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Invariant averagings of locally compact groups

By

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Abstract

A definition of an invariant averaging for a linear representation of a group in a locally convex space is given. Main results: A group H is finite if and only if every linear representation of H in a locally convex space has an invariant averaging. A group H is amenable if and only if every almost periodic representation of H in a quasi-complete locally convex space has an invariant averaging. A locally compact group H is compact if and only if every strongly continuous linear representation of H in a quasi-complete locally convex space has an invariant averaging.

Introduction

The present paper is devoted to a definition and an investigation of an invariant averaging for a linear representation of a group in a locally convex space. Invariant averagings are closely connected with invariant means, vector-valued invariant means, amenable groups, almost periodic functions, almost periodic representations of a group in locally convex spaces and uniformly equicontinuous actions of a group on compacts.

The theory of invariant means for complex-valued bounded functions on a group was founded by von-Neumann [18], [19]. For an arbitrary complete locally convex space L, the theory of L-valued almost periodic functions and L-valued invariant means was developed by von-Neumann and Bochner [20]. The existence of an invariant mean in the space of weakly almost periodic functions was investigated by de-Leew and Glicksberg [14], [15]. Vector-valued invariant means have been used by a number of authors for the study of some vector-valued function spaces, functional equations , a linear topological classification of spaces of continuous functions and for solving stability problems [1], [2], [6], [23]–[25].

It is well known[7], [4], [21]–[23] that the existence of an invariant mean on a locally compact group G is equivalent to many fundamental properties in the harmonic analysis of G. Below another such property will be given in terms of invariant averagings.

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Our paper is organized as follows. In section 1, a definition of an invariant averaging for a linear representation of a group in a locally convex space is given. It is obtained that a group H is finite if and only if every linear representation of H in a locally convex space has an invariant averaging. In section 2, it is proved that a group H is amenable if and only if every almost periodic linear representation of H in a quasi-complete locally convex space has an invariant averaging. In section 3, it is obtained that a locally compact group H is compact if and only if every strongly continuous linear representation of H in a quasi-complete locally convex space has an invariant averaging.

A part of results of section 1 was announced in [11].

1. The concept of an invariant averaging and invariant averagings of finite groups

Let L be a complex locally convex space and $G(L)$ be the group of all continuous linear operators $A : L \to L$ such that A^{-1} exists and is continuous. Let H be a group.

Definition 1.1 ([16, p. 80])**.** A homomorphism $\alpha : H \to G(L)$ will be called a linear representation of a group H in a locally convex space L.

Let α be a linear representation of H in a locally convex space L, $x \in L$. Put $Hx = \{y \in L : y = \alpha(t)x, \quad t \in H\}$. Denote the convex hull of Hx by $Conv\,(Hx)$ and the closure of $Conv\,(Hx)$ in L by $V(x)$.

Definition 1.2. A linear operator $M : L \to L$ will be called an invariant averaging for α if:

(i) $\alpha(t)M(x) = M(\alpha(t)x) = M(x)$ for all $x \in L$ and all $t \in H$;

(ii) $M(x) \in V(x)$ for all $x \in L$.

Put $L^H = \{y \in L : \alpha(t)y = y, \forall t \in H\}$. Then L^H is a closed linear subspace of L. By Definition 1.2, $M(x) \in L^H$ for all $x \in L$, $M(x) = x$ for all $x \in L^H$ and $M(M(x)) = M(x)$ for all $x \in L$. Hence M is a projection operator onto L^H.

Proposition 1.1. *Let $\{K_\tau, \tau \in T\}$ be a family of closed $\alpha(H)$-invariant subspaces of a linear representation α of H in a locally convex space L. Assume that α has an invariant averaging. Then $\sum\limits_T K_\tau^H = (\sum\limits_T K_\tau)^H$, where $\sum$ denotes the algebraic sum of vector subspaces.*

Proof. The inclusion $\sum\limits_T K_\tau^H \subset (\sum\limits_T K_\tau)^H$ is evident. Prove the converse inclusion. Let $x \in (\sum\limits_T K_\tau)^H$. Then there exist elements $x_i \in K_{\tau_i}, i = 1, \ldots, m$, such that $x = \sum_{i=1}^m x_i$. Applying an invariant averaging M to x, we find $x = Mx = \sum_{i=1}^m Mx_i$. Since $Mx_i \in K_{\tau_i}, i = 1, \ldots, m$, we have $x \in \sum\limits_T K_\tau^H$. $\square$

Remark 1. An analog of Proposition 1.1 has an important role in the invariant theory ([17, II.3.2], [13, II.3.2, Theorem (d)], [8], [9], [10, p. 44]).

According to Definition 1.2, $V(x)$ contains an $\alpha(H)$-invariant point for all $x \in L$. It is very important (in particular, in the ergodic theory) to know when $V(x)$ has the unique $\alpha(H)$-invariant point.

Definition 1.3. An invariant averaging M on L will be called continuous if M is continuous on L.

Proposition 1.2. *Let α be a linear representation of H in a locally convex space L. Assume that α has a continuous invariant averaging. Then $V(x)$ contains the unique $\alpha(H)$-invariant point for every $x \in L$.*

Proof. Let M be a continuous invariant averaging for α and $x \in L$. Then $M(x) \in V(x)$ and $M(x) \in L^H$. Let $y \in V(x) \cap L^H$. Then there exists a net $\{y_\nu\}$, $y_\nu \in Conv(Hx)$, such that $\lim y_\nu = y$. Every y_ν has the form

$$y_\nu = \lambda_1^{(\nu)} \alpha(t_1^{(\nu)})x + \cdots + \lambda_{n(\nu)}^{(\nu)} \alpha(t_{n(\nu)}^{(\nu)})x,$$

where $\lambda_i^{(\nu)} \in R$ (R is the field of real numbers), $\lambda_i^{(\nu)} \geq 0$ and $\sum\limits_{i=1}^{n(\nu)} \lambda_i = 1$. Applying the operator M to y_ν, we find

$$My_\nu = \lambda_1^{(\nu)} M\alpha(t_1^{(\nu)})x + \cdots + \lambda_{n(\nu)}^{(\nu)} M\alpha(t_{n(\nu)}^{(\nu)})x = Mx$$

for all ν. Since M is continuous, $y = My = M(\lim y_\nu) = \lim My_\nu = Mx$. ☐

In the following theorem we prove that every linear representation of a finite group in a locally convex space has a continuous invariant averaging.

Theorem 1.1. *For a group H the following conditions are equivalent*:

(*i*) *H is a finite group*;

(*ii*) *every linear representation of H in a locally convex space has an invariant averaging.*

Proof. (*i*) → (*ii*). Let $H = \{t_1, \ldots, t_n\}$ be a finite group, α is a linear representation of H in a locally convex space L and $x \in L$. Consider the operator

$$M(x) = \frac{1}{n}(\alpha(t_1) + \cdots + \alpha(t_n))(x).$$

It is obviously that M is an invariant averaging for α and it is continuous.

(*ii*) → (*i*). Let H be an infinite group. Assume that every linear representation of H in a locally convex space has an invariant averaging.

Let $Q(H)$ be the linear space of all complex functions on H. Denote by $F(H)$ the set of all finite subsets of H. $Q(H)$ is a locally convex space with respect to the topology of the system $\{p_A, A \in F(H)\}$ of semi-norms, where

$$p_A(x) = \max_{t \in A} |x(t)|, x \in Q(H), A \in F(H).$$

Let $Q'(H)$ be the conjugate space of $Q(H)$. $Q'(H)$ is a locally convex space with respect to the w^*-topology. Define the linear representation α of H in $Q(H)$ and the linear representation α' in $Q'(H)$ as follows: $(\alpha(h)x)(t) = x(h^{-1}t), (\alpha'(h)\varphi)(x) = \varphi(\alpha(h^{-1})x)$, where $h \in H, x \in Q(H), \varphi \in Q'(H)$. Put

$$e_t(s) = \begin{cases} 1 & \text{for } t = s, \\ 0 & \text{for } t \neq s \end{cases}$$

for all $t, s \in H$. Then $e_t \in Q(H)$ for all $t \in H$.

Lemma 1.1.
(a) *For $\varphi \in Q'(H)$ the set $\{t \in H : \varphi(e_t) \neq 0\}$ is finite;*
(b) *If $\{t \in H : \varphi(e_t) \neq 0\} = \emptyset$ for $\varphi \in Q'(H)$ then $\varphi = 0$.*

Proof of the Lemma. (a). On account of $\varphi \in Q'(H)$ there exist a semi-norm $p_A, A \in F(H)$, and $c \in R, c > 0$, such that $|\varphi(x)| \leq cp_A(x)$. Then $\varphi(e_t) = 0$ for all $t \notin A$. Hence $\{t \in H : \varphi(e_t) \neq 0\} \subset A$ and the set $\{t \in H : \varphi(e_t) \neq 0\}$ is finite.

(b). Assume that $\varphi \in Q'(H)$ and $\varphi(e_t) = 0$ for all $t \in H$. On account of $\varphi \in Q'(H)$ we have $|\varphi(x)| \leq cp_A(x)$ for some $A \in F(H)$ and some $c \in R, c > 0$. Let $A = \{t_1, \ldots, t_m\}$. Every element $x \in Q(H)$ has the form $x = x_1 + x_2$, where $x_1(t) = x(t_1)e_{t_1} + \cdots + x(t_m)e_{t_m}, x_2 = x - x_1$. Using the inequality $|\varphi(x)| \leq cp_A(x)$, we find $\varphi(x_2) = 0$ and $\varphi(x_1) = x(t_1)\varphi(e_{t_1}) + \cdots + x(t_m)\varphi(e_{t_m}) = 0$. Hence $\varphi(x) = 0$ for all $x \in Q(H)$. The lemma is proved.

According to supposition (ii) of our theorem there exists an invariant averaging M on $Q'(H)$. Let $\varphi \in Q'(H)$ such that $\varphi(1_H) = 1$, where 1_H is the function: $1_H(t) = 1$ for all $t \in H$. Then $M\varphi$ is an $\alpha'(H)$-invariant functional on $Q(H)$ and $(M\varphi)(1_H) = 1$, since $M\varphi \in V(\varphi)$. Hence $M\varphi \neq 0$. According to Lemma 1.1 there exists $s \in H$ such that $(M\varphi)(e_s) \neq 0$. Since $M\varphi$ is $\alpha'(H)$-invariant, we have $M\varphi(e_s) = M\varphi(\alpha(t^{-1})e_s) = M\varphi(e_{ts}) \neq 0$ for all $t \in H$. But it is a contradiction to statement (a) of Lemma 1.1. Hence H is finite. The theorem is completed. □

2. Invariant averagings of amenable groups

Let α be a linear representation of a group H in a quasi-complete locally convex space L.

Definition 2.1. An element $x \in L$ will be called almost periodic if the orbit Hx is precompact in L. A representation α will be called almost periodic if every element of L is almost periodic.

Remark 2. This is a variant of the definition of an almost periodic operator semigroup proposed by K. de Leeuw and I. Glicksberg [14].

Let α be an almost periodic representation of H in a quasi-complete locally convex space L. Then it is known [5, 8.13.4(2)] that $V(x)$ is a compact for all $x \in L$.

Theorem 2.1. *For a group H the following conditions are equivalent*:
(i) *H is an amenable group*;
(ii) *every almost periodic representation of H in a quasi-complete locally convex space has an invariant averaging.*

Proof. $(ii) \to (i)$. Let $B(H)$ be the set of all bounded complex functions on H. $B(H)$ is a Banach space with respect to the norm:

$$\|x\| = \sup_{t \in H} |x(t)|,$$

where $x \in B(H)$. Let $B(H)'$ be the conjugate space of $B(H)$. According to Corollary 2 in [3, ch.III, 3.7] $B(H)'$ is a quasi-complete locally convex space with respect to the w^*-topology. For $\varphi \in B(H)'$ put $(T_s\varphi)(x) = \varphi(x_s)$, where $s \in H, x_s(t) = x(s^{-1}t)$. Then $|(T_s\varphi)(x)| = |\varphi(x_s)| \leq \|\varphi\|\|x_s\| = \|\varphi\|\|x\|$. Hence $\|T_s\varphi\| \leq \|\varphi\|$ for all $s \in H$. We have

$$\|\sum_{i=1}^{n} \lambda_i (T_{s_i}\varphi)(x)\| \leq \sum_{i=1}^{n} \lambda_i \|(T_{s_i}\varphi)(x)\| \leq \sum_{i=1}^{n} \lambda_i \|\varphi\|\|x\| = \|\varphi\|\|x\|$$

for $\lambda_i \in R$ such that $\lambda_i \geq 0$ and $\sum\limits_{i=1}^{n} \lambda_i = 1$. Hence $V(\varphi)$ is bounded in $B(H)'$ and it is compact with respect to the w^*-topology. Then according to supposition (ii) of our theorem there exists an invariant averaging M on $B(H)'$. Let $\mu \in B(H)'$ be a mean that is $\mu(x) \geq 0$ for all $x \in B(H)$, $x \geq 0$ and $\mu(1_H) = 1$, where $1_H(t) = 1$ for all $t \in H$. Then $M(\mu)$ is an H-invariant mean. Therefore H is an amenable group.

$(i) \to (ii)$. Let H be an amenable group, m is a two-sided invariant mean of H and α is an almost periodic representation of H in a quasi-complete locally convex space L.

Let L' be the conjugate space of L. For $x \in L$, $F \in L'$ we consider the function $\psi_x(t) =< F, \alpha(t)x >= F(\alpha(t)x)$ on H. Since the set Hx is precompact and L is a quasi-complete space, $V(x)$ is compact. Hence $\psi_x \in B(H)$. Put $\tilde{m}(F) = m(\psi_x) = m(< F, \alpha(t)x >)$. Then $\tilde{m}$ is a linear functional on L'. We write $\tilde{m}$ in the form $\tilde{m}(F) =< M(x), F >$, $M(x) \in (L')^*$, where $(L')^*$ is the algebraic conjugate space of L'. The mapping $M : L \to (L')^*$ is linear. Prove that $M(x) \in L$ for all $x \in L$.

Let $\Sigma = \{\mu \in B(H)' : \mu(1_H) = 1, \mu(x) \geq 0, \forall x \geq 0\}$ be the set of all means on $B(H)$. For $f \in B(H)$ and $t_i \in H$ put $\delta_{t_i}(f) = f(t_i)$. Let Σ_0 be the set of all $\mu \in \Sigma$ such that: $\mu = \sum\limits_{i=1}^{n} \lambda_i \delta_{t_i}, \lambda_i \geq 0, \sum\limits_{i=1}^{n} \lambda_i = 1$ for some $t_i \in H, i = 1, \ldots, n$. For $\mu \in \Sigma_0$ we consider the operator $M_\mu(x) = \sum\limits_{i=1}^{n} \lambda_i \alpha(t_i)x$, where $\mu = \sum\limits_{i=1}^{n} \lambda_i \delta_{t_i}$. There exists a net $\{\mu_\nu\}, \mu_\nu \in \Sigma_0$, such that $\lim \mu_\nu = m$ in the w^*-topology in $B(H)'$. We have $M_{\mu_\nu}(x) \in V(x)$ for all ν. Put $x_\nu = M_{\mu_\nu}(x)$. On account of compactness of $V(x)$ there exist a subnet $\{y_\tau\}$ of $\{x_\nu\}$ and $x_0 \in V(x)$ such that $\lim y_\tau = x_0$. Then $\lim \mu_\tau = m$, $\mu_\tau(< F, \alpha(t)x >) =< M_{\mu_\tau}(x), F >$,

$\mu_\tau(< F, \alpha(t)x >) \to m(< F, \alpha(t)x >)$ for all $F \in L'$. Using $< M_{\mu_\tau}(x), F > \to < x_0, F >$, we obtain $m(< F, \alpha(t)x >) =< M(x), F >=< x_0, F >$ for all $F \in L'$. Then $M(x) = x_0 \in L$. Thus $M(x) \in L$ and $M(x) \in V(x)$.

Prove $\alpha(s)M(x) = M(\alpha(s)x) = M(x)$. Since m is H-invariant, we find $< F, \alpha(s)M(x) >= m(< F, \alpha(s)\alpha(t)x >) = m(< F, \alpha(t)x >) =< F, M(x) >$. Similarly $< F, M(\alpha(s)x) >= m(< F, \alpha(t)\alpha(s)x >) = m(< F, \alpha(t \cdot s)x >) = m(< F, \alpha(t)x >) =< F, M(x) >$. □

3. Invariant averagings of locally compact groups

Definition 3.1. A linear representation $\alpha : H \to G(L)$ of a topological group H in a locally convex space will be called strongly continuous if $t \to \alpha(t)x$ is a continuous function on H for every $x \in L$.

Our aim in this section is a proof of the following

Theorem 3.1. *For a locally compact group H the following conditions are equivalent*:

(i) *H is compact*;

(ii) *every strongly continuous linear representation of H in a quasi-complete locally convex space has an invariant averaging.*

Proof. The implication $(i) \to (ii)$ is known [12, p. 149].

A proof of the implication $(ii) \to (i)$ consists of some steps. First we give some needful lemmas.

Let H be a topological group, α is a strongly continuous linear representation of H in a complex locally convex space L and L' is the conjugate space of L. We define a linear representation of H in L' as follows:$(\alpha'(t)\varphi)(x) = \varphi(\alpha(t^{-1})x)$.

Lemma 3.1. *α' is a strongly continuous linear representation with respect to the topology $\sigma(L', L)$.*

Proof of the Lemma. It is known that L' is a locally convex space with respect to the topology $\sigma(L', L)$. For $x_1, \ldots, x_n \in L$ and $\varepsilon \in R, \varepsilon > 0$, put $Q(x_1, \ldots, x_n, \varepsilon) = \{\varphi \in L' : |\varphi(x_i)| < \varepsilon,\ \ i = 1, \ldots, n\}$. The family

$$\{Q(x_1, \ldots, x_n, \varepsilon), x_i \in L, \varepsilon \in R, \varepsilon > 0, n \in N\}$$

is a fundamental system of neighborhoods of the zero in L' for the topology $\sigma(L', L)$. From $\alpha'(t)Q(x_1, \ldots, x_n, \varepsilon) = Q(\alpha(t^{-1})x_1, \ldots, \alpha(t^{-1})x_n, \varepsilon)$ and $\alpha'(t^{-1})Q(x_1, \ldots, x_n, \varepsilon) = Q(\alpha(t)x_1, \ldots, \alpha(t)x_n, \varepsilon)$, we obtain that operators $\alpha'(t)$ and $\alpha'(t^{-1})$ are continuous in the the topology $\sigma(L', L)$ for every $t \in H$.

Let φ be a fixed element of L' and $Q(x_1, \ldots, x_n, \varepsilon)$ be an arbitrary neighborhood of the zero in L'. For arbitrary $\varepsilon > 0$ there exists a neighborhood W of the zero in L such that $|\varphi(W)| < \varepsilon$. Since α is strongly continuous, for $x_1, \ldots, x_n \in L$, the neighborhood W of the zero in L and every $t_0 \in H$ there exists a neighborhood U of the unit in H such that $\alpha(Ut_0)x_i - \alpha(t_0)x_i \subset W$ for all $i = 1, \dots, n$. Then $|\varphi(\alpha(Ut_0)x_i - \alpha(t_0)x_i)| < \varepsilon$ for all $i = 1, \dots, n$. Hence $|\alpha'(U^{-1}t_0^{-1})\varphi(x_i) - \alpha'(t_0^{-1})\varphi(x_i)| < \varepsilon$ for all $i = 1, \dots, n$. Then $\alpha'(U^{-1}t_0^{-1})\varphi - \alpha'(t_0^{-1})\varphi \subset Q(x_1, \dots, x_n, \varepsilon)$. This means that the mapping $t \to \alpha'(t)\varphi$ is continuous for every $t_0 \in H$ and every φ. Thus the representation α' is strongly continuous. The lemma is proved.

Let H be a locally compact group. Denote by $K(H)$ the vector space of all complex continuous functions on H with the compact support. Denote the family of compact subsets of H by $T(H)$. For $A \in T(H)$ and $x \in K(H)$ put $p_A(x) = \max_{t\in A} |x(t)|$. According to [5, Theorem 6.31] $K(H)$ is a barrel locally convex space with respect to the family $\{p_A, A \in T(H)\}$.

Denote the vector space of all complex continuous functions on H by $C(H)$. $C(H)$ is a locally convex space with respect to the family $\{p_A, A \in T(H)\}$ of semi-norms. We have $K(H) \subset C(H)$.

Lemma 3.2. *$K(H)$ is dense in $C(H)$.*

Proof of the Lemma. According to statement 0.2.18(2)in [5] for every $A \in T(H)$ there exist a function $e_A \in K(H)$ and a compact neighborhood U of A such that

$$e_A(t) = \begin{cases} 1 & \text{for } t \in A, \\ 0 & \text{for } t \notin U. \end{cases}$$

Let $x \in C(H)$. Then $e_A x \in K(H)$ and $p_A(x - e_A x) = 0$. Therefore for the net $\{e_A x, A \in T(H)\}$ we obtain $\lim_{A\in T(H)} e_A x = x$. The lemma is proved.

Using Lemma 3.2 and [5, Theorem 6.2.4(2)], we obtain the following

Lemma 3.3. *$C(H)$ is a barrel locally convex space.*

We define a linear representation of H in $C(H)$ as follows: $(\alpha(s)x)(t) = x(s^{-1}t), x \in C(H)$.

Lemma 3.4.

(*i*) *The linear representation α in $C(H)$ is strongly continuous*;

(*ii*) *The conjugate space $(C(H))'$ is a quasi-complete locally convex space with respect to the topology $\sigma(L', L)$, where $L = C(H)$*;

(*iii*) *The linear representation α' in $(C(H))'$ is strongly continuous.*

Proof of the Lemma. Put $W_{A,\varepsilon} = \{x \in C(H) : p_A(x) < \varepsilon\}$. For $s \in H$ we have $\alpha(s)W_{A,\varepsilon} = W_{sA,\varepsilon}$. Hence operator $\alpha(t)$ is continuous on L for every $t \in H$. We prove that α is strongly continuous. Let $x \in C(H)$, $s_0 \in H$ and $A \in T(H)$. On account of compactness of A there exists a neighborhood U of the unit of H such that $|x(Us_0^{-1}t) - x(s_0^{-1}t)| < \varepsilon$ for all $t \in A$. Then $p_A(\alpha(Us_0)x - \alpha(s_0)x) < \varepsilon$. This means that the mapping $t \to \alpha(t)x$ is continuous on H for every x. Thus α is strongly continuous.

Using Lemma 3.3 and [3, III.3.7, Corollary 2], we find that the conjugate space $(C(H))'$ is a quasi-complete locally convex space with respect to the $\sigma(L', L)$- topology, where $L = C(H)$. According to Lemma 3.1 the linear representation α' in $(C(H))'$ is strongly continuous with respect to the $\sigma(L', L)$-topology. The lemma is proved.

For $x \in C(H)$ put *supp*$(x) = \{t \in H : x(t) \neq 0\}$.

Lemma 3.5. *Let H be a locally compact topological group, $A \in T(H)$ and U be an arbitrary closed neighborhood of the unit of H. Then there exist a family $\{e_k(t) \in C(H),\ \ k = 0, 1, \dots, n\}$ and elements $t_1, \dots, t_n \in H$ such that:*

(*i*) $\sum_{k=0}^{n} e_k(t) = 1$ *and* $0 \le e_k(t) \le 1$ *for all* $t \in H$, $k = 0, 1, \dots, n$;

(*ii*) $supp(e_0) \cap A = \emptyset$;

(*iii*) $A \subset \cup_{k=1}^{n} t_k U$ *and* $supp(e_k) \subset t_k U$ *for all* $k = 1, \dots, n$.

Proof of the Lemma. We consider the following open covering of the compact set A: $A \subset \cup_{t \in A} tU$. Then there exist $t_1, \dots, t_n \in A$ such that $A \subset \cup_{k=1}^{n} t_k U$. Put $B = H \setminus \cup_{k=1}^{n} t_k U$. H is a completely regular topological space as a separable topological group. Hence for t_k and the neighborhood $t_k U$ there exists a continuous real function $f_k : H \to R$ such that $0 \le f_k(t) \le 1$ for all $t \in H$, $f_k(t_k) = 1$ and $f_k(t) = 0$ for all $t \in H \setminus t_k U$. Put $f'_k(t) = 1 - f_k(t)$. We consider the multiplication

$$(f_1(t) + f'_1(t))(f_2(t) + f'_2(t)) \cdots (f_n(t) + f'_n(t)) = 1.$$

By induction we obtain

$$f_1(t) + f'_1(t) f_2(t) + f'_1(t) f'_2(t) f_3(t) + \cdots + f'_1(t) f'_2(t) \cdots f'_{n-1}(t) f_n(t) \\ + f'_1(t) f'_2(t) \cdots f'_n(t) = 1.$$

Put $e_1 = f_1(t), e_i = f'_1(t) f'_2(t) \cdots f'_{i-1}(t) f_i(t), i = 2, \dots, n; e_0 = f'_1(t) f'_2(t) \cdots f'_n(t)$. Then $supp(e_0) \subset H \setminus \cup_{k=1}^{n} t_k U$ and $supp(e_k) \subset t_k U$ for all $k = 1, \dots n$. The lemma is proved.

Lemma 3.6. *Let H be a locally compact topological group such that there exists a non-zero H-invariant linear continuous functional φ on $C(H)$. Then H is compact.*

Proof of the Lemma. Assume that H is non- compact and φ be a non-zero H-invariant continuous linear functional on $C(H)$. Since φ is continuous, there exist $c \in R, c > 0$, and $A \in T(H)$ such that $|\varphi(f)| \le c p_A(f)$ for all $f \in C(H)$. Then $\varphi(f) = 0$ for $f \in L$ such that $supp(f) \cap A = \emptyset$. Since H is non-compact, there exist $y \in H$ and a closed neighborhood U of the unit of H such that $A \cap yU = \emptyset$.

Since φ is non-zero, $\varphi(f) \ne 0$ for some $f \in C(H)$. According to Lemma 3.5 there exist elements $t_k \in H$ and $e_k \in C(H), k = 1, \dots, n$, satisfying the conditions $(i), (ii), (iii)$ of Lemma 3.5. Then $\varphi(e_j f) \ne 0$ and $supp(e_j(t) f(t)) \subset t_j U$ for some j. Put $z = y t_j^{-1}$. We have $supp(\alpha(z))(e_j(t) f(t)) \subset z t_j U = yU$. Then $supp(\alpha(z)(e_j(t) f(t))) \cap A = \emptyset$. Hence $\varphi(\alpha(z)(e_j f)) = 0$. Since φ is H-invariant, $\varphi(e_j f) = \varphi(\alpha(z)(e_j f)) \ne 0$. It is a contradiction. The lemma is proved.

We continue a proof of the implication $(ii) \to (i)$ of our theorem. Consider a linear representation α' of H in $(C(H))'$. According to Lemma 3.4 $(C(H))'$ is a quasi-complete locally convex space and α' is a strongly continuous linear representation of H in $(C(H))'$. On account of supposition (ii) of our theorem α' has an invariant averaging. Let $a \in H$. Put $\varphi_a(x) = x(a)$ for all $x \in C(H)$. Then $\varphi_a \in (C(H))'$ and $\varphi_a(1_H) = 1$, where 1_H is the unit function of $C(H)$. For every $s \in H$ we find $\alpha'(s)\varphi_a = \varphi_{sa}$ and $\varphi_{sa}(1_H) = 1$. We have $\varphi(1_H) = 1$ for every linear functional φ on $C(H)$ of the form $\varphi = \sum_{k=1}^{n} \lambda_k \alpha'(s_k)\varphi_a$, where $\lambda_k \geq 0$, $\sum_{k=1}^{n} \lambda_k = 1$. Hence $\varphi(1_H) = 1$ for every $\varphi \in V(\varphi_a)$. Let M be an invariant averaging for α'. Then the element $\psi = M(\varphi)$ is H-invariant and $\psi(1_H) = 1$. Therefore ψ is a non-zero H-invariant linear functional on $C(H)$. By Lemma 3.6, H is compact. The theorem is completed. $\square$

We note that a left H-invariant Haar's integral on a locally compact group H is a H-invariant linear functional on $K(H)$. Hence Lemma 3.6 has also the following

Corollary 3.1. *Haar's integral is continuous on $K(H)$ with respect to the topology $\{p_A, A \in T(H)\}$ if and only if H is compact.*

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