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Degeneration of hyperbolic structures on the figure-eight knot complement and points of finite order on an elliptic curve

By

Michihiko Fujii

1. Introduction

In this paper, we consider a hyperbolic structure on a manifold to be a Riemannian metric of constant sectional curvature -1 which is not necessarily complete. A hyperbolic 3-manifold is an orientable 3-manifold with a hyperbolic structure. It is well known that the complement of the figure-eight knot K in the 3-dimensional sphere S^3 admits a complete, finite volume hyperbolic structure σ_∞. By Mostow-Prasad rigidity, such a hyperbolic structure on $S^3 - K$ is unique. Incomplete hyperbolic structures are also of interest. Indeed, Thurston [8] analyzed the flexibility of hyperbolic structures on $S^3 - K$ by allowing incomplete hyperbolic structures. In fact, he showed that there are deformations of σ_∞ on $S^3 - K$ that are not complete hyperbolic structures. Such deformations are holomorphically parametrized by points in an open subset U of a complex affine plane curve C. The curve C and this subset U are given in (1) and (2) below.

When the parameter of deformation becomes close to the boundary ∂U or enters $C - U$, degeneration of hyperbolic structures on $S^3 - K$ occurs. There are cases in which such degeneration results in closed 3-manifolds obtained by Dehn fillings of $S^3 - K$ with other types of geometric structure. In fact, twenty such cases have been identified. In each case, this resultant closed 3-manifold is one of the following types: a Sol-manifold, a $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$-manifold, a Haken manifold which is decomposed into a $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$-manifold and a Euclidean manifold along an embedded torus, or a Euclidean orbifold.

The genus of the curve C is one, because C is of degree four and has two ordinary double points. In fact, we give an explicit form of a birational map from C to a non-singular plane cubic curve in Weierstrass form E, which is given in (4) below. The curve E is an example of what is called an 'elliptic curve defined over $\mathbf{Q}$'. It is well known that any elliptic curve is an abelian group under an addition law. In this paper, we see that there is a concrete

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correspondence between the closed 3-manifolds mentioned above (except the $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$-manifolds) and some points of finite order on the elliptic curve E. In particular, the Haken manifolds mentioned above correspond exactly to the rational points of E, which form a cyclic group of order four.

2. Deformation of hyperbolic structures and hyperbolic Dehn filling

In this section, we briefly describe a concrete treatment of the deformation of hyperbolic structures on 3-manifolds realized by deforming ideal tetrahedra. We also review some well-known results concerning the hyperbolic Dehn filling for the figure-eight knot complement (see Thurston [8], [9], [10], Cooper-Hodgson-Kerckhoff [1] and Neumann-Zagier [7]).

An ideal tetrahedron S is an oriented 3-dimensional simplex in $\mathbf{H}^3 \cup \partial\mathbf{H}^3$ whose vertices are located on $\partial\mathbf{H}^3$. An ideal tetrahedron is determined up to an isometry by a single complex number as follows (see Fig. 1 and Ref. [10]). Consider an oriented 3-dimensional simplex in $\mathbf{H}^3 \cup \partial\mathbf{H}^3$ whose vertices are on $\partial\mathbf{H}^3$. Transform it by application of the unique orientation-preserving isometry of $\mathbf{H}^3$ that sends the first three vertices of the simplex to the points 0, 1 and ∞ in the upper half-space model of $\mathbf{H}^3$. Its congruence class is determined by the position of the last vertex, which is given by some point $z \in \mathbf{C} - \{0, 1\}$. Therefore, we write $S = S(z)$. This complex number z is a parameter of isometry classes of ideal tetrahedra. The Euclidean triangle cut out of any vertex of $S(z)$ by a horosphere section is similar to the triangle in $\mathbf{C}$ with vertices 0, 1 and z. With each edge of $S(z)$ is associated one of the three numbers z, $(z-1)/z$ and $1/(1-z)$. This number is called the modulus of the edge. If z is on the real line, then the ideal tetrahedron $S(z)$ is flattened and contained within a 2-dimensional hyperbolic subspace of $\mathbf{H}^3$. If $\mathrm{Im}(z) > 0$, the map of the simplex preserves orientation. Therefore, an ideal tetrahedron $S(z)$ with $\mathrm{Im}(z) > 0$ is said to be positive. If $\mathrm{Im}(z) < 0$, the map of the simplex reverses orientation, and therefore an ideal tetrahedron $S(z)$ with $\mathrm{Im}(z) < 0$ is said to be negative. When we construct hyperbolic 3-manifolds by gluing the faces of ideal tetrahedra, we usually use positive ones. Note that $S(z)$, $S((z-1)/z)$, $S(1/(1-z))$, $S(1/z)$, $S(z/(z-1))$ and $S(1-z)$ are transformed among each other by orientation-preserving isometries. These orientation-preserving isometries change the order of the vertices of the simplex. By contrast, an ideal tetrahedra $S(\overline{z})$ is transformed to $S(z)$ by the orientation-reversing isometry that is given by the composition of the inverse of the map of the simplex to $S(\overline{z})$ and the map of the simplex to $S(z)$.

In this paper, we denote the figure-eight knot in the 3-dimensional sphere S^3 by K. It is well known that its complement, $S^3 - K$, admits a complete, finite volume hyperbolic structure σ_∞. The hyperbolic 3-manifold $(S^3 - K, \sigma_\infty)$ is obtained by gluing the faces of two ideal tetrahedra $S(z_\infty)$ and $S(w_\infty)$ by orientation-reversing isometries according to the diagram in Fig. 2 (see [8]), where $z_\infty = w_\infty = -\omega^2$ and $\omega = e^{2\pi i/3}$. Thus, we write $(S^3 - K, \sigma_\infty) = S(z_\infty) \cup S(w_\infty)$. Because all the faces of any ideal tetrahedron are congruent,

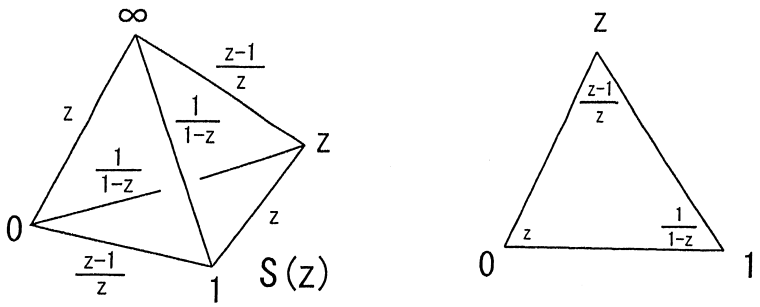

Figure 1.

a complex $S(z) \cup S(w)$ ($\approx S^3 - K$) can be obtained by identifying their faces using orientation-reversing isometries with the same gluing pattern as $S(z_\infty) \cup S(w_\infty)$. The necessary condition that $S(z) \cup S(w)$ becomes a smooth hyperbolic manifold is given by the algebraic equation $z(z-1)w(w-1) = 1$. We denote by C the affine plane curve defined by this equation:

$$C = \{(z,w) \in \mathbf{C}^2 \mid z(z-1)w(w-1) = 1\}. \tag{1}$$

The complete hyperbolic structure σ_∞ corresponds to the point (z_∞, w_∞) on the affine plane curve C. Next, we define the set U as

$$U := \{(z,w) \in \mathbf{C}^2 \mid z(z-1)w(w-1) = 1,\ \mathrm{Im}(z) > 0,\ \mathrm{Im}(w) > 0\}. \tag{2}$$

Therefore, U is an open neighborhood of (z_∞, w_∞) in C that is biholomorphic to the region

$$\{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\} - \left\{ z \in \mathbf{C} \;\middle|\; z = \frac{1}{2} + yi \; \left(y \geq \frac{\sqrt{15}}{2} \right) \right\}.$$

If $(z,w) \in U - \{(z_\infty, w_\infty)\}$, then $S(z) \cup S(w)$ constitutes an incomplete, smooth hyperbolic structure σ on $S^3 - K$. We denote the hyperbolic 3-manifold $(S^3 - K, \sigma)$ by $(S^3 - K)(z,w)$ and call it a deformation of $(S^3 - K, \sigma_\infty)$. The affine plane curve C is called the 'deformation curve' of hyperbolic structures on the figure-eight knot complement $S^3 - K$.

Associated with each point $(z,w) \in U$ is a representation $\rho : \pi_1(S^3 - K) \to \mathrm{PSL}_2(\mathbf{C})$. The image of ρ is not necessarily a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$. Let ρ_∞ denote a representation that corresponds to (z_∞, w_∞). Then the image $\rho_\infty(\pi_1(S^3 - K))$ is a torsion-free discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$ and $\mathbf{H}^3/\rho_\infty(\pi_1(S^3 - K))$ is identically the complete, finite volume hyperbolic 3-manifold $(S^3 - K, \sigma_\infty)$.

Let T be a boundary torus of a sufficiently small tubular neighborhood of K in S^3 and m and l be the standard meridian and longitude of K on T, respectively. Then for each coprime pair of integers (p,q), one can obtain a

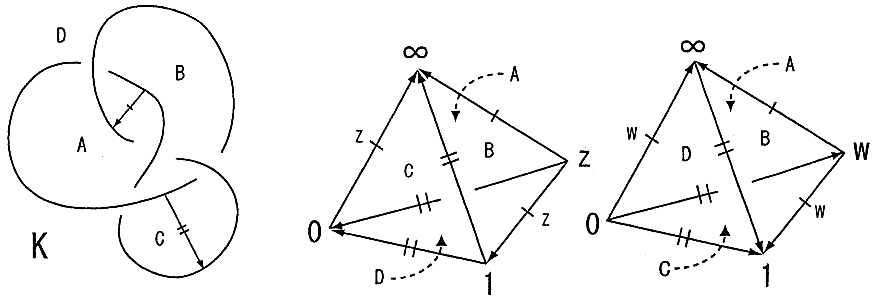

Figure 2.

closed 3-manifold $(S^3-K)_{(p,q)}$ by carrying out the Dehn filling along K while killing the homotopy class of the simple closed curve $pm+ql$. First, we give the following remark.

- If $(p,q)=(\pm 1,0)$, then $(S^3-K)_{(p,q)}$ admits a spherical structure. In fact, both of $(S^3-K)_{(\pm 1,0)}$ are homeomorphic to S^3.

Let (z,w) be a point in U. Then, associated with a representation $\rho:\pi_1(S^3-K)\to \mathrm{PSL}_2(\mathbf{C})$ corresponding to (z,w) is a representation $\mu:\pi_1(T)\to \mathbf{C}^*$. Note that there are two choices of μ according to the orientation of the axis of $\rho(m)$. Following Neumann-Zagier [7], we make a choice as $\mu(m)=(1-z)w$ and $\mu(l)=z^2(1-z)^2$. Then, if $(z,w)\neq(z_\infty,w_\infty)$, we define the generalized Dehn filling coefficient $(p,q)\in\mathbf{R}^2$ by the equation

$$p\log\{(1-z)w\}+q\log\{z^2(1-z)^2\}=2\pi i, \tag{3}$$

where the log is taken such that $-\pi<\arg\le\pi$. Note that if $(z,w)\in U-\{(z_\infty,w_\infty)\}$, there is a unique solution (p,q) to this equation, because $\log\{(1-z)w\}$ is not a real multiple of $\log\{z^2(1-z)^2\}$ if $(z,w)\in U-\{(z_\infty,w_\infty)\}$. If $(z,w)=(z_\infty,w_\infty)$, we stipulate that $(p,q)=\infty$. Now, define the map

$$\psi:U\to\mathbf{R}^2\cup\{\infty\}$$

by

$$\psi((z,w))=(p,q),$$

where (p,q) is taken as above.

The image of the map ψ includes every coprime pair of integers except the following twenty: $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm2,\pm1)$, $(\pm3,\pm1)$, $(\pm4,\pm1)$ (see [8, Sections 4.6 and 4.7]). Thus, if (p,q) is a coprime pair of integers and it is not one of these exceptions, then the closed 3-manifold $(S^3-K)_{(p,q)}$ admits a hyperbolic structure $\overline{\sigma}$. This hyperbolic structure $\overline{\sigma}$ on $(S^3-K)_{(p,q)}$ is identically the completion of the hyperbolic structure σ of $(S^3-K)(z,w)$, where (z, w) is an element of the inverse image of (p, q) under ψ. By contrast, the twenty exceptions listed above do not induce hyperbolic structures. For example, for $(p, q) = (\pm 1, 0)$, $(S^3 - K)_{(p,q)}$ admits a spherical structure, as described above. In each of the other eighteen exceptional cases, there is a point (z_0, w_0) on ∂U that satisfies the equation (3) with the coprime pair of integers in question. Therefore the hyperbolic structure becomes degenerate when (z, w) tends to the point (z_0, w_0) and another type of geometric structure appears on $(S^3 - K)_{(p,q)}$ as follows (see [8, Section 4.9] and [1, Section 5.7]):

- If $(p, q) = (0, \pm 1)$, then $(S^3 - K)_{(p,q)}$ admits a Sol-structure. The image of the corresponding representation $\rho_{(p,q)} : \pi_1(S^3 - K) \to \mathrm{PSL}_2(\mathbf{C})$ is included in $\mathrm{PSL}_2(\mathbf{R})$.
- If $(p, q) = (\pm 1, \pm 1), (\pm 2, \pm 1)$ or $(\pm 3, \pm 1)$, then $(S^3 - K)_{(p,q)}$ admits a $\widetilde{\mathrm{PSL}_2}(\mathbf{R})$-structure. The image of the corresponding representation $\rho_{(p,q)} : \pi_1(S^3 - K) \to \mathrm{PSL}_2(\mathbf{C})$ is included in $\mathrm{PSL}_2(\mathbf{R})$.
- If $(p, q) = (\pm 4, \pm 1)$, then $(S^3 - K)_{(p,q)}$ contains an incompressible torus that splits $(S^3 - K)_{(p,q)}$ into the union of the trefoil knot complement and the non-trivial I-bundle over the Klein bottle, where I denotes the closed interval $[0, 1]$. The trefoil knot complement has a $\widetilde{\mathrm{PSL}_2}(\mathbf{R})$-structure, and the non-trivial I-bundle over the Klein bottle has a Euclidian structure. The images of the representations corresponding to them are included in $\mathrm{PSL}_2(\mathbf{R})$ and $SO(3)$, respectively.

Thurston showed in his lecture notes [8, Section 4.11] that for each coprime pair of integers (p, q), except for $(\pm 4, \pm 1)$ and $(0, \pm 1)$, the 3-manifold $(S^3 - K)_{(p,q)}$ is not Haken.

Consider the case in which the generalized Dehn filling coefficient $(p, q) \in \mathbf{R}^2 - \{(0, 0)\}$ satisfies the conditions that p is not 0 and q/p is rational. Then, choose a coprime pair of integers (α, β) so that $q/p = \beta/\alpha$. With these conditions, in some cases a closed 3-manifold $(S^3 - K)_{(\alpha,\beta)}$ obtained by the Dehn filling of type (α, β) along K has a hyperbolic 3-cone-structure of cone angle $2\pi|\alpha/p|$ with a simple closed curve as its singular locus and in some cases it does not.

Now, consider in particular the case that $q = 0$. In this case, (α, β) is taken as $(\alpha, \beta) = (1, 0)$ if p is positive and $(\alpha, \beta) = (-1, 0)$ if p is negative. Note that $(S^3 - K)_{(\pm 1, 0)}$ are both homeomorphic to S^3, and the singularities of the cone-structures appear along K.

First, let us consider the case of positive p. Analyzing this case, Thurston found that the image of ψ includes the open interval $(\frac{2\pi}{\sin^{-1}(\sqrt{15}/4)}, +\infty)$ of the p-axis. (Note that $2\pi/\sin^{-1}(\sqrt{15}/4) \approx 4.76679\ldots$.) Therefore S^3 admits a hyperbolic 3-cone-structure of cone angle $2\pi/p$ for each $p > \frac{2\pi}{\sin^{-1}(\sqrt{15}/4)}$ with singular locus K. Subsequent to Thurston's work, Hilden, Lozano and Montesinos-Amilibia [5] obtained a one-parameter family of 3-cone-manifolds $\{(S^3, \tau(p))\}_{p \in [3, +\infty]}$ whose sectional curvature increases monotonically from -1 at $p = +\infty$ to 0 at $p = 3$ and whose singular locus is the figure-eight knot K with cone angle $2\pi/p$. The 3-cone-manifold $(S^3, \tau(+\infty))$ coincides with the hyperbolic 3-manifold (S^3, σ_∞). The 3-cone-manifold $(S^3, \tau(3))$ is a Euclidean 3-orbifold with K the singular locus of cone angle $2\pi/3$. For each $p \in (3, +\infty)$, appropriately rescaling the Riemannian metric of $(S^3, \tau(p))$, we obtain a hyperbolic 3-cone-manifold $(S^3, \widehat{\tau}(p))$ whose singular locus is K with cone angle $2\pi/p$. This cone-manifold collapses to a point as $p \searrow 3$. Thus, it can be concluded that the hyperbolic 3-cone-manifold $(S^3, \widehat{\tau}(p))$ is rescaled to converge to the Euclidean 3-orbifold $(S^3, \tau(3))$ as $p \to 3$ (see also Cooper-Hodgson-Kerckhoff [1, Section 5.7]).

Now, consider the case of negative p. In this case, p moves in the open interval $(-\infty, -\frac{2\pi}{\sin^{-1}(\sqrt{15}/4)})$, and the argument given above again holds. Denote by $(S^3, \tau(-3))$ the Euclidean 3-orbifold that is obtained in this case. For each $p \in [-\infty, -3] \cup [3, +\infty]$, there corresponds a representation $\rho_{\widehat{\tau}(p)} : \pi_1(S^3 - K) \to \mathrm{PSL}_2(\mathbf{C})$. With the Euclidean 3-orbifolds $(S^3, \tau(\pm 3))$ are associated representations $\rho_{\tau(\pm 3)} : \pi_1(S^3 - K) \to SO(3)$. Applying the conjugation operation, it is found that the representation $\rho_{\widehat{\tau}(p)}$ converges to $\rho_{\tau(\pm 3)}$ as $p \to \pm 3$.

3. A birational map from the deformation curve C to an elliptic curve in Weierstrass form

Let V be the projective completion of the complex affine plane curve C. Then $V = \{(x_0 : x_1 : x_2) \in \mathbf{CP}^2 \mid x_1(x_1 - x_0)x_2(x_2 - x_0) = x_0^4\}$. Next, denote by A and B the points at infinity of C that correspond to the points $(0 : 0 : 1)$ and $(0 : 1 : 0)$ on V, respectively. The points $(0 : 0 : 1)$ and $(0 : 1 : 0)$ are ordinary double points. Thus the genus of C is one.

The curve C is transformed to a non-singular cubic curve in Weierstrass form E under the following birational map:

$$
\begin{array}{c}
C : z(z-1)w(w-1) = 1 \\
\Big\Updownarrow {\scriptstyle z=u,\ w=\frac{1}{v}} \\
u^2 v + v^2 - uv - u^2 + u = 0 \\
\Big\Updownarrow {\scriptstyle u=\frac{X_1}{X_0},\ v=\frac{X_2}{X_0}} \\
X_1^2 X_2 + X_0 X_2^2 - X_0 X_1 X_2 - X_0 X_1^2 + X_0^2 X_1 = 0 \\
\Big\Updownarrow {\scriptstyle X_0=-x_0-x_1,\ X_1=x_2,\ X_2=-x_1} \\
-x_0 x_1^2 - x_1^3 + x_0 x_1 x_2 + x_0 x_2^2 + x_0^2 x_2 = 0 \\
\Big\Updownarrow {\scriptstyle x=\frac{x_1}{x_0},\ y=\frac{x_2}{x_0}} \\
E : y^2 + xy + y = x^3 + x^2
\end{array}
\tag{4}
$$

The curve E is the elliptic curve referred to as $15A8$ in the table of Cremona [2].

For any elliptic curve, we can define an addition law. Representing the addition operation so defined by $+$, the points on such a curve form an abelian group under $+$, with some point $\mathcal{O}$ as the identity element. The addition law applying to the elliptic curve $E : y^2 + xy + y = x^3 + x^2$ is stated explicitly in the appendix. We choose the point at infinity of E as the identity element. Using the addition law on E, we consider the operation of addition applied to points on C. Note that this addition law on C depends on the choice of the birational transformations from C to E, and it is determined up to a translation on E.

Let $E(\mathbf{Q})$ be the Mordell-Weil group over $\mathbf{Q}$ that is the subset consisting of all rational points of E. $E(\mathbf{Q})$ is a subgroup of the elliptic curve E, and on E it is isomorphic to $\mathbf{Z}/4\mathbf{Z}$, the cyclic group of order 4 (see [2]). Under the birational map given in (4), the singular point A (resp., B) of the deformation curve C is transformed to two non-singular points A_1 and A_3 (resp., B_2 and B_4) on E. The four points A_1, B_2, A_3 and B_4 on E form the Mordell-Weil group $E(\mathbf{Q})$. In fact, we have $A_1 = (0, -1)$, $2A_1 = B_2 = (-1, 0)$, $3A_1 = A_3 = (0, 0)$ and $4A_1 = B_4 = \mathcal{O}$.

4. Points of finite order on the elliptic curve E and Dehn filling on the figure-eight knot

4.1. Rational points on E and Haken manifolds

Culler and Shalen [3] investigated the character varieties of representations of fundamental groups of hyperbolic 3-manifolds. One of the assertions that they proved is that in a hyperbolic 3-manifold there corresponds an incompressible surface to each ideal point of a complex affine algebraic curve in its character variety. Subsequently, Yoshida [12] constructed an explicit realization of the general theory formulated by Culler and Shalen for the special case of hyperbolic one-cusp 3-manifolds that are decomposed into ideal tetrahedra. Instead of the character varieties, he used deformation curves of hyperbolic structures given by ideal tetrahedral decompositions. He also defined ideal points of deformation curves and their slopes. Our complex affine plane curve C is an example of such deformation curves. He showed that there are four ideal points on our deformation curve C and explicitly constructed incompressible surfaces whose boundary slopes are equal to those of the four ideal points. These ideal points are located on the boundary of U and they correspond to $(p, q) = (\pm 4, \pm 1)$. In fact, when these incompressible surfaces are capped with disks, they form the incompressible tori mentioned in Section 2.

Let a_1, b_2, a_3 and b_4 denote the ideal points of C corresponding to $(4, -1)$, $(-4, -1)$, $(4, 1)$ and $(-4, 1)$, respectively. It can be shown that the four rational points A_1, B_2, A_3 and B_4 on E correspond respectively to the four ideal points a_1, b_2, a_3 and b_4 of C under the rational map given in (4). In fact, the points A_1, B_2, A_3 and B_4 are ideal points in the sense of Culler and Shalen. Therefore, on hyperbolic structures of the figure-eight knot complement $S^3 - K$, the rational points of its deformation curve correspond exactly to the closed Haken 3-manifolds $(S^3 - K)_{(\pm 4, \pm 1)}$ that contain incompressible surfaces resulting from non-trivial splittings of $\pi_1(S^3 - K)$.

4.2. Points of finite order on E and Sol manifolds

Let us consider the point $t_1 := (\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on C. This point is located on the boundary ∂U of U. If $(p,q) = (0,1)$ and $(z,w) = (\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$, then p, q, z and w satisfy the equation (3). There are paths in U that individually connect t_1 and the original point (z_∞, w_∞) $(= (-\omega^2, -\omega^2))$. By appropriately choosing one such path, we can observe the simultaneous degeneration of hyperbolic structures on $S^3 - K$ and appearance of the Sol structure on $(S^3 - K)_{(0,1)}$, as discussed in Section 2 (see Refs. [1], [4] and [8] for details). The point t_1 corresponds to the point $T := (\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on E under the birational map given in (4). Therefore the point T on the elliptic curve E can be considered as corresponding to the 3-manifold $(S^3 - K)_{(0,1)}$ with Sol structure that appears through the degeneration of hyperbolic structures on $S^3 - K$.

Using the formulas given in the appendix, we can see that $2T = A_3$, $3T = (\frac{1-\sqrt{5}}{2}, -2+\sqrt{5})$, $4T = B_2$, $5T = (\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$, $6T = A_1$, $7T = (\frac{1+\sqrt{5}}{2}, -2-\sqrt{5})$ and $8T = \mathcal{O}$. The three points $t_3 := (\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$, $t_5 := (\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ and $t_7 := (\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on C correspond respectively to the three points $3T$, $5T$ and $7T$ on E under the birational map given in (4). It is also seen from the equation (3) that t_3, t_5 and t_7 correspond to the generalized Dehn filling coefficients $(0, -1)$, $(0, -1)$ and $(0, 1)$, respectively. Thus, as in the case of T, the point $5T$ can be regarded as corresponding to the Sol structure on $(S^3 - K)_{(0,-1)}$. However, the points t_3 and t_7 do not represent the limits of sequences of points in U, because the two paths connecting the original point (z_∞, w_∞) to t_3 and to t_7 must cross the region $C - U$. To this time, there has been no method proposed to construct hyperbolic structures along a path connecting the points (z_∞, w_∞) and t_3 or t_7. Therefore, at the present time, we cannot regard the points $3T$ and $7T$ as corresponding to geometric 3-manifolds obtained by Dehn filling of the hyperbolic 3-manifold $(S^3 - K, \sigma_\infty)$.

4.3. Points of finite order on E and Euclidean orbifolds

The original point (z_∞, w_∞) $(= (-\omega^2, -\omega^2))$ on the deformation curve C corresponds to the point $R := (\omega^2, -1)$ on E under the birational map given in (4). Also, we can demonstrate that $2R = A_1$ and that the point $5R$ corresponds to $(\overline{z_\infty}, \overline{w_\infty})(= (-\omega, -\omega))$ under the birational map given in (4). Because $\mathrm{Im}(-\omega) < 0$, the ideal tetrahedron $S(-\omega)$ is negative and we can regard $S(-\omega) \cup S(-\omega)$ as a hyperbolic 3-manifold that overlaps $(S^3 - K, \sigma_\infty) = S(-\omega^2) \cup S(-\omega^2)$ by the orientation-reversing isometry. To this time, there has been no method proposed to construct a path consisting of deformations of the hyperbolic structure σ_∞ that connects $(S^3 - K)(z_\infty, w_\infty)$ and $(S^3 - K)(\overline{z_\infty}, \overline{w_\infty})$. Therefore, at the present time, we cannot assume that the hyperbolic 3-manifold $(S^3 - K)(\overline{z_\infty}, \overline{w_\infty})$ can be obtained by hyperbolic Dehn filling of the original hyperbolic 3-manifold $(S^3 - K)(z_\infty, w_\infty)$.

The point on C corresponding to the point $7R = (\omega^2, -\omega^2)$ on E is $(-\omega, -\omega^2)$. Note that because $\mathrm{Im}(-\omega) < 0$, the point $(-\omega, -\omega^2)$ is not contained in U. However, by considering the equation (3) on the Riemann surface of C, we can choose a path connecting the two points (z_∞, w_∞) and $(-\omega, \omega^2)$ along which the degeneration of hyperbolic structures on $S^3 - K$ occurs. On the endpoint of this path, $(-\omega, \omega^2)$, the Euclidean 3-orbifold $(S^3, \tau(3))$ appears, as described in Section 2. This leads us to believe that $7R$ corresponds to the Euclidean 3-orbifold $(S^3, \tau(3))$ that results from the degeneration of the hyperbolic structures (see Refs. [1] and [4] for details). As in the case of $7R$, we can regard $3R = (\omega, -\omega)$ as corresponding to the Euclidean 3-orbifold $(S^3, \tau(-3))$ given in Section 2.

4.4. Theorem

Summerizing the results obtained above, we have the following theorem.

Theorem. *Let K be the figure-eight knot in the 3-dimensional sphere S^3. Also, let*

$$C : z(z-1)w(w-1) = 1$$

be the deformation curve of hyperbolic structures on the figure-eight knot complement $S^3 - K$. For each coprime pair of integers (p, q), let us denote by $(S^3 - K)_{(p,q)}$ a closed 3-manifold obtained from $S^3 - K$ by a Dehn filling along K that kills the homotopy class of the simple closed curve $pm + ql$, where m and l are the standard meridian and longitude of K in S^3. Then, we have the following:

(i) *The deformation curve C is birationally equivalent to the elliptic curve*

$$E : y^2 + xy + y = x^3 + x^2.$$

The conductor of E is 15 and the Mordell-Weil group $E(\mathbf{Q})$ is isomorphic to $\mathbf{Z}/4\mathbf{Z}$, the cyclic group of order four.

(ii) *The four points $A_1 := (0, -1)$, $B_2 := (-1, 0)$, $A_3 := (0, 0)$ and $B_4 := \mathcal{O}$ on E form the Mordell-Weil group $E(\mathbf{Q})$, where $\mathcal{O}$ denotes the point at infinity of E.*

(iii) *The two points $R := (\omega^2, -1)$ and $T := (\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on E are of order eight and satify the relations $2R = 6T = A_1$, $4R = 4T = B_2$, $6R = 2T = A_3$ and $8R = 8T = B_4 = \mathcal{O}$. Here, ω denotes the complex number $e^{2\pi i/3}$.*

(iv) *There are the following correspondences among the points nR and nT $(n = 1, \ldots, 8)$, excluding $5R$, $3T$ and $7T$, on the elliptic curve E and the 3-manifolds $(S^3 - K)_{(p,q)}$ $((p, q) = \infty, (\pm 4, \pm 1), (\pm 1, 0), (0, \pm 1))$ with geometric structures, where $(S^3 - K)_\infty$ represents the unsurgered manifold $S^3 - K$.*

$$
\begin{array}{rl}
R & \Longleftrightarrow S^3 - K, \ \textit{with a complete, finite volume} \\
 & \quad \textit{hyperbolic structure,} \\
2R = 6T = A_1 & \Longleftrightarrow (S^3 - K)_{(4,-1)}, \ \textit{which contains an incompressible} \\
 & \quad \textit{torus that gives rise to a splitting into a Euclidean} \\
 & \quad \textit{manifold and a } \widetilde{PSL}_2(\mathbf{R})\textit{-manifold,} \\
3R & \Longleftrightarrow (S^3 - K)_{(-1,0)}, \ \textit{with a Euclidean orbifold structure} \\
 & \quad \textit{of cone angle } 2\pi/3 \textit{ along the singular locus } K, \\
4R = 4T = B_2 & \Longleftrightarrow (S^3 - K)_{(-4,-1)}, \ \textit{which contains an incompressible} \\
 & \quad \textit{torus that gives rise to a splitting into a Euclidean} \\
 & \quad \textit{manifold and a } \widetilde{PSL}_2(\mathbf{R})\textit{-manifold,} \\
6R = 2T = A_3 & \Longleftrightarrow (S^3 - K)_{(4,1)}, \ \textit{which contains an incompressible} \\
 & \quad \textit{torus that gives rise to a splitting into a Euclidean} \\
 & \quad \textit{manifold and a } \widetilde{PSL}_2(\mathbf{R})\textit{-manifold,} \\
7R & \Longleftrightarrow (S^3 - K)_{(1,0)}, \ \textit{with a Euclidean orbifold structure} \\
 & \quad \textit{of cone angle } 2\pi/3 \textit{ along the singular locus } K, \\
8R = 8T = B_4 = \mathcal{O} & \Longleftrightarrow (S^3 - K)_{(-4,1)}, \ \textit{which contains an incompressible} \\
 & \quad \textit{torus that gives rise to a splitting into a Euclidean} \\
 & \quad \textit{manifold and a } \widetilde{PSL}_2(\mathbf{R})\textit{-manifold,} \\
T & \Longleftrightarrow (S^3 - K)_{(0,1)}, \ \textit{with a Sol-structure,} \\
5T & \Longleftrightarrow (S^3 - K)_{(0,-1)}, \ \textit{with a Sol-structure.}
\end{array}
$$

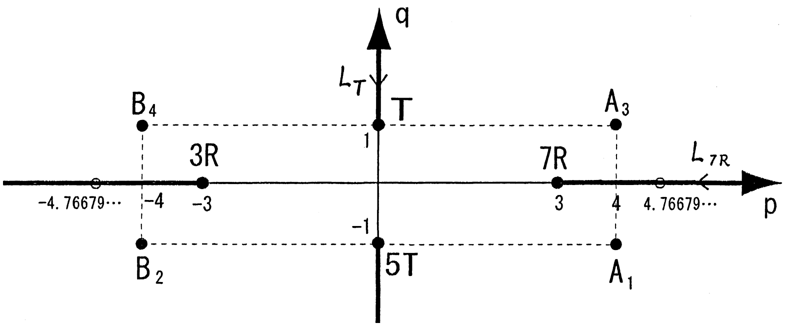

Figure 3.

Remark. The correspondence in (iv) is not canonical, because it depends on the choice of the birationl maps from C to E. It is determined up to a translation of E.

Appendix

In this appendix, we explicitly state the addition law on the elliptic curve $E : y^2 + xy + y = x^3 + x^2$ (see Knapp [6] or Ueno [11]). The elliptic curve E

has a unique point at infinity. Denote this point by $\mathcal{O}$. If we identify $(x, y) \in E$ with the point $(1 : x : y)$ on the corresponding projective variety of E, then $\mathcal{O} = (0 : 0 : 1)$. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points on E. The following are formulas to compute $P + Q$:

- *If* $x_1 \neq x_2$, *then*
$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3),$$
where
$$\begin{aligned} x_3 &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right) - 1 - x_1 - x_2, \\ y_3 &= -\left(\frac{y_2 - y_1}{x_2 - x_1} + 1\right) x_3 - \left(\frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}\right) - 1. \end{aligned}$$
- *If* $x_1 = x_2$ *and* $y_1 + y_2 + x_2 + 1 = 0$, *then*
$$(x_1, y_1) + (x_1, y_2) = \mathcal{O}.$$
- *If* $x_1 = x_2$, $y_1 = y_2$ *and* $y_1 + y_2 + x_2 + 1 = 0$, *then*
$$2(x_1, y_1) = \mathcal{O}.$$
- *If* $x_1 = x_2$, $y_1 = y_2$ *and* $y_1 + y_2 + x_2 + 1 \neq 0$, *then*
$$2(x_1, y_1) = (x_3, y_3),$$
where
$$\begin{aligned} x_3 &= \left(\frac{3x_1^2 + 2x_1 - y_1}{2y_1 + x_1 + 1}\right)^2 + \left(\frac{3x_1^2 + 2x_1 - y_1}{2y_1 + x_1 + 1}\right) - 1 - 2x_1, \\ y_3 &= -\left(\frac{3x_1^2 + 2x_1 - y_1}{2y_1 + x_1 + 1} + 1\right) x_3 - \left(\frac{-x_1^3 - y_1}{2y_1 + x_1 + 1}\right) - 1. \end{aligned}$$
- $$-(x_1, y_1) = (x_1, -x_1 - y_1 - 1).$$

The points on E form an abelian group under this addition law. The point $\mathcal{O}$ is the identity element of the group.

Department of Mathematics
Graduate School of Science
Kyoto University
Sakyo-ku, Kyoto 606-8502
Japan
e-mail: mfujii@math.kyoto-u.ac.jp

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