On the motive of Kummer varieties associated to $\Gamma_1(7)$ – Supplement to the paper: The modularity of certain non-rigid Calabi-Yau threefolds (by R. Livné and N. Yui)

By

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Abstract

In their paper [LY] Livné and Yui discuss several examples of non-rigid Calabi-Yau varieties which admit semi-stable $K3$-fibrations with 6 singular fibres over a base which is a rational modular curve. They also establish the modularity of the L-function of these examples. The purpose of this note is to point out that the examples which were listed in [LY] but which do not lead to semi-stable fibrations are still modular in the sense that their L-function is associated to modular forms. We shall treat the case associated to the group $\Gamma_1(7)$ in detail, but our technique also works in the other cases given in [LY]. We shall also make some comments concerning the Kummer construction for fibre products of elliptic surfaces in general.

1. Introduction

In their paper [LY] Livné and Yui consider Calabi-Yau varieties which possess a non-constant semi-stable $K3$-fibration with 6 singular fibres, which is the minimal number (Arakelov-Yau bound) of singular fibres of such a fibration ([STZ]). In this case the base curve must be a rational modular curve. They start with the list of the (up to conjugacy) 9 possible torsion free genus 0 congruence subgroups of index 24 of $\mathrm{PSL}(2,\mathbb{Z})$. These 9 cases separate into two types, depending on whether the group is a subgroup of $\Gamma(2)$ or, equivalently, whether the curve of 2-torsion points decomposes into four sections (4 cases) or not (5 cases). The first situation leads to semi-stable $K3$ fibrations by performing the following Kummer construction: if Y is the universal elliptic curve and E is an elliptic curve then $X' = (Y \times E)/\iota$ (where ι is the map given by $x \mapsto -x$) is a singular threefold. The fibre structure of Y induces a fibration on X', whose general fibre is a (singular) Kummer surface. Blowing up the fixed point set of ι gives the desired smooth Calabi-Yau variety X.

Livné and Yui also show that these non-rigid Calabi-Yau varieties are modular in the sense that their *L*-function is modular. The point of this note is to make some general statements about Kummer fibrations and to point out that the modularity statement is also true in the remaining cases. Here we shall restrict ourselves to the $\Gamma_1(7)$ case which is particularly interesting. The same method, however, also applies to the other cases. The main difference to the examples treated in [LY] is that the curve of non-zero 2-torsion points is an elliptic curve. This leads to extra contributions in the motive of X.

2.   The Kummer construction and topological considerations

Let $r : Y \to S$ and $r' : Y' \to S$ be two relatively minimal elliptic fibrations with a section over the same base curve. We assume these to be semi-stable, i.e. all fibres are of type $I_n, n \geq 0$. The involutions ι_Y resp. $\iota_{Y'}$ which, on the general fibre, are given by $x \mapsto -x$ extend to Y, resp. Y'. We label the components of a I_n-fibre cyclically by $e_0, \ldots, e_{n-1}$ and in such a way that e_0 corresponds to the component which meets the 0-section. Then ι acts by $e_i \mapsto e_{-i}$. Hence, if n is even, two components are fixed, namely e_0 and $e_{n/2}$, and the others are exchanged pairwise, whereas for n odd only one component is fixed, namely e_0. Let $O, B \subset Y$, resp. $O', B' \subset Y'$ be the 0-section and the closure of the non-zero 2-torsion points. These are smooth curves. If n is even, B meets e_0 and $e_{n/2}$ transversally in 1, resp. 2 smooth points, whereas if n is odd then B meets e_0 transversally in one point, as well as the intersection of $e_{(n-1)/2}$ and $e_{(n+1)/2}$. In this case the map $B \to S$ is branched.

Let $W = Y \times_S Y'$. This variety is singular and has nodes (A_1-singularities) exactly at pairs of nodes of the singular fibres. If, say, $Y' = E \times \mathbb{P}^1$ for an elliptic curve E (and this will be the situation in our example), then W is smooth. We consider the diagonal involution

$$\iota = (\iota_Y, \iota_{Y'}) : W \to W.$$

Let $\tilde{W}$ be the (big) resolution of W, i.e., the variety where all nodes are replaced by quadrics. The involution ι lifts to $\tilde{W}$ where we shall denote it by the same letter. The *(singular) Kummer family* associated to the pair (Y, Y') is defined by

$$X' = \tilde{W}/\iota.$$

This variety is always singular, even if W is smooth. The general fibre is a Kummer surface with 16 nodes. In the analytic category one can also consider small resolutions $\hat{W}$ of W where the double points are replaced by a $\mathbb{P}^1$. These small resolutions are not necessarily projective and it is possible that W does not possess a small projective resolution. Moreover, even if a small projective resolution $\hat{W}$ exists, then it is not clear that ι lifts to $\hat{W}$. For a discussion of this issue see [Sch]. In view of the arithmetic applications we have in mind we are not concerned with the existence of small projective resolutions at this point and hence we will work with either W if this is smooth or with a big resolution $\tilde{W}$ otherwise.

The singularities of X' come from fixed points of ι on W. One has to distinguish two cases, namely:

(1) There is no fibre $I_n \times I_m$ with both n, m odd.

(2) There is such a fibre.

The fixed locus of ι is given by

$$D := (O + B) \times_S (O' + B') \subset W.$$

In the case (1) the curve D does not go through any of the singularities of W, in case (2) it does. In the first case one obtains a desingularization of X' in a way exactly like the usual desingularization of Kummer surfaces: Blowing up $\tilde{W}$ along the curve D replaces D by a $\mathbb{P}^1$-bundle over D whose rulings have relative degree -2. We shall denote the resulting 3-fold by Z. Then

$$X = Z/\iota$$

is a smooth variety, fibred over the base whose general fibre is a smooth $K3$-surface. Alternatively we could have obtained X by blowing up X' along its double curve. We call X the *smooth Kummer fibration* associated to the pair (Y, Y').

In the second case the situation is as follows. Locally, near a singular point we can choose (analytic) coordinates x, y on Y and u, v on Y' such that the projection onto the base is given by $(x, y) \mapsto xy$ and $(u, v) \mapsto uv$. Then the fibre product $Y \times_S Y'$ is locally isomorphic to the subvariety given by $xy - uv = 0$. We obtain $\tilde{W}$ by blowing up the A_1-singularity given by $xy - uv = 0$, thus inserting a quadric Q. The strict transform of B is a smooth curve meeting Q in two points. The involution ι on Q has 4 fixed points. The first two of these lie on the strict transform of B, the other two are isolated singularities. In the quotient X the latter two points give rise to isolated singularities of type $V_{\frac{1}{2}(1,1,1)}$, i.e. to cones over Veronese surfaces. These can be resolved by a $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$. In particular, these singularities are rational, and in an arithmetic context they will not contribute in an essential way to the motive of X.

We are interested in examples where X is Calabi-Yau. For the examples which we shall consider we shall now assume that $r : Y \to \mathbb{P}^1$ is a $K3$-surface and that $Y' = E \times \mathbb{P}^1$, where E is an elliptic curve. Note that $W = Y \times E$ and the curve D are smooth.

We denote the exceptional $\mathbb{P}^1$-bundle over D by V. Moreover, we denote by n_+ the rank of the $(+1)$-eigenspace of $NS(Y)$ with respect to ι_Y and by n_- the rank of the (-1)-eigenspace. Let $g(D)$ be the sum of the genera of the components of D and let $c(D)$ be the number of components of D.

Theorem 2.1. *Let* $r : Y \to \mathbb{P}^1$ *be a $K3$-surface and* $Y' = E \times \mathbb{P}^1$. *Then the associated Kummer variety X is a Calabi-Yau threefold whose Hodge numbers are as follows*:

(i) $h^{00}(X) = 1$, $h^{10}(X) = h^{01}(X) = 0$,

(ii) $h^{20}(X) = h^{02}(X) = 0$, $h^{11}(X) = \rho(X) = n_+ + 1 + c(D)$,

(iii) $h^{30}(X) = h^{03}(X) = 1$, $h^{12}(X) = h^{21}(X) = 1 + n_- + g(D)$.
Moreover, for the Euler numbers

$$e(X) = \frac{3}{2}e(D)$$

and

$$n_+ - n_- = \frac{3}{4}e(D) - c(D) + g(D) = \frac{1}{4}e(D).$$

Proof. In this case W is smooth and there exists a (up to a scalar) unique 3-form on W which descends to X. As in the case of Kummer surfaces one shows that this form has no zeroes on X and hence $\omega_X = \mathcal{O}_X$ and thus $h^{30}(X) = h^{03}(X) = 1$. Since there is no invariant 1-form on W we can also conclude that $h^{10}(X) = h^{01}(X) = 0$ and hence by Serre duality $h^{20}(X) = h^{02}(X) = 0$. This shows that X is a Calabi-Yau threefold. One immediately obtains from the Hodge diamond that

$$e(X) = 2(h^{11}(X) - h^{12}(X)).$$

In order to compute the Betti numbers of X we first note that it is easy to determine the Betti numbers of W from those of Y and E via the Künneth formula. Let $\pi : Z \to W$ be the blow-up along D. Then

$$H^*(Z) \cong \pi^* H^*(W) \oplus H^*(V)/(\pi_{|V})^*(H^*(D)).$$

Moreover, recall that $H^*(V)$ is generated by the tautological class as a ring over $H^*(D)$. From this one obtains

$$h^0(Z) = h^0(W),\ h^1(Z) = h^1(W)$$
$$h^2(Z) = h^2(W) + c(D),\ h^3(Z) = h^3(W) + h^1(D).$$

For the Euler characteristic of X we thus find

$$e(X) = \frac{1}{2}(e(W) - e(D)) + e(V)$$
$$= \frac{1}{2}(e(W) - e(D)) + 2e(D)$$
$$= \frac{1}{2}(e(W) + 3e(D))$$
$$= \frac{3}{2}e(D)$$

where the last equality follows from $e(W) = 0$.

In order to compute $h^2(X) = \rho(X)$ we have to determine the ι-invariant divisors on Z. The divisors on Y are spanned by a general fibre F, the 0-section O, and the components e_i^j of the singular fibres which do not meet the 0-section. The divisors on W are spanned by taking the product of these divisors times E and by $Y \times \{0\}$. The pullback of this divisor under π and the components

of the exceptional divisors are clearly invariant under ι. This accounts for the summand $1 + c(D)$ in the formula for $h^{11}(X) = \rho(X)$. If $l \in \mathrm{NS}(Y)^\pm$ then $l \times E$ is invariant (anti-invariant) under ι and altogether this shows the formula for $h^{11}(X)$. It remains to determine the invariant part of $H^3(Z)$. We shall first treat the contribution from $W = Y \times E$. Let $T \subset H^2(Y)$ be the rank 2 subspace spanned by the transcendental cycles. Then $T \otimes H^1(E)$ is 4-dimensional and ι acts by -1 on both T and $H^1(E)$. This is clear for $H^1(E)$. If it did not act by -1 on T then $h^{20}(X) \neq 0$ resp. there would be no 3-form on X. Hence ι acts by $+1$ on $T \otimes H^1(E)$ and this contributes to $H^3(X)$. The same argument applies to $\mathrm{NS}(X)^- \otimes H^1(E)$. Moreover $H^3(V)$ is invariant under ι and in total we find that

$$ h^3(X) = 4 + 2n_- + 2g(D). $$

Since we already know that $h^{30}(X) = h^{03}(X) = 1$ this proves the claim for $h^{12}(X)$. The final assertion follows from subtracting (iii) from (ii). $\qquad \square$

Remark 1. The middle cohomology of X fits into an exact sequence

$$ 0 \to T \otimes H^1(E) \to H^3(X, \mathbb{C}) \to (\mathrm{NS}(Y)^- \otimes H^1(E)) \oplus H^3(V) \to 0. $$

Proposition 2.1. *Let* $r : Y \to S$ *be an elliptic fibration with semi-stable fibres. The numbers* n_+ *and* n_- *are given by*

$$ n_+ - n_- = 2 + \#\{\text{singular fibres } I_n \text{ with } n > 1 \text{ even}\} $$

and $n_+ + n_- = \rho(Y)$.

Proof. The Nèron-Severi group $\mathrm{NS}(Y)$ is generated by a general fibre F, the 0-section O and the components e_i^j of the singular fibres I_n where j runs through the cusps and $i = 0, \ldots, n-1$. Then ι acts on these components by $e_i^j \mapsto e_{-i}^j$. This means the following: if n is odd then this fibre contributes equally to n_+ and n_-, whereas for even n we have one more invariant component (namely $e_{n/2}^j$), than odd components. Since F and the 0-section O are clearly invariant we obtain the first claim. The second claim follows since $n_+ + n_- = 2 + \rho(Y)$. $\qquad \square$

3. The example associated to $\Gamma_1(7)$

From now on we shall concentrate on the example associated to the group $\Gamma_1(7)$. We note, however, that this method can also be applied to the other examples in [LY] which do not lead to semi-stable $K3$-fibrations.

Let $Y = S(\Gamma_1(7))$. Then Y has 6 singular fibres, namely 3 of type I_1 and 3 of type I_7. By results of Igusa, one knows that the moduli problem for $\Gamma_1(N)$ for $N > 3$ can be represented by a smooth scheme $Y_1(N)$ over $\mathbb{Z}[1/N]$ (see [KM, Introduction] and [KM, Table (10.9.6), p. 308]). Since we shall need this, we shall briefly sketch the construction of a smooth model defined over $\mathbb{Q}$.

Tate [Tate, p. 195] (and also [K, case 15., Table 3, p. 217]) gives the following equation for the universal elliptic curve with a group of sections of order 7

$$(3.1) \qquad y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2.$$

Note that after a change of variables $y \mapsto y - \frac{1}{2}(1 + t - t^2)x - \frac{1}{2}(t^2 - t^3)$ this has the Weierstrass form
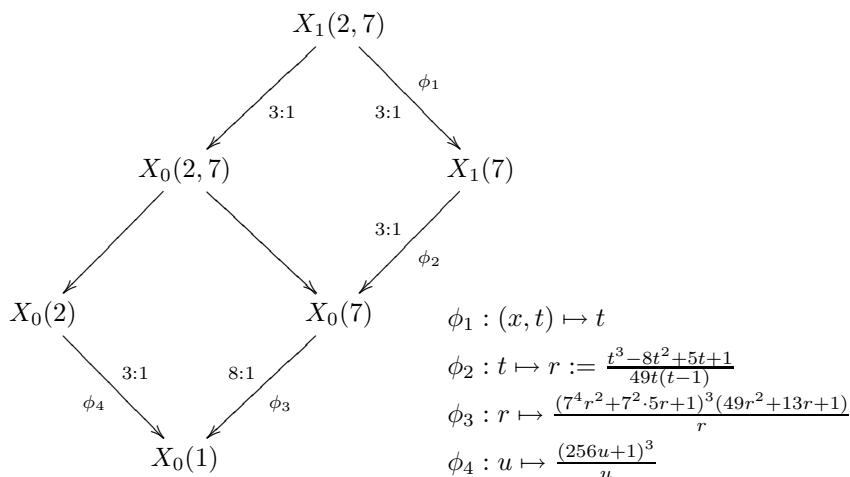
$$(3.2) \quad 4y^2 = 4x^3 + (t^4 - 6t^3 + 3t^2 + 2t + 1)x^2 + 2t^2(t^3 - 2t^2 + 1)x + t^4(t-1)^2.$$

This shows that the curve B of non-zero 2-torsion points is given by the following equation (which was first pointed out to us by N. Yui)

$$(3.3) \quad 4x^3 + (t^4 - 6t^3 + 3t^2 + 2t + 1)x^2 + 2t^2(t^3 - 2t^2 + 1)x + t^4(t-1)^2 = 0.$$

Lemma 3.1. *The curve B of non-zero 2-torsion points of Y is an elliptic curve defined over $\mathbb{Q}$, with conductor 14. The Mellin transform of the L-series of this curve is the unique weight 2 level 14 modular newform, $\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$.*

Proof. The curve B parametrizes elliptic curves with a point of order 7 and a point of order 2. This is an irreducible family, and by the Hurwitz formula it has genus 1. In particular, B is isomorphic to the modular curve $X_1(2,7) = \Gamma_1(2) \cap \Gamma_1(7) \setminus \mathfrak{h}^*$ (where $\mathfrak{h}^*$ is the upperhalf complex plane union the cusps). There is a degree 3 natural map from this curve to $X_0(2,7) = \Gamma_0(2) \cap \Gamma_0(7) \setminus \mathfrak{h}^*$ which parametrizes elliptic curves together with a subgroup of order 2 and a subgroup of order 7. This curve is isomorphic to $X_0(14)$, which is an elliptic curve. Hence the map from B to $X_0(14)$ must be an isogeny of degree 3. The fact that the map from $X_1(2,7)$ to $X_0(14)$ is defined over $\mathbb{Q}$ follows, viewing $X_1(2,7)$ as a fibre product, from the fact that the maps $X_1(7) \to X_0(7)$ ([E, (4.24)]) and $X_0(2) \to X_0(1)$, in the following diagram, are defined over $\mathbb{Q}$.

$$\phi_1 : (x,t) \mapsto t$$
$$\phi_2 : t \mapsto r := \frac{t^3 - 8t^2 + 5t + 1}{49t(t-1)}$$
$$\phi_3 : r \mapsto \frac{(7^4 r^2 + 7^2 \cdot 5r + 1)^3 (49r^2 + 13r + 1)}{r}$$
$$\phi_4 : u \mapsto \frac{(256u + 1)^3}{u}$$

In the diagram x and t are modular functions satisfying the relation (3.3), and $u = (\eta(2\tau)/\eta(\tau))^{24}$ and $r = (\eta(7\tau)/\eta(\tau))^4$ are hauptmodules for $\Gamma_0(2)$ and $\Gamma_0(7)$ respectively, taken from [CN, Table 3]. The maps given in the diagram may be found by comparison of $\mathbb{Q}$-expansions, and the fact that the degree of the rational functions equals the index of the corresponding groups. This implies that B has the same L-series as $X_0(14)$, which is the given modular form (listed in [M]). $\qquad\square$

As a surface (3.1) is isomorphic to $S(\Gamma_1(7))$ away from the singular fibres. This example is discussed in more detail by Elkies [E, §4.2], where the parameter t is given explicitly as a Hauptmodul for $\Gamma_1(7)$. The j-invariant of this elliptic curve (for both (3.1) and (3.2)) is

$$\frac{(t^2 - t + 1)^3 (t^6 - 11t^5 + 30t^4 - 15t^3 - 10t^2 + 5t + 1)^3}{t^7 (t-1)^7 (t^3 - 8t^2 + 5t + 1)},$$

which implies that, after resolution of singularities, this model has I_7 fibres at $t = 0, 1, \infty$, and I_1 fibres at the roots of $t^3 - 8t^2 + 5t + 1 = 0$.

The surface (3.1) is singular, but it is important for us to have a smooth resolution defined over $\mathbb{Q}$, so we now give an outline of an explicit desingularization of (3.1). First, we make the change of variables

(3.4) $\qquad x = x't^2, y = y't^2(t-1), z = x'/(t-1) + y' + z',$

which is invertible on the fibres for $t \neq 0, 1, \infty$. This gives us

(3.5) $\qquad t(t-1)x(x-y)(y+z) + (t-1)(x-y-z)yz + t(x-y)xz = 0$

(where we write x, y, z instead of x', y', z'). This surface (3.5) has 8 singular points, namely $P_1 = (1 : 0 : 1), P_2 = (1 : 1 : 0), P_3 = (1 : 0 : 0)$ on the fibre $t = 0$, $Q_1 = (0 : 0 : 1), Q_2 = (0 : 1 : 0), Q_3 = (1 : 1 : 0)$ over $t = 1$ and $R_1 = (0 : 0 : 1)$ and $R_2 = (0 : 1 : -1)$ over $t = \infty$. Note that these points also lie on the closure of the sections given by the points of order 7. These points may be resolved by a sequence of blow ups in the singular points. The first blow up replaces P_1, P_2, Q_1, Q_2 and R_1 by one line, and the other points by 2 lines. The resulting surface still has one singular point, namely a point on the singular fibre over $t = \infty$, which is infinitesimally close to R_2. Blowing up once more in this point gives rise to a further line and the resulting surface is then smooth. It has three singular fibres of type I_1 and three of type I_7 and is isomorphic to $S(\Gamma_1(7))$.

The locus which is blown up is defined over $\mathbb{Q}$, and so the resulting surface is also defined over $\mathbb{Q}$ (see the discussion in the proof of Proposition 3.3). Moreover, going through the above sequence of blow ups shows that all components of the singular fibres are defined over $\mathbb{Q}$. This gives us the following result.

Proposition 3.1. *The group* $\mathrm{NS}(Y)$ *can be generated by classes of curves defined over* $\mathbb{Q}$.

Proof. As already mentioned (in the proof of Proposition 2.1), NS(Y) is generated by a general fibre, the zero section, and components of the singular fibres not meeting this section. Since all components of the singular fibres and the zero-section are defined over $\mathbb{Q}$ this gives the result. □

Proposition 3.2. *The Mellin transform of the L-series of the summand of the middle cohomology of Y corresponding to the transcendental lattice is given by $(\eta(\tau)\eta(7\tau))^3$, the unique normalized weight 3 modular form for $\Gamma_1(7)$, where $\eta(\tau) = \prod_{n>0}(1 - \exp(2\pi i n\tau))$ is the Dedekind η-function.*

Sketch of proof. We first note that modularity as such can be deduced from Livné's result [Liv2, Example 1.6]. One may also apply Serre and Deligne's [D] methods, (details given by Conrad [C]), as is done in a related situation in [SY]. Using the formula for the dimensions of spaces of cusp forms and applying the transformation properties of the Dedekind η-function, one shows that the corresponding form is as given.

Alternatively, in order to determine the modular form, we can use the Lefschetz trace formula to compute the values of the trace of Frob$_p$ on $H^2(Y, \mathbb{Q}_\ell)$ by counting points on Y (using e.g., MAGMA [BCP]). These may be compared with coefficients of $(\eta(\tau)\eta(7\tau))^3$. We want to use Livné's result [Liv1] to show that the two representations coincide up to semi-simplification. In order to apply this result we first observe that the determinant is $\chi_{-7}\chi_\ell^2$ (again cf. [Liv2, Example 1.6] and [LY, (2.1)]). Here χ_ℓ is the cyclotomic character and χ_{-7} is the quadratic character associated to the field $\mathbb{Q}(\sqrt{-7})$. We also need that the number of points on Y is even. This depends on the curve B having an even number of points. Since B and $X_0(14)$ are isogenous over $\mathbb{Q}$ by Lemma 3.1 it is enough to consider the latter curve. Looking at a Weierstrass equation one observes that this curve has a non-trivial 2-torsion point over $\mathbb{Q}$, and this is enough to conclude that the number of points over $\mathbb{F}_p$ is even for almost all p. This approach also essentially uses Serre and Deligne's work for the existence of a modular Galois representation corresponding to $(\eta(\tau)\eta(7\tau))^3$. □

Remark 2. The form g_3 is related to the unique newform h_2 of weight 2 and level 49 as follows: in terms of representations (at least up to semi-simplification) Sym$^2(h_2) = g_3 \otimes \chi_{-7}\chi_\ell$.

For what follows we need to understand the exceptional locus of the blow up along the fixed point set of ι.

Proposition 3.3. *The exceptional locus $V \subset Z$ has the following properties:*

(i) *V consists of 8 components. Of these 4 are Hirzebruch surfaces Σ_2 and 4 are isomorphic to $B \times \mathbb{P}^1$.*

(ii) *If the 2-torsion points of E are defined over $\mathbb{Q}$, then so are the components of V. In this case the isomorphism of the 4 non-rational components with $B \times \mathbb{P}^1$ is defined over $\mathbb{Q}$.*

Proof. We first note that the fix curve D of ι decomposes as

$$D = (O + B) \times_{\mathbb{P}^1} (O_1 + O_2 + O_3 + O_4),$$

where the O_i correspond to the four 2-torsion points of E, and B is an elliptic curve by Lemma 3.1.

We start with the geometric statements. Let D' be one of the components of D. Then the component V' of V which lies over D' is isomorphic to the projectivized normal bundle $\mathbb{P}(N_{D'/W}) = \mathbb{P}(N_{D'/Z})$. Since Y is a $K3$-surface, and hence has trivial canonical bundle, the adjunction formula shows that $N_{D'/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ if D' is rational and $N_{D'/Y} \cong \mathcal{O}_{D'}$ if D' is elliptic. Hence $N_{D'/W} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ or $N_{D'/W} \cong 2\mathcal{O}_{D'}$.

Now assume that the 2-torsion points of E are defined over $\mathbb{Q}$. Then the same is true for all components of the curve D. The fact that the components of the exceptional divisor are then defined over $\mathbb{Q}$, follows from the general fact that blowing up is compatible with base change (cf. the proof of [Liu, Theorem 1.1.9]). In the case of the non-rational components we saw from the adjunction formula that the normal bundle is trivial, i.e. we have isomorphisms $N_{D'(\mathbb{C})} \cong 2\mathcal{O}_{D'(\mathbb{C})}$ and, in particular, $H^0(\mathcal{H}om_{\mathcal{O}_{D'(\mathbb{C})}}(N_{D'(\mathbb{C})}, \mathcal{O}_{D'(\mathbb{C})}))$ is 2-dimensional. Since cohomology commutes with the flat base extension $\mathbb{Q} \hookrightarrow \mathbb{C}$ ([Ha, Proposition III.9.3]) this implies that the $\mathbb{Q}$ vector space $H^0(\mathcal{H}om_{\mathcal{O}_{D'(\mathbb{Q})}}(N_{D'(\mathbb{Q})}, \mathcal{O}_{D'(\mathbb{Q})}))$ is also 2-dimensional and in particular there is a homomorphism $N_{D'(\mathbb{Q})} \to 2\mathcal{O}_{D'(\mathbb{Q})}$ which, after tensoring with $\mathbb{C}$, is an isomorphism. Using once more flatness of $\mathbb{Q} \hookrightarrow \mathbb{C}$ it follows that $N_{D'(\mathbb{Q})} \cong 2\mathcal{O}_{D'(\mathbb{Q})}$. $\qquad\square$

Theorem 3.1. *Assume that the elliptic curve E, as well as its 2-torsion points, are defined over $\mathbb{Q}$. Then the ι-invariant part of the middle cohomology of the blow up Z of $Y(\Gamma_1(7)) \times E$ along the fixed locus of ι is modular. More precisely*

$$L(H^3_{\acute{e}t}(Z)^\iota, s) \overset{\circ}{=} L(g_3 \otimes g_2^E, s) L(g_2^E, s-1)^9 L(g_2^B, s-1)^4,$$

where g_3 is the weight 3 form associated to the transcendental lattice T and g_2^E and g_2^B are the weight 2 cusp forms associated to the elliptic curves E and B respectively. In terms of the Dedekind η-function, we have

$$g_3(\tau) = (\eta(\tau)\eta(7\tau))^3 = q - 3q^2 + 5q^4 - 7q^7 - 3q^8 + 9q^9 - 6q^{11} + \cdots,$$
$$g_2^B(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) = q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + \cdots.$$

(Here $\overset{\circ}{=}$ denotes equality of the L-series up to finitely many primes, and $q = \exp(2\pi i \tau)$.)

Proof. We first note that $n_+(Y) = 11$ and $n_-(Y) = 9$. This follows from Proposition 2.1 and the fact that there are 3 fibres of type I_1 and 3 of type I_7. Now recall from Remark 1, that we have the following exact sequence of cohomology groups

$$0 \to T \otimes H^1(E) \to H^3(Z, \mathbb{C})^\iota \to (\mathrm{NS}(Y)^- \otimes H^1(E)) \oplus H^3(V) \to 0,$$

where

$$H^3(V) \cong H^3(E \times \mathbb{P}^1)^4.$$

By Proposition 3.1 the standard generators of $\mathrm{NS}(Y)$ are defined over $\mathbb{Q}$. Moreover, by Proposition 3.3 all components of the exceptional locus V are defined over $\mathbb{Q}$ and there are 4 components with non-vanishing third cohomology, each isomorphic to $E \times \mathbb{P}^1$ (over $\mathbb{Q}$). Hence the above sequence can be read as a sequence of étale cohomology groups and this implies the assertion. We saw that g_2^B and g_3 have the given η-product expressions in Proposition 3.2 and Lemma 3.1. $\qquad\square$

As an immediate application we obtain the L-series of the middle cohomology of the Kummer variety X.

Corollary 3.1. *Assume that the elliptic curve E as well as its 2-torsion points are defined over $\mathbb{Q}$. Then the Kummer variety $X = X(\Gamma_1(7))$ is modular, in the sense that its L-series of its middle cohomology is given in terms of modular forms as follows, where g_2^E, g_2^B and g_3 are as in Theorem* 3.1.

$$L(H^3_{\acute{e}t}(X), s) \doteq L(H^3_{\acute{e}t}(Z)^\iota, s) \doteq L(g_3 \otimes g_2^E, s) L(g_2^E, s-1)^9 L(g_2^B, s-1)^4.$$

Proof. Since Z and the involution ι are defined over $\mathbb{Q}$, general theory implies that X is also defined over $\mathbb{Q}$ and $L(H^3_{\acute{e}t}(X), s) \doteq L(H^3_{\acute{e}t}(Z)^\iota, s)$. $\qquad\square$

Remark 3. It follows from the work of Kim and Shahidi [KS] that $L(g_3 \otimes g_2^E, s)$ has the expected analytic properties.

4. Point counting

We will now make a verification of the above result (Corollary 3.1) about the L-series of the Kummer variety X by a counting argument. Suppose that we have q-expansions

$$g_2^B = \sum_{n \geq 1} b_n q^n, \ g_2^E = \sum_{n \geq 1} c_n q^n, \ \text{ and } g_3 = \sum_{n \geq 1} a_n q^n,$$

where g_2^B, g_2^E and g_3 are as in Theorem 3.1. The following table gives us the traces on the cohomology of X predicted by Corollary 3.1, Proposition 3.1, and the fact that ι is defined over $\mathbb{Q}$.

	dimension	trace of Frob_p
H_6	1	p^3
H_5	0	0
H_4	20	$20p^2$
H_3	30	$a_p c_p + 9p c_p + 4p b_p$
H_2	20	$20p$
H_1	0	0
H_0	1	1.

From the Lefschetz fixed trace formula this tells us that

$$(4.1) \qquad n_p = p^3 + 20p^2 - (a_p c_p + 9pc_p + 4pb_p) + 20p + 1.$$

We will now compute the number of points in another way and show that we still get the same result.

In the following our notation remains as in Section 2. We will suppose that the characteristic is not 2 or 3, and that all two-torsion points of E are defined over $\mathbb{Q}$. Suppose that Y and E are given by the following Weierstrass equations in affine space,

$$Y : y^2 = p_1(x,t), E : y_2^2 = p_2(x_2),$$

where

$$p_1(x,t) := x^3 + (t^4 - 6t^3 + 3t^2 + 2t + 1)x^2 + 8t^2(t^3 - 2t^2 + 1)x + 16t^4(t-1)^2$$

is obtained from (3.2) by scaling x and y. The involution on the product is now given by

$$\iota : (x, y, x_2, y_2, t) \mapsto (x, -y, x_2, -y_2, t).$$

Now define a rational map $Y \times E \to \mathbb{A}^5$ by

$$\phi : (x, y, x_2, y_2, t) \mapsto (x, y^2, x_2, yy_2, t).$$

Then the points in the image of this map lie on a variety X' having equation

$$(4.2) \qquad X' : w = p_1(x,t), w_2^2 = p_1(x,t)p_2(x_2),$$

where $\mathbb{A}^5$ is taken to have coordinates x, w, x_2, w_2, t. Since ϕ is surjective onto X', and identifies exactly points identified by ι, X' gives a model of an affine piece of the quotient, away from the $I_7 \times E$ fibres. The situation at the $I_7 \times E$ fibres is as follows. Let $e_0, e_1, \ldots, e_6$ be the components of the I_7 fibres enumerated as usual in such a way that ι acts by $e_i \mapsto e_{-i}$. Then the products $e_i \times \mathbb{P}^1$ for $i = 1, 6$ and $i = 2, 5$ are identified and give rise to two copies of $E \times \mathbb{P}^1$ in the quotient. The component $e_0 \times E$ is mapped to itself and ι has 8 fixed points on this surface. The resulting quotient is a rational surface with 8 nodes which are resolved by blowing up the fixed locus of ι. Finally, the two surfaces $e_3 \times E$ and $e_4 \times E$ are interchanged. Since, however, the intersection point of e_3 and e_4 is fixed it follows that the resulting quotient is a non-normal surface which is singular along a $\mathbb{P}^1$.

We can now count the number of points over $\mathbb{F}_p$ on the Kummer variety X and find the following expression

$$(4.3) \qquad n_p = \#X' + \#X_\infty + 6\#A + 3\#B + \#C + 2\#F + \#V - \#D,$$

where the terms of this equation have the following meaning. First of all $\#X'$ are all the points on the affine model counted by the Legendre symbol, i.e.

$$(4.4) \qquad \#X' = \sum_{x, x_2, t \in \mathbb{F}_p} \left(\left(\frac{p_1(x,t)p_2(x_2)}{p} \right) + 1 \right).$$

A is the surface $\mathbb{A}^1 \times E$. This term counts the contribution from the ruled surfaces coming from the components $e_i \times E, i = 1, 2$ from the three I_7 fibres. B comes from the component $e_3 \times E$ and C comes from $e_0 \times E$. The terms for A and B are counted three times, corresponding to the three fibres over $t = 0, 1, \infty$. The term C refers to $t = \infty$ only, since the corresponding terms for $t = 0, 1$ are already taken care of by the term X' (up to a correction given by F, see below). Note that we have taken care to count the fibres over the intersection points $e_i \cap e_{i+1}$ only once. Recall that

$$(4.5) \qquad \#E(\mathbb{F}_p) = \sum_{x_2 \in \mathbb{F}_p} \left(\left(\frac{p_2(x_2)}{p} \right) + 1 \right) + 1 = p + 1 - c_p.$$

We now find from our geometric discussion and the fact that everything is defined over $\mathbb{Q}$ that

$$(4.6) \qquad \#A = \#(\mathbb{A}^1 \times E) = p(p - c_p + 1)$$

and

$$(4.7) \qquad \#B = \#(\mathbb{A}^1 \times E) - \#E + \#\mathbb{P}^1 = p(p - c_p + 1) + c_p.$$

The surface which is the blow up of the quotient of $e_0 \times E$ is a smooth rational surface with Picard number 10. Moreover the Nèron-Severi group is generated by elements defined over $\mathbb{Q}$. Hence it has $1 + 10p + p^2$ points. It contains 8 lines which belong to the exceptional divisor V. Removing these, but counting the points on D, we get the contribution

$$(4.8) \qquad \#C = p^2 + 2p + 1.$$

This is counted once, namely for the I_7-fibre over ∞. The term F comes from the following effect. The surfaces coming from $e_0 \times E$ over the $t = 0, 1$ are counted in the expression given by the Legendre symbol. However, this needs a correction. In this affine model the elliptic fibre is a nodal cubic and over the node we have a $\mathbb{P}^1$. However, when we blow up this node gets resolved and we have two fibres isomorphic to E which are identified by the involution. This means that we need the correction term

$$(4.9) \qquad \#F = \#E - \#\mathbb{P}^1 = -c_p.$$

For the exceptional locus we find

$$(4.10) \qquad \#V - \#D = 4p(\#B + \#\mathbb{P}^1) = 4p(p - b_p + 1 + p + 1).$$

This can be seen as follows: the ruled surfaces over the elliptic components of D are isomorphic (over $\mathbb{Q}$) to $B \times \mathbb{P}^1$ and the other components are Hirzebruch surfaces Σ_2 which have a basis of the Nèron-Severi group which is defined over $\mathbb{Q}$. Finally X_∞ means points where $t \neq \infty$, but x or $x_2 = \infty$. We find that

$$(4.11) \qquad \#X_\infty = \#(Y_{t \neq \infty} \cup (E \times \mathbb{A}^1))/\iota = 2\#(\mathbb{P}^1 \times \mathbb{A}^1) - \#\mathbb{A}^1 = 2p^2 + p.$$

We can now rewrite the number of points as

$$
(4.12) \quad n_p = \sum_{x,x_2,t\in\mathbb{F}_p}\left[\left(\left(\frac{p_1}{p}\right)+1\right)\left(\left(\frac{p_2}{p}\right)+1\right)-\left(\frac{p_1}{p}\right)-\left(\frac{p_2}{p}\right)\right]+
$$
$$
+\,\#X_\infty+6\#A+3\#B+\#C+2\#F+\#V-\#D,
$$

where $p_1=p_1(x,t)$ and $p_2=p_2(x_2)$. Finally note that from the Lefschetz fixed point theorem, bearing in mind that $\rho(Y)=20$, and that NS(Y) is generated by classes of curves defined over $\mathbb{Q}$ (Proposition 3.1), we have

$$
(4.13) \quad \#Y(\mathbb{F}_p)=\sum_{x,t\in\mathbb{F}_p}\left(\left(\frac{p_1(x,t)}{p}\right)+1\right)+20p=p^2+20p+a_p+1.
$$

Note that in these expressions we have taken into account the points at infinity on E and on the fibres of Y, and the components of the I_7 fibres of Y, which are not counted by the factor involving the Legendre symbol. Using (4.5) and (4.13) together with (4.12), we obtain

$$
\begin{aligned}
n_p =\;& (\#E(\mathbb{F}_p)-1)(\#Y(\mathbb{F}_p)-20p)\\
& -(\#E(\mathbb{F}_p)-1-p)p^2-(\#Y(\mathbb{F}_p)-20p-p^2)p\\
& +\#X_\infty+6\#A+3\#B+\#C+2\#F+\#V-\#D\\
=\;& (p-c_p)(p^2+a_p+1)-(p-c_p-p)p^2-(p^2+a_p+1-p^2)p\\
& +(2p^2+p)+6p(p+1-c_p)+3(p(p+1-c_p)+c_p)\\
& +(p^2+2p+1)-2c_p+4p(p+1-b_p+p+1)\\
=\;& p^3+20p^2-(a_pc_p+9pc_p+4pb_p)+20p+1,
\end{aligned}
$$

which, as predicted, is the same as the number of points given by (4.1).

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References

[BCP] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system* I, *The user language*, J. Symbolic Comput. **24**-3–4 (1997), 235–265.

[C] B. Conrad, *Modular forms, cohomology, and the Ramanujan conjecture*, in preparation.

[CN] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11**-3 (1979), 308–339.

[Cr] J. E. Cremona, Algorithms for modular elliptic curves, Second edition, Cambridge Univ. Press, Cambridge, 1997.

[D] P. Deligne, *Formes modulaires et représentations de* GL(2). In Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 55–105. Lecture Notes in Math. **349**, Springer, Berlin, 1973.

[E] N. D. Elkies, *The Klein quartic in number theory*, In The eightfold way, 51–101, Cambridge Univ. Press, Cambridge, 1999.

[Ha] R. Hartshorne, Algebraic Geometry, Springer Verlag, 1977.

[KM] N. M. Katz and B. Mazur, Arithmetic moduli of elliptic curves, Ann. of Math. Stud., **108**, Princeton Univ. Press, Princeton, NJ, 1985.

[KS] H. H. Kim and F. Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2. With an appendix by C. J. Bushnell and G. Henniart. Ann. of Math. (2) **155**-3 (2002), 837–893.

[K] D. S. Kubert, *Universal bounds on the forsion of elliptic curves*, Proc. London Math. Soc. (3) **33**-2 (1976), 193–237.

[Liu] Q. Liu, Algebraic geometry and arithmetic curves, Oxford University Press, 2002.

[Liv1] R. Livné, *Cubic exponential sums and Galois representations*, in Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), 247–261, Contemp. Math. **67**, Amer. Math. Soc., Providence, RI, 1987.

[Liv2] ——, *Motivic orthogonal two-dimensional representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Israel J. of Math. **92** (1995), 149–156.

[LY] R. Livné and N. Yui, *The modularity of certain non-rigid Calabi-Yau threefolds*, J. Math. Kyoto Univ. **45**-4 (2005), 645–665.

[M] Y. Martin, *Multiplicative η-quotients*, Trans. Amer. Math. Soc. **348**-12 (1996), 4825–4856.

[SY] M.-H. Saito and N. Yui, *The modularity conjecture for rigid Calabi-Yau threefold over Q*, J. Math. Kyoto Univ. **41**-2 (2001), 403–419.

[Sch]   C. Schoen, *On fiber products of rational elliptic surfaces with section*, Math. Z. **197**-2, 177–199 (1988).

[STZ]   X. Sun, S.-L. Tan and K. Zuo, *Families of K3 surfaces over curves satisfying the equality of Arakelov-Yau's type and modularity*, math.AG/0205213.

[Sil]   J. H. Silverman, The arithmetic of elliptic curves, Springer, New York, 1986.

[Tate] J. T. Tate, *The arithmetic of elliptic curves*, Invent. Math. **23** (1974), 179–206.