J. Math. Kyoto Univ. (JMKYAZ)
45-4 (2005), 759–780

Representations of $\mathrm{SU}(p,q)$ and CR geometry I

By

Wei Wang

Abstract

The CR geometry is applied to the representation theory of the group $\mathrm{SU}(p,q)$. We prove that the kernel of the CR Yamabe operator on a CR manifold M is a representation of the conformal CR automorphism group of M. So we can construct a representations of $\mathrm{SU}(p,q)$ on the kernel of the CR Yamabe operator on the projective hyperquadric $\overline{Q}_{p,q}$. This is a complex version of Kobayashi-Orsted's model of the minimal irreducible unitary representation $\varpi^{p,q}$ of $\mathrm{SO}(p,q)$ on $S^{p-1} \times S^{q-1}$.

1. Introduction

Conformal geometry on pseudo-Riemannian manifolds can be applied to the representation theory of the group $\mathrm{SO}(p,q)$ (cf. [3], [13], [14], [15], [16] and references therein). Kostant used the conformal invariance of the vanishing of scalar curvature on 6 dimensional manifolds to explore the minimal representation of $\mathrm{SO}(4,4)$ in [16]. Recently, T. Kobayashi and B. Orsted [13], [14], [15] gave a geometric and intrinsic model of the minimal irreducible unitary representation $\varpi^{p,q}$ of $\mathrm{SO}(p,q)$ on $S^{p-1} \times S^{q-1}$ and on various pseudo-Riemannian manifolds which are conformally equivalent, by using the Yamabe operator. They also gave branching formulae and unitarization of various models. In this paper, we use CR geometry to realize representations of $\mathrm{SU}(p,q)$.

The geometry of strictly pseudoconvex CR manifolds has many parallels with Riemannian geometry [1], and there is a far reaching analogue between conformal geometry and CR geometry. Jerison and Lee gave a table [11] summarizing some important parallels. More generally, strictly k-pseudoconvex CR manifolds correspond to the pseudo-Riemannian manifolds. A nondegenerate contact form on a CR manifold plays the role of a metric in pseudo-Riemannian geometry. In CR geometry, there is a natural connection, called the Webster connection, associated to a contact form θ on a CR manifold. We can define conformal contact forms in CR geometry and develop conformal geometry on CR manifolds. For example we can define a CR Yamabe operator and have similar transformation formula [11]. The kernel of the CR Yamabe operator

2000 *Mathematics Subject Classification*(*s*). Primary 43A65; Secondary 32V20
Received March 2, 2005

on a CR manifold M is proved to be a representation of the conformal CR automorphism group of M. The group $\mathrm{SU}(p,q)$ can be realized as the conformal CR automorphism group of the projective hyperquadric $\overline{Q}_{p,q}$ and so the space of solutions to the CR Yamabe equation on the projective hyperquadric is a representation of $\mathrm{SU}(p,q)$.

In Section 2, we collect some basic facts about CR geometry and the Heisenberg group. In Section 3, we prove a transformation formula for the CR Yamabe operator under CR transformations and so the kernel of CR Yamabe operator on a CR manifold M is a representation of the conformal CR automorphism group of M. In Section 4, we find some solutions to the CR Yamabe equations. In Section 5, we show $\mathrm{SU}(p,q)$ acting as conformal CR transformations on the projective hyperquadric $\overline{Q}_{p,q}$ and construct a representation of $\mathrm{SU}(p,q)$ on the kernel of CR Yamabe operator on it. Its connection to the degenerate principal series representations is mentioned. Representation of $\mathrm{SU}(p,q)$ on the Heisenberg group $\mathbb{H}^{p-1,q-1}$ is also considered.

The Heisenberg group $\mathbb{H}^{p-1,q-1}$ (or equivalently, the hypersurface $Q'_{p,q}$) and its compactification, the projective hyperquadric $\overline{Q}_{p,q}$, correspond to the Euclidean space $\mathbf{R}^{p-1,q-1}$ and its compactification $S^{p-1}\times S^{q-1}$, respectively. Compared to the Yamabe operators on $S^{p-1}\times S^{q-1}$ and on the Euclidean space, the analysis of the CR Yamabe operators on the projective hyperquadric $\overline{Q}_{p,q}$ and on the Heisenberg group is much more complicated. Unitarization and other properties of these representations will be given in the second part.

For other rank-1 Lie groups $\mathrm{Sp}(1)\mathrm{Sp}(n+1,1)$ and F_4^{-20}, there exist quaternionic and octanionic CR geometries. For example, we have corresponding Webster connections, corresponding conformal geometry, corresponding Yamabe operators, etc. (cf. [2]). It is interesting to study the representation theories of $\mathrm{Sp}(1)\mathrm{Sp}(n+1,1)$ (more generally, of $\mathrm{Sp}(p,q)$) and F_4^{-20} by using corresponding conformal geometries.

Acknowledgement. The results were presented at RIMS Symposium 'Representation theory of groups and extension of harmonic analysis' 2005. I would like to thank Professor Toshiyuki Kobayashi for his hospitality and for valuable discussions. This work is supported by National Nature Science Foundation in China.

2. Preliminaries on CR Geometry

We collect some basic facts about CR geometry and the Heisenberg group in this section (cf. [11] and [17]).

Let M be a real $(2n+1)$-dimensional orientable C^∞ manifold. A *CR structure* on M is an n-dimensional complex subbundle $T_{1,0}M$ of the complexified tangent bundle $\mathbf{C}TM$ satisfying $T_{1,0}M\cap T_{0,1}M=\{0\}$, where $T_{0,1}M=\overline{T_{1,0}M}$, and the integrability condition: $[Z_1,Z_2]\in C^\infty(M,T_{1,0}M)$ whenever $Z_1,Z_2\in C^\infty(M,T_{1,0}M)$. Set

$$H=\mathrm{Re}\{T_{1,0}M\oplus T_{0,1}M\}, \tag{2.1}$$

the $2n$-dimensional *real horizontal subbundle* of TM. H carries a *complex structure* $J : H \longrightarrow H$ satisfying $J^2 = -\mathrm{id}_H$ and $T_{1,0} = \ker(J - i \cdot \mathrm{id}_{\mathbf{C}H})$, $T_{0,1} = \ker(J + i \cdot \mathrm{id}_{\mathbf{C}H})$. When M is a codimension-1 submanifold of the complex manifold W, M has an *induced CR structure* defined by

$$T_{1,0}M = \mathbf{C}TM \cap T_{1,0}W, \tag{2.2}$$

if $\dim(T_{1,0}M)_x$ =const. for each $x \in M$, where $T_{1,0}W$ is the holomorphic tangential space of the complex manifold W.

A mapping $f : (M_1, T_{1,0}M_1) \longrightarrow (M_2, T_{1,0}M_2)$ is called a *Cauchy-Riemann mapping* (or a *CR mapping*) if

$$f_* T_{1,0}M_1 \subset T_{1,0}M_2, \tag{2.3}$$

where f_* is the tangential mapping of f. If f is invertible, f and f^{-1} are both CR mappings, f is called a *CR diffeomorphism.*

Let $E \subset T^*M$ denote the 1-dimensional real line bundle $H^\perp$. Namely, any section of E annihilates H. Because we assume M to be orientable and the complex structure J induces an orientation on H, E has a globally non-vanishing section θ. A globally non-vanishing section θ of E is called a *pseudohermitian structure* on $(M, T_{1,0}M)$. We call the triple $(M, T_{1,0}M, \theta)$ a *pseudohermitian manifold.* An 1-form θ on M is called a *contact form* if $\theta \wedge (d\theta)^n$ is non-vanishing on M.

We say $\tilde\theta$ is *conformal* to θ if $\tilde\theta = \phi^{\frac{4}{Q-2}}\theta$ for some positive smooth function ϕ on M, where $Q = \dim M + 1$ is the *homogeneous dimension* of M. A CR mapping between two pseudohermitian manifolds, $f : (M_1, T_{1,0}M_1, \theta_1) \longrightarrow (M_2, T_{1,0}M_2, \theta_2)$, is called *conformal* if $f^*\theta_2 = \phi^{\frac{4}{Q-2}}\theta_1$ for some positive smooth function ϕ on M_1.

If f is a CR diffeomorphism, then $f_*H_1 = H_2$, where H_1 and H_2 are real horizontal subbundles of TM_1 and TM_2, respectively. Then, $f^*\theta_2$ is a globally non-vanishing section of $E_1 = H_1^\perp$ if θ_2 is a globally non-vanishing section of $E_2 = H_2^\perp$. Note that E is 1-dimensional. Thus, given any globally non-vanishing section θ_1 of E_1, we have

$$f^*\theta_2 = \phi\theta_1, \tag{2.4}$$

for some non-vanishing function ϕ on M_1. So, a CR diffeomorphism f is conformal up to a sign and the CR geometry is the complex counterpart of the conformal geometry.

We can define a Hermitian form on $T_{1,0}M$ associated to a pseudohermitian structure θ by

$$L_\theta(V, \overline{W}) = -id\theta(V \wedge \overline{W}), \tag{2.5}$$

which is called *the Levi form* of θ. This can also be written as

$$L_\theta(V, \overline{W}) = d\theta(V \wedge J\overline{W}). \tag{2.6}$$

In this form $L_\theta(\cdot,\cdot)$ extends by complex linearity to a symmetric form on $\mathbf{C}H$ which is real on H. If the Levi form has k positive eigenvalues and $n-k$ negative eigenvalues, $(M, T_{1,0}M, \theta)$ is said to be *strictly k-pseudoconvex*. The inner product $L_\theta(\cdot,\cdot)$ determines an isomorphism $H^* \cong H$, which in turn determines a dual form $L_\theta^*(\cdot,\cdot)$ on H^*. $L_\theta^*(\cdot,\cdot)$ can be naturally extended to T^*M. This defines a norm $|\omega|_\theta$ on the space of real 1−forms ω by

$$|\omega|_\theta^2 = L_\theta^*(\omega,\omega) = 2\sum_{j=1}^{n} |\omega(Z_j)|^2, \tag{2.7}$$

where $Z_1, \ldots, Z_n$ form an orthonormal basis for $T_{1,0}M$ with respect to the Levi form $L_\theta(\cdot,\cdot)$.

In [21], Webster showed that there exists a natural connection on the bundle $T_{1,0}M$ adapted to a pseudohermitian structure θ. For a pseudohermitian structure θ on a strictly k-pseudoconvex CR manifold $(M, T_{1,0}M, \theta)$, there is a unique vector field T, which is transversal to H, defined by

$$\theta(T) = 1, \qquad d\theta(T \wedge \cdot) = 0. \tag{2.8}$$

Let θ^α be *an admissible coframe*, i.e. $(1,0)$−forms θ^α form a basis for $T_{1,0}^*$ such that $\theta^\alpha(T) = 0$ for all $\alpha = 1, \ldots, n$. The integrability condition implies

$$d\theta = i g_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} \tag{2.9}$$

for some Hermitian matrix of functions $(g_{\alpha\overline{\beta}})$, which is nondegenerate and has k positive eigenvalues and $n-k$ negative eigenvalues if $(M, T_{1,0}M, \theta)$ is strictly k-pseudoconvex. In this case, θ is a contact form, i.e. $\theta \wedge (d\theta)^n$ is nowhere vanishing. Here and in the following, we will use the convention that sum over repeated indices. Webster showed that there are uniquely determined 1-forms ${\omega_\alpha}^\beta$ and τ^β on M satisfying

$$\begin{cases} d\theta^\beta = \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta \\ \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha} = dg_{\alpha\overline{\beta}} \\ \tau_\alpha \wedge \theta^\alpha = 0, \end{cases} \tag{2.10}$$

where we use $(g_{\alpha\overline{\beta}})$ to raise and lower indices, e.g. $\omega_{\alpha\overline{\beta}} = {\omega_\alpha}^\gamma g_{\gamma\overline{\beta}}$. Let

$${\Omega_\beta}^\alpha = d{\omega_\beta}^\alpha - {\omega_\beta}^\gamma \wedge {\omega_\gamma}^\alpha. \tag{2.11}$$

Webster showed that ${\Omega_\beta}^\alpha$ could be written as

$${\Omega_\beta}^\alpha = {R_\beta}^\alpha{}_{\rho\overline{\sigma}}\theta^\rho \wedge \theta^{\overline{\sigma}} + {W_\beta}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\overline{\rho}}\theta^{\overline{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha \tag{2.12}$$

The *Webster-Ricci tensor* of $(M, T_{1,0}M, \theta)$ has components $R_{\alpha\overline{\beta}} = R_\rho{}^\rho{}_{\alpha\overline{\beta}}$. The *Webster scalar curvature* is

$$R_\theta = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}. \tag{2.13}$$

The CR Yamabe problem is to find a contact form $\tilde{\theta} = u^{\frac{4}{Q-2}}\theta, u > 0$, which is conformal to the given contact form θ, such that $R_{\tilde{\theta}} \equiv$constant. This problem is considered by D. Jerison and L. Lee [11] for strictly pseudoconvex CR manifolds and completely solved recently by N. Gamara and R. Yacoub [7], [8].

A pseudohermitian manifold $(M, T_{1,0}M, \theta)$ has a natural volume form

$$\psi_\theta = (-1)^{n-k}\theta \wedge (d\theta)^n, \tag{2.14}$$

which is nowhere vanishing because M is strictly k-pseudoconvex. It induces an L^2 inner product on functions

$$\langle u, v\rangle_\theta = \int_M u\overline{v}\psi_\theta, \tag{2.15}$$

and an L^2 inner product on sections of H^*,

$$\langle \omega, \eta\rangle_\theta = \int_M L^*_\theta(\omega, \eta)\psi_\theta. \tag{2.16}$$

For $u \in C^\infty(M)$, we define a section $d_b u$ of H^* by

$$d_b u = pr \circ du, \tag{2.17}$$

where $pr : T^*M \longrightarrow H^*$ is the restriction map. We can define *the SubLaplacian* $\Box_\theta$ associated to a strictly k-pseudoconvex contact form θ by

$$\langle \Box_\theta u, v\rangle_\theta = \frac{1}{2}\langle d_b u, d_b v\rangle_\theta. \tag{2.18}$$

Since evidently, $|\theta|_\theta = 0$, $L^*_\theta(\cdot,\cdot)$ is degenerate on T^*M and so the operator $\Box_\theta$ is a degenerate ultrahyperbolic operator. Let $\{W_1, \ldots, W_n\}$ be a local basis of $T_{1,0}M$ dual to an admissible coframe $\{\theta^\alpha\}$, i.e. $\theta^\alpha(W_\beta) = \delta_{\alpha\beta}$. Denote covariant differentiations $u_\alpha = W_\alpha u$, $u_{\overline{\alpha}} = W_{\overline{\alpha}} u$, $u_{\alpha\overline{\beta}} = W_{\overline{\beta}} W_\alpha u - \omega_\alpha{}^\gamma(W_{\overline{\beta}})W_\gamma u$, $u_{\overline{\alpha}\beta} = W_\beta W_{\overline{\alpha}} u - \omega_{\overline{\alpha}}{}^{\overline{\gamma}}(W_\beta)W_{\overline{\gamma}} u$, $u_\alpha{}^\gamma = u_{\alpha\overline{\beta}} g^{\gamma\overline{\beta}}$, $u_{\overline{\alpha}}{}^{\overline{\gamma}} = u_{\overline{\alpha}\beta} g^{\beta\overline{\gamma}}$, where $(g^{\alpha\overline{\beta}})$ is the inverse of $(g_{\alpha\overline{\beta}})$.

Proposition 2.1 (Proposition 4.10 in [17]). *If $u \in C_0^\infty(M)$, then,*

$$\Box_\theta u = -u_\alpha{}^\alpha - u_{\overline{\alpha}}{}^{\overline{\alpha}}. \tag{2.19}$$

Define a product on $\mathbf{C}^{n+2}$ by

$$(\zeta, \xi)_{p,q} = \sum_{j=0}^{n+1} \varepsilon_j \zeta_j \overline{\xi}_j, \tag{2.20}$$

where $n+2 = p+q$, and

$$\varepsilon_j = \begin{cases} 1, & \text{for} \quad j = 0, 1, \ldots, p-1, \\ -1, & \text{for} \quad j = p, \ldots, p+q-1. \end{cases} \tag{2.21}$$

We denote $(\zeta,\zeta)_{p,q}$ by $|\zeta|^2_{p,q}$ for $\zeta \in \mathbf{C}^{n+2}$. Similarly, we define a product on $\mathbf{C}^n$ by

$$(z,w)_{p-1,q-1} = \sum_{\alpha=1}^{n} \varepsilon_\alpha z_\alpha \overline{w}_\alpha. \tag{2.22}$$

We also denote $(z,z)_{p-1,q-1}$ by $|z|^2_{p-1,q-1}$ for $z \in \mathbf{C}^n$.

The simplest CR manifold is *the Heisenberg group* $\mathbb{H}^{p-1,q-1}$, whose underlying manifold is $\mathbf{C}^{p+q-2} \times \mathbf{R}$, with coordinates (z,t). Its multiplication is given by

$$(z,t)\cdot(z',t') = (z+z', t+t' + 2\mathrm{Im}(z,z')_{p-1,q-1}). \tag{2.23}$$

It's obvious that $(z,t)^{-1} = (-z,-t)$. On the Heisenberg group $\mathbb{H}^{p-1,q-1}$, there are the following CR transformations:

(1) *dilations*:

$$\delta_a(z,t) = (az, a^2t), \qquad a > 0; \tag{2.24}$$

(2) *left translations*:

$$\tau_{(z,t)} : (z',t') \longrightarrow (z,t)\cdot(z',t'), \qquad (z,t),(z',t') \in \mathbb{H}^{p-1,q-1}; \tag{2.25}$$

(3) *unitary transformations*:

$$U_A : (z,t) \longrightarrow (Az,t), \qquad A \in U(p-1,q-1); \tag{2.26}$$

and the inversions. The vector fields

$$Z_\alpha = \frac{\partial}{\partial z_\alpha} + i\varepsilon_\alpha \overline{z}_\alpha \frac{\partial}{\partial t}, \tag{2.27}$$

$\alpha = 1,\ldots,n$, are left invariant vector fields on $\mathbb{H}^{p-1,q-1}$. The *standard CR structure* on the Heisenberg group $\mathbb{H}^{p-1,q-1}$ is given by the subbundle

$$T_{1,0}\mathbb{H}^{p-1,q-1} = \mathrm{span}_{\mathbf{C}}\{Z_1,\ldots,Z_n\}. \tag{2.28}$$

Let

$$\theta_{\mathbb{H}^{p-1,q-1}} = dt + \sum_{\alpha=1}^{n} i\varepsilon_\alpha (z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha) \tag{2.29}$$

be the *standard contact form* on $\mathbb{H}^{p-1,q-1}$, which is also left invariant. Note that $\theta_{\mathbb{H}^{p-1,q-1}}(Z_\alpha) = 0$ for each α. Since $\delta^*_\lambda \theta_{\mathbb{H}^{p-1,q-1}} = \lambda^2 \theta_{\mathbb{H}^{p-1,q-1}}$,

$$\delta^*_\lambda(\theta_{\mathbb{H}^{p-1,q-1}} \wedge (d\theta_{\mathbb{H}^{p-1,q-1}})^n) = \lambda^{2n+2}\theta_{\mathbb{H}^{p-1,q-1}} \wedge (d\theta_{\mathbb{H}^{p-1,q-1}})^n, \tag{2.30}$$

which means *the homogeneous dimension of* $\mathbb{H}^{p-1,q-1}$ is

$$Q = 2(p+q-2)+2 = 2n+2. \tag{2.31}$$

Since $d\theta_{\mathbb{H}^{p-1,q-1}} = 2i\sum_{\alpha=1}^{n}\varepsilon_\alpha dz_\alpha\wedge d\overline{z}_\alpha$, we can choose $\{\theta^\alpha = dz^\alpha\}$ as an admissible coframe ($T=\frac{\partial}{\partial t}$), and so $(g_{\alpha\overline{\beta}}) = (2\varepsilon_\alpha\delta_{\alpha\beta})$. Then, $\{Z_\alpha\}$, defined by (2.27), is a dual frame since $\theta^\alpha(Z_\beta)=\delta_{\alpha\beta}$. Then, by the formula for the SubLaplacian in Proposition 2.1,

$$\Box_{\theta_{\mathbb{H}^{p-1,q-1}}} = -\frac{1}{2}\sum_{\alpha=1}^{p-1}(Z_\alpha\overline{Z}_\alpha+\overline{Z}_\alpha Z_\alpha)+\frac{1}{2}\sum_{\alpha=p}^{p+q-2}(Z_\alpha\overline{Z}_\alpha+\overline{Z}_\alpha Z_\alpha). \tag{2.32}$$

Let us consider a real hypersurface $Q'_{p,q}$ in $\mathbf{C}^{n+1}$ defined by equation

$$\mathrm{Im} z_0 = |z|^2_{p-1,q-1}, \qquad z\in\mathbf{C}^n, \quad z_0\in\mathbf{C}, \tag{2.33}$$

which is the boundary of the *Siegel upper half space*

$$\mathcal{S} = \left\{(z_0,z)\in\mathbf{C}\times\mathbf{C}^n; \mathrm{Im}\ z_0 > |z|^2_{p-1,q-1}\right\}. \tag{2.34}$$

The *Cayley transformation* C is defined by

$$w_0 = \frac{z_0-i}{z_0+i}, \qquad w_\alpha = \frac{2z_\alpha}{z_0+i}, \tag{2.35}$$

which transforms the hypersurface $Q'_{p,q}$ into the hyperquadric $Q_{p,q}$,

$$Q_{p,q} = \left\{w=(w_0,w'); w_0\in\mathbf{C}, w'\in\mathbf{C}^n, |w_0|^2+|w|^2_{p-1,q-1}=1\right\}. \tag{2.36}$$

Now introduce homogeneous coordinates ζ_j, $j=0,\dots,n+1$. By equations

$$z_j = \frac{\zeta_j}{\zeta_{n+1}}, \qquad j=0,\dots,n, \tag{2.37}$$

$\mathbf{C}^{n+1}$ is embedded as an open subset of the complex projective space $\mathbf{CP}^{n+1}$ of dimension $n+1$. In the homogeneous coordinates, $Q_{p,q}$ is embedded as an open subset of the *projective hyperquadric*

$$\overline{Q}_{p,q} = \left\{\zeta=(\zeta_0,\dots,\zeta_{n+1})\in\mathbf{C}P^{n+1}; |\zeta|^2_{p,q}=0\right\}. \tag{2.38}$$

Projective hyperquadric $\overline{Q}_{p,q}$ is the compactification of $Q_{p,q}$ in $\mathbf{CP}^{n+1}$. The hypersurface $Q'_{p,q}$ and the projective hyperquadric $\overline{Q}_{p,q}$ have induced CR structures by (2.2) from complex manifolds $\mathbf{C}^{n+1}$and $\mathbf{CP}^{n+1}$, respectively.

In $\mathbf{C}^{n+2}$, let

$$\begin{aligned} V_0 &= \left\{\zeta\in\mathbf{C}^{n+2}|(\zeta,\zeta)_{p,q}=0\right\},\\ V_- &= \left\{\zeta\in\mathbf{C}^{n+2}|(\zeta,\zeta)_{p,q}<0\right\}. \end{aligned} \tag{2.39}$$

Let $\pi:\mathbf{C}^{n+2}\setminus\{0\}\longrightarrow\mathbf{CP}^{n+1}$ be the canonical projection onto the complex projective space. Then, $\mathbf{H}^{n+1}_{\mathbf{C}}=\pi(V_-)$ is *the complex hyperbolic space* and the group $\mathrm{U}(p,q)$ is a subgroup of $\mathrm{GL}(n+2,\mathbf{C})$ whose elements preserving the Hermitian form $(\cdot,\cdot)_{p,q}$ defined by (2.20). The action of $\mathrm{U}(p,q)$ on V_- induces an action on $\mathbf{H}_{\mathbf{C}}^{n+1}$ with kernel isomorphic to S^1. Set $\mathrm{PU}(p,q) = \mathrm{U}(p,q)/\text{kernel}$. $\mathrm{SU}(p,q)$ is the group of unimodular transformations preserving the Hermitian form (2.20). Its center K consists of $n+2$ transformations:

$$\zeta_j \longrightarrow \eta\zeta_j, \qquad \eta^{n+2} = 1, \qquad j = 0, \ldots, n+1. \tag{2.40}$$

Then $\mathrm{SU}(p,q)/K$ acts on $\pi(V_0) = \overline{Q}_{p,q}$ effectively and $\mathrm{PU}(p,q) = \mathrm{SU}(p,q)/K$. It is well known that $\mathrm{Aut}_{CR}\pi(V_0) = \mathrm{PU}(p,q)$ [5].

3. Representations realized as conformal CR diffeomorphisms

The transformation formula for the Webster scalar curvatures under conformal changes of pseudohermitian structures is proved by L. Lee in Proposition 5.15 in [17] for any nondegenerate pseudohermitian structure.

Proposition 3.1 (Proposition 5.15 in [17]). *Let $(M, T_{1,0}M, \theta)$ be a pseudohermitian manifold with* $\dim M = 2n+1$. *The Webster scalar curvature $R_{\tilde\theta}$ associated with the pseudohermitian structure $\tilde\theta = e^{2f}\theta$ is*

$$R_{\tilde\theta} = e^{-2f}(R_\theta + 2(n+1)\Box_\theta f - 4n(n+1)f_\alpha f^\alpha). \tag{3.1}$$

Let $Q = 2n+2$. If we take $f = \frac{2}{Q-2}\log u$ for $u > 0$ in the above proposition, we can write the transformation formula in the following form.

Corollary 3.1. *Let $(M, T_{1,0}M, \theta)$ be a pseudohermitian manifold with* $\dim M = 2n+1$. *The Webster scalar curvature $R_{\tilde\theta}$ associated with the pseudohermitian structure $\tilde\theta = u^{\frac{4}{Q-2}}\theta$ satisfies*

$$b_n \Box_\theta u + R_\theta u = R_{\tilde\theta} u^{\frac{Q+2}{Q-2}}, \tag{3.2}$$

where $b_n = 2 + \frac{2}{n}$.

Proof. By the definitions of covariant differentiations, we have that

$$f_\alpha = \frac{1}{n}\frac{u_\alpha}{u}, \qquad f^\alpha = \frac{1}{n}\frac{u^\alpha}{u}, \tag{3.3}$$

and

$$\Box_\theta f = -\frac{1}{n}\left(\frac{1}{u}\left(u_\alpha{}^\alpha + u_{\overline{\alpha}}{}^{\overline{\alpha}}\right) - \frac{1}{u^2}\left(u_\alpha u^\alpha + u_{\overline{\alpha}}u^{\overline{\alpha}}\right)\right), \tag{3.4}$$

by the formula for the SubLaplacian in Proposition 2.1. Substituting (3.3) and (3.4) into (3.1) and noting that

$$u_{\overline{\alpha}}u^{\overline{\alpha}} = g_{\overline{\alpha}\beta}u^\beta u^{\overline{\alpha}} = u^\beta u_\beta = u_\alpha u^\alpha \tag{3.5}$$

by using $(g_{\alpha\overline{\beta}})$ to raise and lower indices, the result follows. □

The following is a transformation formula for the SubLaplacians under a conformal CR transformation. See [18] for the corresponding transformation formula for Laplacian under a conformal transformation in the pseudo-Riemannian case.

Proposition 3.2. *Let $(M_1, T_{1,0}M_1)$ and $(M_2, T_{1,0}M_2)$ be two CR manifolds with strictly k-psedoconvex pseudohermitian structure θ_1 and θ_2, respectively. Suppose $\Phi : (M_1, T_{1,0}M_1) \longrightarrow (M_2, T_{1,0}M_2)$ is a CR diffeomorphism with $\Phi^*\theta_2 = u^{\frac{4}{Q-2}}\theta_1$ for some positive smooth function u on M_1. Then*

$$\Box_{\theta_1}(u \cdot \Phi^* f) - u^{\frac{Q+2}{Q-2}}\Phi^*\left(\Box_{\theta_2} f\right) = \Box_{\theta_1} u \cdot \Phi^* f, \tag{3.6}$$

for any smooth real function f on M_2.

Proof. For real 1-forms ω_1 and ω_2, we have the symmetry $L^*_\theta(\omega_1, \omega_2) = L^*_\theta(\omega_2,\ \omega_1)$ and so $\langle\omega_1, \omega_2\rangle_\theta = \langle\omega_2, \omega_1\rangle_\theta$ by the definition of inner product (2.16). Note that

$$\begin{aligned} 2\left\langle\Box_{\theta_1}(u\Phi^* f), g\right\rangle_{\theta_1} &= \left\langle d_b(u\Phi^* f), d_b g\right\rangle_{\theta_1} \\ &= \left\langle d_b u, \Phi^* f \cdot d_b g\right\rangle_{\theta_1} + \left\langle d_b(\Phi^* f), u d_b g\right\rangle_{\theta_1} \\ &= \left\langle d_b u, d_b(\Phi^* f \cdot g)\right\rangle_{\theta_1} - \left\langle d_b u, d_b(\Phi^* f) \cdot g)\right\rangle_{\theta_1} + \left\langle d_b(\Phi^* f), u d_b g\right\rangle_{\theta_1} \\ &= 2\left\langle\Box_{\theta_1} u, \Phi^* f \cdot g\right\rangle_{\theta_1} + \left\langle d_b(\Phi^* f), u d_b g - g d_b u\right\rangle_{\theta_1}, \end{aligned} \tag{3.7}$$

for any smooth real function g on M_1. Let us calculate the second term in the right side of (3.7). Since

$$\Phi^*(d\theta_2) = d(\Phi^*\theta_2) = u^{\frac{4}{Q-2}} d\theta_1 + \frac{4}{Q-2} u^{\frac{4}{Q-2}-1} du \wedge \theta_1, \tag{3.8}$$

we find that

$$\Phi^*\psi_{\theta_2} = (-1)^{n-k}\Phi^*(\theta_2 \wedge (d\theta_2)^n) = (-1)^{n-k} u^{\frac{4(n+1)}{Q-2}} \theta_1 \wedge (d\theta_1)^n = u^{\frac{2Q}{Q-2}}\psi_{\theta_1}, \tag{3.9}$$

by $\theta_1 \wedge du \wedge \theta_1 = 0$. Consequently,

$$\langle\Phi^* f_1, \Phi^* f_2\rangle_{\theta_1} = \left\langle (u \circ \Phi^{-1})^{-\frac{2Q}{Q-2}} f_1, f_2 \right\rangle_{\theta_2} \tag{3.10}$$

for two smooth functions f_1 and f_2 on M_2.

Since Φ is a CR diffeomorphism, we have $\Phi_* H_1 = H_2$, where H_1 and H_2 are real horizontal subbundles of TM_1 and TM_2, respectively. Dually, we have

$$\Phi^* \circ pr_2 = pr_1 \circ \Phi^*, \tag{3.11}$$

where pr_j is the restriction mapping $T^*M_j \longrightarrow H^*_j$, $j = 1, 2$, and so

$$d_b(\Phi^* u) = pr_1(d(\Phi^* u)) = pr_1(\Phi^*(du)) = \Phi^*(pr_2(du)) = \Phi^*(d_b u). \tag{3.12}$$

By using (3.8), we have

$$\begin{aligned} L_{\theta_2}(\Phi_* V, \Phi_*\overline{W}) &= -i d\theta_2(\Phi_* V \wedge \Phi_*\overline{W}) = -i\Phi^*(d\theta_2)(V \wedge\ \overline{W}) \\ &= -i u^{\frac{4}{Q-2}} d\theta_1(V \wedge\ \overline{W}) = u^{\frac{4}{Q-2}} L_{\theta_1}(V, \overline{W}) \end{aligned} \tag{3.13}$$

for any $V, W \in T_{1,0}M_1$. Consequently, $L_{\theta_2}(\Phi_* V_1, \Phi_* V_2) = u^{\frac{4}{Q-2}} L_{\theta_1}(V_1, V_2)$ for any $V_1, V_2 \in H_1$, the real horizontal subspace of TM_1. Dually, we get

$$L^*_{\theta_1}(\Phi^*(\omega_1), \Phi^*(\omega_2))(x) = u^{\frac{4}{Q-2}} L^*_{\theta_2}(\omega_1, \omega_2)(\Phi(x)), \tag{3.14}$$

for $\omega_1, \omega_2 \in H_2^*$ and any $x \in M_1$. Now by using the pull back properties (3.9), (3.10), (3.12) and (3.14), we find that

$$\begin{aligned}
&\langle d_b(\Phi^* f), u d_b g - g d_b u\rangle_{\theta_1} \\
&= \int_{M_1} L^*_{\theta_1}\left(\Phi^*(d_b f), u d_b g - g d_b u\right)(x)\psi_{\theta_1}(x) \\
&= \int_{M_1} u^{-2} L^*_{\theta_2}\left(d_b f, \left(\Phi^{-1}\right)^*(u d_b g - g d_b u)\right)(\Phi(x))\left(\Phi^*\psi_{\theta_2}\right)(x) \\
&= \int_{M_2} \left(u \circ \Phi^{-1}\right)^{-2} L^*_{\theta_2}\left(d_b f, \left(\Phi^{-1}\right)^*(u d_b g - g d_b u)\right)\psi_{\theta_2} \\
&= \left\langle d_b f, d_b\left(\left(\Phi^{-1}\right)^*\left(u^{-1} g\right)\right)\right\rangle_{\theta_2} \\
&= 2\left\langle \Box_{\theta_2} f, \left(\Phi^{-1}\right)^*\left(u^{-1} g\right)\right\rangle_{\theta_2} \\
&= 2\left\langle u^{\frac{Q+2}{Q-2}} \Phi^*\left(\Box_{\theta_2} f\right), g\right\rangle_{\theta_1}.
\end{aligned} \tag{3.15}$$

Equation (3.6) follows from (3.7) and (3.15). The proposition is proved. □

Now define the *CR Yamabe operator* to be

$$\widetilde{\Box}_\theta = b_n \Box_\theta + R_\theta, \tag{3.16}$$

where $b_n = 2 + \frac{2}{n}$, R_θ is the Webster scalar curvature (2.13). This operator in the positive-definite case has already appeared in the work [19] of N. Stanton. The transformation formula for the CR Yamabe operator is a consequence of Corollary 3.1 and Proposition 3.2 as follows.

Proposition 3.3. *Under the same assumption as in Proposition* 3.2, *we have that*

$$\widetilde{\Box}_{\theta_1}(u \cdot \Phi^* f) = u^{\frac{Q+2}{Q-2}} \Phi^*\left(\widetilde{\Box}_{\theta_2} f\right), \tag{3.17}$$

for any smooth function f on M_2.

Suppose $(M_1, T_{1,0}M_1, \theta_1)$ and $(M_2, T_{1,0}M_2, \theta_2)$ are two pseudohermitian manifolds of homogeneous dimension Q. Let conformal CR mapping $\Phi : (M_1, T_{1,0}M_1, \theta_1) \longrightarrow (M_2, T_{1,0}M_2, \theta_2)$ be a local diffeomorphism such that

$$\Phi^*\theta_2 = \Omega^2\theta_1, \tag{3.18}$$

for some positive function Ω on M_1. We can define *twisted pull back*

$$\Phi^*_\lambda : C^\infty(M_2) \longrightarrow C^\infty(M_1), \qquad f \longmapsto \Omega^\lambda(\Phi^* f), \tag{3.19}$$

and write the twisted pull back for $\lambda = \frac{Q-2}{2}$ as $\tilde{\Phi}^* = \Phi^*_{\frac{Q-2}{2}}$.

Let G be a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$. We write the action of $h \in G$ as $L_h : (M, T_{1,0}M, \theta) \longrightarrow (M, T_{1,0}M, \theta)$, $x \longmapsto L_h x$. There exists a positive valued function $\Omega(h,x)$ for $h \in G$ and $x \in M$ such that

$$L_h^* \theta = \Omega(h, \cdot)^2 \theta. \tag{3.20}$$

We have the *cocycle formula* for $\Omega(\cdot,\cdot)$.

Proposition 3.4. *For $h_1, h_2 \in G$ and $x \in M$, we have*

$$\Omega(h_1 h_2, x) = \Omega(h_1, L_{h_2} x)\Omega(h_2, x). \tag{3.21}$$

Proof. By the definition of Ω, we have

(3.22)
$$\Omega(h_1 h_2, \cdot)^2 \theta = L^*_{h_1 h_2}\theta = L^*_{h_2} L^*_{h_1}\theta = L^*_{h_2}(\Omega(h_1, \cdot)^2\theta) = \Omega(h_2, \cdot)^2 \Omega(h_1, L_{h_2}\cdot)^2\theta.$$

The proposition follows. □

Now for $\lambda \in \mathbf{C}$, we can define a representation ϖ_λ of the group G on $C^\infty(M)$ as follows. For $h \in G$, $f \in C^\infty(M)$ and $x \in M$, let

$$\left(\varpi_\lambda(h^{-1})f\right)(x) = \Omega(h,x)^\lambda f(L_h x). \tag{3.23}$$

Proposition 3.4 assures that $\varpi_\lambda(h_1 h_2) = \varpi_\lambda(h_1)\varpi_\lambda(h_2)$, i.e., ϖ_λ is a representation of G. By Proposition 3.3, we have

$$\Omega^{\frac{Q+2}{2}} \Phi^* \left(\widetilde{\Box}_\theta f\right) = \widetilde{\Box}_\theta \left(\Omega^{\frac{Q-2}{2}} \Phi^* f\right). \tag{3.24}$$

for a conformal CR diffeomorphism $\Phi : (M, T_{1,0}M, \theta) \longrightarrow (M, T_{1,0}M, \theta)$ and any smooth function f on M. Thus, $\widetilde{\Box}_\theta f = 0$ if and only if $\widetilde{\Box}_\theta \left(\Omega^{\frac{Q-2}{2}} \tilde{\Phi}^* f\right) = 0$. In summary, we have the following theorem.

Theorem 3.1. *Suppose G is a Lie group acting as conformal CR diffeomorphisms on a pseudohermitian manifold $(M, T_{1,0}M, \theta)$ of homogeneous dimension Q. Then,*

(1) *the CR Yamabe operator $\widetilde{\Box}_\theta$ is an intertwining operator from $\varpi_{\frac{Q-2}{2}}$ to $\varpi_{\frac{Q+2}{2}}$.*

(2) *The kernel* $\ker \widetilde{\Box}_\theta$ *is a subrepresentation of G through $\varpi_{\frac{Q-2}{2}}$.*

We can obtain the functoriality of our representations as in the pseudo-Riemannian case in Proposition 2.6 in [13]. We omit the details.

4. The CR Yamabe operator on the hypersurface $Q'_{p,q}$

Let $\xi \longrightarrow \widetilde{\xi}$ denote the canonical projection of $\mathbf{C}^{n+2} \setminus \{0\}$ into the complex projective space $\mathbf{C}P^{n+1}$. It is easy to see that the transformation

$$I(z_0, z_1, \ldots, z_n) = \left(\frac{z_0 - i}{2}, z, \frac{z_0 + i}{2} \right)^{\sim}, \tag{4.1}$$

maps the hypersurface $Q'_{p,q}$ defined by (2.33) into the projective hyperquadric $\overline{Q}_{p,q}$ (2.38). Define a 1-form

$$\theta = \frac{\sum_{j=0}^{n+1} i\varepsilon_j (\xi_j d\overline{\xi}_j - \overline{\xi}_j d\xi_j)}{\sum_{j=0}^{p-1} |\xi_j|^2}, \tag{4.2}$$

on $\mathbf{C}^{n+2} \setminus \{\xi \in \mathbf{C}^{n+2}; \xi_0 = \cdots = \xi_{p-1} = 0\}$. Since this form is invariant under the homogeneous transformation $(\xi_0, \ldots, \xi_{n+1}) \longrightarrow (c\xi_0, \ldots, c\xi_{n+1})$ for $c \in \mathbf{C}$, it induces a 1-form on an open set $\mathbf{CP}^{n+1} \setminus \{\xi \in \mathbf{CP}^{n+1}; \xi_0 = \cdots = \xi_{p-1} = 0\}$. So it induces a 1-form on the projective hyperquadric $\overline{Q}_{p,q}$ in (2.38) since $\overline{Q}_{p,q} \cap \{\xi \in \mathbf{CP}^{n+1}; \xi_0 = \cdots = \xi_{p-1} = 0\} = \emptyset$. This is actually a contact form on $\overline{Q}_{p,q}$ (see Remark 5.1). We denote it by $\theta_{\overline{Q}_{p,q}}$. The hyperquadric $Q_{p,q}$ in (2.36) has a contact form

$$\theta_{Q_{p,q}} = \sum_{\alpha=0}^{n} i\varepsilon_\alpha (z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha), \tag{4.3}$$

(here we use variables z_α instead of w_α, $\alpha = 0, \ldots, n$, in the definition of $Q_{p,q}$ in (2.36)) and the hypersurface $Q'_{p,q}$ in (2.33) has a contact form

$$\theta_{Q'_{p,q}} = \sum_{\alpha=1}^{n} i\varepsilon_\alpha (z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha) + \frac{1}{2}(d\overline{z}_0 + dz_0). \tag{4.4}$$

Contact forms (4.3) and (4.4) are actually

$$i(\overline{\partial} - \partial) r \tag{4.5}$$

for corresponding defining functions r of $Q_{p,q}$ and $Q'_{p,q}$, respectively. Let $T_{0,1}Q_{p,q}$ and $T_{0,1}\overline{Q}_{p,q}$ be the induced CR structures, i.e., $T_{0,1}Q_{p,q} = T_{0,1}\mathbf{C}^{n+1} \cap \mathbf{C}TQ_{p,q}$, $T_{0,1}\overline{Q}_{p,q} = T_{0,1}\mathbf{CP}^{n+1} \cap \mathbf{C}T\overline{Q}_{p,q}$.

Proposition 4.1.

$$I^*\theta_{\overline{Q}_{p,q}} = \frac{1}{\frac{1}{4}|z_0 - i|^2 + \sum_{j=1}^{p-1} |z_j|^2} \theta_{Q'_{p,q}}, \tag{4.6}$$

on the hypersurface $Q'_{p,q}$.

Proof. This can be checked by simple calculation. $\square$

The vector fields

$$Z_\alpha = \frac{\partial}{\partial z_\alpha} + 2i\varepsilon_\alpha \overline{z}_\alpha \frac{\partial}{\partial z_0}, \tag{4.7}$$

$\alpha = 1, \ldots, n$, are complex tangential vectors of hyperquadric $Q'_{p,q}$ in (2.33), namely, $Z_\alpha r = 0$, where $r = \mathrm{Im} z_0 - |z|^2_{p-1,q-1}$, the defining function of hypersurface $Q'_{p,q}$. They span $T_{1,0}(Q'_{p,q})$ (note that $T_{1,0}(Q'_{p,q}) \subset \ker \theta_{Q'_{p,q}}$).

Note that $d\theta_{Q'_{p,q}} = \sum_{\alpha=1}^n 2i\varepsilon_\alpha dz_\alpha \wedge d\overline{z}_\alpha$, i.e., $(g_{\alpha\overline{\beta}}) = (2\varepsilon_\alpha \delta_{\alpha\beta})$. We can choose an admissible coframe $\theta^\alpha = dz_\alpha$. It is easy to see the corresponding 1-forms ${\omega_\alpha}^\beta = 0$ and $\tau^\beta = 0$. So $Q'_{p,q}$ has vanishing curvature. Then, $\{Z_\alpha\}$, defined by (4.7), is a dual frame since $\theta^\alpha(Z_\beta) = \delta_{\alpha\beta}$. Hence,

$$\square_{\theta_{Q'_{p,q}}} = -\frac{1}{2}\sum_{\alpha=1}^{p-1}(Z_\alpha \overline{Z}_\alpha + \overline{Z}_\alpha Z_\alpha) + \frac{1}{2}\sum_{\alpha=p}^{p+q-2}(Z_\alpha \overline{Z}_\alpha + \overline{Z}_\alpha Z_\alpha), \tag{4.8}$$

by the formula for the SubLaplacian in Proposition 2.1.

Proposition 4.2. *Let $S_0 = \sum_{j=0}^{n+1} a_j |\xi_j|^2$ with $a_j = \varepsilon_j$ or 0, but $a_0 = 1$ and $a_{n+1} = 0$. Then the function*

$$S(z_0, z) = S_0\left(\frac{z_0 - i}{2}, z, \frac{z_0 + i}{2}\right) \tag{4.9}$$

on hypersurface $Q'_{p,q}$ satisfies where it is positive

$$\widetilde{\square}_{\theta_{Q'_{p,q}}} S^{-\frac{Q-2}{4}} = \frac{n+1}{2}\left(\sum_{j=1}^n 2a_j\varepsilon_j - n\right) S^{-\frac{Q+2}{4}}, \tag{4.10}$$

where $Q = 2n+2$.

Proof. Direct manipulation gives

$$Z_j S^{-\frac{Q-2}{4}} = -\frac{n}{2} S^{-\frac{n}{2}-1}\left(\frac{1}{2} i\varepsilon_j \overline{z}_j(\overline{z}_0 + i) + a_j \overline{z}_j\right), \tag{4.11}$$

$\overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - 2i\varepsilon_j z_j \frac{\partial}{\partial \overline{z}_0}$ and

(4.12)

$$\begin{aligned}
\overline{Z}_j Z_j S^{-\frac{Q-2}{4}} &= -\frac{n}{2} S^{-\frac{n}{2}-1}\left(\frac{1}{2} i\varepsilon_j \overline{z}_0 - \frac{1}{2}\varepsilon_j + a_j + |z_j|^2\right) \\
&\quad - \frac{n}{2}\left(-\frac{n}{2} - 1\right) S^{-\frac{n}{2}-2}\left(\frac{1}{2} i\varepsilon_j \overline{z}_j(\overline{z}_0 + i) + a_j \overline{z}_j\right)\left(-\frac{1}{2} i\varepsilon_j z_j(z_0 - i) + a_j z_j\right) \\
&= -\frac{n}{2} S^{-\frac{n}{2}-1}\left(\frac{1}{2} i\varepsilon_j \overline{z}_0 - \frac{1}{2}\varepsilon_j + a_j + |z_j|^2\right) \\
&\quad + \frac{n}{2}\left(\frac{n}{2} + 1\right) S^{-\frac{n}{2}-2}\left(\frac{1}{4}|z_j|^2|z_0 - i|^2 + a_j^2|z_j|^2 - a_j\varepsilon_j|z_j|^2 + a_j\varepsilon_j|z_j|^2 \mathrm{Im} z_0\right).
\end{aligned}$$

Since $a_j\varepsilon_j = a_j^2$ for each j by the definition of a_j, the bracket in the second sum in the right hand side of (4.12) is $\frac{1}{4}|z_j|^2|z_0-i|^2 + a_j\varepsilon_j|z_j|^2|z|^2_{p-1,q-1}$ by the defining equation of $Q'_{p,q}$ and

$$\sum_{j=1}^{n}\varepsilon_j\left(\frac{1}{4}|z_j|^2|z_0-i|^2 + a_j\varepsilon_j|z_j|^2|z|^2_{p-1,q-1}\right) = |z|^2_{p-1,q-1}S(z_0,z), \tag{4.13}$$

by the definitions of S and $|z|^2_{p-1,q-1}$. Thus, by multiplying ε_j in both sides of (4.12), summing over j and adding its conjugate, we find that

$$\sum_{j=1}^{n}\varepsilon_j(\overline{Z}_jZ_j + Z_j\overline{Z}_j)S^{-\frac{Q-2}{4}} = -\frac{n}{2}\left(\sum_{j=1}^{n}2a_j\varepsilon_j - n\right)S^{-\frac{n}{2}-1}, \tag{4.14}$$

which is equivalent to (4.10) by (4.8) and $\widetilde{\Box}_{\theta_{Q'_{p,q}}} = b_n\Box_{\theta_{Q'_{p,q}}}$. □

If we choose a_j so that $\sum_{j=1}^{n}2a_j\varepsilon_j = n$, then $S^{-\frac{Q-2}{4}}$ is a solution of the CR Yamabe equation on hypersurface $Q'_{p,q}$.

Corollary 4.1. *The scalar curvature of the projective quadric $\overline{Q}_{p,q}$ with contact form $\theta_{\overline{Q}_{p,q}}$ is $\frac{n+1}{2}(p-q)$.*

Proof. Since $R_{\theta_{Q'_{p,q}}} = 0$, we have $R_{\theta_{\overline{Q}_{p,q}}} = S^{\frac{Q+2}{4}}\cdot\widetilde{\Box}_{\theta_{Q'_{p,q}}}S^{-\frac{Q-2}{4}}$ with $a_0 = \cdots = a_{p-1} = 1$, $a_p = \cdots = a_{n+1} = 0$ by the transformation formula (3.2) and the conformal relation in Proposition 4.1. Then the result follows from Proposition 4.2. □

See [18] for the corresponding proposition in the Euclidean case.

5. Representations on the projective hyperquadric $\overline{Q}_{p,q}$

Define open subsets of $\mathbf{CP}^{n+1}$

$$O_j = \{\xi = (\xi_0,\ldots,\xi_{n+1}) \in \mathbf{CP}^{n+1}; \xi_j \neq 0\}, \tag{5.1}$$

$j = 0,\ldots,n+1$. Then $\{O_0,\ldots,O_{n+1}\}$ is a covering of $\mathbf{CP}^{n+1}$. Define a diffeomorphism

$$I_j : O_j \longrightarrow \mathbf{C}^{n+1}, \tag{5.2}$$

given by

$$\xi \mapsto z = (z_1,\ldots,z_n), \qquad z_l = \xi_l/\xi_j, \qquad l = 0,\ldots,j-1,j+1,\ldots,n+1. \tag{5.3}$$

Then, I_{n+1} maps $\overline{Q}_{p,q}\cap O_{n+1}$ to $Q_{p,q}$ isomorphically. The inverse I_j^{-1} of I_j provides the embedding

$$I_j^{-1} : \mathbf{C}^{n+1} \longrightarrow \mathbf{CP}^{n+1}, \qquad z \mapsto (z_0,\ldots,z_{j-1},1,z_{j+1},\ldots,z_{n+1}). \tag{5.4}$$

Thus, under the identification I_{n+1}^{-1}, $\mathrm{SU}(p,q)$ acts on $\mathbf{C}^{n+1}$ and so on the hyperquadric $Q_{p,q}$ as

$$g(z) = \left(\frac{g(z,1)_0}{g(z,1)_{n+1}}, \dots, \frac{g(z,1)_n}{g(z,1)_{n+1}} \right), \tag{5.5}$$

for $g \in \mathrm{SU}(p,q)$ and $z \in Q_{p,q}$, where we denote by $g(z,1)_j$ the j-th component of $g(z,1) \in \mathbf{C}^{n+2}$. Let us calculate the conformal factors of elements of $\mathrm{SU}(p,q)$ acting on the hyperquadric $Q_{p,q}$. We have calculated the conformal factors of elements of $\mathrm{SU}(n+1,1)$ acting on the unit sphere S^{2n+1} in [20].

Proposition 5.1. *For $g \in \mathrm{SU}(p,q)$ and $z \in Q_{p,q}$, we have*

$$g^* \theta_{Q_{p,q}}(z) = \frac{1}{|g(z,1)_{n+1}|^2} \theta_{Q_{p,q}}(z). \tag{5.6}$$

Proof. Note that

$$dg(z) = \left(\dots, \frac{g(dz,0)_j}{g(z,1)_{n+1}} - \frac{g(z,1)_j g(dz,0)_{n+1}}{g(z,1)_{n+1}^2}, \dots \right), \tag{5.7}$$

by the linearity of g. We have, for $z \in Q_{p,q}$,

$$\begin{aligned} \sum_{j=0}^{n} \varepsilon_j g(z)_j d\overline{g(z)}_j &= \sum_{j=0}^{n} \varepsilon_j \frac{g(z,1)_j \overline{g}(d\overline{z},0)_j}{|g(z,1)_{n+1}|^2} - \sum_{j=0}^{n} \frac{\varepsilon_j |g(z,1)_j|^2 \overline{g}(d\overline{z},0)_{n+1}}{|g(z,1)_{n+1}|^2 \overline{g}(\overline{z},1)_{n+1}} \\ &= \sum_{j=0}^{n+1} \varepsilon_j \frac{g(z,1)_j \overline{g}(d\overline{z},0)_j}{|g(z,1)_{n+1}|^2}, \end{aligned} \tag{5.8}$$

by

$$\sum_{j=0}^{n} \varepsilon_j |g(z,1)_j|^2 - |g(z,1)_{n+1}|^2 = \sum_{j=0}^{n} \varepsilon_j |z_j|^2 - 1 = 0 \tag{5.9}$$

for $z \in Q_{p,q}$ and $g \in \mathrm{SU}(p,q)$, which preserves the Hermitian product (2.20). By differentiation the first equation in (5.9), which holds for any $z \in \mathbf{C}^{n+1}$, with respect to $\overline{z}$, we get

$$\sum_{j=0}^{n} \varepsilon_j g(z,1)_j \overline{g}(d\overline{z},0)_j - g(z,1)_{n+1} \overline{g}(d\overline{z},0)_{n+1} = \sum_{j=0}^{n} \varepsilon_j z_j d\overline{z}_j. \tag{5.10}$$

Then,

$$\sum_{j=0}^{n} \varepsilon_j g(z)_j d\overline{g(z)}_j = \frac{\sum_{j=0}^{n} \varepsilon_j z_j d\overline{z}_j}{|g(z,1)_{n+1}|^2}. \tag{5.11}$$

(5.6) follows from the definition of $\theta_{Q_{p,q}}$ in (4.3). This complete the proof of the proposition. $\square$

Define the *light cone* to be

$$\Xi := \{\xi \in \mathbf{C}^{n+2}; |\xi|_{p,q} = 0\} \setminus \{0\}, \tag{5.12}$$

and

$$\Sigma := \left\{ \xi \in \mathbf{C}^{n+2}; \sum_{j=0}^{p-1} |\xi_j|^2 = \sum_{j=p}^{p+q-1} |\xi_j|^2 = 1 \right\} \simeq S^{2p-1} \times S^{2q-1}. \tag{5.13}$$

The multiplicative group $\mathbf{R}_+^\times$ acts on Ξ as a dilation and the quotient space $\Xi/\mathbf{R}_+^\times$ is identified with Σ. By definition, $\Xi/\mathbf{C}^\times \simeq \Sigma/S^1 \simeq \overline{Q}_{p,q}$. Because the action of $\mathrm{SU}(p,q)$ on $\mathbf{C}^{n+2}$ commutes with that of $\mathbf{C}^\times$, we can define the action of $\mathrm{SU}(p,q)$ on the quotient space $\Xi/\mathbf{C}^\times$, and also on $\overline{Q}_{p,q}$ through the above diffeomorphism. This action will be denoted by

$$L_h : \overline{Q}_{p,q} \longrightarrow \overline{Q}_{p,q}, \quad \xi \mapsto L_h\xi, \tag{5.14}$$

for $h \in \mathrm{SU}(p,q), \xi \in \overline{Q}_{p,q}$.

For $a \in \mathbf{C}$, denote by $S^a(\Xi)$ the space of smooth function on Ξ homogeneous of degree a, i.e.

$$S^a(\Xi) = \{f \in C^\infty(\Xi); f(t\xi) = t^a f(\xi), \xi \in \Xi, t \in \mathbf{R}_+^\times\}. \tag{5.15}$$

A character ψ of $\mathbf{C}^\times$ has the form

$$\psi(t) = |t|^a \left(\frac{t}{|t|}\right)^m, \tag{5.16}$$

for some $a \in \mathbf{C}^\times, m \in \mathbf{Z}$, which can be formally written as

$$\psi(t) = \psi^{\alpha,\beta}(t) = t^\alpha \bar{t}^\beta, \tag{5.17}$$

with $\alpha + \beta = a$ and $\alpha - \beta = m$. We see that a pair (α, β) can occur if and only if $\alpha - \beta$ is an integer. For such a pair, we define $S^{\alpha,\beta}(\Xi) \subset S^a(\Xi)$ to be the $\psi^{\alpha,\beta}$ eigenspace for $\mathbf{C}^\times$. Then, we have a decomposition

$$S^a(\Xi) = \sum_{\substack{\alpha+\beta=a,\\ \alpha-\beta\in\mathbf{Z}}} S^{\alpha,\beta}(\Xi). \tag{5.18}$$

Let $\nu : \Xi \longrightarrow \mathbf{R}_+$ be defined by

$$\nu(\xi) = \left(\sum_{j=0}^{p-1} |\xi_j|^2\right)^{\frac{1}{2}} = \left(\sum_{j=p}^{p+q-1} |\xi_j|^2\right)^{\frac{1}{2}}. \tag{5.19}$$

Proposition 5.2. *For $g \in \mathrm{SU}(p,q)$ and $\xi \in \overline{Q}_{p,q}$, we have*

$$g^*\theta_{\overline{Q}_{p,q}}(\xi) = \frac{1}{\nu(g(\xi))^2}\theta_{\overline{Q}_{p,q}}(\xi), \tag{5.20}$$

if we require the coordinates of ξ satisfying $\sum_{j=0}^{p-1}|\xi_j|^2 = 1$.

Proof. Note that

$$\overline{Q}_{p,q} = \bigcup_{j=p}^{n+1}\left(O_j \cap \overline{Q}_{p,q}\right). \tag{5.21}$$

We only show (5.20) for $\xi \in O_{n+1} \cap \overline{Q}_{p,q}$. For $\xi \in O_j \cap \overline{Q}_{p,q}$, $j = p,\ldots,n$, (5.20) follows just by the permutation of coordinates $z_j \leftrightarrow z_{n+1}$.

By the definitions of $\theta_{\overline{Q}_{p,q}}$ in (4.2) and $\theta_{Q_{p,q}}$ in (4.3) and the definition of the embedding I_{n+1}^{-1} in (5.4), we find that $I_{n+1}^{-1}(Q_{p,q}) = O_{n+1} \cap \overline{Q}_{p,q}$ and

$$\left(\left(I_{n+1}^{-1}\right)^*\theta_{\overline{Q}_{p,q}}\right)(z) = \frac{1}{\sum_{j=0}^{p-1}|z_j|^2}\theta_{Q_{p,q}}(z), \tag{5.22}$$

for $z \in Q_{p,q}$. By the formula for $g^*\theta_{Q_{p,q}}$ in Proposition 5.1 and the definition of group SU(p,q) acting on $Q_{p,q}$ in (5.5), we have

$$\begin{aligned} g^*\left(\frac{1}{\sum_{j=0}^{p-1}|z_j|^2}\theta_{Q_{p,q}}\right)(z) &= \frac{1}{\sum_{j=0}^{p-1}\left|\frac{g(z,1)_j}{g(z,1)_{n+1}}\right|^2}\cdot\frac{1}{|g(z,1)_{n+1}|^2}\theta_{Q_{p,q}}(z) \\ &= \frac{1}{\sum_{j=0}^{p-1}|g(z,1)_j|^2}\theta_{Q_{p,q}}(z) \\ &= \frac{\sum_{j=0}^{p-1}|z_j|^2}{\sum_{j=0}^{p-1}|g(z,1)_j|^2}\cdot\frac{\theta_{Q_{p,q}}(z)}{\sum_{j=0}^{p-1}|z_j|^2}. \end{aligned} \tag{5.23}$$

Consequently, by substituting $z_j = \xi_j/\xi_{n+1} = I_{n+1}(\xi)$,

$$g^*\theta_{\overline{Q}_{p,q}}(\xi) = \frac{\sum_{j=0}^{p-1}|\xi_j|^2}{\sum_{j=0}^{p-1}|g(\xi)_j|^2}\theta_{\overline{Q}_{p,q}}(\xi), \tag{5.24}$$

for $\xi \in O_{n+1}$. Now (5.20) follows from this equation. The proposition is proved. $\square$

Remark 5.1. (5.22) implies $\theta_{\overline{Q}_{p,q}}(\xi)$ being a contact form for $\xi \in O_{n+1} \cap \overline{Q}_{p,q}$ since $\theta_{Q_{p,q}}$ is a contact form on $Q_{p,q}$. This is because $\theta_{Q_{p,q}} \wedge (d\theta_{Q_{p,q}})^n$ is non-vanishing on $Q_{p,q}$ by $d\theta_{Q_{p,q}} = \sum_{\alpha=0}^{n} 2i\varepsilon_\alpha dz_\alpha \wedge d\overline{z}_\alpha$ by the definition of $\theta_{Q_{p,q}}$ in (4.3). Similar formulae show $\theta_{\overline{Q}_{p,q}}(\xi)$ being a contact form for $\xi \in O_j \cap \overline{Q}_{p,q}$, $j = p,\ldots,n$. Therefore, $\theta_{\overline{Q}_{p,q}}$ is a contact form on $\overline{Q}_{p,q}$.

Proposition 5.3. *$S^{-\frac{\lambda}{2},-\frac{\lambda}{2}}(\Xi)$ is isomorphic to $\left(\varpi_\lambda, C^\infty\left(\overline{Q}_{p,q}\right)\right)$ as $\mathrm{U}(p,q)$ modules.*

Proof. For $f \in S^{-\frac{\lambda}{2},-\frac{\lambda}{2}}(\Xi)$, $g \in \mathrm{U}(p,q)$ and $\xi \in \overline{Q}_{p,q}$ with $\sum_{j=0}^{p-1}|\xi_j|^2 = 1$,

$$f(g(\xi)) = f\left(\nu(g(\xi))\frac{g(\xi)}{\nu(g(\xi))}\right) = \nu(g(\xi))^{-\lambda} f(L_g\xi) = \left(\varpi_\lambda\left(g^{-1}\right)f\right)(\xi), \tag{5.25}$$

since $\Omega(g,\xi) = \nu(g(\xi))^{-1}$ by (5.20), where $\sum_{j=0}^{p-1}\left|\frac{g(\xi)_j}{\nu(g(\xi))}\right|^2 = 1$. $\square$

Define the representation $(\varpi^{p,q}, V^{p,q})$ to be $\left(\varpi_{\frac{Q-2}{2}}, \ker\widetilde{\Box}_{\theta_{\overline{Q}_{p,q}}}\right)$.

Let us identify $S^{\alpha,\beta}(\Xi)$ with degenerate principal series representations in the standard notations (cf. [6]). Let $G = \mathrm{U}(p,q)$ and let K be maximal compact subgroup $\mathrm{U}(p-1)\times\mathrm{U}(q-1)$ of G and let $H = \mathrm{U}(1)\times\mathrm{U}(p-1,q)$. Let

$$L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & O_n & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{5.26}$$

where O_n is the zero matrix of size n. Let A denote the group of matrix $a_t = \exp(tL)$, $t \in \mathbf{R}$, and let M be its centralizer in H. The Lie algebra $\mathfrak{g}$ of G can be decomposed into a direct sum $\sum_{j=-2}^{2}\mathfrak{g}_j$, where $\mathfrak{g}_j$ are eigenspaces of the operator $\mathrm{ad}L$: $\mathrm{ad}L|_{\mathfrak{g}_j} = j\cdot\mathrm{id}$. Denote by $N = \exp\mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent subalgebra $\mathfrak{n} = \mathfrak{g}_1 + \mathfrak{g}_2$, generated by positive roots. Then,

$$P^{\max} = MAN \tag{5.27}$$

is a maximal parabolic subgroup of G. Let χ_l, $l = 0,1,\ldots$, be the one-dimensional representation of the group M given by

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} \longrightarrow e^{-il\theta}, \tag{5.28}$$

with $v \in U(p-1,q-1)$, $\theta \in \mathbf{R}$. We define $\mathcal{F}$ to be the $\mathcal{C}^\infty$ or $\mathcal{D}'$ valued degenerate principal series by

$$\mathcal{F} - \mathrm{Ind}_{P^{\max}}^{G}(\chi_l \otimes \mathbf{C}_\lambda) = \{f \in \mathcal{F}(G); f(gma_tn) = \chi_l(m^{-1})e^{-(\lambda+\rho)t}f(g)\}, \tag{5.29}$$

where $\rho = p+q-1$.

Let M_0 be the subgroup of M, consisting of the matrices as in (5.28) but with $\theta = 0$. Then, G/M_0N can be identified with the light cone Ξ. We have an isomorphism of $G-$modules

$$\mathcal{F} - \mathrm{Ind}_{P^{\max}}^{G}(\chi_l \otimes \mathbf{C}_\lambda) \simeq S^{-\frac{\lambda+\rho+l}{2},-\frac{\lambda+\rho-l}{2}}(\Xi). \tag{5.30}$$

Since $(\varpi^{p,q}, V^{p,q})$ is a subrepresentation of $S^{-\frac{Q-2}{4},-\frac{Q-2}{4}}(\Xi)$ by Proposition 5.3 and $\frac{Q-2}{4} = \frac{\rho-1}{2}$, we have

Corollary 5.1. *$(\varpi^{p,q}, V^{p,q})$ is a subrepresentation of $S^{-\frac{\rho-1}{2},-\frac{\rho-1}{2}}(\Xi)$, or equivalently, of $C^{\infty}-\mathrm{Ind}_{P^{\max}}^{G}(\chi_0\otimes \mathbf{C}_{-1})$*

Now consider representations on the Heisenberg group $\mathbb{H}^{p-1,q-1}$.

Lemma 5.1. *Under the Cayley transformation C (2.35), we have*

$$C^*\theta_{Q_{p,q}}(z)=\frac{4}{|z_0+i|^2}\theta_{Q'_{p,q}}(z) \tag{5.31}$$

for $z\in Q'_{p,q}$.

Proof. Direct calculation gives

$$\begin{aligned}\overline{w}_\alpha dw_\alpha &= \frac{4\overline{z}_\alpha dz_\alpha}{|z_0+i|^2}-\frac{4|z_\alpha|^2dz_0}{|z_0+i|^2(z_0+i)},\\ \overline{w}_0dw_0&=\frac{2i(\overline{z}_0+i)dz_0}{|z_0+i|^2(z_0+i)},\end{aligned} \tag{5.32}$$

by the Cayley transformation (2.35). Noting that $\sum_{\alpha=1}^{n}\varepsilon_\alpha|z_\alpha|^2=\frac{1}{2i}(z_0-\overline{z}_0)$, and summing (5.32) over α, we find that

$$\begin{aligned}&\sum_{\alpha=0}^{n} i\varepsilon_\alpha(w_\alpha d\overline{w}_\alpha-\overline{w}_\alpha dw_\alpha)\\ &=\frac{4}{|z_0+i|^2}\left(\sum_{\alpha=1}^{n} i\varepsilon_\alpha(z_\alpha d\overline{z}_\alpha-\overline{z}_\alpha dz_\alpha)+\frac{1}{2}(dz_0+d\overline{z}_0)\right),\end{aligned} \tag{5.33}$$

by direct calculation, which is exactly (5.31). ☐

We can identify $\mathbb{H}^{p-1,q-1}$ with the hypersurface $Q'_{p,q}$ by the map

(5.34)

$$\tilde{\iota}:\mathbb{H}^{p-1,q-1}\longrightarrow Q'_{p,q},\qquad (z_1,\ldots,z_n,t)\mapsto\left(t+i\sum_{\alpha=1}^{n}\varepsilon_\alpha|z_\alpha|^2,z_1,\ldots,z_n\right).$$

Denote $\iota=C\circ\tilde{\iota}:\mathbb{H}^{p-1,q-1}\longrightarrow Q_{p,q}$, where C is the Cayley transformation (2.35). We can calculate the conformal factor $\Omega\left(g,z;\mathbb{H}^{p-1,q-1}\right)$, defined by

$$g^*\theta_{\mathbb{H}^{p-1,q-1}}(z)=\Omega\left(g,z;\mathbb{H}^{p-1,q-1}\right)^2\theta_{\mathbb{H}^{p-1,q-1}}(z), \tag{5.35}$$

by using Lemma 5.1 and Proposition 5.1. We omit the details.

Since $\mathbb{H}^{p-1,q-1}$ is flat, i.e. it has vanishing curvature, its CR Yamabe operator is

$$\widetilde{\Box}_{\theta_{\mathbb{H}^{p-1,q-1}}}=-\frac{b_n}{4}\sum_{\alpha=1}^{n}\varepsilon_\alpha(X_\alpha^2+Y_\alpha^2), \tag{5.36}$$

where

$$X_\alpha = \frac{\partial}{\partial x_\alpha} + 2\varepsilon_\alpha y_\alpha \frac{\partial}{\partial t}, \qquad Y_\alpha = \frac{\partial}{\partial y_\alpha} - 2\varepsilon_\alpha x_\alpha \frac{\partial}{\partial t}, \tag{5.37}$$

$z_\alpha = x_\alpha + iy_\alpha$, $\alpha = 1, \ldots, n$. Then, $Z_\alpha = \frac{1}{2}(X_\alpha + iY_\alpha)$ for each α. Since group $\mathrm{SU}(p,q)$ acts on $\mathbb{H}^{p-1,q-1}$ as "meromorphic" CR transformations (the inversions are not continuous), we obtain a 'representaion' with parameter $\lambda \in \mathbf{C}$

$$\varpi_{\lambda,\mathbb{H}^{p-1,q-1}}(g^{-1})f(z) = \Omega\left(g, z; \mathbb{H}^{p-1,q-1}\right)^\lambda f(L_g z). \tag{5.38}$$

$C^\infty(\mathbb{H}^{p-1,q-1})$ is not a real representation since it is unstable under the action $\varpi_{\lambda,\mathbb{H}^{p-1,q-1}}\ (g^{-1})$. The maximal parabolic subgroup is

$$\overline{P^{\max}} = A^{\max} M^{\max} \overline{N^{\max}} = (\mathbf{R}_+^\times \times SU(p-1, q-1)) * \mathbb{H}^{p-1,q-1}. \tag{5.39}$$

When restricted to its maximal parabolic subgroup $\overline{P^{\max}}$, the representation has a simple form as follows

$$\begin{aligned} (\varpi_\lambda(\delta_a)f)(z) &= a^\lambda f(\delta_a^{-1} z), && a \in \mathbf{R}_+, \\ (\varpi_\lambda(U_A)f)(z) &= f(U_A^{-1} z), && A \in SU(p-1, q-1), \\ (\varpi_\lambda(\tau_w)f)(z) &= f(\tau_w^{-1} z), && w \in \mathbb{H}^{p-1,q-1}. \end{aligned} \tag{5.40}$$

Remark 5.2. When $X_\alpha = \frac{\partial}{\partial x_\alpha} + 2\lambda_\alpha y_\alpha \frac{\partial}{\partial t}$, $Y_\alpha = \frac{\partial}{\partial y_\alpha} - 2\lambda_\alpha x_\alpha \frac{\partial}{\partial t}$ with $\lambda_\alpha > 0$ for each α, the Green function of the "wave operator" $\sum_{\alpha=1}^{p-1}(X_\alpha^2 + Y_\alpha^2) - \sum_{\alpha=p}^{n}(X_\alpha^2 + Y_\alpha^2)$ has been constructed by T. Godoy and L. Saal in [9] and also by D.-C. Chang and J. Tie in [4]. Their method can be applied to our SubLaplacian $\widetilde{\Box}_{\theta_{\mathbb{H}}}$ (5.36)–(5.37) with small modifications.

Department of Mathematics
Zhejiang University (Xixi campus)
Zhejiang 310028
P. R. China
e-mail: wangf@mail.hz.zj.cn

References

[1] M. Beals, C. Fefferman, and R. Grossman, *Strictly pseudoconvex domains in* $\mathbf{C}^n$, Bull. Amer. Math. Soc. (N.S.) **8** (1983), 125–322.

[2] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000), 1–109.

[3] T. Branson, *Group representaions arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987), 199–291.

[4] D.-C. Chang and J. Tie, *Some differential operators related to the non-isotropic Heisenberg sub-Laplacian*, Math. Nachr. **221** (2001), 19–39.

[5] S. S. Chern and J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[6] G. Van Dijk and Y. Sharshov, *The Plancherel formula for line bundles on complex hyperbolic spaces*, J. Math. Pures Appl. **79** (2000), 451–473.

[7] N. Gamara, *CR Yamabe conjecture-the conformally flat case*, Pacific J. Math. **201** (2001), 121–175.

[8] N. Gamara and R. Yacoub, *The CR Yamabe conjecture the case* $n = 1$, Journal of the European Mathematical Society **3** (2001), 105–137.

[9] T. Godoy and L. Saal, L^2 *spectral decomposition on the Heisenberg group associated to the action of* U(p, q), Pacific J. Math. **193**-2 (2000), 327–353.

[10] R. Howe and E.-C. Tan, *Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations*, Bull. Amer. Math. Soc. **28** (1993), 1–74.

[11] D. S. Jerison, and L. M. Lee, *The Yamabe problem on CR manifolds*, J. Differential Geom. **25** (1987), 167–196.

[12] T. Kobayashi, *Singular unitary representations and discrete series for indefinite Stiefel manifolds* $\mathrm{U}(p, q; F)/\mathrm{U}(p - m, q; F)$, Mem. Amer. Math. Soc. **95**-462 (1992), 1–106.

[13] T. Kobayashi and B. Orsted, *Ananlysis on the minimal representation of* $O(p, q)$ I*, Realizationvia congormal geometry*, Adv. in Math. **180** (2003), 486–512.

[14] ——, *Ananlysis on the minimal representation of* $O(p, q)$ II*, Branching law*, Adv. in Math. **180** (2003), 513–550.

[15] ——, *Ananlysis on the minimal representation of* $O(p, q)$ III*, Ultrahyperbolic equations on* $\mathbb{R}^{p-1,q-1}$, Adv. in Math. **180** (2003), 551–595.

[16] B. Kostant, *The vanishing scalar curvature and the minimal unitary representation of* SO(4, 4), In: Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progr. Math. **92**, A. Connes et al. (eds.), Birkhäuser, Boston, 1990, pp. 85–124.

[17] L. M. Lee, *The Fefferman metric and pseudohermitian invariant*, Trans. Amer. Math. Soc. **296** (1986), 411–429.

[18] B. Orsted, *Conformally invariant differential equations and projective grometry*, J. Funct. Anal. **44** (1981), 1–23.

[19] N. Stanton, *Spectral invariants of CR manifolds*, Michigan Math. J. **36**-2 (1989), 267–288.

[20] W. Wang, *Canonical contact forms on spherical CR manifolds*, Journal of the European Mathematical Society **5** (2003), 245–273.

[21] S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), 25–41.