly,* PB_n *acts on* Hol_d^* *freely.*
(ii) *There is a commutative diagram*

$$
\begin{array}{ccccc}
\mathrm{PB}_n & \xrightarrow{s'_d} & \mathrm{Hol}_d^* & \xrightarrow{p'_d} & \mathrm{Hol}_d^*/\mathrm{PB}_n \\
{\scriptstyle g_n}\downarrow\cap & & {\scriptstyle i'_d}\downarrow & & {\scriptstyle q_d}\downarrow\cong \\
\mathrm{PGL}_{n+1}(\mathbb{C}) & \xrightarrow{s_d} & \mathrm{Hol}_d & \xrightarrow{p_d} & X_d^n
\end{array}
\tag{4}
$$

where two horizontal sequences are principal fibrations and q_d *is a homeomorphism.*

Proof. Since (i) is clear, it is sufficient to show (ii). It suffices to show that the natural projection q_d is a homeomorphism. Because $\mathrm{PGL}_{n+1}(\mathbb{C})$ acts on $\mathbb{C}\mathrm{P}^n$ transitively, the induced map q_d is surjective. Since $(f\cdot \mathrm{PGL}_{n+1}(\mathbb{C}))\cap \mathrm{Hol}_d^* = \mathrm{PB}_n\cdot f$ for any $f\in \mathrm{Hol}_d^*$, q_d is injective. If we identify these spaces by q_d, it is easy to see that the topologies coincide. □

Theorem 3.1. $\pi_1(X_d^n) = \mathbb{Z}/d^n$.

Proof. Because q_d is a homeomorphism, it is sufficient to show that the fundamental group $\pi_1(\mathrm{Hol}_d^*/\mathrm{PB}_n)$ is isomorphic to $\mathbb{Z}/d^n$. Consider the principal fibration sequence $\mathrm{PB}_n \xrightarrow{s'_d} \mathrm{Hol}_d^* \xrightarrow{p'_d} \mathrm{Hol}_d^*/\mathrm{PB}_n$. Since we choose the point $\psi_d^{n,n} = [z_0^d : z_1^d : \cdots : z_n^d]$ as the base point of Hol_d^*, the map s'_d can be represented by the matrix multiplication $s'_d(A) = [z_0^d : z_d^d : \cdots : z_n^d]\cdot A$. Hence, it follows from the definition of h'_d that there is a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Hol}_1^* & \xrightarrow{h'_d} & \mathrm{Hol}_d^* & \longrightarrow & \mathrm{Hol}_d^*/\mathrm{Hol}_1^* \\
{\scriptstyle \beta_n}\downarrow\cong & & =\downarrow & & {\scriptstyle \gamma_d}\downarrow\cong \\
\mathrm{PB}_n(\mathbb{C}) & \xrightarrow{s'_d} & \mathrm{Hol}_d^* & \xrightarrow{p'_d} & \mathrm{Hol}_d^*/\mathrm{PB}_n
\end{array}
$$

where two horizontal sequences are principal fibration sequences and the natural projection γ_d is a homeomorphism. So this induces a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{\times d^n} & \mathbb{Z} & & & & \\
\| & & \| & & & & \\
\pi_1(\mathrm{Hol}_1^*) & \xrightarrow{h'_{d*}} & \pi_1(\mathrm{Hol}_d^*) & \longrightarrow & \pi_1(\mathrm{Hol}_d^* \,/\, \mathrm{Hol}_1^*) & \longrightarrow & 0 \\
\beta_{n*}\downarrow\cong & & =\downarrow & & \gamma_{d*}\downarrow\cong & & \\
\pi_1(\mathrm{PB}_n) & \xrightarrow{s'_{d*}} & \pi_1(\mathrm{Hol}_d^*) & \xrightarrow{p'_{d*}} & \pi_1(\mathrm{Hol}_d^* /\mathrm{PB}_n) & \longrightarrow & 0
\end{array}$$

where two horizontal sequences are exact. It follows from Corollary 2.1 that h'_{d*} is a multiplication by d^n. Hence, $\mathbb{Z}/d^n = \pi_1(\mathrm{Hol}_d^* /\mathrm{PB}_n)$. $\square$

Corollary 3.1. (i) *There is a short exact sequence*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(\mathrm{PB}_n) & \xrightarrow{s'_{d*}} & \pi_1(\mathrm{Hol}_d^*) & \xrightarrow{p'_{d*}} & \pi_1(\mathrm{Hol}_d^* /\mathrm{PB}_n) & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
& & \mathbb{Z} & \xrightarrow{\mu'_{d^n}} & \mathbb{Z} & \xrightarrow{\rho'} & \mathbb{Z}/d^n & &
\end{array}$$

where $\mu'_{d^n} : \mathbb{Z} \to \mathbb{Z}$ is a multiplication by d^n and $\rho' : \mathbb{Z} \to \mathbb{Z}/d^n$ is a natural projection.
(ii) *There is a short exact sequence*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(\mathrm{PGL}_{n+1}(\mathbb{C})) & \xrightarrow{s_{d*}} & \pi_1(\mathrm{Hol}_d) & \xrightarrow{p_{d*}} & \pi_1(X_d^n) & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
& & \mathbb{Z}/(n+1) & \xrightarrow{\mu_{d^n}} & \mathbb{Z}/(n+1)d^n & \xrightarrow{\rho} & \mathbb{Z}/d^n & &
\end{array}$$

where $\mu_{d^n} : \mathbb{Z}/(n+1) \to \mathbb{Z}/(n+1)d^n$ is a multiplication by d^n and $\rho : \mathbb{Z}/(n+1)d^n \to \mathbb{Z}/d^n$ is a natural projection.

Proof. Since the assertion (i) easily follows from the proof of Theorem 3.1, we shall prove (ii). Consider the following commutative diagram induced from (4):

$$\begin{array}{ccccccccc}
& & \mathbb{Z} & \xrightarrow{\mu'_{d^n}} & \mathbb{Z} & \xrightarrow{\rho'} & \mathbb{Z}/d^n & & \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & \pi_1(\mathrm{PB}_n) & \xrightarrow{s'_{d*}} & \pi_1(\mathrm{Hol}_d^*) & \xrightarrow{p'_{d*}} & \pi_1(\mathrm{Hol}_d^* /\mathrm{PB}_n) & \longrightarrow & 0 \\
& & g_{n*}\downarrow & & i'_{d*}\downarrow & & q_{d*}\downarrow\cong & & \\
& & \pi_1(\mathrm{PGL}_{n+1}(\mathbb{C})) & \xrightarrow{s_{d*}} & \pi_1(\mathrm{Hol}_d) & \xrightarrow{p_{d*}} & \pi_1(X_d^n) & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
& & \mathbb{Z}/(n+1) & \longrightarrow & \mathbb{Z}/(n+1)d^n & \longrightarrow & \mathbb{Z}/d^n & &
\end{array}$$

where all horizontal sequences are exact.

Since i'_{d*} can be identified with the natural projection homomorphism $\mathbb{Z} \to \mathbb{Z}/(n+1)d^n$ by Corollary 2.2, p_{d*} can be also identified with the natural projection $\rho : \mathbb{Z}/(n+1)d^n \to \mathbb{Z}/d^n$ and we have $\mathrm{Ker}\, p_{d*} \cong \mathbb{Z}/(n+1)$. Hence s_{d*} must be injective and we obtained the assertion (ii). $\square$

Proof of Theorem B. It follows from Theorem 3.1 that it suffices to show that there is a fibration $(*)$ (up to homotopy). Let $\iota : X^n_d \to B\mathbb{Z}/d^n = K(\mathbb{Z}/d^n, 1)$ and $\iota' : \mathrm{Hol}_d \to B\mathbb{Z}/(n+1)d^n = K(\mathbb{Z}/(n+1)d^n, 1)$ denote the maps which represent the generators of $H^1(X^n_d, \mathbb{Z}/d^n) \cong \mathbb{Z}/d^n$ and $H^1(\mathrm{Hol}_d, \mathbb{Z}/(n+1)d^n) \cong \mathbb{Z}/(n+1)d^n$, respectively. Then it follows from (ii) of Corollary 3.1 and [[2], (2.1)] that there is a homotopy commutative diagram

$$
\begin{array}{ccccc}
F & \longrightarrow & \widetilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{p}_d} & \tilde{X}^n_d \\
{\scriptstyle\pi}\downarrow & & \downarrow & & \downarrow \\
\mathrm{PGL}_{n+1}(\mathbb{C}) & \xrightarrow{s_d} & \mathrm{Hol}_d & \xrightarrow{p_d} & X^n_d \\
{\scriptstyle\iota_1}\downarrow & & {\scriptstyle\iota'}\downarrow & & {\scriptstyle\iota}\downarrow \\
B\mathbb{Z}/(n+1) & \xrightarrow{B\mu_{d^n}} & B\mathbb{Z}/(n+1)d^n & \xrightarrow{B\rho} & B\mathbb{Z}/d^n
\end{array}
\tag{5}
$$

where all horizontal and vertical sequences are fibration sequences. It suffices to show that there is a homotopy equivalence $F \simeq SU(n+1)$.

First, using (ii) of Corollary 3.1 and the diagram chasing of (5), we can easily show that $\iota_{1*} : \pi_1(\mathrm{PGL}_{n+1}(\mathbb{C})) \xrightarrow{\cong} \pi_1(B\mathbb{Z}/(n+1)) = \mathbb{Z}/(n+1)$ is an isomorphism. Hence, F is connected. Next, if we consider the fibration sequence $F \to \widetilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \xrightarrow{\tilde{p}_d} \tilde{X}^n_d$, we have $\pi_1(F) = 0$. Hence, $\pi : F \to \mathrm{PGL}_{n+1}(\mathbb{C})$ is a universal covering, and we have a homotopy equivalence $F \simeq SU(n+1)$. $\square$

4. Appendix

The following result may be well-known, but for completeness of this paper we shall give its proof.

Lemma A.1. *If $1 \le k \le n$, the sequence*

$$\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{i'_d} \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{ev_d} \mathbb{C}\mathrm{P}^n$$

is a fibration sequence.

Proof. Let $p, p' \in \mathbb{C}\mathrm{P}^n$ be any two points. Then there is $A \in \mathrm{PGL}_{n+1}(\mathbb{C})$ such that $p' \cdot A = p$. Define the map $\phi_A : ev_d^{-1}(p') \to \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ by the matrix multiplication $\phi_A([f_0 : \cdots : f_n]) = [f_0 : \cdots : f_n] \cdot A$. If $[f_0 : \cdots : f_n] \in ev_d^{-1}(p')$, $ev_d(\phi_A([f_0 : \cdots : f_n])) = [f_0(\mathbf{e}_n) : \cdots : f_n(\mathbf{e}_n)] \cdot A = p' \cdot A = p$. Hence, $\mathrm{Im}\, \phi_A \subset ev_d^{-1}(p)$, and the map ϕ_A induces the map $\phi_A : ev_d^{-1}(p') \to ev_d^{-1}(p)$.

In this case, an easy computation shows that $\phi_A^{-1} = \phi_{A^{-1}}$, and so that $\phi_A : ev_d^{-1}(p') \xrightarrow{\cong} ev_d^{-1}(p)$ is a homeomorphism. That is, any two fibers of ev_d are homeomorphic. It remains to show that the local triviality hold.

Let $p \in \mathbb{C}\mathrm{P}^n$ be any point. Then there exists a triple (U, V, ϕ) satisfying the following conditions:

(i) $U \subset \mathbb{C}\mathrm{P}^n$ is an open neighborhood of p, and V is a subset of $\mathrm{PGL}_{n+1}(\mathbb{C})$.

(ii) $\phi : U \xrightarrow{\cong} V$ is a homeomorphism such that $\mathbf{e}_n \cdot \phi(q) = q$ for any point $q \in U$.

Define the map $\Phi : U \times \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ by the matrix multiplication $\Phi(q, [f_0 : \cdots : f_n]) = [f_0 : \cdots : f_n] \cdot \phi(q)$. Since

$$ev_d(\Phi(q, [f_0 : \cdots : f_n])) = [f_0(\mathbf{e}_k) : \cdots : f_n(\mathbf{e}_k)] \cdot \phi(q) = \mathbf{e}_n \cdot \phi(q) = q \in U,$$

Im Φ is contained in $ev_d^{-1}(U)$, and Φ may be regarded as the map

$$\Phi : U \times \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to ev_d^{-1}(U).$$

Moreover, there is a commutative diagram

$$\begin{array}{ccc}
U \times \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\Phi} & ev_d^{-1}(U) \\
{\scriptstyle \pi_1}\downarrow & & {\scriptstyle ev_d}\downarrow \\
U & \xrightarrow{=} & U
\end{array}$$

where π_1 denotes the first projection. Now, define the map $\Psi : ev_d^{-1}(U) \to U \times \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ by $\Psi(f) = (f(\mathbf{e}_k), f \cdot \phi(ev_d(f))^{-1})$. Direct computation shows that $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$. Hence, $\Phi : U \times \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{\cong} ev_d^{-1}(U)$ is a homeomorphism. $\square$

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