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Sharp lower bound for the lifespan of systems of semilinear wave equations with multiple speeds*

By

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Abstract

We consider the Cauchy problem for a system of semilinear wave equations with multiple propagation speeds in three space dimensions. We assume that quadratic terms in its nonlinearity consist of products of unknowns whose speeds are different from each other. It is known that some classical solution to the above type of system may blow up in finite time. In this article, we get the sharp lower bound for the lifespan of classical solutions to this kind of system.

1. Introduction

In this paper, we investigate lifespan estimates for the Cauchy problem to systems of semilinear wave equations in three space dimensions. We define

$$\Box_c = \partial_t^2 - c^2\Delta_x \text{ for } c > 0.$$

We consider

$$\begin{cases} \Box_{c_i} u_i(t,x) = F_i\big(u(t,x)\big) \text{ for } (t,x) \in (0,\infty)\times\mathbb{R}^3 & (1 \le i \le m), \\ u_i(x) = \varepsilon f_i(0,x),\ \partial_t u_i(0,x) = \varepsilon g_i(x) \text{ for } x \in \mathbb{R}^3 & (1 \le i \le m), \end{cases} \tag{1.1}$$

where c_i $(1 \le i \le m)$ are positive constants, $u = (u_i)_{1\le i\le m}$, and ε is a small and positive parameter.

For a while, we assume that the initial data $f = (f_i)_{1\le i\le m}$ and $g = (g_i)_{1\le j\le m}$ are compactly supported and sufficiently smooth functions.

We define T_ε by

$$\begin{aligned} T_\varepsilon &= T_\varepsilon(F,f,g) \\ &= \sup\{T \in (0,\infty);\ (1.1) \text{ possesses a } C^2\text{-solution } u \text{ for } 0 \le t < T\}, \end{aligned} \tag{1.2}$$

which is called lifespan of classical solutions.

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First we consider the single case $m = 1$:

$$\begin{cases} \Box u = F(u) & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u = \varepsilon f,\ \partial_t u = \varepsilon g & \text{at } t = 0 \end{cases} \tag{1.3}$$

where $\Box = \Box_1 = \partial_t^2 - \Delta_x$, $F(u) = |u|^{p-1}u$ or $|u|^p$ with $p > 1$. It is well known that the critical power is $1 + \sqrt{2}$ for this kind of semilinear wave equation. If $p > 1 + \sqrt{2}$, (1.1) possesses a global C^2–solution for any f and g, namely $T_\varepsilon = \infty$, provided that ε is sufficiently small. On the other hand, when $1 < p \le 1 + \sqrt{2}$, there exist some f and g such that the solution to (1.3) blows up in finite time, that is $T_\varepsilon < \infty$, no matter how small ε is (see Asakura [1], John [2], Kubota [7], Schaeffer [10], Sideris [11], Strauss [12], Tsutaya [14] for example). The lifespan estimate for the case $1 < p \le 1 + \sqrt{2}$ is also studied well, and when p is the critical value $1 + \sqrt{2}$, it is known that for any f and g we have $T_\varepsilon \ge \exp(C_1\varepsilon^{-p(p-1)})$ with some positive constant C_1. This lower bound is sharp in the sense that $T_\varepsilon \le \exp(C_2\varepsilon^{-p(p-1)})$ holds for some f, g and C_2 (see Lindblad [9], Takamura [13] and Zhou [15]).

Next we turn our attention to the system (1.1) with $m \ge 2$. It is known that difference of the propagation speeds makes the critical power lower than $1 + \sqrt{2}$ in some cases, but such nonlinearity is unstable under the perturbation of nonlinearity of higher order. To explain the situation clearly, let us consider the following example:

$$\begin{cases} \Box_{c_1} u_1 = \alpha_1 u_1 u_2 + \beta_1 u_2^3, \\ \Box_{c_2} u_2 = \alpha_2 u_1 u_2 + \beta_2 u_1^3, \end{cases} \tag{1.4}$$

where α_j, $\beta_j \in \mathbb{R}$ $(j = 1, 2)$. When $\alpha_1 = \alpha_2 = 0$, there exists a global solution to (1.4) for small data, as is expected from $3 > 1 + \sqrt{2}$. When $\beta_1 = \beta_2 = 0$, blow-up of solutions may occur in general. But if we further assume $c_1 \ne c_2$, (1.4) admits a global solution for small data, although the power of nonlinearity here is 2 which is less than $1 + \sqrt{2}$ (see Kubo–Ohta [6]). Now we are led to a question: Does a global solution exist for the case $c_1 \ne c_2$ no matter how we choose the coefficients α_j and β_j $(j = 1, 2)$? The answer is no by the following result of Kubo–Ohta [6] [*1]: Let $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$. When $c_1 > c_2$, we have $T_\varepsilon = \infty$ for any f and g. On the other hand, when $c_1 < c_2$, we have the upper bound of lifespan

$$T_\varepsilon \le \exp(C_1\varepsilon^{-3}) \tag{1.5}$$

for some f and g, where C_1 is a constant depending on f and g. This upper bound seems somewhat different from the critical case for the single equation

[*1] More precisely, they have proved the result for C^0–solutions to the system

$$\begin{cases} \Box_{c_1} u_1 = |u_1 u_2|, \\ \Box_{c_2} u_2 = |u_1|^3, \end{cases}$$

but our example here also can be treated by their method without any change in their proof.

(1.3). Note that what we get by formally substituting 2 for p of $\exp(C\varepsilon^{-p(p-1)})$, the sharp bound for (1.3) with the critical power, is $\exp(C\varepsilon^{-2})$, which is shorter than $\exp(C\varepsilon^{-3})$. Therefore it is interesting to see whether (1.5) is sharp or not.

For that purpose, we want to investigate the lower bound for the lifespan of solutions to systems of the type (1.4). For $\kappa \geq 0$, we define

$$X^\kappa = \left\{ (f,g) \in C^3(\mathbb{R}^3;\mathbb{R}^m) \times C^2(\mathbb{R}^3;\mathbb{R}^m); \right.$$
$$\left. \sum_{|\alpha|\leq 3} \|\partial_x^\alpha f\|_{2+\kappa} + \sum_{|\alpha|\leq 2} \|\partial_x^\alpha g\|_{2+\kappa} < \infty \right\},$$

where

$$\|\phi\|_\rho = \sup_{x\in\mathbb{R}^3} (1+|x|)^\rho |\phi(x)|.$$

Our main result is the following:

Theorem 1.1. *Assume that F_i $(1 \leq i \leq m)$ can be written as*

$$F_i(u) = \sum_{(j,k)\in R} \alpha_{ijk} u_j u_k + H_i(u), \tag{1.6}$$

where

$$R = \left\{ (j,k) \in \{1,\dots,m\}^2; c_j \neq c_k \right\}, \tag{1.7}$$

H_i is a C^2 function satisfying

$$H_i(u) = O(|u|^3) \text{ near } u = 0, \tag{1.8}$$

and $\alpha_{ijk} \in \mathbb{R}$ are constants. Suppose $(f,g) \in X^\kappa$ with $\kappa > 1$. Then there exist positive constants ε_0 and C, depending on f and g, such that for any $\varepsilon \in (0,\varepsilon_0]$ the Cauchy problem (1.1) *admits a unique solution $u \in C^2([0,T_\varepsilon) \times \mathbb{R}^3;\mathbb{R}^m)$ with*

$$T_\varepsilon \geq \exp(C\varepsilon^{-3}). \tag{1.9}$$

Remark. (1) If we are just looking for C^0–solutions, we can also treat the nonlinearity like

$$F_i(u) = \sum_{(j,k)\in R} \alpha_{ijk}|u_j u_k| + \sum_{j,k,l=1}^m \beta_{ijkl}|u_j u_k u_l|$$

to get the same lower bound for the lifespan.
(2) The above theorem shows that (1.5) is sharp. Conversely, our lower bound (1.9) is also sharp because of (1.5).

The main idea in our proof of Theorem 1.1 is to regard (1.1) with (1.6) as a perturbation of a nonlinear system which possesses a global solution of nice behavior. The proof will be given in Section 3.

2. Basic decay estimates

In what follows, we always suppose that c_j $(1 \le j \le m)$ are the positive constants which appeared in (1.1).

In this section we state basic decay estimates for wave equations. For that purpose, we introduce some weights. We define

$$w_+(t,r) = 1 + t + r \tag{2.1}$$

for $(t,r) \in [0,\infty) \times [0,\infty)$, and

$$w_c(t,r) = 1 + |\, ct - r\,| \tag{2.2}$$

for $c \ge 0$ and $(t,r) \in [0,\infty) \times [0,\infty)$.

We also define

$$w_-(t,r) = \min_{1 \le j \le m} w_{c_j}(t,r). \tag{2.3}$$

Note that there exists a positive constant C such that

$$w_{c_j}(t,r) w_{c_k}(t,r) \ge C w_+(t,r) w_-(t,r) \tag{2.4}$$

holds for any $(t,r) \in [0,\infty) \times [0,\infty)$, provided $c_j \ne c_k$.

For a positive constant c and a continuous function $\Phi = \Phi(t,x)$, we define

$$L_c[\Phi](t,x) = \int_0^t \frac{1}{4\pi c^2 (t-\tau)} \left(\int_{|x-y|=c(t-\tau)} \Phi(\tau,y)\, dS_y \right) d\tau, \tag{2.5}$$

where dS_y is the surface element of a sphere whose center and radius are x and $c(t-\tau)$, respectively. For a positive constant c, a C^1-function $\phi = \phi(x)$ and a continuous function $\psi = \psi(x)$, we also define

$$\begin{aligned} U_c[\phi,\psi](t,x) =& \partial_t \left(\frac{1}{4\pi c^2 t} \int_{|x-y|=ct} \phi(y)\, dS_y \right) \\ & + \frac{1}{4\pi c^2 t} \int_{|x-y|=ct} \psi(y)\, dS_y, \end{aligned} \tag{2.6}$$

where dS_y denotes the surface element of a sphere whose center and radius are x and ct, respectively.

It is well known that the classical solution v to

$$\begin{cases} \Box_c v(t,x) = \Phi(t,x) \text{ in } [0,\infty) \times \mathbb{R}^3, \\ v(0,x) = \phi(x),\ (\partial_t v)(0,x) = \psi(x) \text{ for } x \in \mathbb{R}^3 \end{cases} \tag{2.7}$$

can be written as

$$v(t,x) = U_c[\phi,\psi](t,x) + L_c[\Phi](t,x). \tag{2.8}$$

Conversely, if $\phi \in C^3$, $\psi \in C^2$, and $\partial_x^\alpha \Phi \in C$ ($|\alpha| \le 2$), then v defined by (2.8) is the classical solution to (2.7).

We start with a decay estimate for homogeneous wave equations.

Lemma 2.1. *Let $c > 0$ and $\kappa \ge 1$. Then we have*

$$w_+(t,|x|)w_c(t,|x|)^\kappa \, |U_c[\phi,\psi](t,x)| \le C\left(\|\phi\|_{2+\kappa} + \|\partial_x\phi\|_{2+\kappa} + \|\psi\|_{2+\kappa}\right) \tag{2.9}$$

for any $(t,x) \in [0,\infty) \times \mathbb{R}^3$. Here C is a positive constant.

For the proof, see Asakura [1] for instance.

Next we state a decay estimate for inhomogeneous wave equations.

Lemma 2.2. *Let c be a positive constant. Suppose $\delta \ge 0$ and $\kappa > 0$. Then we have*

$$\begin{aligned} &w_+(t,|x|)w_c(t,|x|)^\kappa \, |L_c[\Phi](t,x)| \\ &\qquad \le C\Psi_\delta(t) \sup_{(\tau,y)\in[0,t]\times\mathbb{R}^3} |y| w_+(\tau,|y|)^{1+\kappa} w_-(\tau,|y|)^{1+\delta} |\Phi(\tau,y)| \end{aligned} \tag{2.10}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^3$, where

$$\Psi_\delta(t) = \begin{cases} \log(2+t) & \textit{when } \delta = 0, \\ 1 & \textit{when } \delta > 0. \end{cases}$$

Proof. This kind of decay estimate is studied by many authors (see the references cited in Section 1; see also Katayama [4] and [5], and Kubota–Yokoyama [8]). But for the sake of completeness, we give a proof here. Without loss of generality, we may assume $c = 1$. Then, we can express $L_1[\Phi]$ as

$$L_1[\Phi](t,x) = \frac{1}{4\pi r}\int_0^t d\tau \int_{|r-(t-\tau)|}^{r+t-\tau} \lambda d\lambda \int_0^{2\pi} \Phi\Big(\tau, \lambda\Theta\big(\tau,\lambda,\varphi;t,x\big)\Big) d\varphi, \tag{2.11}$$

where $r = |x|$ and Θ is some S^2-valued function (see John [3] for the detail).

Set

$$I_{\kappa,\delta}(t,r) = \frac{1}{r}\int_0^t d\tau \int_{|r-(t-\tau)|}^{r+t-\tau} w_+(\tau,\lambda)^{-(1+\kappa)} w_-(\tau,\lambda)^{-(1+\delta)} d\lambda. \tag{2.12}$$

Our task is to prove

$$I_{\kappa,\delta}(t,r) \le C\Psi_\delta(t) w_+(t,r)^{-1} w_1(t,r)^{-\kappa}, \tag{2.13}$$

because (2.11) and (2.13) yield (2.10) immediately.

Assume $r \geq 2\big(1+\max_{j=1,\ldots,m} c_j\big)t$. Then we have $w_1(t,r) \geq Cw_+(t,r)$. Noting that we also have $w_-(\tau,\lambda) \geq Cw_+(\tau,\lambda)$ for (τ,λ) satisfying $|r-(t-\tau)| \leq \lambda \leq r+t-\tau$ and $0 \leq \tau \leq t$, we obtain

$$\begin{aligned}
I_{\kappa,\delta}(t,r) \leq & I_{\kappa,0}(t,r) \leq \frac{C}{r}\int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} w_+(\tau,\lambda)^{-(2+\kappa)}d\lambda \\
\leq & \frac{C}{r}\int_0^t (1+r-t+2\tau)^{-(1+\kappa)}d\tau \\
\leq & \frac{Ct}{r} w_1(t,r)^{-(1+\kappa)} \leq Cw_+(t,r)^{-1}w_1(t,r)^{-\kappa},
\end{aligned}$$

which implies (2.13).

Now we assume $r \leq 2\big(1+\max_{j=1,\ldots,m} c_j\big)t$. For $j=1,\ldots,m$, we introduce

$$I_{j,\kappa,\delta}(t,r) = \frac{1}{r}\int_0^t d\tau \int_{|r-(t-\tau)|}^{r+t-\tau} w_+(\tau,\lambda)^{-(1+\kappa)} w_{c_j}(\tau,\lambda)^{-(1+\delta)}d\lambda. \tag{2.14}$$

We are going to prove that each $I_{j,\kappa,\delta}$ is bounded by the right-hand side of (2.13). Once we prove it, we immediately find (2.13) true, because we have

$$I_{\kappa,\delta}(t,r) \leq \sum_{j=1}^m I_{j,\kappa,\delta}(t,r). \tag{2.15}$$

By setting

$$p=\tau+\lambda,\ q=\lambda-c_j\tau, \tag{2.16}$$

we get

$$I_{j,\kappa,\delta}(t,r) = \frac{1}{(c_j+1)r}\int_{|t-r|}^{t+r}(1+p)^{-(1+\kappa)}dp\int_{p_j}^{p}(1+|q|)^{-(1+\delta)}dq, \tag{2.17}$$

where $2p_j=(1-c_j)p+(1+c_j)(r-t)$. Noting that we have $p_j \geq -c_j p$ for $p \geq |t-r|$, we obtain

$$\begin{aligned}
I_{j,\kappa,\delta}(t,r) \leq & \frac{C}{r}\int_{|t-r|}^{t+r}(1+p)^{-(1+\kappa)}dp\int_{-c_jp}^{p}(1+|q|)^{-(1+\delta)}dq \\
\leq & \frac{C}{r}\Psi_\delta(t)\int_{|t-r|}^{t+r}(1+p)^{-(1+\kappa)}dp.
\end{aligned} \tag{2.18}$$

Since we have

$$\int_{|t-r|}^{t+r}(1+p)^{-(1+\kappa)}dp \leq (t+r-|t-r|)w_1(t,r)^{-(1+\kappa)} \leq 2rw_1(t,r)^{-(1+\kappa)},$$

(2.18) leads to

$$I_{j,\kappa,\delta}(t,r) \leq C\Psi_\delta(t)w_1(t,r)^{-(1+\kappa)}. \tag{2.19}$$

When $r \le t/2$, or when $r \le 1/2$, (2.19) implies (2.13) because we have $w_+(t,r) \le Cw_1(t,r)$ for such t and r.

On the other hand, if $r \ge t/2$ and $r \ge 1/2$, since we have $r \ge Cw_+(t,r)$ and

$$\int_{|t-r|}^{t+r} (1+p)^{-(1+\kappa)} dp \le Cw_1(t,r)^{-\kappa}$$

for such t and r, (2.13) follows from (2.18). This completes the proof. $\square$

3. Proof of the main theorem

In this section, we will give a proof of Theorem 1.1.

Using the weights w_+ and w_c defined in Section 2, we introduce

$$\|\psi(t,\cdot)\|_{s,\kappa} = \sum_{j=1}^{m} \sum_{|\alpha|\le s} \sup_{x\in\mathbb{R}^3} \left| w_+(t,|x|) w_{c_j}(t,|x|)^\kappa \partial_x^\alpha \psi_j(t,x)\right| \tag{3.1}$$

for a sufficiently smooth function $\psi = (\psi_j)_{1\le j\le m}$, a non-negative integer s and a positive constant κ. We also define

$$\|\psi\|_{s,\kappa,T} = \sup_{t\in[0,T)} \|\psi(t,\cdot)\|_{s,\kappa}, \tag{3.2}$$

for $0 < T \le \infty$. Here ψ, s and κ are as in the above.

First we consider the following system:

$$\begin{cases} \square_{c_i} v_i = G_i(v) \ (1 \le i \le m), \\ v = \varepsilon f, \ \partial_t v = \varepsilon g \text{ at } t = 0, \end{cases} \tag{3.3}$$

where

$$G_i(v) = \sum_{(j,k)\in R} \alpha_{ijk} v_j v_k.$$

Here α_{ijk} and R are defined as in Theorem 1.1.

For $\kappa \ge 1$, $B > 0$ and $0 < T \le \infty$, we define

$$\begin{aligned} Y_T^\kappa(B) = \bigl\{ & v \in C\bigl([0,T)\times\mathbb{R}^3\bigr); \\ & \partial_x^\alpha v \in C\bigl([0,T)\times\mathbb{R}^3\bigr) \text{ for } |\alpha| \le 2, \text{ and } \|v\|_{2,\kappa,T} \le B \bigr\}. \end{aligned} \tag{3.4}$$

For $v \in Y_T^\kappa(B)$ with some $B > 0$ and $T > 0$, we introduce

$$V[v](t,x) = \bigl(L_{c_i}[G_i(v)](t,x)\bigr)_{1\le i\le m}.$$

Note that $\partial_x^\alpha V[v]$ with $|\alpha| \le 2$ belongs to $C\bigl([0,\infty)\times\mathbb{R}^3\bigr)$.

Lemma 3.1. *Let $M > 0$ and $\kappa > 1$. If a positive parameter ε is sufficiently small, then we have*

$$V[v] \in Y_\infty^\kappa(M\varepsilon/2) \tag{3.5}$$

for any $v \in Y_\infty^\kappa(M\varepsilon)$, and

$$\|V[v] - V[\widetilde{v}]\|_{2,\kappa,\infty} \leq \frac{1}{2}\|v - \widetilde{v}\|_{2,\kappa,\infty} \tag{3.6}$$

holds for any v, $\widetilde{v} \in Y_\infty^\kappa(M\varepsilon)$.

Proof. If $c_j \neq c_k$, remembering (2.4), we get

$$\begin{aligned} |\partial_x^\alpha(v_j v_k)(t,x)| \leq& C w_+(t,|x|)^{-2} w_{c_j}(t,|x|)^{-\kappa} w_{c_k}(t,|x|)^{-\kappa} \|v\|_{2,\kappa,\infty}^2 \\ \leq& C w_+(t,|x|)^{-2-\kappa} w_-(t,|x|)^{-\kappa} M^2\varepsilon^2 \end{aligned} \tag{3.7}$$

for any $(t,x) \in [0,\infty) \times \mathbb{R}^3$ and any multi-index α with $|\alpha| \leq 2$, provided $v \in Y_\infty^\kappa(M\varepsilon)$. Therefore Lemma 2.2 with $\delta = \kappa - 1 (> 0)$ implies

$$w_+(t,|x|) w_{c_i}(t,|x|)^\kappa \left|\partial_x^\alpha L_{c_i}[G_i(v)](t,x)\right| \leq C M^2 \varepsilon^2 \ (1 \leq i \leq m, |\alpha| \leq 2) \tag{3.8}$$

for any $(t,x) \in [0,\infty) \times \mathbb{R}^3$, and this yields

$$\|V[v]\|_{2,\kappa,\infty} \leq C M^2 \varepsilon^2 \tag{3.9}$$

for any $v \in Y_\infty^\kappa(M\varepsilon)$.

Just in the same manner, we obtain

$$\|V[v] - V[\widetilde{v}]\|_{2,\kappa,\infty} \leq C M \varepsilon \|v - \widetilde{v}\|_{2,\kappa,\infty} \tag{3.10}$$

for any $v, \widetilde{v} \in Y_\infty^\kappa(M\varepsilon)$. Choose $\varepsilon_1 = 1/(2CM)$. Then (3.9) and (3.10) imply the desired results for $\varepsilon \in (0, \varepsilon_1]$. □

Using Lemma 3.1, we obtain the following:

Proposition 3.1. *Assume $(f,g) \in X^\kappa$ with some $\kappa > 1$. Then there exists a positive constant ε_1 such that* (3.3) *admits a unique global solution*

$$v = (v_j)_{1 \leq j \leq m} \in C^2\left([0,\infty) \times \mathbb{R}^3; \mathbb{R}^m\right),$$

provided $\varepsilon \in (0, \varepsilon_1]$. Moreover the above solution v satisfies

$$\|v\|_{2,\kappa,\infty} \leq M\varepsilon \tag{3.11}$$

with some positive constant M.

Proof. Define a sequence of functions $\{v^{(N)}\}_{N=0}^\infty$ by

$$v^{(0)} = \left(U_{c_i}[\varepsilon f_i, \varepsilon g_i]\right)_{1 \leq i \leq m}, \tag{3.12}$$

$$v^{(N)} = v^{(0)} + V[v^{(N-1)}] \text{ for } N \geq 1. \tag{3.13}$$

From Lemma 2.1, we see

$$\|v^{(0)}\|_{2,\kappa,\infty} \leq C\varepsilon \left(\sum_{|\alpha|\leq 3} \|\partial_x^\alpha f\|_{2+\kappa} + \sum_{|\alpha|\leq 2} \|\partial_x^\alpha g\|_{2+\kappa} \right).$$

Therefore, if we choose sufficiently large M, we see $v^{(0)} \in Y_\infty^\kappa(M\varepsilon/2)$. From (3.5) in Lemma 3.1, we inductively find $v^{(N)} \in Y_\infty^\kappa(M\varepsilon)$ for all $N \geq 1$, provided that ε is sufficiently small. We also see that $\{v^{(N)}\}_{N=0}^\infty$ is a Cauchy sequence in $Y_\infty^\kappa(M\varepsilon)$ from (3.6) in Lemma 3.1 for small ε. Therefore $v^{(N)}$ converges to some $v \in Y_\infty^\kappa(M\varepsilon)$ as $N \to \infty$, and passing to the limit in (3.13), we find

$$v = v^{(0)} + V[v].$$

Now it is easy to see that this v is the classical solution to (3.3). $\square$

Finally we are going to prove Theorem 1.1. Let u be a solution to (1.1) with $F_i(u) = G_i(u) + H_i(u)$, and v be the solution to (3.3) satisfying $\|v\|_{2,\kappa,\infty} \leq M\varepsilon$ with $\kappa > 1$. Set $z = u - v$, and we see that z satisfies

$$\begin{cases} \square_{c_i} z_i = \widetilde{G}_i(v,z) + H_i(v+z) \quad (1 \leq i \leq m), \\ z(0,x) = \partial_t z(0,x) = 0, \end{cases} \tag{3.14}$$

where

$$\widetilde{G}_i(v,z) = G_i(v+z) - G_i(v) = \sum_{(j,k)\in R} \alpha_{ijk} \left(v_j z_k + z_j v_k + z_j z_k \right).$$

Conversely, if z is a solution to (3.14), then $u = v + z$ is the solution to (1.1). Therefore, it suffices for the proof of Theorem 1.1 to show that there exists a solution $w \in C^2\big([0,T) \times \mathbb{R}^3\big)$ to (3.14) as far as T satisfies $T \leq \exp(C_1\varepsilon^{-3})$ with some positive constant C_1.

We define a mapping Z by

$$Z[z] = \Big(L_{c_i}[\widetilde{G}_i(v,z) + H_i(v+z)] \Big)_{1\leq i\leq m}.$$

Lemma 3.2. *There exist three positive constants ε_0, C_1 and C_2 such that*

$$\varepsilon \leq \varepsilon_0,\ B \geq C_2 M^3 \ \textit{and}\ T \leq \exp(C_1\varepsilon^{-3})$$

imply

$$Z[z] \in Y_T^1\left(B\varepsilon^3\right) \tag{3.15}$$

for any $z \in Y_T^1\left(B\varepsilon^3\right)$, and

$$\|Z[z] - Z[\widetilde{z}]\|_{2,1,T} \leq \frac{1}{2}\|z - \widetilde{z}\|_{2,1,T} \tag{3.16}$$

for any $z, \widetilde{z} \in Y_T^1\left(B\varepsilon^3\right)$.

Proof. Let $1 \le i \le m$ and α be a multi-index with $|\alpha| \le 2$ in what follows. Suppose that ε is small so that we have $M\varepsilon + B\varepsilon^3 \ll 1$. Then we have

$$|\partial_x^\alpha H_i(v+z)| \le C\left\{\left(\sum_{|\beta|\le 2} |\partial_x^\beta v|\right)^3 + \left(\sum_{|\beta|\le 2} |\partial_x^\beta z|\right)^3\right\},$$

which leads to

$$\begin{aligned} |\partial_x^\alpha H_i(v+z)(t,x)| \le & C w_+(t,|x|)^{-3} w_-(t,|x|)^{-3\kappa} M^3\varepsilon^3 \\ & + w_+(t,|x|)^{-3} w_-(t,|x|)^{-3} B^3\varepsilon^9 \\ \le & C w_+(t,|x|)^{-3} w_-(t,|x|)^{-3}(M^3 + B^3\varepsilon^6)\varepsilon^3 \end{aligned} \tag{3.17}$$

for $(t,x) \in [0,T) \times \mathbb{R}^3$ and $z \in Y_T^1(B\varepsilon^3)$. Applying Lemma 2.2 with $\kappa = 1$ and $\delta = 2$, we obtain

$$w_+ w_{c_i} |\partial_x^\alpha L_{c_i}[H_i(v+z)]| \le C(M^3 + B^3\varepsilon^6)\varepsilon^3 \tag{3.18}$$

in $[0,T) \times \mathbb{R}^3$ for any $z \in Y_T^1(B\varepsilon^3)$. Similarly we have

$$w_+ w_{c_i} |\partial_x^\alpha L_{c_i}[H_i(v+z) - H_i(v+\widetilde{z})]| \le C(M^2\varepsilon^2 + B^2\varepsilon^6)\|\!|z - \widetilde{z}|\!\|_{2,1,T} \tag{3.19}$$

in $[0,T) \times \mathbb{R}^3$ for any z, $\widetilde{z} \in Y_T^1(B\varepsilon^3)$.

Now we are going to treat terms in $\widetilde{G}_i(v,w)$. We assume $c_j \ne c_k$. Then we have

$$|\partial_x^\alpha (v_j z_k)| \le w_+^{-2} w_{c_j}^{-\kappa} w_{c_k}^{-1} \|\!|v|\!\|_{2,\kappa,\infty} \|\!|z|\!\|_{2,1,T} \le C w_+^{-3} w_-^{-\kappa} MB\varepsilon^4 \tag{3.20}$$

in $[0,T) \times \mathbb{R}^3$ for any $z \in Y_T^1(B\varepsilon^3)$. Since $\kappa > 1$, Lemma 2.2 implies

$$w_+ w_{c_i} \, |\partial_x^\alpha L_{c_i}[v_j z_k + z_j v_k]| \le CMB\varepsilon^4 \tag{3.21}$$

in $[0,T) \times \mathbb{R}^3$ for $z \in Y_T^1(B\varepsilon^3)$, and we also get

$$w_+ w_{c_i} |\partial_x^\alpha L_{c_i}[v_j(z_k - \widetilde{z}_k) + (z_j - \widetilde{z}_j)v_k]| \le CM\varepsilon \|\!|z - \widetilde{z}|\!\|_{2,1,T} \tag{3.22}$$

in $[0,T) \times \mathbb{R}^3$ for any z, $\widetilde{z} \in Y_T^1(B\varepsilon^3)$.

Similarly to (3.7), we get

$$|\partial_x^\alpha (z_j z_k)| \le C w_+^{-3} w_-^{-1} B^2\varepsilon^6 \tag{3.23}$$

in $[0,T) \times \mathbb{R}^3$ for $z \in Y_T^1(B\varepsilon^3)$. Now, Lemma 2.2 with $\kappa = 1$ and $\delta = 0$ gives us

$$w_+ w_{c_i} |\partial_x^\alpha L_{c_i}[z_j z_k]| \le C \log(2+T) B^2\varepsilon^6 \tag{3.24}$$

in $[0,T) \times \mathbb{R}^3$ for any $z \in Y_T^1(B\varepsilon^3)$. In the same manner, we obtain

$$w_+ w_{c_i} |\partial_x^\alpha L_{c_i}[z_j z_k - \widetilde{z}_j \widetilde{z}_k]| \le C \log(2+T) B\varepsilon^3 \|\!|z - \widetilde{z}|\!\|_{2,1,T} \tag{3.25}$$

in $[0,T)\times\mathbb{R}^3$ for any z, $\widetilde{z}\in Y_T^1(B\varepsilon^3)$.

From (3.18), (3.21) and (3.24) we find

$$\|Z(z)\|_{2,1,T}\le C\left(M^3+MB\varepsilon+B^3\varepsilon^6+B^2\varepsilon^3\log(2+T)\right)\varepsilon^3 \tag{3.26}$$

for $z\in Y_T^1(B\varepsilon^3)$. (3.19), (3.22) and (3.25) yield

$$\begin{aligned}&\|Z(z)-Z(\widetilde{z})\|_{2,1,T}\\&\quad\le C(M^2\varepsilon^2+M\varepsilon+B^2\varepsilon^6+B\varepsilon^3\log(2+T))\|z-\widetilde{z}\|_{2,1,T}\end{aligned} \tag{3.27}$$

for z, $\widetilde{z}\in Y_T^1(B\varepsilon^3)$.

Choose some B satisfying $B\ge 4CM^3$. Assume that ε is so small to satisfy

$$C(M\varepsilon+B^2\varepsilon^6)\le\frac{1}{2},$$

and that T satisfies

$$\log(2+T)\le\frac{1}{4CB\varepsilon^3} \tag{3.28}$$

in (3.26). Then we get $\|Z(z)\|_{2,1,T}\le B\varepsilon$ and consequently we obtain (3.15). Also, if T satisfies (3.28) and ε is so small that we have

$$C(M^2\varepsilon^2+M\varepsilon+B^2\varepsilon^6)\le\frac{1}{4}$$

in (3.27), then we obtain (3.16). This completes the proof. □

Now we are in a position to conclude the proof of Theorem 1.1. Let ε, B and T be chosen so that (3.15) and (3.16) hold. Then Lemma 3.2 says that Z is a contraction mapping on $Y_T^1(B\varepsilon^3)$, and we see that there exists a unique fixed point $z\in Y_T^1(B\varepsilon^3)$, namely we have $z=Z[z]$. It is easy to verify that this z is the classical solution to (3.14). This completes the proof of Theorem 1.1.

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