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Characteristic cycles of standard modules for the rational Cherednik algebra of type $\mathbb{Z}/l\mathbb{Z}$

By

Toshiro KUWABARA

Abstract

We study the representation theory of the rational Cherednik algebra $H_\kappa = H_\kappa(\mathbb{Z}_l)$ for the cyclic group $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$ and its connection with the geometry of the quiver variety $\mathfrak{M}_\theta(\delta)$ of type $A_{l-1}^{(1)}$.

We consider a functor between the categories of H_κ-modules with different parameters, called the shift functor, and give the condition when it is an equivalence of categories.

We also consider a functor from the category of H_κ-modules with good filtration to the category of coherent sheaves on $\mathfrak{M}_\theta(\delta)$. We prove that the image of the regular representation of H_κ by this functor is the tautological bundle on $\mathfrak{M}_\theta(\delta)$. As a corollary, we determine the characteristic cycles of the standard modules. It gives an affirmative answer to a conjecture given in [Go] in the case of $\mathbb{Z}_l$.

1. Introduction

1.1. Background

The rational Cherednik algebra for the wreath product $\mathbb{Z}_l \wr \mathfrak{S}_n$ of the cyclic group $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$ and the symmetric group $\mathfrak{S}_n$ is defined by [EG]. Let $D(\mathbb{C}^n_{reg})$ be the algebra of algebraic differential operators on $\mathbb{C}^n_{reg} = \{(x_1, \ldots, x_n) \mid x_i \neq 0, x_i^l \neq x_j^l\}$. The rational Cherednik algebra is a subalgebra of the smash product $D(\mathbb{C}^n_{reg})\#(\mathbb{Z}_l \wr \mathfrak{S}_n)$ which is generated by the multiplication of functions, $\mathbb{Z}_l \wr \mathfrak{S}_n$ and the Dunkl operators. The category of modules over the rational Cherednik algebra contains an interesting subcategory called the category $\mathcal{O}$. The category $\mathcal{O}$ is the subcategory of modules on which the Dunkl operators act locally nilpotently. The category $\mathcal{O}$ is a highest weight category in the sense of [CPS].

In this paper, we consider the case of $n = 1$. Our work is motivated by the papers [GS1] and [GS2], in which the case $l = 1$ was considered. We first review this case.

The rational Cherednik algebra $H_c(\mathfrak{S}_n)$ is the algebra with a parameter $c \in \mathbb{R}$. We denote the category $\mathcal{O}$ of $H_c(\mathfrak{S}_n)$ by $\mathcal{O}_c(\mathfrak{S}_n)$. By results of [Op],

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[He] and [BEG], we have a functor called a shift functor (or a Heckman-Opdam shift functor),

$$\widehat{S}_c : H_c(\mathfrak{S}_n)\text{-Mod} \longrightarrow H_{c+1}(\mathfrak{S}_n)\text{-Mod}. \tag{1.1}$$

If the shift functor $\widehat{S}_c$ is an equivalence of categories, we can construct a functor from the category of filtered modules $H_c(\mathfrak{S}_n)$-filt to the category of coherent sheaves $\mathrm{Coh}(\mathrm{Hilb}^n(\mathbb{C}^2))$ on the Hilbert scheme of n points on $\mathbb{C}^2$,

$$\widehat{\Phi}_c : H_c(\mathfrak{S}_n)\text{-filt} \longrightarrow \mathrm{Coh}(\mathrm{Hilb}^n(\mathbb{C}^2)). \tag{1.2}$$

We recall that $\mathrm{Hilb}^n(\mathbb{C}^2)$ is a symplectic resolution of the singularity $\mathbb{C}^{2n}/\mathfrak{S}_n$,

$$\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \longrightarrow \mathbb{C}^{2n}/\mathfrak{S}_n.$$

These functors $\widehat{S}_c$ and $\widehat{\Phi}_c$ are generalized to the other cases by [Mu], [Bo] and [Va].

In [GS1] and [GS2], Gordon and Stafford also considered the images of certain modules by $\widehat{\Phi}_c$. Consider the rational Cherednik algebra $H_c(\mathfrak{S}_n)$ itself as a left $H_c(\mathfrak{S}_n)$-module. Then the corresponding coherent sheaf $\widehat{\Phi}_c(H_c(\mathfrak{S}_n))$ coincides with the Procesi bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$. The Procesi bundle, which was defined in [Ha1], is a vector bundle whose fiber is isomorphic to the regular representation of $\mathfrak{S}_n$. As a corollary of the above result, they described the images of the standard modules by the functor $\widehat{\Phi}_c$ and determined their characteristic cycles. The characteristic cycle $\mathrm{Ch}(M)$ is an invariant of a module M in $\mathcal{O}_c(\mathfrak{S}_n)$, which is the sum of irreducible components of $\mathrm{Supp}\widehat{\Phi}_c(M)$ with multiplicities. The standard modules of $H_c(\mathfrak{S}_n)$ are indexed by partitions λ of n. Denote them by $\Delta_c(\lambda)$. Let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a coordinate system of $\mathbb{C}^{2n}$. The irreducible components of $\pi^{-1}(\{y_1 = \cdots = y_n = 0\})$ are indexed by partitions μ of n. Denote them by $\mathcal{Z}_\mu$. Let $[\,\mathcal{Z}_\mu\,]$ be the homology class given by $\mathcal{Z}_\mu$. One of the main results in [GS2] is

$$\mathrm{Ch}(\Delta_c(\lambda)) = \sum_\mu K_{\lambda\mu}[\,\mathcal{Z}_\mu\,] \tag{1.3}$$

for each partition λ of n. Here $K_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ is the Kostka number.

In the general case, the rational Cherednik algebra $H_h(\mathbb{Z}_l \wr \mathfrak{S}_n)$ has an l-dimensional parameter $h \in \mathbb{R}^l$. The l-multipartitions of n parametrize the standard modules of the category $\mathcal{O}_h(\mathbb{Z}_l \wr \mathfrak{S}_n)$. We have the partial ordering $\trianglerighteq_{rep,h}$ on the set of l-multipartitions of n, which arises from the structure of the highest weight category $\mathcal{O}_h(\mathbb{Z}_l \wr \mathfrak{S}_n)$. This ordering $\trianglerighteq_{rep,h}$ depends on h.

Let $\mathfrak{M}_h(n\delta)$ be the quiver variety of type $A^{(1)}_{l-1}$ with the stability parameter $h \in \mathbb{Q}^l$ and the dimension vector $n\delta = (n, \ldots, n)$. Similarly to the case of $\mathfrak{S}_n$, the l-multipartitions of n parametrize the components of a certain subvariety of $\mathfrak{M}_h(n\delta)$. The action of $\mathbb{T} = (\mathbb{C}^*)^2$ on $\mathfrak{M}_h(n\delta)$ induces a partial ordering $\trianglerighteq_{geom,h}$ on the set of l-multipartitions of n. The ordering $\trianglerighteq_{geom,h}$ also depends on h.

Consider an analogue of the functor (1.2) in the general case. It has been conjectured that the representation theory of $H_h(\mathbb{Z}_l \wr \mathfrak{S}_n)$ is deeply connected with the geometry of $\mathfrak{M}_h(n\delta)$ ([Go], [Va]). The functor

$$\widehat{\Phi}_h : H_h(\mathbb{Z}_l \wr \mathfrak{S}_n)\text{-filt} \longrightarrow \mathrm{Coh}(\mathfrak{M}_h(n\delta))$$

is defined for generic h in [Va]. In [Go], Gordon compared the ordering $\trianglerighteq_{rep,h}$ and the ordering $\trianglerighteq_{geom,h}$, and proved that $\trianglerighteq_{geom,h}$ refines $\trianglerighteq_{rep,h}$. He conjectured an analogue of the identity (1.3) in the general case (Question 10.2 in [Go]).

1.2. Shift functors and their equivalences

In this paper, we consider the rational Cherednik algebra $H_\kappa = H_\kappa(\mathbb{Z}_l)$ in the case of $n = 1$. Here κ is an $(l-1)$-dimensional parameter $\kappa = (\kappa_i)_{i=1,\dots,l-1} \in \mathbb{R}^{l-1}$. Let γ be the element of $\mathbb{Z}_l$ which acts on $\mathbb{C}^*$ by the multiplication of $\zeta = \exp(2\pi\sqrt{-1}/l)$. The algebra H_κ is the subalgebra of $D(\mathbb{C}^*)\#\mathbb{Z}_l$ generated by the coordinate function x, $\gamma \in \mathbb{Z}_l$ and the Dunkl operator $y = (d/dx) + (l/x)\sum_{i=0}^{l-1} \kappa_i \bar{e}_i$ where $\bar{e}_i = (1/l)\sum_{j=0}^{l-1} \zeta^{ij}\gamma^j \in \mathbb{C}\mathbb{Z}_l$. As a vector space, we have

$$H_\kappa = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}\mathbb{Z}_l \otimes_{\mathbb{C}} \mathbb{C}[y].$$

The algebra H_κ is isomorphic to another algebra called the deformed preprojective algebra defined by Crawley-Boevey and Holland in [CBH] and [Ho]. Let $Q = (I, E)$ be the Dynkin quiver of type $A_{l-1}^{(1)}$ such that $I = \{I_0, \dots, I_{l-1}\}$ is the set of vertices and $E = \{F_i : I_{i-1} \to I_i,\ i = 0, \dots, l-1\}$ is the set of arrows. We regard indices for vertices and arrows as integers modulo l. For $\lambda = (\lambda_i)_{i=0,\dots,l-1} \in \mathbb{R}^l$ (or $\mathbb{Z}^l$), we regard the sum $\lambda_i + \lambda_{i+1} + \cdots + \lambda_{j-1}$ as cyclic, i.e.,

$$\lambda_i + \lambda_{i+1} + \cdots + \lambda_{j-1} = \lambda_i + \cdots + \lambda_{l-1} + \lambda_0 + \lambda_1 + \cdots + \lambda_{j-1}$$

if $j < i$. Let $\mathrm{Rep}(Q, \delta) \simeq \mathbb{C}^l$ be the space of representations with the dimension vector $\delta = (1, \dots, 1)$. Set $GL(\delta) = \prod_{i=0}^{l-1} \mathbb{C}^*$ and set $\mathfrak{gl}(\delta) = \bigoplus_{i=0}^{l-1} \mathbb{C}e^{(i)} = Lie(GL(\delta))$. Let $t_0, \dots, t_{l-1} \in \mathbb{C}[\mathrm{Rep}(Q, \delta)]$ be the coordinate functions, and let $\partial_0, \dots, \partial_{l-1} \in D(\mathrm{Rep}(Q, \delta))$ be the corresponding differential operators. Let $\mathbb{R}^l_1$ be the set of $\lambda = (\lambda_i)_{i=0,\dots,l-1} \in \mathbb{R}^l$ such that $\lambda_0 + \cdots + \lambda_{l-1} = 1$. For a parameter $\lambda = (\lambda_i)_{i=0,\dots,l-1} \in \mathbb{R}^l_1$, an algebra $\mathcal{T}_\lambda$ is

$$\mathcal{T}_\lambda = M_l(D(\mathrm{Rep}(Q, \delta)))^{GL(\delta)} \Big/ \sum_{i=0}^{l-1} M_l(D(\mathrm{Rep}(Q, \delta)))^{GL(\delta)}(\tau(e^{(i)}) - \lambda_i),$$

where $\tau(e^{(i)}) = E_{ii} \otimes 1 + 1 \otimes (t_{i+1}\partial_{i+1} - t_i\partial_i) \in M_l(D(\mathrm{Rep}(Q, \delta)))^{GL(\delta)}$. By [Ho, Cor 4.6], this algebra is isomorphic to the deformed preprojective algebra Π_λ defined by [CBH]. Moreover, the algebra $\mathcal{T}_\lambda$ is isomorphic to H_κ for $\lambda_i = \kappa_{i+1} - \kappa_i + (1/l)$. Set $e_i = E_{ii}$. Denote the spherical subalgebra of H_κ by $U_\kappa = \bar{e}_0 H_\kappa \bar{e}_0$. The algebra U_κ is isomorphic to the following subalgebra of $\mathcal{T}_\lambda$

$$\mathcal{A}_\lambda = e_0\mathcal{T}_\lambda e_0 \simeq D(\mathrm{Rep}(Q, \delta))^{GL(\delta)} \Big/ \sum_{i=0}^{l-1} D(\mathrm{Rep}(Q, \delta))^{GL(\delta)}(\iota(e^{(i)}) - \bar{\lambda}_i)$$

where $\iota(e^{(i)}) = t_{i+1}\partial_{i+1} - t_i\partial_i$ and $\bar{\lambda}_i = \lambda_i - \delta_{i0}$. A Dynkin root is a root $\beta = (\beta_i)_{i=0,\dots,l-1} \in \mathbb{Z}^l$ such that $\beta_0 = 0$. Then, the $(\mathcal{A}_\lambda, \mathcal{T}_\lambda)$-bimodule $e_0\mathcal{T}_\lambda$ yields a Morita equivalence between $\mathcal{T}_\lambda$ and $\mathcal{A}_\lambda$ if $\lambda = (\lambda_i)_{i=0,\dots,l-1}$ satisfies $\langle\lambda, \beta\rangle = \sum_{i=0}^{l-1} \lambda_i\beta_i \neq 0$ for all Dynkin roots $\beta = (\beta_i)_{i=0,\dots,l-1} \in \mathbb{Z}^l$.

For $i = 0, \dots, l-1$, the standard module $\Delta_\lambda(i)$ is the following $\mathcal{T}_\lambda$-module:

$$\Delta_\lambda(i) = (\mathcal{T}_\lambda/\mathcal{T}_\lambda A^*)e_i$$

where $A^* = \sum_{i=0}^{l-1} E_{i-1,i} \otimes \partial_i \in \mathcal{T}_\lambda$. We have the isomorphism of vector spaces

$$\Delta_\lambda(i) = \mathbb{C}[A]\mathbf{1}_i$$

where $A = \sum_{i=0}^{l-1} E_{i,i-1} \otimes t_i \in \mathcal{T}_\lambda$ and $\mathbf{1}_i$ is the image of e_i.

Let $\mathbb{Z}_0^l$ be the set of $\theta = (\theta_i)_{i=0,\dots,l-1} \in \mathbb{Z}^l$ such that $\theta_0 + \cdots + \theta_{l-1} = 0$. For $\theta \in \mathbb{Z}_0^l$, let χ_θ be the character of $GL(\delta)$

$$\chi_\theta(g) = \prod_{i=0}^{l-1} (g_i)^{\theta_i}$$

for $g = (g_i)_{i=0,\dots,l-1} \in GL(\delta)$. We define the shift functor $\mathcal{S}_\lambda^\theta$ for $\lambda = (\lambda_i)_{i=0,\dots,l-1} \in \mathbb{R}_1^l$ and $\theta = (\theta_i)_{i=0,\dots,l-1} \in \mathbb{Z}_0^l$,

$$\begin{aligned} \mathcal{S}_\lambda^\theta : \mathcal{A}_\lambda\text{-mod} &\longrightarrow \mathcal{A}_{\lambda+\theta}\text{-mod} \\ N &\mapsto \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} N \end{aligned}$$

where $\mathcal{B}_\lambda^\theta$ is the following $(\mathcal{A}_{\lambda+\theta}, \mathcal{A}_\lambda)$-bimodule of semi-invariants

$$\mathcal{B}_\lambda^\theta = \left[D(\mathrm{Rep}(Q,\delta)) \Big/ \sum_{i=0}^{l-1} D(\mathrm{Rep}(Q,\delta))(\iota(e^{(i)}) - \bar{\lambda}_i) \right]^{GL(\delta),\chi_\theta}.$$

We also define the functor,

$$\begin{aligned} \widehat{\mathcal{S}}_\lambda^\theta : \mathcal{T}_\lambda\text{-mod} &\longrightarrow \mathcal{T}_{\lambda+\theta}\text{-mod}, \\ M &\mapsto \mathcal{T}_\lambda e_0 \otimes_{\mathcal{A}_{\lambda+\theta}} \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0 M. \end{aligned}$$

Since the algebra $\mathcal{T}_\lambda$ has $(l-1)$-dimensional parameter, we have the $(l-1)$-dimensional parameter $\theta \in \mathbb{Z}^l$ for shifting the parameter λ. Thus we have many shift functors for the same $\mathcal{T}_\lambda$ while we have only one shift functor S_c for $H_c(\mathfrak{S}_n)$.

We study the case when the shift functors $\mathcal{S}_\lambda^\theta$ and $\widehat{\mathcal{S}}_\lambda^\theta$ are equivalences of categories. The main difficulty of this question is that we must consider complicated combinatorics which depends on the $(l-1)$-dimensional parameters λ and θ.

Define the following sets of parameters

$$\begin{aligned} &\mathbb{R}_{reg}^l = \{\lambda = (\lambda_i)_{i=0,\dots,l-1} \in \mathbb{R}_1^l \mid \bar{\lambda}_i + \cdots + \bar{\lambda}_{j-1} \neq 0 \quad \text{for all } i \neq j\}, \\ &\mathbb{Z}_{reg}^l = \left\{\theta \in \mathbb{Z}_0^l \;\middle|\; \theta_i + \cdots + \theta_{j-1} \neq 0 \quad \text{for all } i \neq j\right\}, \\ &\mathbb{Z}_\lambda^l = \{\theta = (\theta_i)_{i=0,\dots,l-1} \in \mathbb{Z}_{reg}^l \mid \theta_i + \cdots + \theta_{j-1} < 0 \quad \text{if } \lambda_i + \cdots + \lambda_{j-1} \in \mathbb{Z}_{\leq 0}\}. \end{aligned}$$

The set of parameters $\mathbb{Z}^l_{reg}$ is decomposed into $(l-1)!$ alcoves. If $\lambda \in \mathbb{R}^l_{reg}$ is generic, we have $\mathbb{Z}^l_\lambda = \mathbb{Z}^l_{reg}$. If λ belongs to $\mathbb{R}^l_{reg} \cap \mathbb{Z}^l$, $\mathbb{Z}^l_\lambda$ is one of $(l-1)!$ alcoves in $\mathbb{Z}^l_{reg}$.

Then the following theorem is the first main result of this paper.

Theorem 1.1. *For $\lambda \in \mathbb{R}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$, the shift functors $\mathcal{S}^\theta_\lambda$ and $\widehat{\mathcal{S}}^\theta_\lambda$ are equivalences of categories.*

Moreover, we explicitly determine the images of the standard modules by $\mathcal{S}^\theta_\lambda$.

The parameter $\theta = (\theta_i)_{i=0,\dots,l-1}$ defines a total ordering $\trianglerighteq_\theta$ on the set of indices $\Lambda = \{0, 1, \dots, l-1\}$,

$$i \triangleright_\theta j \Leftrightarrow \theta_i + \cdots + \theta_{j-1} < 0.$$

If θ and θ' belong to the same alcove in $\mathbb{Z}^l_{reg}$, then $\triangleright_\theta$ is equal to $\triangleright_{\theta'}$. Let $\trianglerighteq_{rep,\lambda}$ be the partial ordering on Λ defined as $i \trianglerighteq_{rep,\lambda} j$ if and only if $\mathrm{Hom}_{\mathcal{T}_\lambda}(\Delta_\lambda(j), \Delta_\lambda(i)) \neq 0$. Our total ordering $\trianglerighteq_\theta$ refines the partial ordering $\trianglerighteq_{rep,\lambda}$ when $\theta \in \mathbb{Z}^l_\lambda$, i.e., $\mathrm{Hom}_{\mathcal{T}_\lambda}(\Delta_\lambda(j), \Delta_\lambda(i)) \neq 0$ implies $i \trianglerighteq_\theta j$. Let η_1, $\dots$, η_l be the elements of Λ such that

$$\eta_l \trianglerighteq_\theta \eta_{l-1} \trianglerighteq_\theta \cdots \trianglerighteq_\theta \eta_1. \tag{1.4}$$

Proposition 1.2. *For $i = 1, \dots, l$, we have an isomorphism of $\mathcal{A}_{\lambda+\theta}$-modules*

$$\begin{aligned} e_0\Delta_{\lambda+\theta}(\eta_i) &\longrightarrow \mathcal{S}^\theta_\lambda(e_0\Delta_\lambda(\eta_i)) = \mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i), \\ e_0t_0t_{l-1}\cdots t_{\eta_i}\mathbf{1}_{\eta_i} &\mapsto \tilde{f}_i \otimes e_0t_0t_{l-1}\cdots t_{\eta_i}\mathbf{1}_{\eta_i}. \end{aligned}$$

where

$$\tilde{f}_i = \prod_{j=i+1}^{l-1} (t_{\eta_{j+1}} \cdots t_{\eta_l})^{\theta_{\eta_j}+\theta_{\eta_j+1}+\cdots+\theta_{\eta_{j+1}-1}} \prod_{j=1}^{i-1} (\partial_{\eta_1} \cdots \partial_{\eta_j})^{\theta_{\eta_j}+\theta_{\eta_j+1}+\cdots+\theta_{\eta_{j+1}-1}}.$$

1.3. Construction of a tautological bundle

Next, we consider analogues of the functor (1.2) and determine the image of the regular representation H_κ by this functor.

For $\theta \in \mathbb{Z}^l_{reg}$, the quiver variety $\mathfrak{M}_\theta(\delta)$ with the stability parameter θ can be described as follows.

$$\begin{gathered} \mathfrak{M}_\theta(\delta) = \mathrm{Proj}\, S, \\ S = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} S_m, \qquad S_m = \mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi^m_\theta}. \end{gathered}$$

where $\mu : \mathrm{Rep}(\bar{Q}, \delta) = T^*\mathrm{Rep}(Q, \delta) \longrightarrow \mathfrak{gl}(\delta)^*$ is the moment map. The variety $\mathfrak{M}_\theta(\delta)$ gives a minimal resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_l$,

$$\pi_\theta : \mathfrak{M}_\theta(\delta) \longrightarrow \mathbb{C}^2/\mathbb{Z}_l.$$

For any $\theta \in \mathbb{Z}^l_{reg}$, $\mathfrak{M}_\theta(\delta)$ is isomorphic to the toric variety $X(\Delta)$ defined in Section 3.2.

For $\lambda \in \mathbb{R}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$, we define the functor

$$\begin{aligned} \widehat{\Phi}^\theta_\lambda : \mathcal{T}_\lambda\text{-filt} &\longrightarrow \mathrm{Coh}(\mathfrak{M}_\theta(\delta)) \\ M &\mapsto \Big(\bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{gr}\big(\mathcal{B}^{m\theta}_\lambda \otimes_{\mathcal{A}_\lambda} e_0 M\big)\Big)^{\sim}. \end{aligned} \tag{1.5}$$

as in [Bo].

We define a locally free sheaf $\widetilde{\mathcal{P}}_\theta$ on $\mathfrak{M}_\theta(\delta)$ as follows

$$\widetilde{\mathcal{P}}_\theta = \Big(\bigoplus_{m \in \mathbb{Z}_{\geq 0}} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta), \chi^m_\theta}\Big)^{\sim}.$$

Then $\widetilde{\mathcal{P}}_\theta$ is a tautological bundle of the quiver variety $\mathfrak{M}_\theta(\delta)$. It is an analogue of the Procesi bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$. Although the structure as an algebraic variety of $\mathfrak{M}_\theta(\delta)$ is independent of θ, the tautological bundle $\widetilde{\mathcal{P}}_\theta$ depends on θ.

We have a construction of the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_l$ by the toric variety $X(\Delta)$ (Section 3.2). We prove the following proposition using this construction. This is an analogue of a result of [Ha2] for the Procesi bundle.

Proposition 1.3. *For $m \in \mathbb{Z}_{>0}$, we have*

$$H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)) = \begin{cases} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta), \chi^m_\theta} & (p = 0), \\ 0 & (p \neq 0). \end{cases}$$

where $\mathcal{O}(1)$ is the twisting sheaf of of $\mathfrak{M}_\theta(\delta)$ associated to the homogeneous coordinate ring S (see [Har, p.117]*).*

We make use of this proposition to calculate the (q,t)-dimension of the module $e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta), \chi^m_\theta}$, i.e., the character with respect to the $\mathbb{T}$-action.

Using this result, we obtain the following second main result. Set

$$\widetilde{\mathbb{R}}^l_{reg} = \{(\lambda_i)_{i=0,\dots,l-1} \in \mathbb{R}^l_{reg} \mid \lambda_i + \cdots + \lambda_{j-1} \neq 0 \quad \text{for all } i \neq j\}.$$

Theorem 1.4. *For $\lambda \in \widetilde{\mathbb{R}}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$, we have an isomorphism of coherent sheaves on $\mathfrak{M}_\theta(\delta)$*

$$\widehat{\Phi}^\theta_\lambda(\mathcal{T}_\lambda) \simeq \widetilde{\mathcal{P}}_\theta.$$

As a corollary of Theorem 1.4, we have

Corollary 1.5. *For $\lambda \in \widetilde{\mathbb{R}}^l_{reg}$, $\theta \in \mathbb{Z}^l_\lambda$, we have the isomorphism*

$$\widehat{\Phi}^\theta_\lambda(\Delta_\lambda(i)) \simeq \big(\widetilde{\mathcal{P}}_\theta / \widetilde{\mathcal{P}}_\theta \bar{A}^*\big) e_i$$

where $\bar{A}^* = \sum_{i=0}^{l-1} E_{i-1,i} \otimes \xi_i \in M_l(\mathbb{C}[\mu^{-1}(0)])$ and $\xi_i = \overline{\partial_i}$ is the image of ∂_i in $\mathrm{gr}\, D(\mathrm{Rep}(Q,\delta))^{GL(\delta)} \simeq \mathbb{C}[\mu^{-1}(0)]^{GL(\delta)}$.

1.4. The characteristic cycles of the standard modules

Finally, we determine the characteristic cycles of the standard modules.

The structure of the subvariety $\pi_\theta^{-1}(\{y=0\})$ is well-known: we have

$$\pi_\theta^{-1}(\{y=0\}) = \bigsqcup_{i=0}^{l-1} \mathcal{U}_i^0$$

where $\mathcal{U}_i^0$ is one-dimensional affine subvariety depending on θ defined by (3.6) in Section 3.1. We denote by $\mathcal{U}_i$ the closure of $\mathcal{U}_i^0$. Then, the irreducible components of $\pi_\theta^{-1}(\{y=0\})$ are $\mathcal{U}_0, \ldots, \mathcal{U}_{l-1}$.

The following proposition is the third main result.

Proposition 1.6. *For $i = 1, \ldots, l$, we have*

$$\mathrm{Ch}(\Delta_\lambda(\eta_i)) = \sum_{j=1}^{i} [\,\mathcal{U}_{\eta_j}\,].$$

where η_i is the index defined on (1.4).

This proposition answers a conjecture in [Go] in the case of $\mathbb{Z}_l$. Consider the geometric ordering defined in Section 5.4 of [Go],

$$i \vartriangleright_{geom,\theta} j \quad \text{if } i \neq j \text{ and } \mathcal{U}_i \cap \mathcal{U}_j^0 \neq \emptyset.$$

By the general theory of quiver varieties, if θ and θ' belong to the same alcove in $\mathbb{Z}^l_{reg}$, $\vartriangleright_{geom,\theta}$ is equal to $\vartriangleright_{geom,\theta'}$. Moreover, we show that the geometric ordering $\vartriangleright_{geom,\theta}$ is equal to the ordering $\vartriangleright_\theta$. Thus, we have

$$\eta_l \vartriangleright_{geom,\theta} \eta_{l-1} \vartriangleright_{geom,\theta} \cdots \vartriangleright_{geom,\theta} \eta_1.$$

Therefore, Proposition 1.6 is written as

$$\mathrm{Ch}(\Delta_\lambda(i)) = \sum_{j \trianglelefteq_{geom,\theta} i} [\,\mathcal{U}_j\,].$$

This is an affirmative answer to Question 10.2 of [Go].

The functor $\widehat{\Phi}_\lambda^\theta$ of (1.5) confirms a deep connection between the representation theory of $\mathcal{T}_\lambda$ and the geometry of $\mathfrak{M}_\theta(\delta)$. In fact, to prove Theorem 1.1 and Theorem 1.4, we make use of this connection. Although Proposition 1.2 is purely representation theoretical, the elements $\tilde{f}_i$ are obtained from the information on the geometry of $\mathfrak{M}_\theta(\delta)$.

1.5. Plan of paper

The paper is organized as follows. In Section 3.1, we recall the definition and basic facts about the quiver variety $\mathfrak{M}_\theta(\delta)$. In Section 3.2, we recall the construction of the minimal resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_l$ as a toric variety, and compare it with $\mathfrak{M}_\theta(\delta)$. In Section 3.3, we construct the tautological bundle

$\widetilde{\mathcal{P}}_\theta$. In Section 3.4, we prove Proposition 1.3 by using well-known facts about line bundles on toric varieties. In Section 3.5 and Section 3.6, we calculate the (q,t)-dimension of $\mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta),\chi_\theta}$. This result is used when we prove Theorem 1.4. In Section 4.1 and Section 4.2, we define the rational Cherednik algebra H_κ and the deformed preprojective algebra $\mathcal{T}_\lambda$, and recall their fundamental properties. In Section 4.3, we prepare some conditions for the parameters λ and θ, and define the ordering $\trianglerighteq_\theta$. In Section 4.4, we define the shift functor $\mathcal{S}_\lambda^\theta$ and prove Theorem 1.1. In Section 4.5, we calculate the q-dimension of $\mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0 \mathcal{T}_\lambda$. We use this result to prove Theorem 1.4. In Section 5, we recall the definition of the functor $\widehat{\Phi}_\lambda^\theta$. In Section 6.2, we prove Theorem 1.4. In Section 6.3, we prove Corollary 1.5 and Proposition 1.6.

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2. Preliminaries

2.1. Basic notations

Fix an integer $l \in \mathbb{Z}_{>0}$. We define two sets of parameters. Let $\mathbb{Z}_0^l$ be the set of $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l$ such that $\theta_0 + \cdots + \theta_{l-1} = 0$. Let $\mathbb{R}_1^l$ be the set of $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}^l$ such that $\lambda_0 + \cdots + \lambda_{l-1} = 1$.

Set $\gamma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ to be the element of $SL_2(\mathbb{C})$ where $\zeta = \exp(2\pi\sqrt{-1}/l)$. Let $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$ be the finite subgroup of $SL_2(\mathbb{C})$ generated by the element γ. Denote the group ring of $\mathbb{Z}_l$ over the field $\mathbb{C}$ by $\mathbb{C}\mathbb{Z}_l$. For $i = 0, \ldots, l-1$, let $\bar{e}_i$ be the idempotent $\bar{e}_i = (1/l) \sum_{j=0}^{l-1} \zeta^{ij} \gamma^j \in \mathbb{C}\mathbb{Z}_l$. Then we have $\bar{e}_i \bar{e}_j = \delta_{ij} \bar{e}_i$, and $\mathbb{C}\mathbb{Z}_l = \bigoplus_{i=0}^{l-1} \mathbb{C}\bar{e}_i$. For $i = 0, \ldots, l-1$, let $L_i = \mathbb{C}\mathbf{1}_i$ be the one-dimensional irreducible representation of $\mathbb{Z}_l$ on which $\bar{e}_j$ acts by $\bar{e}_j \mathbf{1}_i = \delta_{ij} \mathbf{1}_i$.

For a group G and a G-module M, we denote by M^G the G-invariant subspace of M. For a character χ of the group G, we denote by $M^{G,\chi}$ the semi-invariant subspace of M belonging to the character χ, i.e.,

$$M^{G,\chi} = \{v \in M \mid gv = \chi(g)v \quad (g \in G)\}.$$

For a $\mathbb{C}$-algebra R, let $M_l(R)$ be the $l \times l$ matrix algebra whose elements have coefficients in R, i.e.

$$M_l(R) \simeq M_l(\mathbb{C}) \otimes_{\mathbb{C}} R.$$

Let E_{ij} be the matrix in $M_l(R)$ such that the (i',j')-entry of E_{ij} is given by $\delta_{ii'}\delta_{jj'}$.

For an algebra R, denote by R-Mod the category of R-modules. Let R-mod be the full subcategory of R-Mod whose objects are finitely generated over R.

Fix an affine variety X. Let $\mathbb{C}[X]$ be the ring of polynomial functions on X. Let $D(X)$ be the ring of algebraic differential operators on X. For an $\mathcal{O}_X$-module $\mathcal{F}$ and a point $x \in X$, we denote the stalk of $\mathcal{F}$ at x by $\mathcal{F}_x$. We define its fiber at x by $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ where $\mathbb{C} = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and $\mathfrak{m}_x$ is a unique maximal ideal of $\mathcal{O}_{X,x}$. We denote by $\mathrm{Qcoh}(X)$ the category of quasi-coherent sheaves on X and by $\mathrm{Coh}(X)$ the category of coherent sheaves on X.

2.2. Quivers

A quiver $Q = (I,E)$ is a pair of a set of vertices I and a set of arrows E equipped with two maps in, $out : E \to I$. We assume I and E are finite sets. Let $Q^* = (I, E^*)$ be the quiver with the same set of vertices I and the set of arrows $E^* = \{\alpha^* \mid \alpha \in E\}$ where α^* is an arrow such that $in(\alpha^*) = out(\alpha)$ and $out(\alpha^*) = in(\alpha)$. Let $\bar{Q}$ be the quiver $(I, E \sqcup E^*)$.

A representation of a quiver $Q = (I,E)$ is a pair $(V, (\phi_\alpha)_{\alpha\in E})$ of an I-graded vector space $V = \bigoplus_{i\in I} V_i$ and a set of linear maps $\phi_\alpha : V_{out(\alpha)} \to V_{in(\alpha)}$. For an I-graded vector space $V = \bigoplus_{i\in I} V_i$, its dimension vector is $\underline{\dim} V = (\dim V_i)_{i\in I} \in (\mathbb{Z}_{\geq 0})^I$. For a dimension vector $v = (v_i)_{i\in I}$, the space of representation $\mathrm{Rep}(Q,v)$ is the following space:

$$\mathrm{Rep}(Q,v) = \bigoplus_{\alpha\in E} \mathrm{Hom}(\mathbb{C}^{v_{out(\alpha)}}, \mathbb{C}^{v_{in(\alpha)}}).$$

We identify a point $(\phi_\alpha)_{\alpha\in E}$ of $\mathrm{Rep}(Q,v)$ and a representation $(\bigoplus_{i\in I} \mathbb{C}^{v_i}, (\phi_\alpha)_{\alpha\in E})$.

Fix a dimension vector $v = (v_i)_{i\in I}$. Let $GL(v)$ be the Lie group $\prod_{i\in I} GL_{v_i}(\mathbb{C})$ and let $\mathfrak{gl}(v)$ be the Lie algebra of $GL(v)$: $\mathfrak{gl}(v) = \bigoplus_{i\in I} \mathfrak{gl}_{v_i}(\mathbb{C})$. The group $GL(v)$ acts naturally on $\mathrm{Rep}(Q,v)$.

Let $\langle\ ,\ \rangle : \mathbb{R}^I \times \mathbb{R}^I \longrightarrow \mathbb{R}$ be the bilinear form $\langle \lambda, \mu\rangle = \sum_{i\in I} \lambda_i\mu_i$ for $\lambda = (\lambda_i)_{i\in I}$, $\mu = (\mu_i)_{i\in I}$.

2.3. Quiver of type $A^{(1)}_{l-1}$

In this paper, we consider the McKay quiver associated to a group $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$ with cyclic orientation. In other words, it is a Dynkin quiver of type $A^{(1)}_{l-1}$ with cyclic orientation. Let $Q = (I,E)$ be a quiver with $I = \{I_0, \ldots, I_{l-1}\}$ as the set of vertices and $E = \{F_i : I_{i-1} \to I_i \mid i = 0, \ldots, l-1\}$ as the set of arrows. We regard indices for vertices and arrows as integers modulo l, i.e., we consider $I_{-1} = I_{l-1}$, $F_l = F_0$ and so on.

$$\begin{tikzcd} & & I_0 \arrow[dll, "F_1"'] & \\ I_1 \arrow[r, "F_2"'] & I_2 \arrow[rr, dotted] & & I_{l-1} \arrow[ul, "F_0"'] \end{tikzcd}$$

We identify $\mathbb{Z}^I = \mathbb{Z}^l$ and the root lattice of type $A_{l-1}^{(1)}$, and identify $\mathbb{R}^I = \mathbb{R}^l$ and the dual of the Cartan subalgebra. For $i = 0, \ldots, l-1$, let $\epsilon_i \in \mathbb{Z}^l$ be the standard coordinate vector corresponding to the vertex I_i. Under the above identification, we regard $\epsilon_0, \ldots, \epsilon_{l-1}$ as simple roots. Let $\delta = (1, \ldots, 1)$ be the minimal positive imaginary root. A root $\beta \in \mathbb{Z}^l$ is called a Dynkin root when $\langle \beta, \epsilon_0 \rangle = 0$. When a positive root $\beta = (\beta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l$ satisfies $\beta \neq \delta$ and $\beta_i \leq \delta_i = 1$ for all $i = 0, \ldots, l-1$, we write $\beta < \delta$. For $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}^I = \mathbb{R}^l$ (or $\mathbb{Z}^I = \mathbb{Z}^l$), we regard the sum $\lambda_i + \lambda_{i+1} + \cdots + \lambda_{j-1}$ as cyclic, i.e.,

$$\lambda_i + \lambda_{i+1} + \cdots + \lambda_{j-1} = \lambda_i + \cdots + \lambda_{l-1} + \lambda_0 + \lambda_1 + \cdots + \lambda_{j-1}$$

if $j < i$.

3. Quiver varieties

3.1. Definition of quiver varieties

In this subsection we review the definition and fundamental properties of quiver varieties which were introduced by Nakajima in [Na].

Define the quiver $Q = (I, E)$ as in Section 2.2. The space of representations

$$\mathrm{Rep}(\bar{Q}, \delta) = \{(a_i, b_i)_{i=0,\ldots,l-1} \mid a_i, b_i \in \mathbb{C}\} \simeq \mathbb{C}^{2l}$$

is a symplectic manifold with the symplectic form $\sum_{i=0}^{l-1} db_i \wedge da_i$. We have the symplectic action of $GL(\delta)$ on $\mathrm{Rep}(\bar{Q}, \delta)$, and the corresponding moment map is

$$\begin{aligned} \mu : \mathrm{Rep}(\bar{Q}, \delta) &\longrightarrow \mathfrak{gl}(\delta)^* \simeq \mathbb{C}^l, \\ (a_i, b_i)_{i=0,\ldots,l-1} &\mapsto (a_i b_i - a_{i+1} b_{i+1})_{i=0,\ldots,l-1}. \end{aligned}$$

Let $t_0, \ldots, t_{l-1}$ and $\xi_0, \ldots, \xi_{l-1} \in \mathbb{C}[\mathrm{Rep}(\bar{Q}, \delta)]$ be the coordinate functions such that $t_i((a_j, b_j)_{j=0,\ldots,l-1}) = a_i$ and $\xi_i((a_j, b_j)_{j=0,\ldots,l-1}) = b_i$ for $(a_j, b_j)_{j=0,\ldots,l-1} \in \mathrm{Rep}(\bar{Q}, \delta)$. Then we have

$$\mathbb{C}[\mu^{-1}(0)] = \mathbb{C}[t_0, \ldots, t_{l-1}, \xi_0, \ldots, \xi_{l-1}] \,/\, (t_i \xi_i - t_{i+1} \xi_{i+1})_{i=0,\ldots,l-1}.$$

The group $GL(\delta)$ acts on $\mathbb{C}[\mu^{-1}(0)]$ as follows.

$$\begin{aligned} g \cdot t_i &= g_i^{-1} g_{i-1} t_i, \\ g \cdot \xi_i &= g_i g_{i-1}^{-1} \xi_i \end{aligned}$$

for $i = 0, \ldots, l-1$ and $g = (g_k)_{k=0,\ldots,l-1} \in GL(\delta)$.

Fix a parameter $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}_0^l$ called a stability parameter. For a representation $(V, (a_i, b_i)_{i=0,\ldots,l-1})$ of $\bar{Q}$ with the dimension vector δ, we call it θ-semistable if $\langle \underline{\dim} W, \theta \rangle \leq 0$ for any subrepresentation W of V. Define the subset $\mu^{-1}(0)_\theta$ of $\mu^{-1}(0) \subset \mathrm{Rep}(\bar{Q}, \delta)$:

$$\mu^{-1}(0)_\theta = \big\{(a_i, b_i)_{i=0,\ldots,l-1} \in \mu^{-1}(0) \;\big|\; (a_i, b_i)_{i=0,\ldots,l-1} \text{ is a } \theta\text{-semistable representation.}\big\}.$$

It is a Zariski open subset of $\mu^{-1}(0)$. For p, $q \in \mu^{-1}(0)_\theta$, we denote $p \sim q$ when the closures of $GL(\delta)$-orbits intersect in $\mu^{-1}(0)_\theta$. Then $\sim$ is an equivalence relation. Then we define the quiver variety $\mathfrak{M}_\theta(\delta)$ as follows:

$$\mathfrak{M}_\theta(\delta) = \mu^{-1}(0)_\theta / \sim .$$

For a point $(a_i, b_i)_{i=0,\dots,l-1} \in \mu^{-1}(0)_\theta$, we denote by $[a_i, b_i]_{i=0,\dots,l-1}$ the corresponding point of $\mathfrak{M}_\theta(\delta)$.

Remark 3.1. Although our definition of the quiver variety is different from one in [Na], $\mathfrak{M}_\theta(\delta)$ coincides with $\mathfrak{M}_{(\zeta_\mathbb{C},\zeta_\mathbb{R})}(\mathbf{v}, \mathbf{w})$ with $\zeta_\mathbb{C} = 0$, $\zeta_\mathbb{R} = \theta$, $\mathbf{v} = \delta$ and $\mathbf{w} = \epsilon_0$. This definition is the same as one of [Na2, Section 4].

For $\theta = 0 = (0, \dots, 0)$, we have

$$\mathfrak{M}_0(\delta) \simeq \mathbb{C}^2/\mathbb{Z}_l$$

(see [CS]).

Proposition 3.2 ([Kr], [Na2]). *If a stability parameter $\theta = (\theta_i)_{i=0,\dots,l-1} \in \mathbb{Z}_0^l$ satisfies $\langle\theta, \beta\rangle \neq 0$ for all positive roots β which satisfy $\beta < \delta$, $\mathfrak{M}_\theta(\delta)$ is nonsingular and we have a minimal resolution of Kleinian singularities of type A_{l-1}:*

$$\pi_\theta : \mathfrak{M}_\theta(\delta) \longrightarrow \mathfrak{M}_0(\delta) \simeq \mathbb{C}^2/\mathbb{Z}_l.$$

In this paper, we always consider the case when the stability parameter $\theta \in \mathbb{Z}_0^l$ satisfies $\langle\theta, \beta\rangle \neq 0$ for all positive roots β which satisfy $\beta < \delta$. Set

(3.1)
$$\mathbb{Z}^l_{reg} = \left\{\theta \in \mathbb{Z}_0^l \;\middle|\; \langle\theta, \beta\rangle \neq 0 \quad \text{for all positive roots } \beta \text{ which satisfy } \beta < \delta\right\}.$$

For a stability parameter $\theta = (\theta_i)_{i=0,\dots,l-1} \in \mathbb{Z}^l_{reg}$, define the following graded commutative algebra

$$S = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} S_m,$$
$$S_m = \mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi_\theta^m},$$

where χ_θ is the character of $GL(\delta)$ given by $\chi_\theta(g) = \prod_{i=0}^{l-1}(g_i)^{\theta_i}$ for $g = (g_i)_{i=0,\dots,l-1} \in GL(\delta)$. The injective homomorphism $S_0 \to S$ induces the morphism of schemes

$$\operatorname{Proj} S \longrightarrow \operatorname{Spec} S_0 \simeq \mathbb{C}^2/\mathbb{Z}_l.$$

We have the following construction of quiver varieties.

Proposition 3.3 ([CS], [Na2]). *As schemes over $\mathbb{C}^2/\mathbb{Z}_l$, we have the following isomorphism:*

$$\mathfrak{M}_\theta(\delta) \simeq \operatorname{Proj} S.$$

The above construction induces the twisting sheaf on the scheme $\mathfrak{M}_\theta(\delta) \simeq \operatorname{Proj} S$ which we denote by $\mathcal{O}(1)$ (see [Har, p.117]).

Fix a stability parameter $\theta = (\theta_i)_{i=0,\dots,l-1} \in \mathbb{Z}^l_{reg}$. The two-dimensional torus $\mathbb{T} = (\mathbb{C}^*)^2$ acts on the quiver variety $\mathfrak{M}_\theta(\delta)$ as follows: for $[a_i, b_i]_{i=0,\dots,l-1} \in \mathfrak{M}_\theta(\delta)$ and $(m_1, m_2) \in \mathbb{T}$,

$$(m_1, m_2)[a_i, b_i]_{i=0,\dots,l-1} = [m_1 a_i, m_2 b_i]_{i=0,\dots,l-1}.$$

The group $\mathbb{T}$ acts on $\mathbb{C}^2$ by $(m_1, m_2)(a, b) \mapsto (m_1 a, m_2 b)$ and it induces the $\mathbb{T}$-action on $\mathbb{C}^2/\mathbb{Z}_l$. Moreover π_θ is $\mathbb{T}$-equivariant. The variety $\mathfrak{M}_\theta(\delta)$ has l $\mathbb{T}$-fixed points $p_0, \dots, p_{l-1}$ where $p_i = [a_j, b_j]_{j=0,\dots,l-1}$ is given as follows:

$$\begin{aligned} &a_i = 0, \quad b_i = 0, \\ &a_j = 0, \quad b_j \neq 0 \quad \text{if } \theta_i + \theta_{i+1} + \cdots + \theta_{j-1} < 0, \\ &a_j \neq 0, \quad b_j = 0 \quad \text{if } \theta_i + \theta_{i+1} + \cdots + \theta_{j-1} > 0. \end{aligned} \tag{3.2}$$

Note that we have $\theta_i + \theta_{i+1} \cdots + \theta_{j-1} \neq 0$ for all $i \neq j$ by the assumption $\theta \in \mathbb{Z}^l_{reg}$.

Define the ordering $\trianglerighteq_{geom,\theta}$ on the set $\Lambda = \{0, \dots l-1\}$ by

$$i \triangleright_{geom,\theta} j \iff \theta_i + \cdots + \theta_{j-1} < 0. \tag{3.3}$$

Since we take $\theta \in \mathbb{Z}^l_{reg}$, the ordering $\trianglerighteq_{geom,\theta}$ is a total ordering.

Set $\eta_1, \dots, \eta_l$ be the indices in Λ such that

$$\eta_l \triangleright_{geom,\theta} \eta_{l-1} \triangleright_{geom,\theta} \cdots \triangleright_{geom,\theta} \eta_1. \tag{3.4}$$

By (3.2) and (3.3), for $p_{\eta_i} = [a_j, b_j]_{j=0,\dots,l-1}$, we have

$$\#\{j \in \Lambda \mid b_j \neq 0\} = i - 1. \tag{3.5}$$

Proposition 3.4. *For $i = 1, \dots, l$, the fixed point $p_{\eta_i} = [a_j, b_j]_{j=0,\dots,l-1}$ is given by*

$$\begin{aligned} &a_{\eta_i} = 0, \quad b_{\eta_i} = 0, \\ &a_{\eta_j} = 0, \quad b_{\eta_j} \neq 0 \qquad \textit{for } j < i, \\ &a_{\eta_j} \neq 0, \quad b_{\eta_j} = 0 \qquad \textit{for } j \geq i. \end{aligned}$$

Proof. By (3.3) and (3.4), we have

$$\theta_{\eta_i} + \cdots + \theta_{\eta_j - 1} < 0 \qquad \text{for } j = 1, \dots, i-1.$$

Thus we have

$$a_{\eta_j} = 0, \quad b_{\eta_j} \neq 0 \qquad \text{for } j < i.$$

By (3.5), we have

$$a_{\eta_j} \neq 0, \quad b_{\eta_j} = 0 \qquad \text{for } j \geq i.$$

□

For $i = 0, \ldots, l-1$, we define the following one-dimensional affine subvariety of $\mathfrak{M}_\theta(\delta)$

(3.6)

$$\mathcal{U}_i^0 = \left\{ [a_j, b_j]_{j=0,\ldots,l-1} \;\middle|\; \begin{array}{ll} b_i = 0, & \\ a_j = 0, b_j \neq 0 & \text{if } \theta_i + \cdots + \theta_{j-1} < 0, \\ a_j \neq 0, b_j = 0 & \text{if } \theta_i + \cdots + \theta_{j-1} > 0. \end{array} \right\} \subset \mathfrak{M}_\theta(\delta).$$

Clearly $p_i \in \mathcal{U}_i^0$ for all $i = 0, \ldots, l-1$. We denote by $\mathcal{U}_i$ the closure of $\mathcal{U}_i^0$. By Proposition 3.4, $\mathcal{U}_{\eta_i}$ is given as follows.

$$\mathcal{U}_{\eta_i}^0 = \left\{ [a_j, b_j]_{j=0,\ldots,l-1} \;\middle|\; \begin{array}{ll} b_{\eta_i} = 0, & \\ a_{\eta_j} = 0,\ b_{\eta_j} \neq 0 & \text{for } j < i, \\ a_{\eta_j} \neq 0,\ b_{\eta_j} = 0 & \text{for } j \geq i. \end{array} \right\}. \tag{3.7}$$

The ordering $\trianglerighteq_{geom,\theta}$ is related with the $\mathbb{T}$-action on $\mathfrak{M}_\theta(\delta)$ as follows. We denote $p_i \to p_j$ when there is a point $[a_k, b_k]_{k=0,\ldots,l-1} \in \pi_\theta^{-1}(0)$ such that

$$\lim_{m \to 0} (m^{-1}, m)[a_k, b_k]_{k=0,\ldots,l-1} = p_i,$$

and

$$\lim_{m \to 0} (m, m^{-1})[a_k, b_k]_{k=0,\ldots,l-1} = p_j.$$

By Proposition 3.4 and (3.7), taking $[a_k, b_k]_{k=0,\ldots,l-1}$ on $\mathcal{U}_{\eta_i}$, we have $p_{\eta_i} \to p_{\eta_{i-1}}$ for each $i = 2, \ldots, l$.

The structure of the subvariety $\pi_\theta^{-1}(\{y = 0\})$ is well-known (see [Sl, Lecture 1]). The subvariety $\pi_\theta^{-1}(\{y = 0\})$ is the disjoint union of $\mathcal{U}_i^0$:

$$\pi_\theta^{-1}(\{y = 0\}) = \bigsqcup_{i=0}^{l-1} \mathcal{U}_i^0$$

The irreducible components of $\pi_\theta^{-1}(\{y = 0\})$ are $\mathcal{U}_0, \ldots, \mathcal{U}_{l-1}$.

For $i = 1, \ldots, l-1$, we have

$$\mathcal{U}_{\eta_{i+1}} \cap \mathcal{U}_{\eta_i}^0 = \{p_{\eta_i}\}$$

and $\mathcal{U}_{\eta_j}$ does not intersect with $\mathcal{U}_{\eta_i}$ unless $j = i+1, i, i-1$.

3.2. Quiver varieties vs. toric varieties

In this subsection, we compare two constructions of the minimal resolution of Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_l$: i.e. as a quiver variety and as a toric variety.

Let $N = \mathbb{Z}^2$, $M = \mathrm{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^2$ and let

$$\langle\ ,\ \rangle : M \times N \longrightarrow \mathbb{Z}$$

be the natural pairing. Set $v_i = (1, i) \in N$ for $0 \leq i \leq l$. Let $\sigma_i = \mathbb{R}_{\geq 0} v_i + \mathbb{R}_{\geq 0} v_{i-1}$ be a 2-dimensional cone for $i = 1, \ldots, l$. Let Δ be the fan with the 2-dimensional cones σ_i for $i = 1, \ldots, l$ and the 1-dimensional cones $\mathbb{R}_{\geq 0} v_i$ for

$i = 0, \ldots, l$. Then the toric variety $X(\Delta)$ associated to the fan Δ gives the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_l$:

$$X(\Delta) \longrightarrow \mathbb{C}^2/\mathbb{Z}_l. \tag{3.8}$$

For $i = 1, \ldots, l$, let $M_i = M \cap \check{\sigma}_i$ be the semigroup where $\check{\sigma}_i = \mathbb{R}_{\geq 0}(i, -1) + \mathbb{R}_{\geq 0}(1-i, 1)$ is the dual cone of σ_i. Let $R_i = \mathbb{C}M_i$ be the group ring of M_i and let $X_i = \operatorname{Spec} R_i$. The toric variety $X(\Delta)$ has the open covering $X(\Delta) = \bigcup_{i=1}^{l} X_i$.

Let $u = (1, 0)$, $v = (0, 1)$ be the basis of the lattice M. Then, $R_i = \mathbb{C}[u^i v^{-1}, u^{1-i} v]$. Let $xy = u$, $y^l = v$. Then we have

$$R_i = \mathbb{C}[x^i y^{i-l}, x^{1-i} y^{l+1-i}].$$

The natural embedding $\mathbb{C}[x, y]^{\mathbb{Z}_l} \to R_i$ induces the morphism $X_i \to \mathbb{C}^2/\mathbb{Z}_l$ of (3.8).

Let $\mathfrak{m}_i = (x^i y^{i-l}, x^{1-i} y^{l+1-i}) \subset R_i$ be the maximal ideal of R_i. It is the maximal ideal corresponding to the unique $\mathbb{T}$-fixed point in X_i.

Fix a stability parameter $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l_{reg}$. We consider the quiver variety $\mathfrak{M}_\theta(\delta)$ defined in Section 3.1. By Proposition 3.2, we also have the minimal resolution

$$\mathfrak{M}_\theta(\delta) \longrightarrow \mathbb{C}^2/\mathbb{Z}_l.$$

Thus we have an isomorphism of algebraic varieties

$$\mathfrak{M}_\theta(\delta) \simeq X(\Delta).$$

We construct this isomorphism explicitly.

For $i = 1, \ldots, l$, let

$$R'_i = \mathbb{C}\left[\frac{t_{\eta_{l-i+1}} \cdots t_{\eta_l}}{\xi_{\eta_1} \cdots \xi_{\eta_{l-i}}}, \frac{\xi_{\eta_1} \cdots \xi_{\eta_{l-i+1}}}{t_{\eta_{l-i+2}} \cdots t_{\eta_l}}\right]$$

where $\eta_1, \ldots, \eta_l$ are the indices defined on (3.4). Note that the polynomials in R'_i have no poles at the fixed point $p_{\eta_{l-i+1}}$ by Proposition 3.4.

Then we have an open covering

$$\mathfrak{M}_\theta(\delta) = \bigcup_{i=1}^{l} X'_i, \qquad X'_i = \operatorname{Spec} R'_i.$$

Thus, for $i = 1, \ldots, l$, we define an isomorphism,

$$\begin{aligned} &R'_i \longrightarrow R_i, \\ &t_j \mapsto x \qquad (j = 0, \ldots, l-1), \\ &\xi_j \mapsto y \qquad (j = 0, \ldots, l-1). \end{aligned}$$

This induces the isomorphism of algebraic varieties

$$\mathfrak{M}_\theta(\delta) \longrightarrow X(\Delta). \tag{3.9}$$

Set

$$\mathfrak{m}'_i = \left(\frac{t_{\eta_{l-i+1}} \cdots t_{\eta_l}}{\xi_{\eta_1} \cdots \xi_{\eta_{l-i}}}, \frac{\xi_{\eta_1} \cdots \xi_{\eta_{l-i+1}}}{t_{\eta_{l-i+2}} \cdots t_{\eta_l}} \right). \tag{3.10}$$

This is a maximal ideal of R'_i. This ideal corresponds to the $\mathbb{T}$-fixed point $p_{\eta_{l-i+1}} \in X'_i$.

3.3. Tautological bundles

In this subsection, we define tautological bundles on the quiver variety $\mathfrak{M}_\theta(\delta)$ and give their explicit construction.

A tautological bundle is defined as follows. Consider a vector bundle of rank l on $\mu^{-1}(0)_\theta$ whose fiber is isomorphic to the representation of $\bar{Q}$ given by $(a_i, b_i)_{i=0,\dots,l-1}$ for each point $(a_i, b_i)_{i=0,\dots,l-1} \in \mu^{-1}(0)_\theta$. If the vector bundle descends to a vector bundle $\widetilde{\mathcal{P}}_\theta$ on $\mathfrak{M}_\theta(\delta)$, we call $\widetilde{\mathcal{P}}_\theta$ a tautological bundle.

To construct a tautological bundle, consider the matrix algebra

$$M_l(\mathbb{C}[\mu^{-1}(0)]) \simeq M_l(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\mu^{-1}(0)].$$

The group $GL(\delta)$ acts on $M_l(\mathbb{C})$ by

$$g \cdot E_{ij} = g_i g_j^{-1} E_{ij} \tag{3.11}$$

for $g = (g_k)_{k=0,\dots,l-1} \in GL(\delta)$ and $0 \le i,\, j \le l-1$. The group $GL(\delta)$ acts on $\mathbb{C}[\mu^{-1}(0)]$. Thus $GL(\delta)$ acts on $M_l(\mathbb{C}[\mu^{-1}(0)])$. We define a graded S-module

$$\mathcal{P}_\theta = \bigoplus_{m \in \mathbb{Z}_{\ge 0}} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta), \chi_\theta^m}. \tag{3.12}$$

where $e_0 = E_{00}$. Let $\widetilde{\mathcal{P}}_\theta$ be the sheaf associated to $\mathcal{P}_\theta$. We show that $\widetilde{\mathcal{P}}_\theta$ is a direct sum of l line bundles. For $i = 0, \dots, l-1$ let $\widetilde{\mathcal{L}}_i$ be the sheaf associated to the following graded S-module $\mathcal{L}_i$,

$$\mathcal{L}_i = \bigoplus_{m \in \mathbb{Z}_{\ge 0}} \mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi_\theta^m \chi_{\tau_i}} \tag{3.13}$$

where $\tau_i = \epsilon_i - \epsilon_0 \in \mathbb{Z}_0^l$. Consider the i-th column of $\mathcal{P}_\theta$. The group $GL(\delta)$ acts on $E_{0i} \in M_l(\mathbb{C})$ by the character $\chi_{\tau_i}^{-1}$ by (3.11). Thus the coefficients of the i-th column of $\mathcal{P}_\theta$ coincide with $\mathcal{L}_i$. Then, each $\widetilde{\mathcal{L}}_i$ is a line bundle on $\mathfrak{M}_\theta(\delta)$ and we have

$$\widetilde{\mathcal{P}}_\theta = \bigoplus_{i=0}^{l-1} \widetilde{\mathcal{L}}_i.$$

Note that $\mathcal{P}_\theta$ is a right module of the matrix algebra $M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta)}$. Let A_i and $\bar{A}_i^*$ be the following elements of $M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta)}$: $A_i = E_{i,i-1} \otimes t_i$ and $\bar{A}_i^* = E_{i-1,i} \otimes \xi_i$. Then the collection of maps $(A_i, \bar{A}_i^*)_{i=0,\dots,l-1}$ gives an action of $\bar{Q}$ on $\widetilde{\mathcal{P}}_\theta$. Thus we have the following proposition.

Proposition 3.5. *The vector bundle $\widetilde{\mathcal{P}}_\theta$ is a tautological bundle on $\mathfrak{M}_\theta(\delta)$.*

Lemma 3.6. *The module $\mathbb{C}[\mu^{-1}(0)]^{GL(\delta),\chi_\theta^m\chi_{\tau_i}}$ is a $\mathbb{C}[t_0\cdots t_{l-1}]$-free module.*

Proof. Consider the grading of the $\mathbb{C}[t_0\cdots t_{l-1}]$-module $\mathbb{C}[\mu^{-1}(0)]^{GL(\delta),\chi_\theta^m\chi_{\tau_i}}$ defined by the degree

$$\deg t_k = 0, \qquad \deg \xi_k = 1.$$

Since $\mu^{-1}(0) \not\subseteq \bigcup_{i=0}^{l-1}\{t_i = 0\}$, the $\mathbb{C}[t_0\cdots t_{l-1}]$-module $\mathbb{C}[\mu^{-1}(0)]$ is torsion free. Thus, each component with respect to the above grading is a finitely generated torsion free $\mathbb{C}[t_0\cdots t_{l-1}]$-module. The algebra $\mathbb{C}[t_0\cdots t_{l-1}]$ is a one-dimensional polynomial algebra. Therefore a finitely generated $\mathbb{C}[t_0\cdots t_{l-1}]$-torsion-free module is automatically $\mathbb{C}[t_0\cdots t_{l-1}]$-free. □

3.4. Vanishing of higher cohomologies

In the previous subsection, we constructed the tautological bundle $\widetilde{\mathcal{P}}_\theta$. To calculate the higher cohomologies of $\widetilde{\mathcal{P}}_\theta$, we recall well-known facts about line bundles on the toric variety $X(\Delta) \simeq \mathfrak{M}_\theta(\delta)$. Let $\mathrm{Pic}(X(\Delta))$ be the Picard group of $X(\Delta)$. Let D_i be the divisor of $X(\Delta)$ corresponding to $v_i \in N$ for $i = 0, \ldots, l$ as in [Fu, Sec 3.3]. Under the isomorphism (3.9), D_i corresponds to $\mathcal{U}_{\eta_i}$ defined by (3.6) for $i = 1, \ldots, l$. By the general theory of toric varieties, we have the following lemma.

Lemma 3.7 ([Mu] (2.3), [Fu] Prop 3.4). *The Picard group* $\mathrm{Pic}(X(\Delta))$ *is generated by the divisors $D_0, \ldots, D_l$. Moreover their relations in* $\mathrm{Pic}(X(\Delta))$ *are given by*:

$$D_0 + D_1 + \cdots + D_l = 0,$$
$$\sum_{i=1}^{l} iD_i = 0.$$

For $i = 1, \ldots, l-1$, we define the cycle $D(i) = \sum_{j=0}^{i-1}(i-j)D_{l-j} \in \mathrm{Pic}(X(\Delta))$. We have $\mathrm{Pic}(X(\Delta)) = \bigoplus_{i=1}^{l-1}\mathbb{Z}D(i)$.

For $b = (b_i)_{i=1,\ldots,l-1} \in \mathbb{Z}^{l-1}$, let $D(b) = \sum_{i=1}^{l-1} b_i D(i) \in \mathrm{Pic}(X(\Delta))$ as in [Mu]. For each divisor $D \in \mathrm{Pic}(X(\Delta))$, we have the $\mathbb{T}$-invariant line bundle $\mathcal{O}(D)$ on $X(\Delta)$. The following two lemmas are proved in [Mu].

Lemma 3.8 ([Mu], Lemma 2.4). *When $b = (b_k)_{k=1,\ldots,l-1} \in \mathbb{Z}_{\geq 0}^{l-1}$, the space of local sections $H^0(X_j, \mathcal{O}(D(b)))$ is a free R_j-module generated by the element $x^{\sum_{k=l-j+1}^{l-1}(l-k)b_k}y^{-\sum_{k=l-j+1}^{l-1}kb_k}$.*

Proposition 3.9 ([Mu], Lemma 2.1). *If $b = (b_k)_{k=1,\ldots,l-1} \in \mathbb{Z}_{\geq 0}^{l-1}$, then we have*

$$H^p(X(\Delta), \mathcal{O}(D(b))) = 0$$

for $p \neq 0$.

For $b=(b_k)_{k=1,\dots,l-1}\in(\mathbb{Z}_{\geq 0})^{l-1}$, let $\mathcal{O}'(D(b))$ be the line bundle on $X(\Delta)$ such that $H^0(X_j,\mathcal{O}'(D(b)))$ is the free R_j-module generated by the element $x^{\sum_{k=l-j+1}^{l-1}(l-k)b_k}y^{\sum_{k=1}^{l-j}kb_k}$. Namely, as a $\mathbb{T}$-equivariant line bundle

$$\mathcal{O}'(D(b))\simeq\mathcal{O}(D(b))\otimes_{\mathbb{C}}\mathbb{C}_{(0,\sum_{k=1}^{l-1}kb_k)}$$

where $\mathbb{C}_{(a,b)}$ is the one-dimensional vector space with the $\mathbb{T}$-action of weight (a,b).

Fix a stability parameter $\theta=(\theta_i)_{i=0,\dots,l-1}\in\mathbb{Z}^l_{reg}$. In Section 3.3, we defined the line bundle $\widetilde{\mathcal{L}}_i$ on the quiver variety $\mathfrak{M}_\theta(\delta)$ for $i=0,\dots,l-1$ by (3.13).

For $i=0,\dots,l-1$ and $m\in\mathbb{Z}_{\geq 0}$, set

$$b_k^{\theta'}=\theta'_{\eta_k}+\theta'_{\eta_k+1}+\cdots+\theta'_{\eta_{k+1}-1}. \tag{3.14}$$

where $\theta'=(\theta'_k)_{k=0,\dots,l-1}=m\theta+\tau_i\in\mathbb{Z}^l_0$. Note that we have $b_k^{\theta'}\in\mathbb{Z}_{\geq 0}$ for all k.

For $\theta'=(\theta'_k)_{k=0,\dots,l-1}=m\theta+\tau_i\in\mathbb{Z}^l$ where $m\in\mathbb{Z}_{\geq 0}$ and $i=0,\dots,l-1$, set

$$f_j^{\theta'}=\prod_{k=j}^{l-1}(t_{\eta_{k+1}}\cdots t_{\eta_l})^{b_k^{\theta'}}\prod_{k=1}^{j-1}(\xi_{\eta_1}\cdots\xi_{\eta_k})^{b_k^{\theta'}}. \tag{3.15}$$

Note that $f_j^{\theta'}$ does not vanish at the fixed point p_{η_j} by Proposition 3.4.

We show that $f_j^{\theta'}$ belongs to $\mathbb{C}[\mu^{-1}(0)]^{GL(\delta),\chi_{\theta'}}$. We calculate the weight of the function $f_j^{\theta'}$. Because the weight of $\xi_{\eta_1}\cdots\xi_{\eta_k}$ is equal to the weight of $t_{\eta_{k+1}}\cdots t_{\eta_l}$ for all k, the weight of $f_j^{\theta'}$ is independent of j. By $\theta'_0+\cdots+\theta'_{l-1}=0$, we have

$$\begin{aligned}&b_k^{\theta'}+b_{k+1}^{\theta'}+\cdots+b_{l-1}^{\theta'}\\&\quad=(\theta'_{\eta_k}+\cdots+\theta'_{\eta_{k+1}-1})+(\theta'_{\eta_{k+1}}+\cdots+\theta'_{\eta_{k+2}-1})+\cdots+(\theta'_{\eta_{l-1}}+\cdots+\theta'_{\eta_l-1})\\&\quad=\theta'_{\eta_k}+\theta'_{\eta_k+1}+\cdots+\theta'_{\eta_l-1}\end{aligned}$$

Thus, we have

$$f_l^{\theta'}=\prod_{k=1}^{l-1}(\xi_{\eta_1}\cdots\xi_{\eta_k})^{b_k^{\theta'}}=\prod_{k=0,\dots,l-1}\xi_k^{\theta'_k+\theta'_{k+1}+\cdots+\theta'_{\eta_l-1}}.$$

Thus the weight of $f_l^{\theta'}$ is

$$\begin{aligned}&\sum_{k=0}^{l-1}\{(\theta'_k+\cdots+\theta'_{\eta_l-1})\epsilon_k-(\theta'_k+\cdots+\theta'_{\eta_l-1})\epsilon_{k-1}\}\\&\qquad=\sum_{k=0}^{l-1}\{(\theta'_k+\cdots+\theta'_{\eta_l-1})-(\theta'_{k+1}+\cdots+\theta'_{\eta_l-1})\}\epsilon_k=\sum_{k=0}^{l-1}\theta'_k\epsilon_k.\end{aligned}$$

Therefore, $f_j^{\theta'}$ belongs to $\mathbb{C}[\mu^{-1}(0)]^{GL(\delta),\chi_{\theta'}}$ for all j.

Lemma 3.10. *For $1 \le j \le l$ and $0 \le i \le l-1$ and $m \in \mathbb{Z}_{>0}$, $H^0(X'_j, \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m))$ is the free R'_j-module with the generator $f^{\theta'}_{l+1-j}$ with $\theta' = m\theta + \tau_i \in \mathbb{Z}^l_0$.*

Proof. By the definition of $\widetilde{\mathcal{L}}_i$ (3.13), Laurent monomials of $t_0, \ldots, t_{l-1}$, $\xi_0, \ldots, \xi_{l-1}$ with the $GL(\delta)$-character $\chi_{\theta'}$ span $H^0(X'_j, \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m))$ over $\mathbb{C}$.

Since the unique fixed point $p_{\eta_{l-j+1}} \in X'_j$ corresponds to the maximal ideal $\mathfrak{m}'_j \subset R'_j$ defined by (3.10), the generators of $H^0(X'_j, \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m))$ must have no zero at $p_{\eta_{l-j+1}}$.

The Laurent monomial $f^{\theta'}_j$ has no zero and no pole at $p_{\eta_{l-j+1}}$ by Proposition 3.4. On the other hand, let g be a Laurent monomial with the $GL(\delta)$-character $\chi_{\theta'}$. Then, $g/f^{\theta'}_j$ is a product of the following Laurent monomials:

$$(t_0 \cdots t_{l-1})^{\pm 1}, \quad (\xi_0 \cdots \xi_{l-1})^{\pm 1}, \quad (t_0\xi_0)^{\pm 1},$$
$$\left(\frac{t_p t_{p+1} \cdots t_{q-1}}{\xi_q \xi_{q+1} \cdots \xi_{p-1}}\right)^{\pm 1} \qquad \text{for } p \neq q.$$

Therefore, g has either zeros or poles at $p_{\eta_{l-j+1}}$ by Proposition 3.4.

Therefore, the Laurent monomials other than $f^{\theta'}_j$ have either zeros or poles at $p_{\eta_{l-j+1}}$, and $f^{\theta'}_j$ is the generator of $H^0(X'_j, \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m))$ over R'_j. $\square$

We identify $\mathfrak{M}_\theta(\delta)$ and $X(\Delta)$ by the isomorphism (3.9).

Proposition 3.11. *For $i = 0, \ldots, l-1$ and $m \in \mathbb{Z}_{>0}$, we have an isomorphism of $\mathbb{T}$-equivariant line bundles on $\mathfrak{M}_\theta(\delta) \simeq X(\Delta)$: $\widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m) \simeq \mathcal{O}'(D(b))$ where $b = (b^{\theta'}_j)_{j=1,\ldots,l-1}$ and $\theta' = m\theta + \tau_i$.*

Proof. For $j = 1, \ldots, l$, $H^0(X'_j, \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m))$ is a free R'_j-module with the generator $f^{\theta'}_{l-j+1}$. On the other hand $H^0(X_j, \mathcal{O}'(D(b)))$ is a free R_j-module with the generator $x^{\sum_{k=l-j+1}^{l-1}(l-k)b^{\theta'}_k} y^{\sum_{k=1}^{l-j} k b^{\theta'}_k}$. Thus the map given by $t_k \mapsto x$, $\xi_k \mapsto y$ is a $\mathbb{T}$-equivariant isomorphism $\widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m) \simeq \mathcal{O}'(D(b))$. $\square$

By the general theory of toric varieties, we have the following $\mathbb{C}$-basis of $H^0(X(\Delta), \mathcal{O}(D(b)))$ for $b = (b_i)_{i=1,\ldots,l-1} \in \mathbb{Z}^{l-1}_{\geq 0}$. As in [Fu, page 66], set

$$P_{D(b)} = \{m \in M \mid \langle m, v_i \rangle \geq -a_i \quad \text{for } i = 0, \ldots, l\}.$$

where $a_i = b_{l-i+1} + 2b_{l-i+2} + \cdots + (i-1)b_{l-1}$. Then we have

$$H^0(X(\Delta), \mathcal{O}(D(b))) = \bigoplus_{m \in P_{D(b)} \cap M} \mathbb{C} x^{m_1} y^{m_1 + l m_2}.$$

The following lemma is an immediate consequence of this fact.

Lemma 3.12. *As a $\mathbb{C}[x^l, xy]$-module, we have*

$$H^0(X(\Delta), \mathcal{O}'(D(b))) = \sum_{i=1}^{l} \sum_{m=0}^{b_i - 1} \mathbb{C}[x^l, xy] x^{\sum_{j=i+1}^{l}(l-j)b_j + (l-i)(b_i - m)} y^{\sum_{j=1}^{i-1} j b_j + im}$$

where we set $b_l = \infty$*.*

Now we have the following proposition.

Proposition 3.13. *For* $i = 0, \ldots, l-1$ *and* $m \in \mathbb{Z}_{>0}$*, we have*

$$H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m)) = \begin{cases} \mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi_\theta^m \chi_{\tau_i}} & (p = 0), \\ 0 & (p \neq 0). \end{cases}$$

Proof. Set $b = (b_j^{\theta'})_{j=1,\ldots,l-1}$ with $\theta' = m\theta + \tau_i$. By Proposition 3.11, we have an isomorphism of line bundles $\widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m) \simeq \mathcal{O}'(D(b))$. Therefore, by Proposition 3.9, we have the vanishing of the higher cohomologies

$$H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m)) = 0$$

for $p \neq 0$. By the definition of $\widetilde{\mathcal{L}}_i$ at (3.13), it is clear that

$$\mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi_\theta^m \chi_{\tau_i}} \subseteq H^0(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m)).$$

We show the opposite inclusion. By Lemma 3.12, we have

$$\begin{aligned} &H^0(X(\Delta), \mathcal{O}'(D(b))) \\ &= \sum_{k=1}^{l} \sum_{n=0}^{b_k^{\theta'}-1} \mathbb{C}[x^l, xy] x^{\sum_{j=k+1}^{l}(l-j)b_j^{\theta'} + (l-k)(b_k^{\theta'}-n)} y^{\sum_{j=1}^{k-1} j b_j^{\theta'} + kn}. \end{aligned} \tag{3.16}$$

On the other hand, we consider the elements $g_k^{\theta'}(n) \in \mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi_\theta^m \chi_{\tau_i}}$

$$g_k^{\theta'}(n) = (t_{\eta_{k+1}} \cdots t_{\eta_l})^{b_k^{\theta'}-n} \prod_{j=k+1}^{l-1} (t_{\eta_{j+1}} \cdots t_{\eta_l})^{b_j^{\theta'}} (\xi_{\eta_1} \cdots \xi_{\eta_k})^n \prod_{j=1}^{k-1} (\xi_{\eta_1} \cdots \xi_{\eta_j})^{b_j^{\theta'}}$$

for $k = 1, \ldots, l$ and $n = 0, \ldots, b_k^{\theta'} - 1$. Here we set $b_l^{\theta'} = \infty$. The homomorphism given by $t_j \mapsto x$, $\xi_j \mapsto y$ maps the elements $g_k^{\theta'}(n)$ to the generators in (3.16). The isomorphism $\widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m) \simeq \mathcal{O}'(D(b))$ implies that $H^0(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m))$ is isomorphic to $H^0(X(\Delta), \mathcal{O}'(D(b)))$. Thus we have

$$\mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi_\theta^m \chi_{\tau_i}} = H^0(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{L}}_i \otimes \mathcal{O}(m)).$$

□

Corollary 3.14. *For* $m \in \mathbb{Z}_{>0}$*, we have*

$$H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)) = \begin{cases} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta), \chi_\theta^m} & (p = 0), \\ 0 & (p \neq 0). \end{cases}$$

3.5. (q,t)-dimension

Let V be a possibly infinite-dimensional vector space equipped with an action of the torus $\mathbb{T} = (\mathbb{C}^*)^2$. Then, we have the weight space decomposition of V: $V = \bigoplus_{r,s} V_{r,s}$ where $V_{r,s}$ is the weight space which belongs to the weight

$$\begin{aligned} \rho_{r,s} : \mathbb{T} &\longrightarrow \mathbb{C} \\ (m_1, m_2) &\mapsto m_1^r m_2^s \end{aligned}$$

The (q,t)-dimension of V is the following formal series:

$$\dim_{q,t} V = \sum_{r,s} (\dim V_{r,s}) q^r t^s$$

The torus $\mathbb{T}$ acts on $\mathrm{Rep}(\bar{Q}, \delta)$. The action induces an action of $\mathbb{T}$ on $\mathbb{C}[\mu^{-1}(0)]$. The weight spaces with respect to this action are equal to the homogeneous spaces with respect to the following bi-grading on $\mathbb{C}[\mu^{-1}(0)]$.

$$\deg t_i = (1,0), \qquad \deg \xi_i = (0,1),$$

for $i = 0, \ldots, l-1$. We consider the (q,t)-dimensions of $e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta),\chi_\theta^m}$ for $m \in \mathbb{Z}_{\geq 0}$ with respect to this action.

To calculate the (q,t)-dimension of $e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta),\chi_\theta^m}$ for $m \in \mathbb{Z}_{>0}$, we use the following Atiyah-Bott-Lefschetz formula together with Corollary 3.14.

Theorem 3.15 ([Ha1] Theorem 3.1). *Let X be a smooth surface equipped with an action of $\mathbb{T}$, and assume the fixed point set $X^{\mathbb{T}}$ is finite. Let $\mathcal{F}$ be a $\mathbb{T}$-equivariant locally free sheaf on X. For $x \in X^{\mathbb{T}}$, $\mathbb{T}$ acts on $\mathcal{F}(x)$. Suppose that $\mathbb{T}$ acts on the cotangent space at x with weights (v_1, v_2) and (w_1, w_2). Then we have,*

$$\sum_{p \geq 0} (-1)^p \dim_{q,t} H^p(X, \mathcal{F}) = \sum_{x \in X^{\mathbb{T}}} \frac{\dim_{q,t} \mathcal{F}(x)}{(1 - q^{v_1} t^{v_2})(1 - q^{w_1} t^{w_2})}.$$

We will apply the above theorem for $X = \mathfrak{M}_\theta(\delta)$ and $\mathcal{F} = \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)$. Then we have the following theorem.

Theorem 3.16. *For $m \geq 0$, $i = 0, \ldots, l-1$, we have the following identity*

$$\dim_{q,t} \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta),\chi_\theta^m} \Big|_{t=q^{-1}} = \sum_{i=1}^{l} q^{d_i^{m\theta} + (l-i)} \frac{1}{1 - q^{-1}} \tag{3.17}$$

where

$$d_i^\theta = -\theta_0 - 2\theta_1 - \cdots - i\theta_{i-1} + (l-i-1)\theta_i + \cdots + \theta_{l-2}. \tag{3.18}$$

for $i = 0, \ldots, l-1$. Here we set $d_l^\theta = d_0^\theta$.

This will be proved in the next subsection.

3.6. The proof of the theorem

In this subsection, we prove Theorem 3.16. To prove the theorem, we apply Theorem 3.15 for $\mathcal{F} = \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)$ together with Corollary 3.14 for $m \geq 1$. Since we did not prove the vanishing of the cohomologies Corollary 3.14 for $m = 0$, we cannot make use of Theorem 3.15 in this case. On the other hand, in the case of $m = 0$, the space $\mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta)}$ is independent of the stability parameter $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l_{reg}$. Therefore we can easily show (3.17) by a direct calculation.

Set $\bar{A}^* = \bar{A}^*_0 + \bar{A}^*_1 + \cdots + \bar{A}^*_{l-1}$. By the right action on $\mathcal{P}_\theta$ given in Section 3.3, $e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta)}$ is a right $\mathbb{C}[\bar{A}^*]$-module. As the right $\mathbb{C}[\bar{A}^*]$-module, we have

$$\mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta)} = \bigoplus_{i=0}^{l-1} E_{0i} \otimes t_{i+1} \cdots t_{l-1} t_0 \mathbb{C}[\bar{A}^*].$$

Therefore, we have

$$\dim_{q,t} \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta)}\Big|_{t=q^{-1}} = \sum_{i=1}^{l} q^{l-i} \frac{1}{1-q^{-1}}.$$

Consider the case of $m \in \mathbb{Z}_{>0}$. Since, at the fixed point $p_i \in \mathfrak{M}_\theta(\delta)$ for $i = 0, \ldots, l-1$,

$$\dim_{q,t} \left(\widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)\right)(p_i) = \dim_{q,t} \widetilde{\mathcal{P}}_\theta(p_i) \cdot \left(\dim_{q,t} \mathcal{O}(1)(p_i)\right)^m,$$

we need to calculate $\dim_{q,t} \mathcal{O}(1)(p_i)$ and $\dim_{q,t} \widetilde{\mathcal{P}}_\theta(p_i)$.

First we consider the fibers of $\mathcal{O}(1)$ at the fixed points $p_{\eta_1}, \ldots, p_{\eta_l}$.

Lemma 3.17. *We have $\mathcal{O}(1)_{p_{\eta_i}} = \mathcal{O}_{p_{\eta_i}} f_i^\theta$ where f_i^θ is the function defined on* (3.15) *with $\theta' = \theta$, i.e.,*

$$f_i^\theta = \prod_{j=i}^{l-1} (t_{\eta_{j+1}} \cdots t_{\eta_l})^{b_j^\theta} \prod_{j=1}^{i-1} (\xi_{\eta_1} \cdots \xi_{\eta_j})^{b_j^\theta}.$$

Proof. Apply Lemma 3.10 for $i = 0$ and $m = 1$, we have $\widetilde{\mathcal{L}}_0 \otimes \mathcal{O}(1) = \mathcal{O}(1)$, thus we have

$$H^0(X'_i, \mathcal{O}(1)) = R'_i f_i^\theta.$$

The fixed point $p_{\eta_{l-i+1}} \in X'_i$ corresponds to the maximal ideal $\mathfrak{m}'_i \subset R'_i$. Therefore we have

$$\mathcal{O}(1)_{p_{\eta_i}} = \mathcal{O}_{p_{\eta_i}} f_i^\theta.$$

$\square$

Corollary 3.18. *For $i = 1, \ldots, l$, we have*

$$\dim_{q,t} \mathcal{O}(1)(p_{\eta_i})\Big|_{t=q^{-1}} = q^{d^{\theta}_{\eta_i}}.$$

Proof. First we calculate the case of $i = 1$, then we have

$$\begin{aligned} f_1^{\theta} &= \prod_{j=1}^{l-1} (t_{\eta_{j+1}} \cdots t_{\eta_l})^{b_j^{\theta}} \\ &= t_{\eta_1+1}^{\theta_{\eta_1}} t_{\eta_1+2}^{\theta_{\eta_1}+\theta_{\eta_1+1}} \cdots t_{\eta_1+l-1}^{\theta_{\eta_1}+\theta_{\eta_1+1}+\cdots+\theta_{\eta_1+l-2}}. \end{aligned}$$

Consider the degree given by $\deg t_i = 1$ and $\deg \xi_i = -1$ for $i = 0, \ldots, l-1$. The degree of f_1^{θ} is

$$\deg f_1^{\theta} = (l-1)\theta_{\eta_1} + (l-2)\theta_{\eta_1+1} + \cdots + \theta_{\eta_1+l-2} = d^{\theta}_{\eta_1}.$$

Thus the statement of the corollary is valid for $i = 1$. On the other hand, we have $d^{\theta}_{\eta_i} - d^{\theta}_{\eta_{i+1}} = lb_i^{\theta}$ and $\deg f_i^{\theta} - \deg f_{i+1}^{\theta} = lb_i^{\theta}$. Therefore we have $\deg f_i^{\theta} = d^{\theta}_{\eta_i}$ for $i = 2, \ldots, l-1$ by induction on i. $\square$

Next we consider the fibers of $\widetilde{\mathcal{P}}_{\theta}$ at the fixed points. At the fixed point $p_i = [a_j, b_j]_{j=0,\ldots,l-1}$ defined by (3.2), we consider the stalk $(\widetilde{\mathcal{P}}_{\theta})_{p_i} = \bigoplus_{k=0}^{l-1} \mathbb{C}E_{0k} \otimes (\widetilde{\mathcal{L}}_k)_{p_i}$. For $k = 0, \ldots, l-1$, let v_k be the germ of $(\widetilde{\mathcal{P}}_{\theta})_{p_i}$ defined as follows.

$$\begin{aligned} v_k &= E_{0k} \otimes \nu_1 \cdots \nu_k \qquad \text{if } k \leq i-1, \\ v_k &= E_{0k} \otimes \nu'_{k+1} \cdots \nu'_{l-1}\nu'_0 \qquad \text{if } k \geq i \end{aligned}$$

where

$$\nu_j = \begin{cases} t_j^{-1} & (\text{if } a_j \neq 0) \\ \xi_j & (\text{if } b_j \neq 0) \end{cases},$$

$$\nu'_j = \begin{cases} t_j & (\text{if } a_j \neq 0) \\ \xi_j^{-1} & (\text{if } b_j \neq 0) \end{cases}.$$

Note that the group $GL(\delta)$ acts on v_k by the character χ_{τ_k}, and v_k is not zero at p_i.

The stalk $(\widetilde{\mathcal{P}}_{\theta})_{p_i}$ is a free $\mathcal{O}_{p_i}$-module of rank l. As the v_k above for $k = 0, \ldots, l-1$ are clearly linearly independent, we have the following lemma.

Lemma 3.19. *For $i = 0, \ldots, l-1$, the stalk $(\widetilde{\mathcal{P}}_{\theta})_{p_i}$ has the $\mathcal{O}_{p_i}$-basis $\{v_k\}_{k=0}^{l-1}$.*

Corollary 3.20. *For $i = 0, \ldots, l-1$, we have*

$$\dim_{q,t} \widetilde{\mathcal{P}}_{\theta}(p_i)\Big|_{t=q^{-1}} = q^{l-i}\frac{1-q^{-l}}{1-q^{-1}}.$$

Finally, the cotangent space of $\mathfrak{M}_{\theta}(\delta)$ at the fixed points has the following well-known structure.

Lemma 3.21 ([Mu]). *For $i = 1, \dots, l$, the cotangent space of $\mathfrak{M}_\theta(\delta)$ at p_{η_i} has $\mathbb{T}$-action with weights*

$$(v_1, v_2) = (l - i, -i),$$
$$(w_1, w_2) = (-l + i + 1, i + 1).$$

Now we apply Theorem 3.15 for $\mathcal{F} = \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)$ with $m \in \mathbb{Z}_{>0}$ to prove Theorem 3.16.

By Lemma 3.6 and Corollary 3.14, we have the modules $H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m))$ are $\mathbb{C}[t_0 \cdots t_{l-1}]$-free. Thus we have

$$\begin{aligned} \dim_{q,t} \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} & H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)) \\ &= (1 - q^l) \dim_{q,t} H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)). \end{aligned} \tag{3.19}$$

By Corollary 3.14,

$$\begin{aligned} &\dim_{q,t} \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta), \chi_\theta^m} \\ &\qquad = \sum_p (-1)^p \dim_{q,t} \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)). \end{aligned}$$

By Theorem 3.15 together with Corollary 3.18, Corollary 3.20, Lemma 3.21 and (3.19), we have

$$\begin{aligned} &\sum_p (-1)^p \dim_{q,t} \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} H^p(\mathfrak{M}_\theta(\delta), \widetilde{\mathcal{P}}_\theta \otimes \mathcal{O}(m)) \Big|_{t=q^{-1}} \\ &\quad = \sum_{i=1}^{l} (1 - q^l) \frac{\dim_{q,t} \widetilde{\mathcal{P}}_\theta(p_{\eta_i}) \dim_{q,t} \mathcal{O}(m)(p_{\eta_i})}{(1 - v_{p_{\eta_i}}(q,t))(1 - w_{p_{\eta_i}}(q,t))} \Big|_{t=q^{-1}} \\ &\quad = \sum_{i=1}^{l} (1 - q^l) \frac{q^{d_{\eta_i}^{m\theta}} q^{l-\eta_i} (1 - q^{-l})/(1 - q^{-1})}{(1 - q^l)(1 - q^{-l})} \\ &\quad = \sum_{i=1}^{l} q^{d_i^{m\theta} + l - i} \frac{1}{1 - q^{-1}}. \end{aligned}$$

Thus we have (3.17) of Theorem 3.16.

4. Rational Cherednik algebras

4.1. Rational Cherednik algebras

First we introduce the rational Cherednik algebra $H_\kappa = H_\kappa(\mathbb{Z}_l)$ for the group $\mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$ with a parameter $\kappa = (\kappa_0, \dots, \kappa_{l-1}) \in \mathbb{R}^l$. As a vector space, H_κ is given by

$$H_\kappa = \mathbb{C}[x] \otimes \mathbb{C}\mathbb{Z}_l \otimes \mathbb{C}[y].$$

The relations of H_κ are as follows:

$$\gamma x\gamma^{-1} = \zeta^{-1}x,$$
$$\gamma y\gamma^{-1} = \zeta y,$$
$$[y, x] = 1 + l\sum_{i=0}^{l-1}(\kappa_{i+1} - \kappa_i)\bar{e}_i.$$

Here we set $\kappa_l = \kappa_0$ and $\bar{e}_i$ is an idempotent defined in Section 2.1. Note that H_κ depends only on $\kappa_{i+1} - \kappa_i$ for $i = 0, \ldots, l-1$. We have

$$x\bar{e}_i = \bar{e}_{i+1}x, \qquad y\bar{e}_i = \bar{e}_{i-1}y.$$

The polynomial algebras $\mathbb{C}[x]$ and $\mathbb{C}[y]$ are subalgebras of H_κ. Moreover, the smash products $\mathbb{C}[x]\#\mathbb{Z}_l$ and $\mathbb{C}[y]\#\mathbb{Z}_l$ are subalgebras of H_κ. We also define the spherical subalgebra U_κ of H_κ as $U_\kappa = \bar{e}_0H_\kappa\bar{e}_0$.

In [DO], the following homomorphism of algebras from H_κ into the algebra $D(\mathbb{C}^*)\#\mathbb{Z}_l$ was defined,

$$H_\kappa \longrightarrow D(\mathbb{C}^*)\#\mathbb{Z}_l,$$
$$x \mapsto x,$$
$$\gamma \mapsto \gamma,$$
$$y \mapsto D_y = \frac{d}{dx} + \frac{l}{x}\sum_{i=0}^{l-1}\kappa_i\bar{e}_i.$$

This homomorphism is injective. This map is called the Dunkl-Cherednik embedding, and the operator D_y is called the Dunkl operator. Let $\mathcal{O}_\kappa$ be the subcategory of H_κ-mod such that $y \in H_\kappa$ acts locally nilpotently on objects of $\mathcal{O}_\kappa$. By [DO] and [GGOR], $\mathcal{O}_\kappa$ is a highest weight category with index poset $\Lambda = \{0, 1, \ldots, l-1\}$ in the sense of [CPS].

We define the standard modules of $\mathcal{O}_\kappa$. For $i = 0, \ldots, l-1$, we have the irreducible $\mathbb{Z}_l$-modules $L_i = \mathbb{C}\mathbf{1}_i$. Let y act trivially on L_i, so this induces an action of the algebra $\mathbb{C}[y]\#\mathbb{Z}_l$. The algebra $\mathbb{C}[y]\#\mathbb{Z}_l$ is a subalgebra of H_κ, thus we define the standard module $\Delta_\kappa(i)$ as the induced module

$$\Delta_\kappa(i) = H_\kappa \otimes_{\mathbb{C}[y]\#\mathbb{Z}_l} L_i.$$

The following proposition is due to [DO] and [GGOR].

Proposition 4.1. (1) *For each $i = 0, \ldots, l-1$, the standard module $\Delta_\kappa(i)$ has a unique simple quotient which we denote by $L_\kappa(i)$.*

(2) *For any simple object $L \in \mathcal{O}_\kappa$, we have an isomorphism $L \simeq L_\kappa(i)$ for some $i = 0, \ldots, l-1$.*

4.2. Deformed preprojective algebras

Deformed preprojective algebras were first introduced by [CBH]. We use another equivalent definition which was defined by [Ho].

As in Section 2.2 and Section 3.1, let $Q = (I, E)$ be the quiver of type $A^{(1)}_{l-1}$.

We consider the space of representations

$$\mathrm{Rep}(Q, \delta) = \{(a_i)_{i=0,\ldots,l-1} \mid a_i \in \mathbb{C}\} \simeq \mathbb{C}^l$$

with the dimension vector $\delta = (1, \ldots, 1)$. Consider the algebra $D(\mathrm{Rep}(Q, \delta))$ of algebraic differential operators. Let $t_0, \ldots, t_{l-1} \in \mathbb{C}[\mathrm{Rep}(Q, \delta)]$ be the coordinate functions such that $t_i((a_j)_{j=0,\ldots,l-1}) = a_i$. Set $\partial_i = \partial/\partial t_i$. The algebra $D(\mathrm{Rep}(Q, \delta))$ is generated by $t_0, \ldots, t_{l-1}, \partial_0, \ldots, \partial_{l-1}$.

The group $GL(\delta)$ acts on $D(\mathrm{Rep}(Q, \delta))$ by

$$\begin{aligned} g \cdot t_i &= g_i^{-1} g_{i-1} t_i, \\ g \cdot \partial_i &= g_i g_{i-1}^{-1} \partial_i \end{aligned}$$

for $i = 0, \ldots, l-1$ and $g = (g_k)_{k=0,\ldots,l-1} \in GL(\delta)$. The action of $GL(\delta)$ on $\mathrm{Rep}(Q, \delta)$ induces a homomorphism of Lie algebras

$$\iota : \mathfrak{gl}(\delta) \longrightarrow D(\mathrm{Rep}(Q, \delta))^{GL(\delta)}.$$

As a Lie algebra, $\mathfrak{gl}(\delta) = \bigoplus_{i=0}^{l-1} \mathfrak{gl}_1(\mathbb{C}) \simeq \bigoplus_{i=0}^{l-1} \mathbb{C}e^{(i)}$ where $e^{(i)}$ is a natural basis of the i-th component. Then, we have

$$\iota(e^{(i)}) = t_{i+1}\partial_{i+1} - t_i\partial_i.$$

Consider the $l \times l$ matrix algebra $M_l(D(\mathrm{Rep}(Q, \delta)))$. We have an isomorphism

$$M_l(D(\mathrm{Rep}(Q, \delta))) \simeq M_l(\mathbb{C}) \otimes_{\mathbb{C}} D(\mathrm{Rep}(Q, \delta)). \tag{4.1}$$

The group $GL(\delta)$ acts on $M_l(\mathbb{C})$ by (3.11). It also acts on $D(\mathrm{Rep}(Q, \delta))$. Thus $GL(\delta)$ acts diagonally on $M_l(D(\mathrm{Rep}(Q, \delta)))$ through the isomorphism (4.1).

We have the following homomorphism of Lie algebras:

$$\begin{gathered} \tau : \mathfrak{gl}(\delta) \longrightarrow M_l(D(\mathrm{Rep}(Q, \delta)))^{GL(\delta)}, \\ \tau = \varpi \otimes 1 + 1 \otimes \iota, \end{gathered}$$

where $\varpi : \mathfrak{gl}(\delta) \to M_l(\mathbb{C})$ is given by $\varpi(e^{(i)}) = E_{ii}$.

For a parameter $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}^l_1$, we define the deformed preprojective algebra $\mathcal{T}_\lambda$ as

$$\mathcal{T}_\lambda = M_l(D(\mathrm{Rep}(Q, \delta)))^{GL(\delta)} \Big/ \sum_{i=0}^{l-1} M_l(D(\mathrm{Rep}(Q, \delta)))^{GL(\delta)}(\tau(e^{(i)}) - \lambda_i),$$

and define the spherical subalgebra $\mathcal{A}_\lambda$ of $\mathcal{T}_\lambda$ as $\mathcal{A}_\lambda = e_0\mathcal{T}_\lambda e_0$ where $e_i = E_{ii}$ for $i = 0, \ldots, l-1$. It is clear that

$$\mathcal{A}_\lambda = D(\mathrm{Rep}(Q, \delta))^{GL(\delta)} \Big/ \sum_{i=0}^{l-1} D(\mathrm{Rep}(Q, \delta))^{GL(\delta)}(\iota(e^{(i)}) - \bar{\lambda}_i) \tag{4.2}$$

where $\bar{\lambda}_0 = \lambda_0 - 1$ and $\bar{\lambda}_i = \lambda_i$ for $i \neq 0$.

By the proposition of [Ho] together with [CBH], we have the following isomorphisms of algebras.

Proposition 4.2 ([CBH], [Ho]). *For $\lambda = (\lambda_i)_{i=0,\dots,l-1}$, $\lambda_i = \kappa_{i+1} - \kappa_i + 1/l$, we have isomorphisms of algebras*:

$$H_\kappa \simeq \mathcal{T}_\lambda, \qquad U_\kappa \simeq \mathcal{A}_\lambda. \tag{4.3}$$

This isomorphisms are given by

$$\begin{aligned} \mathcal{T}_\lambda &\longrightarrow H_\kappa, \\ A_i = E_{i,i-1} \otimes t_i &\mapsto \bar{e}_i x \bar{e}_{i-1}, \\ A_i^* = E_{i-1,i} \otimes \partial_i &\mapsto \bar{e}_{i-1} y \bar{e}_i, \\ e_i &\mapsto \bar{e}_i. \end{aligned}$$

Set $A = A_0 + A_1 + \cdots + A_{l-1}$, and $A^* = A_0^* + A_1^* + \cdots + A_{l-1}^*$. They correspond to x, $y \in H_\kappa$ under the above isomorphism. We have the following triangular decomposition of $\mathcal{T}_\lambda$.

$$\mathcal{T}_\lambda = \mathbb{C}[A] \otimes_{\mathbb{C}} \left(\bigoplus_{i=0}^{l-1} \mathbb{C} e_i \right) \otimes_{\mathbb{C}} \mathbb{C}[A^*]. \tag{4.4}$$

By the isomorphism (4.3), we identify the rational Cherednik algebra H_κ and the deformed preprojective algebra $\mathcal{T}_\lambda$ with $\lambda_i = \kappa_{i+1} - \kappa_i + (1/l)$. Thus we regard category $\mathcal{O}_\kappa$ as a subcategory of $\mathcal{T}_\lambda$-mod, and denote it by $\mathcal{O}_\lambda$. Then the category $\mathcal{O}_\lambda$ is the subcategory of $\mathcal{T}_\lambda$-mod such that the operator A^* acts locally nilpotently on each object of $\mathcal{O}_\lambda$. We also regard the standard modules of H_κ as $\mathcal{T}_\lambda$-modules. Denote them by $\Delta_\lambda(i)$ for $i = 0, \dots, l-1$. As $\mathcal{T}_\lambda$-modules, we have the natural description of the standard modules

$$\Delta_\lambda(i) = (\mathcal{T}_\lambda / \mathcal{T}_\lambda A^*) e_i. \tag{4.5}$$

By (4.4), we have

$$\Delta_\lambda(i) = \mathbb{C}[A] \mathbf{1}_i$$

as a vector space. By Proposition 4.1, $\Delta_\lambda(i)$ has a unique simple quotient which we denote by $L_\lambda(i)$.

Lemma 4.3. *We have $\mathrm{Hom}_{\mathcal{T}_\lambda}(\Delta_\lambda(j), \Delta_\lambda(i)) \neq 0$ if and only if $\lambda_i + \cdots + \lambda_{j-1} \in \mathbb{Z}_{\leq 0}$. Moreover in this case, $\mathrm{Hom}_{\mathcal{T}_\lambda}(\Delta_\lambda(j), \Delta_\lambda(i))$ is one-dimensional and any non-zero homomorphism from $\Delta_\lambda(j)$ to $\Delta_\lambda(i)$ is injective.*

Proof. To construct a homomorphism from $\Delta_\lambda(j)$ to $\Delta_\lambda(i)$, it is enough to find a vector $v \in \Delta_\lambda(i)$ such that $A^* v = 0$ and $e_k v = \delta_{kj} v$ for $k = 0, \dots, l-1$. Indeed, if we have a non-zero homomorphism $\phi \in \mathrm{Hom}(\Delta_\lambda(j), \Delta_\lambda(i))$, the vector $v = \phi(\mathbf{1}_j)$ satisfies $A^* v = 0$ and $e_k v = \delta_{kj} v$. Conversely, assume there is a vector v such that $A^*v = 0$ and $e_k v = \delta_{kj} v$ for $k = 0, \ldots, l-1$. Then we can define a homomorphism $\phi : \Delta_\lambda(j) \to \Delta_\lambda(i)$ as $\phi(A^m \mathbf{1}_j) = A^m v$ for $m \in \mathbb{Z}_{\geq 0}$. Moreover, v is equal to $A^p \mathbf{1}_i$ and $i + p$ is equivalent to j modulo l.

Assume the above $v = A^p \mathbf{1}_i \in \Delta_\lambda(i)$ exists. By the relation $[A^*, A] = \sum_{k=0}^{l-1} \lambda_k e_k$, we have

$$0 = A^* v = A^*(A^p \mathbf{1}_i) = [A^*, A^p]\mathbf{1}_i = l \sum_{k=0}^{p-1} \lambda_{i+k} \mathbf{1}_i. \tag{4.6}$$

By $\lambda_0 + \cdots + \lambda_{l-1} = 1$ and $i + p \equiv j$ modulo l, we have

$$\sum_{k=0}^{p-1} \lambda_{i+k} = n(\lambda_i + \cdots + \lambda_{i-1}) + (\lambda_i + \cdots + \lambda_{j-1}) = n + (\lambda_i + \cdots + \lambda_{j-1}).$$

where $n = (p - j + i)/l \in \mathbb{Z}_{\geq 0}$. Then, we have $\lambda_i + \cdots + \lambda_{j-1} = -n \in \mathbb{Z}_{\leq 0}$. Conversely, when $-n = \lambda_i + \cdots + \lambda_{j-1} \in \mathbb{Z}_{\leq 0}$, the vector $v = A^{nl+j-i}\mathbf{1}_i$ satisfies $A^* v = 0$ and $e_k v = \delta_{jk} v$. Moreover, since such v is uniquely determined by i, j and n, we have $\mathrm{Hom}_{\mathcal{T}_\lambda}(\Delta_\lambda(j), \Delta_\lambda(i)) = \mathbb{C}$. Obviously we have $\mathcal{T}_\lambda v = \mathbb{C}[A]v$, thus this map is injective. $\square$

For $i \neq j$ such that $\mathrm{Hom}_{\mathcal{T}_\lambda}(\Delta(j), \Delta(i)) \neq 0$. Let $L_\lambda(i, j)$ be the quotient

$$0 \to \Delta_\lambda(j) \to \Delta_\lambda(i) \to L_\lambda(i, j) \to 0.$$

By the above lemma $L_\lambda(i, j)$ is uniquely determined. By the proof of Lemma 4.3, we have $L_\lambda(i, j) \simeq \mathbb{C}[A]\mathbf{1}_i / \mathbb{C}[A]A^{nl+j-i}\mathbf{1}_i$ for some $n \in \mathbb{Z}_{\geq 0}$. Therefore, we have $\langle \underline{\dim} L_\lambda(i, j), \lambda \rangle = \sum_{k=0}^{nl+j-i-1} \lambda_{i+k} = 0$ by (4.6).

Consider the following functors between the categories of modules:

$$\begin{aligned} E_\lambda : \mathcal{T}_\lambda\text{-Mod} &\longrightarrow \mathcal{A}_\lambda\text{-Mod}, \\ M &\mapsto e_0 M, \\ F_\lambda : \mathcal{A}_\lambda\text{-Mod} &\longrightarrow \mathcal{T}_\lambda\text{-Mod}, \\ N &\mapsto \mathcal{T}_\lambda e_0 \otimes_{\mathcal{A}_\lambda} N. \end{aligned}$$

Restricting the functors E_λ and F_λ, we have functors between $\mathcal{T}_\lambda$-mod and $\mathcal{A}_\lambda$-mod. We also denote it by the same symbols E_λ and F_λ.

Proposition 4.4. *If $\langle \lambda, \beta \rangle \neq 0$ for all Dynkin roots $\beta \in \mathbb{Z}^l$, E_λ is an equivalence of categories with quasi-inverse F_λ.*

Proof. The following proof is essentially the same as the argument in the proof of Theorem 3.3 of [GS1].

To prove the equivalence, we show that $\mathcal{T}_\lambda e_0 \otimes_{\mathcal{A}_\lambda} e_o \mathcal{T}_\lambda \simeq \mathcal{T}_\lambda e_0 \mathcal{T}_\lambda = \mathcal{T}_\lambda$ and $e_0 \mathcal{T}_\lambda \otimes_{\mathcal{T}_\lambda} \mathcal{T}_\lambda e_0 \simeq \mathcal{A}_\lambda$. It is clear that $e_0 \mathcal{T}_\lambda \otimes_{\mathcal{T}_\lambda} \mathcal{T}_\lambda e_0 \simeq e_0 \mathcal{T}_\lambda e_0 = \mathcal{A}_\lambda$. Assume that $\mathcal{T}_\lambda e_0 \mathcal{T}_\lambda \neq \mathcal{T}_\lambda$, so then $\mathcal{T}_\lambda e_0 \mathcal{T}_\lambda$ is proper two-sided ideal. By the generalized

Duflo theorem proved by [Gi], $\mathcal{T}_\lambda e_0 \mathcal{T}_\lambda$ annihilates the irreducible module $L_\lambda(i)$ for some $i = 0, \ldots, l-1$. Namely, there is an $i = 0, \ldots, l-1$ such that $e_0 L_\lambda(i) = 0$. If $\Delta_\lambda(i) = L_\lambda(i)$, then $e_0 L_\lambda(i) = e_0 \Delta_\lambda(i) \neq 0$ because we have $\Delta_\lambda(i) = \mathbb{C}[x]\mathbf{1}_i$. Thus, it contradicts the assumption $e_0 L_\lambda(i) = 0$. Assume $\Delta_\lambda(i) \neq L_\lambda(i)$, then there is an exact sequence

$$0 \to \Delta_\lambda(j) \to \Delta_\lambda(i) \to L_\lambda(i) \to 0$$

for some $j \neq i$. Let $\alpha = \underline{\dim} L_\lambda(i)$, then we have $\langle \lambda, \alpha \rangle = 0$ and $\alpha \in \mathbb{Z}^l$ is a root. Moreover, by the assumption $e_0 L_\lambda = 0$, α is a Dynkin root. This is a contradiction. $\square$

4.3. Parameters and orderings

In the next subsection, we define a functor $\mathcal{S}_\lambda^\theta : \mathcal{A}_\lambda\text{-Mod} \to \mathcal{A}_{\lambda+\theta}\text{-Mod}$ called the shift functor. The shift functor $\mathcal{S}_\lambda^\theta$ depends on the parameter $\lambda \in \mathbb{R}_1^l$ and $\theta \in \mathbb{Z}_0^l$. In this paper, we concentrate our attention on the case where $\mathcal{S}_\lambda^\theta$ is an equivalences of categories. In this section we define our space of parameters.

For $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}_1^l$, we have the highest weight category $\mathcal{O}_\lambda$. We have the ordering $\trianglerighteq_{rep,\lambda}$ on the index set $\Lambda = \{0, \ldots, l-1\}$ which arises from the structure of the highest weight category $\mathcal{O}_\lambda$. Namely,

$$i \trianglerighteq_{rep,\lambda} j \Leftrightarrow \mathrm{Hom}_{\mathcal{T}_\lambda}(\Delta_\lambda(j), \Delta_\lambda(i)) \neq 0 \Leftrightarrow \lambda_i + \cdots + \lambda_{j-1} \in \mathbb{Z}_{\leq 0} \tag{4.7}$$

as we proved in Lemma 4.3.

Define

$$\mathbb{R}_{reg}^l = \{\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}_1^l \mid \bar\lambda_i + \cdots + \bar\lambda_{j-1} \neq 0 \quad \text{for all } i \neq j\}. \tag{4.8}$$

where $\bar\lambda_i = \lambda_i - \delta_{i0}$.

Fix $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}_{reg}^l$. Then we have $\bar\lambda_i + \cdots + \bar\lambda_{j-1} < 0$ for all $i \triangleright_{rep,\lambda} j$.

The set of parameters $\mathbb{Z}_{reg}^l$ defined in (3.1) is separated into $(l-1)!$ alcoves by the hyperplanes $\theta_i + \cdots + \theta_{j-1} = 0$ for $i \neq j$. Set

$$\mathbb{Z}_\lambda^l = \{\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}_{reg}^l \mid \theta_i + \cdots + \theta_{j-1} < 0 \quad \text{if } \lambda_i + \cdots + \lambda_{j-1} \in \mathbb{Z}_{\leq 0}\}. \tag{4.9}$$

The set $\mathbb{Z}_\lambda^l$ is a union of alcoves in $\mathbb{Z}_{reg}^l$ depending on λ. If $\lambda \in \mathbb{R}_{reg}^l$ is generic, we have $\mathbb{Z}_\lambda^l = \mathbb{Z}_{reg}^l$. If λ belongs to $\mathbb{R}_{reg}^l \cap \mathbb{Z}^l$, $\mathbb{Z}_\lambda^l$ is one of $(l-1)!$ alcoves in $\mathbb{Z}_{reg}^l$. For $\lambda \in \mathbb{R}_{reg}^l$, $\theta \in \mathbb{Z}_\lambda^l$ and $m \in \mathbb{Z}_{\geq 0}$, we have $\mathbb{Z}_{\lambda+m\theta}^l = \mathbb{Z}_\lambda^l$ and $\trianglerighteq_{rep,\lambda+m\theta}$ is equal to $\trianglerighteq_{rep,\lambda}$.

By the result of [Ro], we have the following theorem and corollary.

Theorem 4.5 ([Ro]). *If $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda'}$ have the same ordering defined by* (4.7)*, then there exists an equivalence of categories $\mathcal{O}_\lambda \simeq \mathcal{O}_{\lambda'}$.*

Corollary 4.6. *For $\lambda \in \mathbb{R}_{reg}^l$ and $\theta \in \mathbb{Z}_\lambda^l$, there is an equivalence of categories between $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda+\theta}$.*

Fix $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}^l_{reg}$ and $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l_\lambda$. Define a new ordering $\trianglerighteq_\theta$ of Λ by

$$i \trianglerighteq_\theta j \Leftrightarrow \theta_i + \cdots + \theta_{j-1} \leq 0. \tag{4.10}$$

It is a total ordering and it refines the ordering $\trianglerighteq_{rep,\lambda}$, i.e., $i \triangleright_{rep,\lambda} j$ implies $i \trianglerighteq_\theta j$. If θ and θ' belong to the same alcove, $\trianglerighteq_\theta$ is equal to $\trianglerighteq_{\theta'}$. If we take θ and θ' from different alcoves, $\trianglerighteq_\theta$ is different from $\trianglerighteq_{\theta'}$.

By (3.3), the ordering $\trianglerighteq_\theta$ is exactly same as the ordering $\trianglerighteq_{geom,\theta}$ defined in Section 3.1. Therefore we have

$$\eta_l \triangleright_\theta \eta_{l-1} \triangleright_\theta \cdots \triangleright_\theta \eta_1. \tag{4.11}$$

The following lemma will be repeatedly used in the next subsection.

Lemma 4.7. *For any $i > j$, we have $\bar{\lambda}_{\eta_j} + \cdots + \bar{\lambda}_{\eta_i - 1} > 0$ or $\bar{\lambda}_{\eta_j} + \cdots + \bar{\lambda}_{\eta_i - 1} \notin \mathbb{Z}$.*

Proof. Assume $\bar{\lambda}_{\eta_j} + \cdots + \bar{\lambda}_{\eta_i - 1} \in \mathbb{Z}$. By (4.7), we have $\eta_i \triangleright_{rep,\lambda} \eta_j$ or $\eta_j \triangleright_{rep,\lambda} \eta_i$. Since $\triangleright_\theta$ refines $\triangleright_{rep,\lambda}$ and we have $\eta_i \triangleright_\theta \eta_j$, the case $\eta_j \triangleright_{rep,\lambda} \eta_i$ cannot occur. Thus we have $\eta_i \triangleright_{rep,\lambda} \eta_j$. Therefore we have $\bar{\lambda}_{\eta_j} + \cdots + \bar{\lambda}_{\eta_i - 1} > 0$ by (4.8). □

4.4. Shift functors

As in [Bo], we define a functor called the shift functor between the two categories of modules of the rational Cherednik algebras with different parameters. Moreover we prove that it gives the equivalence of categories that we discussed in Theorem 4.5 and Corollary 4.6.

Fix a parameter $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}^l_1$ of the rational Cherednik algebra. We take another parameter $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l_0$. Define

$$\mathcal{B}^\theta_\lambda = \left[D(\mathrm{Rep}(Q,\delta)) \Big/ \sum_{i=0}^{l-1} D(\mathrm{Rep}(Q,\delta))(\iota(e^{(i)}) - \bar{\lambda}_i) \right]^{GL(\delta),\chi_\theta} \tag{4.12}$$

where χ_θ is the character of $GL(\delta)$ defined in Section 3.1. It is easy to see that $\mathcal{B}^\theta_\lambda$ has an $(\mathcal{A}_{\lambda+\theta}, \mathcal{A}_\lambda)$-bimodule structure.

Definition 4.8. We define the functor

$$\mathcal{S}^\theta_\lambda : \mathcal{A}_\lambda\text{-Mod} \longrightarrow \mathcal{A}_{\lambda+\theta}\text{-Mod},$$
$$M \mapsto \mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} M.$$

The functor $\mathcal{S}^\theta_\lambda$ is called a shift functor. Restricting $\mathcal{S}^\theta_\lambda$ to the subcategory $\mathcal{A}_\lambda$-mod, we have the functor

$$\mathcal{S}^\theta_\lambda : \mathcal{A}_\lambda\text{-mod} \longrightarrow \mathcal{A}_{\lambda+\theta}\text{-mod}.$$

In Section 4.2, we defined the functors E_λ, F_λ between $\mathcal{T}_\lambda$-Mod and $\mathcal{A}_\lambda$-Mod. Using $E_\lambda : \mathcal{T}_\lambda\text{-Mod} \to \mathcal{A}_\lambda\text{-Mod}$ and $F_{\lambda+\theta} : \mathcal{A}_{\lambda+\theta}\text{-Mod} \to \mathcal{T}_{\lambda+\theta}\text{-Mod}$, we define the shift functor

$$\widehat{\mathcal{S}}^\theta_\lambda = F_{\lambda+\theta} \circ \mathcal{S}^\theta_\lambda \circ E_\lambda : \mathcal{T}_\lambda\text{-Mod} \longrightarrow \mathcal{T}_{\lambda+\theta}\text{-Mod}.$$

We also denote the restricted functor

$$\widehat{\mathcal{S}}^\theta_\lambda : \mathcal{T}_\lambda\text{-mod} \longrightarrow \mathcal{T}_{\lambda+\theta}\text{-mod}$$

by the same symbol $\widehat{\mathcal{S}}^\theta_\lambda$.

Lemma 4.9. *The functor $\widehat{\mathcal{S}}^\theta_\lambda$ restricts to a functor*

$$\widehat{\mathcal{S}}^\theta_\lambda : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_{\lambda+\theta}.$$

Proof. Fix $M \in \mathcal{O}_\lambda$. Since we have $(A^*)^l = (e_0 + \cdots + e_{l-1}) \otimes \partial_0 \cdots \partial_{l-1}$, to prove the lemma, we only need to show $\partial_0 \cdots \partial_{l-1}$ acts locally nilpotently on $\mathcal{S}^\theta_\lambda(e_0 M) = \mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0 M$. Fix $b \in \mathcal{B}^\theta_\lambda$ and $m \in e_0 M$. We have $(\partial_0 \cdots \partial_{l-1})^p m = 0$ for a sufficiently large $p \in \mathbb{Z}_{\geq 0}$. Consider a filtration $\{F_k \mathcal{B}^\theta_\Lambda\}_k$ defined by the degree

$$\deg t_i = 1, \quad \deg \partial_i = 0 \qquad (i = 0, \ldots, l-1).$$

For $b \in F_k \mathcal{B}^\theta_\lambda \backslash F_{k-1} \mathcal{B}^\theta_\lambda$, let q be an integer greater than $k + p$. Then, we have

$$[(\partial_0 \cdots \partial_{l-1})^q, b] = b'(\partial_0 \cdots \partial_{l-1})^p$$

Here $b' = \sum_{j=1}^k \operatorname{ad}(\partial_0 \cdots \partial_{l-1})^j (b') \cdot (\partial_0 \cdots \partial_{l-1})^{q-p-j} \in \mathcal{B}^\theta_\lambda$. Thus, we have

$$\begin{aligned}(\partial_0 \cdots \partial_{l-1})^q b \otimes m &= b(\partial_0 \cdots \partial_{l-1})^q \otimes m + [(\partial_0 \cdots \partial_{l-1})^q, b] \otimes m \\ &= b \otimes (\partial_0 \cdots \partial_{l-1})^q m + b' \otimes (\partial_0 \cdots \partial_{l-1})^p m = 0.\end{aligned}$$

Therefore $\partial_0 \cdots \partial_{l-1}$ acts locally nilpotently on $\mathcal{S}^\theta_\lambda(e_0 M)$. □

Let $\bar{\mathcal{A}}_\lambda$ be the subalgebra of $\mathcal{A}_\lambda$ generated by the elements $t_0 \partial_0$ and $t_0 \cdots t_{l-1}$.

Lemma 4.10. *For $k = 1, \ldots, l-1$, let b^θ_k be the non-negative integer defined by (3.14) with $\theta' = \theta$. For $k = 1, \ldots, l-1$, $n = 0, \ldots, b^\theta_k - 1$, define*

$$\tilde{g}_k(n) = (t_{\eta_{k+1}} \cdots t_{\eta_l})^{b^\theta_k - n} \prod_{j=k+1}^{l-1} (t_{\eta_{j+1}} \cdots t_{\eta_l})^{b^\theta_j} (\partial_{\eta_1} \cdots \partial_{\eta_k})^n \prod_{j=1}^{k-1} (\partial_{\eta_1} \cdots \partial_{\eta_j})^{b^\theta_j}$$

and, for $n \in \mathbb{Z}_{\geq 0}$, define

$$\tilde{g}_l(n) = (\partial_0 \cdots \partial_{l-1})^n \prod_{j=1}^{l-1} (\partial_{\eta_1} \cdots \partial_{\eta_j})^{b^\theta_j}.$$

Then $\{\tilde{g}_k(n)\}_{k,n}$ *generates* $\mathcal{B}_\lambda^\theta$ *as a left* $\bar{\mathcal{A}}_\lambda$*-module.*

Proof. Consider the filtration of $\mathcal{B}_\lambda^\theta$ defined by the order of differential operators in $D(\mathrm{Rep}(Q,\delta))$. By [Bo, (5.1)], the associated graded module is

$$\mathrm{gr}\,\mathcal{B}_\lambda^\theta = \mathbb{C}[\mu^{-1}(0)]^{GL(\delta),\chi_\theta}.$$

Thus the statement of the lemma follows from Proposition 3.13 and Lemma 3.12. □

Proposition 4.11. *For* $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}^l_{reg}$ *and* $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l_\lambda$*, we have*

$$\mathcal{S}_\lambda^\theta(e_0\Delta_\lambda(i)) \simeq e_0\Delta_{\lambda+\theta}(i) \quad \textit{and} \quad \widehat{\mathcal{S}}_\lambda^\theta(\Delta_\lambda(i)) \simeq \Delta_{\lambda+\theta}(i)$$

for all $i = 0, \ldots, l-1$.

Proof. We show that $\mathcal{S}_\lambda^\theta(e_0\Delta_\lambda(\eta_i))$ is isomorphic to $e_0\Delta_{\lambda+\theta}(\eta_i)$ for all $i = 1, \ldots, l$. To prove this, we see the structure of $\mathcal{S}_\lambda^\theta(e_0\Delta_\lambda(\eta_i))$ with the help of the geometric information which we studied in Section 3.4. As a result of it, we can construct the isomorphism $\mathcal{S}_\lambda^\theta(e_0\Delta_\lambda(\eta_i)) \simeq e_0\Delta_{\lambda+\theta}(\eta_i)$ explicitly.

Let $w_k(n) = \tilde{g}_k(n) \otimes e_0t_0t_{l-1}\ldots t_{\eta_i+1}\mathbf{1}_{\eta_i}$ be an element of $\mathcal{S}_\lambda^\theta(e_0\Delta_\lambda(\eta_i)) = \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i)$. By Lemma 4.10, $\{w_k(n)\}_{k,n}$ span the module $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i)$. We show that the vector $w_i(0)$ generates $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i)$ and $(\partial_0\cdots\partial_{l-1})w_k(0) = 0$.

First we show $w_k(n)$ is non-zero when $k < i$ or $k = i$ and $n = 0$. We identify $w_k(0) = w_{k-1}(b_{k-1}-1)$. Then, by a straightforward calculation, we have

$$\begin{aligned}
&(t_0\cdots t_{l-1})w_k(n)\\
&\quad= (t_0\cdots t_{l-1})(t_{\eta_{k+1}}\cdots t_{\eta_l})^{b_k^\theta-n}\prod_{j=k+1}^{l-1}(t_{\eta_{j+1}}\cdots t_{\eta_l})^{b_j^\theta}\\
&\qquad(\partial_{\eta_1}\cdots\partial_{\eta_k})^n\prod_{j=1}^{k-1}(\partial_{\eta_1}\cdots\partial_{\eta_j})^{b_j^\theta}\otimes e_0t_0t_{l-1}\cdots t_{\eta_i}\mathbf{1}_{\eta_i}\\
&\quad= (t_{\eta_{k+1}}\cdots t_{\eta_l})^{b_k^\theta-n+1}\prod_{j=k+1}^{l-1}(t_{\eta_{j+1}}\cdots t_{\eta_l})^{b_j^\theta}(t_{\eta_1}\partial_{\eta_1}\cdots t_{\eta_k}\partial_{\eta_k})\\
&\qquad(\partial_{\eta_1}\cdots\partial_{\eta_k})^{n-1}\prod_{j=1}^{k-1}(\partial_{\eta_1}\cdots\partial_{\eta_j})^{b_j^\theta}\otimes e_0t_0t_{l-1}\cdots t_{\eta_i}\mathbf{1}_{\eta_i}.
\end{aligned}$$

For $1 \leq p \leq k$, we have

$$(t_{\eta_p}\partial_{\eta_p})(\partial_{\eta_1}\cdots\partial_{\eta_k})^{n-1}\prod_{j=1}^{k-1}(\partial_{\eta_1}\cdots\partial_{\eta_j})^{b_j^\theta}\otimes e_0t_0t_{l-1}\cdots t_{\eta_i}\mathbf{1}_{\eta_i}$$
$$= -\Big(\bar{\lambda}_{\eta_p}+\cdots+\bar{\lambda}_{\eta_i-1}+\sum_{q=p}^{k-1}b_q^\theta+n-1\Big)$$
$$(\partial_{\eta_1}\cdots\partial_{\eta_k})^{n-1}\prod_{j=1}^{k-1}(\partial_{\eta_1}\cdots\partial_{\eta_j})^{b_j^\theta}\otimes e_0t_0t_{l-1}\cdots t_{\eta_i}\mathbf{1}_{\eta_i}.$$

Thus we have

$$(t_0\cdots t_{l-1})w_k(n)\left\{\prod_{p=1}^{k}-\Big(\bar{\lambda}_{\eta_p}+\cdots+\bar{\lambda}_{\eta_i-1}+\sum_{q=p}^{k-1}b_q^\theta+n-1\Big)\right\}w_k(n-1).$$

For any $p=1,\ldots,i-1$ we have $\bar{\lambda}_{\eta_p}+\cdots+\bar{\lambda}_{\eta_i-1}>0$ or $\bar{\lambda}_{\eta_p}+\cdots+\bar{\lambda}_{\eta_i-1}\notin\mathbb{Z}$ by Lemma 4.7. Thus the coefficient of the right hand side of this equation is non-zero.

Therefore we have

$$(t_0\cdots t_{l-1})^{\sum_{j=1}^{i-1}b_j^\theta}w_i(0)=Cw_1(0)=C\prod_{j=1}^{l-1}(t_{\eta_{j+1}}\cdots t_{\eta_l})^{b_j^\theta}\otimes e_0t_0t_{l-1}\cdots t_{\eta_i+1}\mathbf{1}_{\eta_i}$$

where $C\in\mathbb{C}\backslash\{0\}$. Since the right hand side of this equation is non-zero, so is $w_i(0)$. Moreover, for $k<i$, $w_k(n)$ is non-zero and it belongs to $\mathbb{C}[t_0\cdots t_{l-1}]w_i(0)$. Then, we have

$$\mathbb{C}[t_0\cdots t_{l-1}]w_i(0)=\mathbb{C}w_i(0)\oplus\bigoplus_{k<i,n}\mathbb{C}w_k(n), \tag{4.13}$$

and it is $\mathbb{C}[t_0\cdots t_{l-1}]$-free.

Next we show $w_k(n)=0$ when $k>i$ or $k=i$ and $n\geq 1$. Inserting $\prod_{j=k+1}^{l}t_{\eta_j}\partial_{\eta_j}$ into the factors of $w_k(n)$, we have

$$\begin{aligned}&(t_{\eta_{k+1}}\cdots t_{\eta_l})^{b_k^\theta-n}\prod_{j=k+1}^{l-1}(t_{\eta_{j+1}}\cdots t_{\eta_l})^{b_j^\theta}\Big(\prod_{j=k+1}^{l}t_{\eta_j}\partial_{\eta_j}\Big)\\&\qquad\times(\partial_{\eta_1}\cdots\partial_{\eta_k})^{n}\prod_{j=1}^{k-1}(\partial_{\eta_1}\cdots\partial_{\eta_j})^{b_j^\theta}\otimes e_0t_0t_{l-1}\cdots t_{\eta_i+1}\mathbf{1}_{\eta_i}\\&=(t_{\eta_{k+1}}\cdots t_{\eta_l})^{b_k^\theta-n+1}\prod_{j=k+1}^{l-1}(t_{\eta_{j+1}}\cdots t_{\eta_l})^{b_j^\theta}(\partial_0\cdots\partial_{l-1})\\&\qquad\times(\partial_{\eta_1}\cdots\partial_{\eta_k})^{n-1}\prod_{j=1}^{k-1}(\partial_{\eta_1}\cdots\partial_{\eta_j})^{b_j^\theta}\otimes e_0t_0t_{l-1}\cdots t_{\eta_i+1}\mathbf{1}_{\eta_i}\\&=0.\end{aligned} \tag{4.14}$$

On the other hand, we have

$$
\begin{aligned}
&(t_{\eta_{k+1}}\cdots t_{\eta_l})^{b_k^\theta - n}\prod_{j=k+1}^{l-1}(t_{\eta_{j+1}}\cdots t_{\eta_l})^{b_j^\theta}\Big(\prod_{j=k+1}^{l} t_{\eta_j}\partial_{\eta_j}\Big)\\
&\times(\partial_{\eta_1}\cdots\partial_{\eta_k})^n\prod_{j=1}^{k-1}(\partial_{\eta_1}\cdots\partial_{\eta_j})^{b_j^\theta}\otimes e_0 t_0 t_{l-1}\cdots t_{\eta_i+1}\mathbf{1}_{\eta_i}\\
&=\left\{\prod_{j=k+1}^{l}-(\bar\lambda_{\eta_j}+\cdots+\bar\lambda_{\eta_i-1})\right\}w_k(n)
\end{aligned}
\tag{4.15}
$$

For $j \neq i$, we have $\bar\lambda_{\eta_j}+\cdots+\bar\lambda_{\eta_i-1}\neq 0$. By (4.14) and (4.15), we have

$$w_k(n) = 0 \tag{4.16}$$

for $k > i$ or $k = i$ and $n \geq 1$.

By Lemma 4.10, we have

$$\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i) = \sum_{k,n}\mathbb{C}[t_0\cdots t_{l-1}]w_k(n).$$

Then, by (4.16), we have

$$\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i) = \mathbb{C}[t_0\cdots t_{l-1}]w_i(0) + \sum_{k<i,n}\mathbb{C}[t_0\cdots t_{l-1}]w_k(n). \tag{4.17}$$

By (4.15), we have $w_k(n) \in \mathbb{C}[t_0\cdots t_{l-1}]w_i(0)$ for $k < i$. Thus we have,

$$\sum_{k<i,n}\mathbb{C}[t_0\cdots t_{l-1}]w_k(n) = \mathbb{C}[t_0\cdots t_{l-1}]w_i(0). \tag{4.18}$$

By (4.17) and (4.18), we have

$$\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i) = \mathbb{C}[t_0\cdots t_{l-1}]w_i(0).$$

Therefore we have $A^* w_i(0) = 0$ and $w_i(0)$ generates the module $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(\eta_i)$. Thus we have $\mathcal{S}_\lambda^\theta(e_0\Delta_\lambda(\eta_i)) \simeq e_0\Delta_{\lambda+\theta}(\eta_i)$.

By Proposition 4.4, $F_{\lambda+\theta}$ is an equivalence of categories. Thus, we have $F_{\lambda+\theta}(e_0\Delta_{\lambda+\theta}(i)) \simeq \Delta_{\lambda+\theta}(i)$. Therefore we have

$$\widehat{\mathcal{S}}_\lambda^\theta(\Delta_\lambda(i)) = F_{\lambda+\theta}\circ\mathcal{S}_\lambda^\theta\circ E_\lambda(\Delta_\lambda(i)) \simeq F_{\lambda+\theta}(e_0\Delta_{\lambda+\theta}(i)) \simeq \Delta_{\lambda+\theta}(i).$$

□

Next, we show that the shift functor $\mathcal{S}_\lambda^\theta$ is an equivalence of categories between $\mathcal{A}_\lambda$-Mod and $\mathcal{A}_{\lambda+\theta}$-Mod.

Lemma 4.12. *For $\lambda \in \mathbb{R}^l_{reg}$, $\theta \in \mathbb{Z}^l_\lambda$ and i, $j = 0, \ldots, l-1$ such that $i \rhd_{rep,\lambda} j$, the shift functor $\widehat{\mathcal{S}}^\theta_\lambda$ sends the exact sequence in $\mathcal{O}_\lambda$,*

$$0 \to \Delta_\lambda(j) \xrightarrow{\varphi} \Delta_\lambda(i) \to L_\lambda(i,j) \to 0$$

to the exact sequence in $\mathcal{O}_{\lambda+\theta}$,

$$0 \to \Delta_{\lambda+\theta}(j) \to \Delta_{\lambda+\theta}(i) \to L_{\lambda+\theta}(i,j) \to 0.$$

Proof. By Proposition 4.11, we have $\widehat{\mathcal{S}}^\theta_\lambda(\Delta_\lambda(k)) = \Delta_{\lambda+\theta}(k)$ for $k = i$, j. Then $\widehat{\mathcal{S}}^\theta_\lambda(\varphi)$ is a homomorphism

$$\widehat{\mathcal{S}}^\theta_\lambda(\varphi) : \Delta_{\lambda+\theta}(j) \longrightarrow \Delta_{\lambda+\theta}(i).$$

By Lemma 4.3, $\widehat{\mathcal{S}}^\theta_\lambda(\varphi)$ is injective and its quotient is $L_{\lambda+\theta}(i,j)$. Since $\widehat{\mathcal{S}}^\theta_\lambda$ is a right exact functor, it implies $\widehat{\mathcal{S}}^\theta_\lambda(L_\lambda(i,j)) \simeq L_{\lambda+\theta}(i,j)$. □

Corollary 4.13. *For $\lambda \in \mathbb{R}^l_{reg}$, $\theta \in \mathbb{Z}^l_\lambda$ and $i = 0, \ldots, l-1$, we have*

$$\widehat{\mathcal{S}}^\theta_\lambda(L_\lambda(i)) \simeq L_{\lambda+\theta}(i).$$

Proposition 4.14. *For $\lambda \in \mathbb{R}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$, the functor $\widehat{\mathcal{S}}^\theta_\lambda$ is an exact functor from $\mathcal{O}_\lambda$ to $\mathcal{O}_{\lambda+\theta}$.*

Proof. Since $\widehat{\mathcal{S}}^\theta_\lambda$ is right exact, to prove the exactness it is enough to show that $\widehat{\mathcal{S}}^\theta_\lambda$ sends injective homomorphisms to injective homomorphisms. Assume there is a non-zero module $M \in \mathcal{O}_\lambda$ such that $\widehat{\mathcal{S}}^\theta_\lambda(M) = 0$. Without loss of generalities, we can suppose that M is irreducible. By Proposition 4.1, M is isomorphic to $L_\lambda(i)$ for some $i = 0, \ldots, l-1$. On the other hand, we have $\widehat{\mathcal{S}}^\theta_\lambda(L_\lambda(i)) \simeq L_{\lambda+\theta}(i)$ by Corollary 4.13. This contradicts the assumption $\widehat{\mathcal{S}}^\theta_\lambda(M) = 0$. □

The following proposition is a result of the general theory of highest weight categories. R. Rouquier suggested it to the author as an approach to proving that $\widehat{\mathcal{S}}^\theta_\lambda$ is an equivalence. using it to prove the equivalence of $\mathcal{S}^\theta_\lambda$. The following proof of the proposition is given by S. Ariki.

Proposition 4.15. *Assume there are two highest weight categories $(\mathcal{O}, \Lambda)$, $(\mathcal{O}', \Lambda')$ which are equivalent to each other. If an exact functor $F : \mathcal{O} \longrightarrow \mathcal{O}'$ preserves the partial orderings of Λ and Λ', and F sends the standard modules of $\mathcal{O}$ to the standard modules of $\mathcal{O}'$, then F is an equivalence of categories.*

Proof. We denote the partial ordering of index poset Λ by $\rhd$. We also denote the standard modules of $\mathcal{O}$ by $\Delta(i)$ and the simple modules of $\mathcal{O}$ by $L(i)$ for $i \in \Lambda$. For $i \in \Lambda$, let $P(i)$ be the projective cover of $L(i)$.

Let $G : \mathcal{O}' \longrightarrow \mathcal{O}$ be an equivalence of categories. Consider the exact functor $F' = G \circ F : \mathcal{O} \longrightarrow \mathcal{O}$. Since G and F preserve the partial orderings of Λ and Λ', so does F'. Since G is the equivalence, F' is an equivalence of categories if and only if F is an equivalence of categories. Therefore we assume $\mathcal{O}' = \mathcal{O}$ and $F : \mathcal{O} \longrightarrow \mathcal{O}$ is an exact functor such that $F(\Delta(i)) \simeq \Delta(i)$ for any $i \in \Lambda$.

First we show that $F(L(i)) \simeq L(i)$ for any $i \in \Lambda$ by induction on i. If i is minimal in Λ, we have $L(i) = \Delta(i)$. Thus we have $F(L(i)) \simeq L(i)$. Assume $F(L(j)) \simeq L(j)$ for all $j \lhd i$. Consider the exact sequence

$$0 \to N(i) \to \Delta(i) \to L(i) \to 0.$$

In the composition factors of $N(i)$, only $L(j)$ with $j \lhd i$ appears. Thus $F(N(i))$ and $N(i)$ has the same composition factors by the hypothesis of the induction. Therefore we have $F(L(i)) \simeq L(i)$.

Second, we have

$$\operatorname{Ext}^n(M, N) \simeq \operatorname{Ext}^n(F(M), F(N)) \tag{4.19}$$

for any M, N and n by inductions on the length of M and N. In particular, F is fully faithful.

By (4.19), we have $\operatorname{Ext}^1(F(P(i)), L(j)) = 0$ for any i, $j \in \Lambda$. Moreover, we have

$$\operatorname{Ext}^1(F(P(i)), M) = 0$$

for any $i \in \Lambda$ and M by induction on the length of M. Thus $F(P(i))$ is a projective object in $\mathcal{O}$. Since, $\operatorname{End}(F(P(i))) \simeq \operatorname{End}(P(i))$ is a local ring, $F(P(i))$ is indecomposable. On the other hand, $F(P(i))$ has $F(L(i)) \simeq L(i)$ as its quotient. Therefore, we have $F(P(i)) \simeq P(i)$ for all $i \in \Lambda$.

Let A be a finite dimensional algebra such that $\mathcal{O} \simeq A$-mod. Since $F(P(i)) \simeq P(i)$ for all $i \in \Lambda$, we have $F(A) \simeq A$.

Therefore we have

$$F(M) \simeq F(A \otimes_A M) \simeq F(A) \otimes_A M \simeq M.$$

for any $M \in \mathcal{O}$. Therefore F is an equivalence of categories. □

Remark 4.16. Since we proved $\widehat{\mathcal{S}}^\theta_\lambda(L_\lambda(i)) \simeq L_{\lambda+\theta}(i)$ in Corollary 4.13, we actually do not need the first part of the above proof.

Theorem 4.17. *For $\lambda \in \mathbb{R}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$, the shift functor $\widehat{\mathcal{S}}^\theta_\lambda : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_{\lambda+\theta}$ is an equivalence of categories.*

Proof. By Corollary 4.6, we have an equivalence $\mathcal{O}_\lambda \simeq \mathcal{O}_{\lambda+\theta}$. By Proposition 4.14 $\widehat{\mathcal{S}}^\theta_\lambda$ is an exact functor. The assumption of Proposition 4.15 is satisfied for $F = \widehat{\mathcal{S}}^\theta_\lambda$ by Proposition 4.11. Then $\widehat{\mathcal{S}}^\theta_\lambda$ is an equivalence of categories. □

Corollary 4.18. *The shift functor $\mathcal{S}_\lambda^\theta$ is an equivalence of categories between $\mathcal{A}_\lambda$-Mod and $\mathcal{A}_{\lambda+\theta}$-Mod, and $\widehat{\mathcal{S}}_\lambda^\theta$ is an equivalence of categories between $\mathcal{T}_\lambda$-Mod and $\mathcal{T}_{\lambda+\theta}$-Mod.*

Proof. This proof is essentially same as the proof of Theorem 3.3 in [GS1].

To prove the equivalence, we show that $\mathcal{B}_{\lambda+\theta}^{-\theta} \otimes_{\mathcal{A}_{\lambda+\theta}} \mathcal{B}_\lambda^\theta \simeq \mathcal{A}_\lambda$ and $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} \mathcal{B}_{\lambda+\theta}^{-\theta} \simeq \mathcal{A}_{\lambda+\theta}$. Assume that $I := \mathcal{B}_{\lambda+\theta}^{-\theta} \otimes_{\mathcal{A}_{\lambda+\theta}} \mathcal{B}_\lambda^\theta \simeq \mathcal{B}_{\lambda+\theta}^{-\theta}\mathcal{B}_\lambda^\theta \neq \mathcal{A}_\lambda$. Then I is a proper two-sided ideal of $\mathcal{A}_\lambda$. By the generalized Duflo theorem proved in [Gi], I annihilates a irreducible module $e_0 L_\lambda(i)$ for some $i = 0, \ldots, l-1$. However, by Theorem 4.17, we have

$$I e_0 L_\lambda(i) \simeq \mathcal{B}_{\lambda+\theta}^{-\theta} \otimes_{\mathcal{A}_{\lambda+\theta}} \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0 L_\lambda(i) \simeq e_0 L_\lambda(i).$$

Therefore $I = \mathcal{A}_\lambda$. The second isomorphism can be proved similarly. □

In the rest of this subsection, we consider the $(\mathcal{A}_{\lambda+\theta}, \mathcal{T}_\lambda)$-bimodule $\mathcal{S}_\lambda^\theta(e_0\mathcal{T}_\lambda) = \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$.

First we define a space

$$\mathcal{M}_\lambda = M_l(D(\mathrm{Rep}(Q,\delta))) \Big/ \sum_{i=0}^{l-1} M_l(D(\mathrm{Rep}(Q,\delta)))(\tau(e^{(i)}) - \lambda_i).$$

Then, consider the following natural maps:

$$\begin{aligned} \phi_1 : e_0\mathcal{T}_\lambda = e_0\mathcal{M}_\lambda^{GL(\delta)} &\longrightarrow e_0\mathcal{M}_\lambda, \\ \phi_2 : e_0\mathcal{A}_\lambda = e_0\mathcal{T}_\lambda e_0 &\longrightarrow e_0\mathcal{M}_\lambda, \\ \phi_3 : \mathcal{B}_\lambda^\theta = e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta} e_0 &\longrightarrow e_0\mathcal{M}_\lambda. \end{aligned}$$

The above maps ϕ_1, ϕ_2 and ϕ_3 are clearly injective. Then, these injective maps induce a map

$$\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda \longrightarrow \mathcal{B}_\lambda^\theta e_0\mathcal{T}_\lambda \subset e_0\mathcal{M}_\lambda.$$

Clearly the image of this map is inside the subspace $e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta}$, thus we have the following map:

$$\begin{aligned} \Theta : \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda &\longrightarrow e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta}, \\ b \otimes m &\mapsto \phi_3(b)\phi_1(m). \end{aligned} \tag{4.20}$$

It is a homomorphism of $(\mathcal{A}_{\lambda+\theta}, \mathcal{T}_\lambda)$-bimodules.

Lemma 4.19 ([Bo], Lemma 6.8)**.** *Let B be a left Ore domain and P an (A, B)-bimodule which yields Morita equivalence between A and B. If P' is torsion free A-module, then every surjective homomorphism $P \longrightarrow P'$ is isomorphism.*

Set

$$\widetilde{\mathbb{R}}_{reg}^l = \{(\lambda_i)_{i=0,\ldots,l-1} \in \mathbb{R}_{reg}^l \mid \lambda_i + \cdots + \lambda_{j-1} \neq 0 \quad \text{for all } i \neq j\}. \tag{4.21}$$

Lemma 4.20. *For $\lambda \in \widetilde{\mathbb{R}}^l_{reg}$, $\theta \in \mathbb{Z}^l_\lambda$, the homomorphism Θ is injective.*

Proof. Set $\mathcal{A}^{(i)}_\lambda = e_i \mathcal{T}_\lambda e_i$ for each $i = 0, \ldots, l-1$. Each $\mathcal{A}^{(i)}_\lambda$ is a left Ore domain. The $(\mathcal{T}_\lambda, \mathcal{A}^{(i)}_\lambda)$-bimodule $\mathcal{T}_\lambda e_i$ yields a Morita equivalence when $\lambda \in \widetilde{\mathbb{R}}^l_{reg}$ by the same argument as in Proposition 4.4 for the $(\mathcal{A}_\lambda, \mathcal{T}_\lambda)$-bimodule $e_0\mathcal{T}_\lambda$. Therefore $\mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0 \mathcal{T}_\lambda e_i = \mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0 \mathcal{T}_\lambda \otimes_{\mathcal{T}_\lambda} \mathcal{T}_\lambda e_i$ yields a Morita equivalence between $\mathcal{A}_{\lambda+\theta}$ and $\mathcal{A}^{(i)}_\lambda$. The module $\mathcal{B}^\theta_\lambda e_0 \mathcal{T} e_i$ is a torsion free $\mathcal{A}_{\lambda+\theta}$-module. Applying Lemma 4.19 to $P = \mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0 \mathcal{T}_\lambda e_i$ and $P' = \mathcal{B}^\theta_\lambda e_0 \mathcal{T}_\lambda e_i$, we have $\mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0 \mathcal{T}_\lambda e_i \simeq \mathcal{B}^\theta_\lambda e_0 \mathcal{T}_\lambda e_i \subseteq e_0 \mathcal{M}^{GL(\delta), \chi_\theta}_\lambda e_i$ for $i = 0, \ldots, l-1$. Therefore we have $\mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0 \mathcal{T}_\lambda \simeq \mathcal{B}^\theta_\lambda e_0 \mathcal{T}_\lambda \subseteq e_0 \mathcal{M}^{GL(\delta), \chi_\theta}_\lambda$. $\square$

4.5. q-dimension of representations

In this subsection we calculate the q-dimension of the module $\mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} \mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0 \mathcal{T}_\lambda$. This result is used in Section 6.2 to prove our main theorem, Theorem 6.1.

Consider the Euler operator

$$\mathrm{eu}_\lambda = \sum_{i=0}^{l-1} A_i A_i^* - \sum_{i=0}^{l-1} c_i(\lambda) e_i \in \mathcal{T}_\lambda$$

where $c_i(\lambda) \in \mathbb{R}$ such that $c_{i+1}(\lambda) - c_i(\lambda) = l\kappa_{i+1} - l\kappa_i = l\lambda_i - 1$ and $c_0(\lambda) + c_1(\lambda) + \cdots + c_{l-1}(\lambda) = 1$.

For $\mathbf{1}_i \in \Delta_\lambda(i)$, eu_λ acts as follows:

$$\mathrm{eu}_\lambda \mathbf{1}_i = -c_i(\lambda) \mathbf{1}_i. \tag{4.22}$$

The following lemma is proved by a straightforward calculation.

Lemma 4.21.
(1). $[\mathrm{eu}_\lambda, A_i] = A_i$.
(2). $[\mathrm{eu}_\lambda, A_i^*] = -A_i^*$.
(3). $[\mathrm{eu}_\lambda, e_i] = 0$.

A $\mathcal{T}_\lambda$-module (or $\mathcal{A}_\lambda$-module) M is called a graded module if M has a vector space decomposition $M = \bigoplus_m M_m$ such that $A_i M_m \subseteq M_{m+1}$, $A_i^* M_m \subseteq M_{m-1}$ and $e_i M_m \subseteq M_m$ for all i.

For a graded module $M = \bigoplus_m M_m$ and $k \in \mathbb{R}$, let $M[k]$ be the graded module shifted degree by k, i.e.

$$(M[k])_m = M_{m+k}.$$

For a module $M \in \mathcal{O}_\lambda$, M has a vector space decomposition $M = \bigoplus_m M_m$ where M_m is the generalized eigenspace for an eigenvalue m with respect to eu_λ. By Lemma 4.21, this decomposition makes M a graded module. This grading is called the canonical grading.

For a standard module $\Delta_\lambda(i)$ ($i = 0, \ldots, l-1$), let $\widetilde{\Delta}_\lambda(i)$ be a graded module which is isomorphic to $\Delta_\lambda(i)$ as an ungraded module, and

$$\widetilde{\Delta}_\lambda(i) = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \left(\widetilde{\Delta}_\lambda(i)\right)_m, \qquad \left(\widetilde{\Delta}_\lambda(i)\right)_m = \mathbb{C}A^m\mathbf{1}_i.$$

Considering $\Delta_\lambda(i)$ to be a graded $\mathcal{T}_\lambda$-module with the canonical grading, we have

$$\Delta_\lambda(i) \simeq \widetilde{\Delta}_\lambda(i)[-c_i(\lambda)] \tag{4.23}$$

as a graded $\mathcal{T}_\lambda$-module by (4.22).

For a $(\mathcal{T}_{\lambda'}, \mathcal{T}_\lambda)$-bimodule M, let ${}_{\lambda'}\mathrm{eu}_\lambda$ be the operator on M:

$${}_{\lambda'}\mathrm{eu}_\lambda(m) = \mathrm{eu}_{\lambda'} \cdot m - m \cdot \mathrm{eu}_\lambda.$$

for $m \in M$. We have the decomposition of M,

$$M = \bigoplus_n M_n$$

where M_n is the generalized eigenspace for an eigenvalue n with respect to the operator ${}_{\lambda'}\mathrm{eu}_\lambda$. By this decomposition, M is a graded module. This grading is called the adjoint grading.

The $(\mathcal{T}_\lambda, \mathcal{T}_\lambda)$-bimodule $\mathcal{T}_\lambda$ has the decomposition

$$\mathcal{T}_\lambda = \bigoplus_{n\in\mathbb{Z}} (\mathcal{T}_\lambda)_n$$

where $(\mathcal{T}_\lambda)_n$ is the eigenspace for an eigenvalue $n \in \mathbb{Z}$ with respect to the operator ${}_\lambda\mathrm{eu}_\lambda$. This grading coincides with the grading given by the degree,

$$\deg E_{ij} \otimes t_k = 1, \quad \deg E_{ij} \otimes \partial_k = -1.$$

For an $(\mathcal{A}_{\lambda'}, \mathcal{A}_\lambda)$-bimodule M, let ${}_{\lambda'}\mathrm{eu}_\lambda$ be the operator on M,

$${}_{\lambda'}\mathrm{eu}_\lambda(m) = e_0\mathrm{eu}_{\lambda'}e_0 \cdot m - m \cdot e_0\mathrm{eu}_\lambda e_0.$$

Then, the operator ${}_{\lambda'}\mathrm{eu}_\lambda$ gives M the structure of a graded module. We also call this grading the adjoint grading.

Consider the following decomposition of $\mathcal{B}_\lambda^\theta$,

$$\mathcal{B}_\lambda^\theta = \bigoplus_{n\in\mathbb{Z}} \left(\mathcal{B}_\lambda^\theta\right)_n$$

where $\left(\mathcal{B}_\lambda^\theta\right)_n$ is the eigenspace for an eigenvalue n with respect to the operator ${}_{\lambda+\theta}\mathrm{eu}_\lambda$. By the above decomposition, $\mathcal{B}_\lambda^\theta$ is a graded module. This grading coincides with the grading given by the degree,

$$\deg t_k = 1, \quad \deg \partial_k = -1.$$

In the rest of this subsection, we assume $\lambda \in \widetilde{\mathbb{R}}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$.

Lemma 4.22. *For $i = 0, \ldots, l-1$, we consider the grading of the module*

$$\mathcal{S}^\theta_\lambda(\widetilde{\Delta}_\lambda(i)) = \mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0\widetilde{\Delta}_\lambda(i)$$

induced from the adjoint grading of $\mathcal{B}^\theta_\lambda$ and the grading of $\widetilde{\Delta}_\lambda(i)$. Then we have

$$\mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0\widetilde{\Delta}_\lambda(i) \simeq e_0\widetilde{\Delta}_{\lambda+\theta}(i)[d^\theta_i]$$

where d^θ_i are the integers defined on (3.18).

Proof. We have the two gradings on the standard modules $\Delta_\lambda(i)$ and $\Delta_{\lambda+\theta}(i)$. Let $\deg_{can}$ be the degree defined by the canonical grading and let $\deg_\lambda$ be the degree defined by the grading of $\widetilde{\Delta}_\lambda(i)$. For the other graded modules, let deg be the degree of the grading of each module.

We have

$$\mathcal{B}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} e_0\Delta_\lambda(i) \simeq e_0\Delta_{\lambda+\theta}(i)$$

as ungraded modules by Proposition 4.11. Considering the canonical grading, we have

$$\deg_{can}(b \otimes v) = \deg b + \deg_{can} v$$

for $b \in \mathcal{B}^\theta_\lambda$ and $v \in e_0\widetilde{\Delta}_\lambda(i)$.

By (4.23), we have

$$\begin{aligned}\deg_{can}(v) &= \deg_\lambda(v) - c_i(\lambda)\\ \deg_{can}(v') &= \deg_{\lambda+\theta}(v') - c_i(\lambda+\theta)\end{aligned}$$

for $v \in \Delta_\lambda(i)$ and $v' \in \Delta_{\lambda+\theta}(i)$. Therefore we have

$$\begin{aligned}\deg_{\lambda+\theta}(b \otimes v) &= \deg_{can}(b \otimes v) + c_i(\lambda+\theta)\\ &= \deg(b) + \deg_{can}(v) + c_i(\lambda+\theta)\\ &= \deg(b) + \deg_\lambda(v) + c_i(\lambda+\theta) - c_i(\lambda)\\ &= \deg(b \otimes v) + c_i(\lambda+\theta) - c_i(\lambda).\end{aligned}$$

On the other hand, we can easily obtain

$$\begin{gathered}d^\theta_{i+1} - d^\theta_i = l\theta_i = (c_{i+1}(\lambda+\theta) - c_i(\lambda+\theta)) - (c_{i+1}(\lambda) - c_i(\lambda)),\\ d^\theta_0 + d^\theta_1 + \cdots + d^\theta_{l-1} = 0.\end{gathered}$$

Thus we obtain

$$c_i(\lambda+\theta) - c_i(\lambda) = d^\theta_i.$$

□

For a graded module $M=\bigoplus_m M_m$, define the q-dimension of M as a formal series

$$\dim_q M=\sum_m(\dim M_m)q^m.$$

By the above lemma, we have the q-dimension of $\mathcal{S}_\lambda^\theta(e_0\widetilde{\Delta}_\lambda(i))=\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\widetilde{\Delta}_\lambda(i)$,

$$\dim_q\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\widetilde{\Delta}_\lambda(i)=q^{d_i^\theta+l-i}\frac{1}{1-q^l}$$

for $i=0,\ldots,l-1$. Here we set $d_l^\theta=d_0^\theta$.

The adjoint gradings of $\mathcal{B}_\lambda^\theta$ and $\mathcal{T}_\lambda$ induce the gradings of the $(\mathcal{A}_{\lambda+\theta},\mathcal{T}_\lambda)$-bimodule $\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}\mathcal{T}_\lambda$. They also induce the grading of the left $\mathcal{A}_{\lambda+\theta}$-module $\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}\mathcal{T}_\lambda\otimes_{\mathbb{C}[A^*]}\mathbb{C}$ and the grading of the right $\mathcal{T}_\lambda$-module $\mathbb{C}\otimes_{\mathbb{C}[t_0\cdots t_{l-1}]}\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}\mathcal{T}_\lambda$.

By (4.5), we have the following natural isomorphism as graded $\mathcal{A}_{\lambda+\theta}$-modules

$$\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda\otimes_{\mathbb{C}[A^*]}\mathbb{C}\simeq\bigoplus_{i=0}^{l-1}\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\widetilde{\Delta}_\lambda(i)$$

By the above equations, we have

$$\dim_q\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda\otimes_{\mathbb{C}[A^*]}\mathbb{C}=q^{d_0^\theta}\frac{1}{1-q^l}+\sum_{i=1}^{l-1}q^{d_i^\theta+l-i}\frac{1}{1-q^l}.$$

Lemma 4.23 ([GS1], Theorem A.1)**.** *Let R be a connected $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra. Let P be an R-module that is both graded and projective. Then P is a graded-free R-module in the sense that P has a free basis of homogeneous elements.*

Lemma 4.24.

(1). *The module $\mathcal{S}_\lambda^\theta(e_0\mathcal{T}_\lambda)=\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda$ is graded-free as a left $\mathbb{C}[t_0\cdots t_{l-1}]$-module and graded-free as a right $\mathbb{C}[A^*]$-module.*

(2). *$\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda\otimes_{\mathbb{C}[A^*]}\mathbb{C}$ is a finitely generated, graded-free $\mathbb{C}[t_0\cdots t_{l-1}]$-module.*

(3). *$\mathbb{C}\otimes_{\mathbb{C}[t_0\cdots t_{l-1}]}\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda$ is a finitely generated, graded-free right $\mathbb{C}[A^*]$-module.*

Proof. The following proof is essential the same as the proof of [GS1, Lemma 6.11].

(1) By Proposition 4.4 and Corollary 4.18, $\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda$ is projective as a left $\mathcal{A}_{\lambda+\theta}$-module and a right $\mathcal{T}_\lambda$-module. By the structure of the graded module $\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda$ which is defined above, $\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda$ is graded as a left $\mathbb{C}[t_0\cdots t_{l-1}]$-module and a right $\mathbb{C}[A^*]$-module. By Lemma 4.23, $\mathcal{B}_\lambda^\theta\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda$ is graded-free as a $\mathbb{C}[t_0\cdots t_{l-1}]$-module and a right $\mathbb{C}[A^*]$-module.

(2) By (4.5) and Proposition 4.11, we have

$$\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda \otimes_{\mathbb{C}[A^*]} \mathbb{C} \simeq \bigoplus_{i=0}^{l-1} e_0\Delta_{\lambda+\theta}(i) \simeq \mathbb{C}[t_0 \cdots t_{l-1}] \otimes \mathbb{C}\mathbb{Z}_l.$$

Therefore, $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda \otimes_{\mathbb{C}[A^*]} \mathbb{C}$ is graded-free.

(3) First, we show that $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is a finitely generated right module over $R = (\mathbb{C}[t_0 \cdots t_{l-1}])^{\mathrm{op}} \otimes_{\mathbb{C}} \mathbb{C}[A^*]$. By Lemma 4.20, $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda \subset e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta}$. Thus $\operatorname{gr}\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda \subset \operatorname{gr} e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta} = e_0M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta),\chi_\theta}$, which is certainly a noetherian $\mathbb{C}[t_0 \cdots t_{l-1}] \otimes \mathbb{C}[A^*]$-module. The $\mathbb{C}[t_0 \cdots t_{l-1}] \otimes \mathbb{C}[A^*]$-module structure of $\operatorname{gr}\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is the one induced from the R-module structure of $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$. Therefore, $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is finitely generated.

By (2), $\Sigma = \{A^*, (t_0 \cdots t_l - 1)\}$ is a regular sequence for the right R-module $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$. In particular, if $\mathfrak{n} = A^*R + (t_0 \cdots t_{l-1})R$, then Σ is a regular sequence for the $R_\mathfrak{n}$-module $(\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda)_\mathfrak{n}$. By the Auslander-Buchsbaum formula [Ma, Ex. 4, p.114], $(\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda)_\mathfrak{n}$ is free as a $R_\mathfrak{n}$-module.

Finally, $\mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]} \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is a finitely generated, graded $\mathbb{C}[A^*]$-module and so corresponds to a $\mathbb{C}^*$-equivariant coherent sheaf on $\mathbb{C}$. Therefore, the locus where $\mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]} \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is not free is a $\mathbb{C}^*$-stable closed subvariety of $\mathbb{C}$. If this locus is non-empty, it must contain $0 \in \mathbb{C}$. By the conclusion of the last paragraph, the stalk at $0 \in \mathbb{C}$ of $\mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]} \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is free. Therefore, $\mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]} \mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ must be free. Since $\mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]}$ $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is graded module, it is graded free by Lemma 4.23. □

By Lemma 4.24, a homogeneous $\mathbb{C}[A^*]$-basis of $\mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]} \otimes B_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is given by a homogeneous $\mathbb{C}$-basis of $\mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]} \otimes B_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda \otimes_{\mathbb{C}[A^*]} \mathbb{C}$. Therefore we have the following proposition.

Proposition 4.25. *We have*

$$\dim_q \mathbb{C} \otimes_{\mathbb{C}[t_0\cdots t_{l-1}]} \otimes B_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda = \sum_{i=1}^{l} q^{d_i^\theta + (l-i)} \frac{1}{1-q^{-1}}. \tag{4.24}$$

5. Gordon-Stafford functors

5.1. $\mathbb{Z}$-algebras

In this section, we define the functor $\widehat{\Phi}_\lambda^\theta$ of (1.5) as Boyarchenko defined it in [Bo]. First we review the definition and basic properties of $\mathbb{Z}$-algebras.

Definition 5.1. A lower-triangular $\mathbb{Z}$-algebra B is an algebra such that

(1). B is bigraded by $\mathbb{Z}$ in the following way: $B = \bigoplus_{i\geq j\geq 0} B_{ij}$.

(2). The multiplication of B is defined in matrix fashion, i.e., B satisfies $B_{ij}B_{jk} \subseteq B_{ik}$ for $i \geq j \geq k \geq 0$ but $B_{ij}B_{lk} = 0$ if $j \neq l$.

(3). B_{ii} is an unital subalgebra for all $i \in \mathbb{Z}_{\geq 0}$.

We also define graded modules of lower-triangular $\mathbb{Z}$-algebras. Let B be a lower-triangular $\mathbb{Z}$-algebra. A graded B-module is $\mathbb{Z}_{\geq 0}$-graded left B-module $M = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M_i$, such that $B_{ij} M_j \subseteq M_i$ for all $i \geq j \geq 0$ and $B_{ij} M_k = 0$ if $j \neq k$. Homomorphisms of graded B-modules are defined to be graded homomorphisms of degree zero.

We denote the category of graded B-modules by B-Grmod, and denote the subcategory of finitely generated graded B-modules by B-grmod.

A graded module $M = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M_i \in B$-Grmod is bounded if $M_i = 0$ all but finitely many $i \in \mathbb{Z}_{\geq 0}$, and torsion if it is a direct limit of bounded modules. We denote the subcategory of torsion modules in B-Grmod by B-Tor, and the subcategory of bounded modules in B-grmod by B-tor. The corresponding quotient categories are written $B\text{-Qgr} = B\text{-Grmod}/B\text{-Tor}$ and $B\text{-qgr} = B\text{-grmod}/B\text{-tor}$.

For a graded commutative algebra $S = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} S_m$, we define a lower-triangular $\mathbb{Z}$-module $\widehat{S} = \bigoplus_{i \geq j \geq 0} \widehat{S}_{ij}$ where $\widehat{S}_{ij} = S_{i-j}$ for $i \geq j \geq 0$. Define the categories S-Grmod, $\ldots$, $\widehat{S}$-qgr in the usual manner. Then, as in Section 5.3 of [GS1], we have equivalences of categories:

$$
\begin{gathered}
S\text{-Qgr} \longrightarrow \widehat{S}\text{-Qgr}, \\
S\text{-qgr} \longrightarrow \widehat{S}\text{-qgr}, \\
M = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M_i \mapsto M = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M_i.
\end{gathered}
\tag{5.1}
$$

We define another example of $\mathbb{Z}$-algebra called a Morita $\mathbb{Z}$-algebra. Suppose we have countably many Morita equivalent algebras $\{B_i\}_{i \in \mathbb{Z}_{\geq 0}}$ and B_{ij} is a (B_i, B_j)-bimodule which yields Morita equivalences for $i > j \geq 0$. Moreover, suppose we have an isomorphism $B_{ij} \otimes_{B_j} B_{jk} \simeq B_{ik}$ for $i \geq j \geq k \geq 0$. Set $B_{ii} = B_i$ and define the Morita $\mathbb{Z}$-algebra B to be $B = \bigoplus_{i \geq j \geq 0} B_{ij}$. Note that our definition of Morita $\mathbb{Z}$-algebras is same as one of [GS1], and it requires a stronger condition than one of [Bo].

Lemma 5.2 (GS1, Lemma 5.5). *Assume B_0 is noetherian, then*

(1). *Each finitely generated graded left B-module is noetherian.*

(2). *The association $\phi : M \mapsto \bigoplus_{i \in \mathbb{Z}_{\geq 0}} B_{i0} \otimes_{B_0} M$ induces an equivalence of categories between B_0-mod and B-qgr.*

5.2. Construction of the functor

For a stability parameter $\theta \in \mathbb{Z}^l_{reg}$, set

$$
S = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} S_m, \qquad S_m = \mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi_\theta^m}
$$

as defined in Section 3.1. By Proposition 3.3, we have an isomorphism $\mathfrak{M}_\theta(\delta) \simeq \operatorname{Proj} S$.

Let $\widehat{S} = \bigoplus_{i \geq j \geq 0} \widehat{S}_{ij}$ be the lower-triangular $\mathbb{Z}$-algebra obtained from the above graded algebra S. By (5.1), we have an equivalence of categories $\widehat{S}$-qgr $\simeq$ $\mathrm{Coh}(\mathfrak{M}_\theta(\delta))$.

Fix $\lambda \in \mathbb{R}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$. Recall the algebra $\mathcal{A}_\lambda$ of (4.2) and the bimodule $\mathcal{B}^\theta_\lambda$ of (4.12). Set $B_i = \mathcal{A}_{\lambda+i\theta}$ for $i \in \mathbb{Z}_{\geq 0}$ and set $B_{ij} = \mathcal{B}^{(i-j)\theta}_{\lambda+j\theta}$ for $i > j \geq 0$. By Corollary 4.18, B_{ij} is a (B_i, B_j)-bimodule which yields a Morita equivalence.

Proposition 5.3. *For $\lambda \in \mathbb{R}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$, we have the isomorphism*

$$\mathcal{B}^{(i-j)\theta}_{\lambda+j\theta} \otimes_{\mathcal{A}_{\lambda+j\theta}} \mathcal{B}^{(j-k)\theta}_{\lambda+k\theta} \simeq \mathcal{B}^{(i-k)\theta}_{\lambda+k\theta}.$$

Proof. We apply Lemma 4.19 for the algebras $A = \mathcal{A}_{\lambda+i\theta}$, $B = \mathcal{A}_{\lambda+k\theta}$ and the modules $P = \mathcal{B}^{(i-j)\theta}_{\lambda+j\theta} \otimes_{\mathcal{A}_{\lambda+j\theta}} \mathcal{B}^{(j-k)\theta}_{\lambda+k\theta}$, $P' = \mathcal{B}^{(i-k)\theta}_{\lambda+k\theta}$. It is clear that B is a left Ore domain, P' is torsion free and P yields an equivalence of categories by Corollary 4.18. We have the surjective homomorphism

$$\begin{aligned} P = \mathcal{B}^{(i-j)\theta}_{\lambda+j\theta} \otimes_{\mathcal{A}_{\lambda+j\theta}} \mathcal{B}^{(j-k)\theta}_{\lambda+k\theta} &\longrightarrow P' = \mathcal{B}^{(i-k)\theta}_{\lambda+k\theta} \\ b_1 \otimes b_2 &\mapsto b_1 b_2 \end{aligned}$$

Thus, by the above lemma, it is an isomorphism. $\square$

By the above proposition, we have the Morita $\mathbb{Z}$-algebra $B = \bigoplus_{i \geq j \geq 0} B_{ij}$ where $B_{ii} = B_i = \mathcal{A}_{\lambda+i\theta}$. By Lemma 5.2, we have an equivalence of categories

$$\begin{aligned} \mathcal{A}_\lambda\text{-mod} &\longrightarrow B\text{-qgr}, \\ M &\mapsto \widetilde{M} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathcal{B}^{m\theta}_\lambda \otimes_{\mathcal{A}_\lambda} M. \end{aligned}$$

The algebra $\mathcal{A}_\lambda$ and the bimodule $\mathcal{B}^\theta_\lambda$ are filtered by the order of differential operators in $D(\mathrm{Rep}(Q, \delta))$, and we have

$$\begin{aligned} \mathrm{gr}\, \mathcal{A}_\lambda &\simeq \mathbb{C}[\mu^{-1}(0)]^{GL(\delta)} = S_0, \\ \mathrm{gr}\, \mathcal{B}^{m\theta}_\lambda &\simeq \mathbb{C}[\mu^{-1}(0)]^{GL(\delta), \chi^m_\theta} = S_m. \end{aligned}$$

The filtration induces the filtration on the Morita $\mathbb{Z}$-algebra $B = \bigoplus_{i \geq j \geq 0} B_{ij}$. Thus we have the following theorem as in [GS1] and [Bo].

Theorem 5.4. *For $\lambda \in \mathbb{R}^l_{reg}$, $\theta \in \mathbb{Z}^l_\lambda$, define $\mathbb{Z}$-algebras B and $\widehat{S}$ as above. Then*:

(1). *There is an equivalence of categories $\mathcal{A}_\lambda$-mod $\simeq B$-qgr.*

(2). *There is an isomorphism of lower-triangular $\mathbb{Z}$-algebras $\mathrm{gr}\, B \simeq \widehat{S}$.*

(3). *We have an equivalence of categories $\widehat{S}$-qgr $\simeq \mathrm{Coh}(\mathfrak{M}_\theta(\delta))$.*

Let $\mathcal{T}_\lambda$-Filt, $\mathcal{A}_\lambda$-Filt be categories of filtered modules. Given $(M, \Lambda) \in \mathcal{T}_\lambda$-Filt (resp. $\in \mathcal{A}_\lambda$-Filt), we say Λ is a good filtration on M if $\mathrm{gr}_\Lambda M$ is a finitely generated $\mathrm{gr}\, \mathcal{T}_\lambda$-module (resp. $\mathrm{gr}\, \mathcal{A}_\lambda$-module). We denote the subcategories of good filtered modules by $\mathcal{T}_\lambda$-filt and $\mathcal{A}_\lambda$-filt.

Let $(M,\Lambda)\in\mathcal{A}_\lambda$-Filt. Then each module $M(i)=\mathcal{B}_\lambda^{i\theta}\otimes_{\mathcal{A}_\lambda}M$ is filtered by the tensor product filtration

$$\Lambda^k M(i)=\sum_{l\in\mathbb{Z}}F^l\mathcal{B}_\lambda^{i\theta}\otimes\Lambda^{k-l}M$$

where F is the filtration of $\mathcal{B}_\lambda^{i\theta}$. Therefore, the B-module $\widetilde{M}=\bigoplus_{i\in\mathbb{Z}_{\geq 0}}M(i)$ is filtered and we have the graded $\widehat{S}$-module $\operatorname{gr}\widetilde{M}=\bigoplus_{i\in\mathbb{Z}_{\geq 0}}\operatorname{gr}M(i)$ associated to $\widetilde{M}$. For a graded B-module with filtration $(\widetilde{M},\Lambda)$, we call Λ is a good filtration if $\operatorname{gr}_\Lambda\widetilde{M}$ is a finitely generated $\widehat{S}$-module. The following lemma is due to [GS2].

Lemma 5.5 ([GS2], Lemma 2.5)**.** *If Λ is a good filtration of M, then the induced filtration Λ on $\widetilde{M}$ is also good.*

As in [Bo], we define the functor

$$\Phi_\lambda^\theta:\mathcal{A}_\lambda\text{-Filt}\longrightarrow\operatorname{Qcoh}(\mathfrak{M}_\theta(\delta)),$$
$$(M,\Lambda)\mapsto\operatorname{gr}_\Lambda\widetilde{M}.$$

By restricting Φ_λ^θ to $\mathcal{A}_\lambda$-filt, we have a functor

$$\Phi_\lambda^\theta:\mathcal{A}_\lambda\text{-filt}\longrightarrow\operatorname{Coh}(\mathfrak{M}_\theta(\delta)).$$

We also define the functor from $\mathcal{T}_\lambda$-Filt and $\mathcal{T}_\lambda$-filt

$$\widehat{\Phi}_\lambda^\theta=\Phi_\lambda^\theta\circ E_\lambda:\mathcal{T}_\lambda\text{-Filt}\longrightarrow\operatorname{Qcoh}(\mathfrak{M}_\theta(\delta)),$$
$$\widehat{\Phi}_\lambda^\theta=\Phi_\lambda^\theta\circ E_\lambda:\mathcal{T}_\lambda\text{-filt}\longrightarrow\operatorname{Coh}(\mathfrak{M}_\theta(\delta)).$$

We call the above functors Φ_λ^θ and $\widehat{\Phi}_\lambda^\theta$ the Gordon-Stafford functors.

6. Construction of a tautological bundle

6.1. Main theorem

Now we consider our main theorem. We determine the image of the module $e_0\mathcal{T}_\lambda$ by the functor Φ_λ^θ.

Recall the set of parameter $\widetilde{\mathbb{R}}^l_{reg}$ of (4.21). Fix the parameters $\lambda=(\lambda_i)_{i=0,\dots,l-1}\in\widetilde{\mathbb{R}}^l_{reg}$ and $\theta=(\theta_i)_{i=0,\dots,l-1}\in\mathbb{Z}^l_\lambda$. Consider an $\mathcal{A}_\lambda$-module $e_0\mathcal{T}_\lambda$. Considering the filtration by the order of differential operators in $D(\operatorname{Rep}(Q,\delta))$, e_0T_λ is a filtered A_λ-module with good filtration. The following theorem is the main result of this paper.

Theorem 6.1. *For $\lambda\in\widetilde{\mathbb{R}}^l_{reg}$, $\theta\in\mathbb{Z}^l_\lambda$ and $m\in\mathbb{Z}_{\geq 0}$, we have the isomorphism*

$$\operatorname{gr}\mathcal{B}_\lambda^{m\theta}\otimes_{\mathcal{A}_\lambda}e_0\mathcal{T}_\lambda\simeq e_0M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta),\chi_\theta^m}.$$

Therefore we have

$$\Phi_\lambda^\theta(e_0\mathcal{T}_\lambda)\simeq\widetilde{\mathcal{P}}_\theta$$

as coherent sheaves on $\mathfrak{M}_\theta(\delta)$.

In the next subsection, we will give the proof of the above theorem.

6.2. Proof of the main theorem

In this subsection, we complete the proof of our main theorem, Theorem 6.1.

First, we construct the homomorphism of Theorem 6.1. The module $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is filtered by the tensor product filtration. The module $e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta}$ is filtered by the order of differential operators in $D(\mathrm{Rep}(Q,\delta))$. Clearly, the homomorphism Θ of (4.20) is a homomorphism of filtered modules. Thus we have the associated homomorphism between the associated graded modules

$$\mathrm{gr}\,\Theta : \mathrm{gr}\left(\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda\right) \longrightarrow \mathrm{gr}\, e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta} = e_0 M_l(\mathbb{C}[\mu^{-1}(0)])^{GL(\delta),\chi_\theta}.$$

This homomorphism was what we wanted to construct.

The $(\mathcal{A}_{\lambda+\theta}, \mathcal{T}_\lambda)$-bimodule $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ is graded by the grading induced from the adjoint gradings of $\mathcal{B}_\lambda^\theta$ and $e_0\mathcal{T}_\lambda$. The $(\mathcal{A}_{\lambda+\theta}, \mathcal{T}_\lambda)$-bimodule $e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta}$ is graded by the adjoint grading with respect to the operator ${}_{\lambda+\theta}\mathrm{eu}_\lambda$.

Lemma 6.2. *The homomorphism Θ is homogeneous with respect to the gradings of $\mathcal{B}_\lambda^\theta \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda$ and $e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta}$.*

By Lemma 4.20 we have the injective homomorphism

$$\Theta : \mathcal{B}_\lambda^{m\theta} \otimes_{\mathcal{A}_\lambda} e_0\mathcal{T}_\lambda \simeq \mathcal{B}_\lambda^{m\theta} e_0\mathcal{T}_\lambda \longrightarrow e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}.$$

To complete the proof of theorem, we need to prove the equality of the inclusion $\mathcal{B}_\lambda^\theta e_0\mathcal{T}_\lambda \subseteq e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}$. The following proof is essentially the same as the proof [GS1, Section 6.17].

Lemma 6.3. *The modules $\mathcal{B}_\lambda^{m\theta} e_0\mathcal{T}_\lambda$ and $e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}$ are free as left $\mathbb{C}[t_0 \cdots t_{l-1}]$ modules.*

Lemma 6.4. *We have an equality of localized spaces*

$$(\mathcal{B}_\lambda^{m\theta} e_0\mathcal{T}_\lambda)[(t_0 \cdots t_{l-1})^{-1}] = e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}[(t_0 \cdots t_{l-1})^{-1}].$$

Proof. It is clear that

$$(\mathcal{B}_\lambda^{m\theta} e_0\mathcal{T}_\lambda)[(t_0 \cdots t_{l-1})^{-1}] \subseteq e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}[(t_0 \cdots t_{l-1})^{-1}].$$

Fix an arbitrary element $\sum_{i=0}^{l-1} f_i \otimes E_{0i} \in e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}$ where $f_i \in \mathcal{B}_\lambda^{m\theta+\tau_i}$. We show by induction that, for any $f \otimes E_{0i} \in F_n e_0\mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m} e_i$, there exists $p \in \mathbb{Z}_{\geq 0}$ such that

$$(t_0 \cdots t_{l-1})^p f \otimes E_{0i} \in \mathcal{B}_\lambda^{m\theta} e_0\mathcal{T}_\lambda. \tag{6.1}$$

For $n = 0$, we have

$$F_0 e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m} = e_0 M_l(\mathbb{C}[t_0, \ldots, t_{l-1}])^{GL(\delta),\chi_\theta^m}.$$

Thus we have

$$(t_0 \cdots t_{l-1}) f = (t_{i+1} \cdots t_{l-1} t_0 f)(t_1 \cdots t_i).$$

We have $(t_{i+1} \cdots t_{l-1} t_0 f) \in \mathcal{B}_\lambda^{m\theta}$ and $(t_1 \cdots t_i) \otimes E_{0i} \in e_0 \mathcal{T}_\lambda e_i$. Therefore we have (6.1) for $n = 0$.

Assume we have (6.1) for $n < n_0$. For $f \otimes E_{0i} \in F_{n_0} e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m} e_i$, we have

$$(t_0 \cdots t_{l-1}) f \otimes E_{0i} = f(t_0 \cdots t_{l-1}) \otimes E_{0i} + [t_0 \cdots t_{l-1}, f] \otimes E_{0i}.$$

The second term $[t_0 \cdots t_{l-1}, f] \otimes E_{0i}$ belongs to $F_{n_0-1} e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m} e_i$. By the hypothesis of the induction, there exists $p \in \mathbb{Z}_{\geq 0}$ such that

$$(t_0 \cdots t_{l-1})^p [t_0 \cdots t_{l-1}, f] \otimes E_{0i} \in \mathcal{B}_\lambda^{m\theta} e_0 \mathcal{T}_\lambda.$$

We have $(t_0 \cdots t_{l-1})^p (f t_{i+1} \cdots t_{l-1} t_0) \in \mathcal{B}_\lambda^{m\theta}$ and $(t_1 \cdots t_i) \otimes E_{0i} \in e_0 \mathcal{T}_\lambda e_i$. Therefore, we have

$$\begin{aligned}
&(t_0 \cdots t_{l-1})^{p+1} f \otimes E_{0i} \\
&\quad = (t_0 \cdots t_{l-1})^p f (t_{i+1} \cdots t_{l-1} t_0)(t_1 \cdots t_i) \otimes E_{0i} \\
&\qquad + (t_0 \cdots t_{l-1})^p [t_0 \cdots t_{l-1}, f] \otimes E_{0i} \in \mathcal{B}_\lambda^{m\theta} e_0 \mathcal{T}_\lambda.
\end{aligned}$$

Therefore we have (6.1) for $n = n_0$. □

Let $\{a_{gp}\}_{g,p}$ be a $\mathbb{C}[t_0 \cdots t_{l-1}]$-free basis of $\mathcal{B}_\lambda^{m\theta} e_0 \mathcal{T}_\lambda$ such that a_{gp} is a homogeneous vector of degree g. Let $\{b_{gq}\}_{g,q}$ be a $\mathbb{C}[t_0 \cdots t_{l-1}]$-free basis of $e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}$ such that b_{gq} is a homogeneous vector of degree g.

By Theorem 3.16 and Proposition 4.25, we have the equality

$$\begin{aligned}
\dim_q \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} \mathcal{B}_\lambda^{m\theta} e_0 \mathcal{T}_\lambda &= \sum_{i=1}^{l} q^{d_{i+1}^{m\theta} + (l-i-1)} \frac{1}{1 - q^{-1}} \\
&= \dim_q \mathbb{C} \otimes_{\mathbb{C}[t_0 \cdots t_{l-1}]} e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}.
\end{aligned} \tag{6.2}$$

By (6.2), we have:

(†1). For any $g \in \mathbb{R}$, $\{a_{gp}\}_p$ and $\{b_{gq}\}_q$ have finite cardinality.

(†2). There is $T \in \mathbb{R}$ such that there is no nonzero a_{gp} and b_{gq} when $g > T$.

(†3). For any $g \in \mathbb{R}$ $\#\{a_{gp}\}_p$ is equal to $\#\{b_{gq}\}_q$.

We show that we can adjust the basis $\{b_{gq}\}_q$ to be equal to the basis $\{a_{gp}\}_p$ by a downward induction on g. By (†3), we have $\{a_{gp}\}_p = \{b_{gq}\}_q = \emptyset$ for $g > T$.

Let $-\infty < G \leq T$, suppose that $\{b_{gq}\}_q = \{a_{gp}\}_p$ for all $g > G$ by induction. Suppose that there exists an element b_{Gq_0} which does not belong to

$\{a_{gp}\}_p$. By Lemma 6.4, there exists an integer $n \in \mathbb{Z}_{\geq 0}$ such that we have a homogeneous equation

$$(6.3) \qquad (t_0 \cdots t_{l-1})^n b_{Gq_0} = \sum_{g<G,p} c_{gp} a_{gp} + \sum_p c_{Gp} a_{Gp} + \sum_{g>G,q} c'_{gq} b_{gq}$$

where each c_{gp}, $c'_{gq} \in \mathbb{C}[t_0 \cdots t_{l-1}]$. Note that we use the hypothesis of the induction $\{a_{gp}\}_p = \{b_{gq}\}_q$ for $g > G$. Since $\mathcal{B}_\lambda^{m\theta} e_0 \mathcal{T}_\lambda \subseteq e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}$, we can write each $a_{gp} = \sum_{h,q} d_{gp}^{hq} b_{hq}$ for some $d_{gp}^{hq} \in \mathbb{C}[t_0 \cdots t_{l-1}]$. Thus we obtain a homogeneous equation

$$(6.4) \qquad (t_0 \cdots t_{l-1})^n b_{Gq_0} = \sum_{g<G,p,h,q} c_{gp} d_{gp}^{hq} b_{hq} + \sum_{p,h,q} c_{Gp} d_{Gp}^{hq} b_{hq} \sum_{g>G,q} c'_{gq} b_{gq}$$

The above equations (6.3), (6.4) are homogeneous of degree $G + ln$. By (6.3), $\deg c_{gp} \geq ln$ for each g and p. Thus the b_{gq} in the first two terms of the right hand side of (6.4) has degree $\leq G$. Since $\{b_{gq}\}_{g,q}$ are $\mathbb{C}[t_0 \cdots t_{l-1}]$-basis, the third term $\sum_{g>G,q} c'_{gq} b_{gq}$ is actually zero in (6.3), (6.4).

Now consider where b_{Gq_0} appears on the right hand side of (6.4). For $g < G$, (6.3) implies $\deg c_{gp} > ln$ for each p, thus there is no b_{Gq_0} in the first term of the right hand side of (6.4). Thus b_{Gq_0} appears only in the second term of the right hand side of (6.4). Since $\{b_{gq}\}_{g,q}$ is $\mathbb{C}[t_0 \cdots t_{l-1}]$-basis, there is nonzero $c_{Gp} d_{Gp}^{Gq_0} b_{Gq_0}$ in the second term of the right hand side of (6.4). In this case by (6.3) we have $\deg c_{Gp} = nl$. Hence $d_{Gp}^{Gq_0} \in \mathbb{C}\backslash\{0\}$, and we have

$$a_{Gp} = d_{Gp}^{Gq_0} b_{Gq_0} + \sum_{(h,q)\neq(G,q_0)} d_{Gp}^{hq} b_{hq}.$$

Thus we can replace b_{Gq_0} by a_{Gp} in our basis of $e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}$. By (†3), $\{a_{Gp}\}_p$ and $\{b_{Gq}\}_q$ have the same cardinality. After a finite number of steps, we have $\{b_{Gq}\}_q \subseteq \{a_{Gp}\}_p$ and hence $\{b_{Gq}\}_q = \{a_{Gp}\}_p$. This completes the induction on g. Thus, we have $\{b_{gq}\}_q = \{a_{gp}\}_p$ for all g. It implies the equality $\mathcal{B}_\lambda^{m\theta} e_0 \mathcal{T}_\lambda = e_0 \mathcal{M}_\lambda^{GL(\delta),\chi_\theta^m}$.

6.3. Characteristic cycles

In this section, we determine the characteristic cycle of the standard module $\Delta_\lambda(i)$ in $\mathfrak{M}_\theta(\delta)$ for $i = 0, \ldots, l-1$.

Fix the parameters $\lambda = (\lambda_i)_{i=0,\ldots,l-1} \in \widetilde{\mathbb{R}}^l_{reg}$ and $\theta = (\theta_i)_{i=0,\ldots,l-1} \in \mathbb{Z}^l_\lambda$. For $M \in A_\lambda$-filt, we consider $\widetilde{M} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathcal{B}_\lambda^{m\theta} \otimes_{\mathcal{A}_\lambda} M$ and the coherent sheaf $\Phi_\lambda^\theta(M) = (\operatorname{gr} \widetilde{M})^\sim$. Let the associated radical ideal $N_{\widetilde{M}}$ to be the radical ideal $N_{\widetilde{M}} = \sqrt{\operatorname{ann}_S \operatorname{gr} \widetilde{M}}$. The characteristic variety of M is

$$\operatorname{Char}(M) = \operatorname{Supp}(\Phi_\lambda^\theta(M)) = \mathcal{V}(N_{\widetilde{M}})$$

where $\mathcal{V}(I)$ is closed subvariety associated to ideal I. Note that $\operatorname{Char}(M) \subseteq \pi_\theta^{-1}(\{y = 0\}) = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_{l-1}$ for $M \in \mathcal{O}_\lambda$.

Let $\mathrm{Min}\widetilde{M}$ be a set of prime ideals minimal over $N_{\widetilde{M}}$. If $P \in \mathrm{Min}\widetilde{M}$, let $n_{\widetilde{M},P}$ be the length of the S_P-module $(\mathrm{gr}\,\widetilde{M})_P$. Define the characteristic cycle $\mathrm{Ch}(M)$ of M to be

$$\mathrm{Ch}(M) = \sum_{P \in \mathrm{Min}\widetilde{M}} n_{\widetilde{M},P} \mathcal{V}(P).$$

Let $\mathrm{Min}'\widetilde{M}$ be a subset of $\mathrm{Min}\widetilde{M}$ consisting of those prime ideals P for which $\dim \mathcal{V}(P) = \dim \mathcal{V}(N_{\widetilde{M}})$. Define the restricted characteristic cycle $\mathrm{rCh}(M)$ to be

$$\mathrm{rCh}(M) = \sum_{P \in \mathrm{Min}'\widetilde{M}} n_{\widetilde{M},P} \mathcal{V}(P).$$

The following lemmas are proved in [GS2].

Lemma 6.5. *The characteristic varieties, the characteristic cycles and the restricted characteristic cycles are independent of the choice of good filtrations.*

Lemma 6.6. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated $\mathcal{A}_\lambda$-modules. Then one of the following cases occurs*:

(1). $\dim \mathrm{Char}(A) < \dim \mathrm{Char}(B) = \dim \mathrm{Char}(C)$ *and* $\mathrm{rCh}(B) = \mathrm{rCh}(C)$.

(2). $\dim \mathrm{Char}(A) = \dim \mathrm{Char}(B) > \dim \mathrm{Char}(C)$ *and* $\mathrm{rCh}(B) = \mathrm{rCh}(A)$.

(3). $\dim \mathrm{Char}(A) = \dim \mathrm{Char}(B) = \dim \mathrm{Char}(C)$ *and* $\mathrm{rCh}(B) = \mathrm{rCh}(A) + \mathrm{rCh}(C)$.

By Theorem 6.1, we have the isomorphism

$$\Phi_\lambda^\theta(e_0 \mathcal{T}_\lambda) \simeq \widetilde{\mathcal{P}}_\theta.$$

By the isomorphism of $\mathcal{A}_\lambda$-modules

$$e_0 \mathcal{T}_\lambda / e_0 \mathcal{T}_\lambda A^* \simeq \bigoplus_{i=0}^{l-1} e_0 \Delta_\lambda(i),$$

we have

$$\widetilde{\mathcal{P}}_\theta / \widetilde{\mathcal{P}}_\theta \bar{A}^* \simeq \Phi_\lambda^\theta(e_0 \mathcal{T}_\lambda / e_o \mathcal{T}_\lambda A^*) \simeq \bigoplus_{i=0}^{l-1} \Phi_\lambda^\theta(e_0 \Delta_\lambda(i)) \tag{6.5}$$

as coherent sheaves on $\mathfrak{M}_\theta(\delta)$. Thus we can determine the characteristic cycles from (6.5).

Proposition 6.7. *For $\lambda \in \widetilde{\mathbb{R}}^l_{reg}$ and $\theta \in \mathbb{Z}^l_\lambda$, we have an isomorphism of coherent sheaves*

$$\Phi_\lambda^\theta(e_0 \Delta_\lambda(i)) \simeq (\widetilde{\mathcal{P}}_\theta / \widetilde{\mathcal{P}}_\theta \bar{A}^*) e_i.$$

Proposition 6.8. *For $\lambda \in \widetilde{\mathbb{R}}^l_{reg}$, $\theta \in \mathbb{Z}^l_\lambda$ and $i = 0, \ldots, l-1$, we have*

$$\mathrm{Ch}_\theta(e_0\Delta_\lambda(i)) \simeq [\,\mathcal{U}_i\,] + \sum_{\theta_i+\cdots+\theta_{j-1}<0} [\,\mathcal{U}_j\,] = \sum_{i \trianglerighteq_\theta j} [\,\mathcal{U}_j\,] = \sum_{i \trianglerighteq_{geom,\theta} j} [\,\mathcal{U}_j\,].$$

Proof. Since the sheaf $\widetilde{\mathcal{P}}_\theta$ is the tautological bundle on $\mathfrak{M}_\theta(\delta)$ defined by (3.12), The action of $\bar{A}^*$ on a fiber $\widetilde{\mathcal{P}}_\theta([a_k, b_k]_{k=0,\ldots,l-1}) \simeq \bigoplus_{i=0}^{l-1} \mathbb{C}E_{0i}$ is given by the right-multiplication of a matrix $\sum_{k=0}^{l-1} b_k E_{k-1,k}$. By the definition of $\mathcal{U}_j$, we have $b_i = 0$ if and only if $\theta_i + \cdots + \theta_{j-1} < 0$ at $[a_k, b_k]_{k=0,\ldots,l-1} \in \mathcal{U}_j$. Therefore

$$(\widetilde{\mathcal{P}}_\theta/\widetilde{\mathcal{P}}_\theta\bar{A}^*)([a_k, b_k]_{k=0,\ldots,l-1}) = \bigoplus_{\theta_i+\cdots+\theta_{j-1}<0} \mathbb{C}E_{0,i}$$

at any point $[a_k, b_k]_{k=0,\ldots,l-1} \in \mathcal{U}_j$.

By Proposition 6.7, we have

$$\Phi^\theta_\lambda(e_0\Delta(\lambda)(i)) \simeq (\widetilde{\mathcal{P}}_\theta/\widetilde{\mathcal{P}}_\theta\bar{A}^*)e_i.$$

Therefore we have

$$\mathrm{Ch}_\theta(e_0\Delta(\lambda)(i)) \simeq [\,\mathcal{U}_i\,] + \sum_{\theta_i+\cdots+\theta_{j-1}<0} [\,\mathcal{U}_j\,].$$

By the definition of $\trianglerighteq_\theta$ and the definition of $\trianglerighteq_{geom,\theta}$, we have

$$i \triangleright_\theta j \Leftrightarrow i \triangleright_{geom,\theta} j \Leftrightarrow \theta_i + \cdots + \theta_{j-1} < 0.$$

Thus we have the statement of the proposition. □

Note that Proposition 6.8 is an affirmative answer to Question 10.2 of [Go] in the case of $G = \mathbb{Z}/l\mathbb{Z}$.

Corollary 6.9. *Taking the reduced characteristic cycle* $\mathrm{rCh}_\theta(M)$ *of a module $M \in \mathcal{O}_\lambda$ induces an isomorphism of vector spaces*

$$\mathrm{rCh}_\theta : K(\mathcal{O}_\lambda) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^0(\pi_\theta^{-1}(\{y = 0\}), \mathbb{C}).$$

Corollary 6.10. *If $\eta_i \triangleright_{rep,\lambda} \eta_j$ and there is no $i > k > j$ such that $\eta_i \triangleright_{rep,\lambda} \eta_k$, then*

$$\mathrm{rCh}_\theta(e_0L_\lambda(\eta_i)) = [\,\mathcal{U}_{\eta_i}\,] + [\,\mathcal{U}_{\eta_{i-1}}\,] + \cdots + [\,\mathcal{U}_{\eta_{j-1}}\,].$$

Proof. By the assumption, we have an exact sequence

$$0 \to e_0\Delta_\lambda(\eta_j) \to e_0\Delta_\lambda(\eta_i) \to e_0L_\lambda(\eta_i) \to 0$$

of $\mathcal{A}_\lambda$-modules. By Lemma 6.6, we have

$$\mathrm{rCh}_\theta(e_0\Delta_\lambda(\eta_i)) = \mathrm{rCh}_\theta(e_0\Delta_\lambda(\eta_j)) + \mathrm{rCh}_\theta(e_0L_\lambda(\eta_i)).$$

Therefore we have the statement of the corollary. □

Department of Mathematics
Graduate School of Science
Kyoto University
Kyoto 606-8502, Japan
e-mail: toshiro@math.kyoto-u.ac.jp

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