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The global profile of blow-up at space infinity in semilinear heat equations

By

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Abstract

We consider semilinear heat equations on $\mathbb{R}^N$ and discuss the blow-up of solutions that occurs only at space infinity. We give sufficient conditions for such phenomena, and study the global profile of solutions at the blow-up time. Among other things, we establish a nearly optimal upper bound for the blow-up profile, which shows that the profile $u(x,T)$ cannot grow too fast as $|x| \to \infty$. We also prove that such blow-up is always complete.

1. Introduction

In this paper we consider the initial value problem for a semilinear heat equation of the form

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $f : [0,\infty) \to [0,\infty)$ is a locally Lipschitz function, and discuss the blow-up of solutions that occurs at space infinity. The equation appears, for example, as a model for combustion, in which the term $f(u)$ represents heating due to an exothermic reaction. We assume certain growth conditions on f, as specified in (A1), (A2) below. Typical examples include:

$$\begin{aligned} f(u) &= u^p && (p > 1), \\ f(u) &= e^{\alpha u} && (\alpha > 0), \\ f(u) &= u(\log(1+u))^\beta && (\beta > 2). \end{aligned}$$

We say that the solution of (1.1) *blows up* in finite time if there is some $T = T(u_0) < \infty$ such that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty$$

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and $T(u_0)$ is called the *blow-up time* of the solution u with the initial value u_0. We define the *blow-up set* by

$$B(u_0) = \Big\{ a \in \mathbb{R}^N \Big| \limsup_{x \to a,\, t \nearrow T} |u(x,t)| \to \infty \Big\}.$$

Each element of $B(u_0)$ is called a *blow-up point* of u. It is easily seen that $B(u_0)$ is a closed set and that $a \notin B(u_0)$ if and only if the solution remains bounded as $t \nearrow T$ in a neighborhood of a. Therefore, by the standard regularity theorems for the parabolic equations, $u(\cdot,t)$ remains bounded in $C^{2+\alpha}_{loc}(\mathbb{R}^N \setminus B(u_0))$ as $t \nearrow T$ for some $0 < \alpha < 1$, which then implies the boundedness of $u_t(\cdot,t)$ in $C^{\alpha}_{loc}(\mathbb{R}^N \setminus B(u_0))$. Consequently, the pointwise limit

$$u(x,T) := \lim_{t \nearrow T} u(x,t)$$

exists for every $x \in \mathbb{R}^N \setminus B(u_0)$ and it belongs to $C^{2+\alpha}_{loc}(\mathbb{R}^N \setminus B(u_0))$. We call this limit the *global blow-up profile* of u.

We say that the solution of (1.1) *blows up at space infinity* if there exist sequences $\{x_m\}_{m=1}^{\infty} \subset \mathbb{R}^N$ and $\{t_m\}_{m=1}^{\infty} \subset (0,T)$ such that

$$|x_m| \to \infty, \quad t_m \nearrow T, \qquad |u(x_m,t_m)| \to \infty \quad \text{as} \quad m \to \infty. \tag{1.2}$$

We say that u blows up *only at space infinity* if, in addition to (1.2), the following holds for any compact set $K \subset \mathbb{R}^N$:

$$\limsup_{t \nearrow T(u_0)} \|u(\cdot,t)\|_{L^\infty(K)} < \infty. \tag{1.3}$$

In this case, the global blow-up profile $u(x,T) := \lim_{t\to T} u(x,t)$ is defined for every $x \in \mathbb{R}^N$, and belongs to $C^{2+\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. It is not difficult to see that the combination of (1.2), (1.3) is equivalent to the following two conditions:

(a) u blows up in finite time;

(b) $B(u_0) = \emptyset$.

In this paper, by "blow-up at space infinity", we always mean that (a), (b) hold.

Let us recall known results on blow-up at space infinity. Lacey [8] considered a one-dimensional problem on the half-line and constructed examples of solutions that blow up (only) at space infinity. He also obtained results of the blow-up profile. Giga and Umeda [6] considered the equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$ and derived sufficient conditions for blow-up at space infinity. Roughly speaking, they showed that blow-up at space infinity occurs if the initial data u_0 satisfies $0 \le u_0 \le M$, $u_0 \to M$ as $|x| \to \infty$ for some constant $M > 0$. Shimojō [17] improved their result by using a different method borrowed from [11], [12]. More precisely, he showed that blow-up at space infinity occurs under a weaker assumption on the initial data, namely the conditions (1.7), (1.8) in Theorem 1.1 below. More recently Giga and Umeda [7] extended the result of [17] to a more general equation of the form $u_t = \Delta u + f(u)$, where the nonlinearity $f(u)$

satisfies certain growth conditions; it also gives a result on "blow-up direction" (See also [15], [16] for quasilinear equations, which generalized the results of [7]). For the continuation problem, Galaktionov and Vazquez [5] proved that blow-up at space infinity is always complete, provided that the initial data is nonnegative and radially symmetric.

The aim of the present paper is to give a simple proof for blow-up at space infinity and, more importantly, study the global profile of solutions at the blow-up time, particularly the growth rate of $u(x,T)$ as $|x| \to \infty$. We also prove that the blow-up at space infinity which occurs under the conditions (1.7), (1.8) is always complete. This means that the solutions can not be extended beyond the blow-up time as a mild solution.

Throughout this paper, we assume that $f(\sigma)$ is C^2 in $\sigma > 0$ and C^1 in $\sigma \geq 0$, and that the following conditions (A1) and (A2) hold:

(A1) $f(0) \geq 0$, $f'(\sigma) > 0$, and $f''(\sigma) > 0$ for $\sigma > 0$;

(A2) There exist $K > 0$ and a C^2 function $F : [K, \infty) \to (0, \infty)$ such that $F'(\sigma) > 0, F''(\sigma) > 0$ for $\sigma > K$, and

$$\int_K^\infty \frac{d\sigma}{F(\sigma)} < \infty, \tag{1.4}$$

$$f'(\sigma)F(\sigma) - f(\sigma)F'(\sigma) \geq \mu F(\sigma)F'(\sigma) \quad \text{for} \quad \sigma \geq K. \tag{1.5}$$

Remark 1. The same hypothesis as (A2) appears in the paper of Friedman and McLeod [2]. For technical reasons we also require that $f''(\sigma) > 0$ ($\sigma > 0$), although these may be weakened for the inequalities to hold with σ sufficiently large in Theorem 1.1 and Theorem 1.2 below.

Since the inequality (1.4) and the positivity of f imply

$$\int_l^\infty \frac{d\sigma}{f(\sigma)} < \infty, \quad (\text{ for any } l > 0)$$

the solution of the following ODE blows up in finite time, where M is any positive constant:

$$U' = f(U), \quad U(0) = M \ (> 0). \tag{1.6}$$

We denote this blow-up time by $T(M)$.

Now let us state our theorems. The first is a preliminary result which gives a sufficient condition for blow-up at space infinity:

Theorem 1.1 (Blow-up at infinity). *Suppose that f satisfies* (A1) *and* (A2). *Let $u_0 \in C(\mathbb{R}^N)$ satisfy*

$$0 \leq u_0 \leq M,\ u_0 \not\equiv M \tag{1.7}$$

for some constant $M > 0$. *Assume also that there exist a sequence of points* $a_1, a_2, a_3, \cdots \in \mathbb{R}^N$ *with* $|a_n| \to \infty$ *as* $n \to \infty$, *and a sequence* $0 < R_1 < R_2 < \cdots \to \infty$ *such that*

$$\lim_{n\to\infty} \|u_0 - M\|_{L^1(B_{R_n}(a_n))} = 0, \tag{1.8}$$

where $B_R(a)$ *is a ball of radius* R *centered at* a. *Then the solution of* (1.1) *has the following properties*:

(i) $T(u_0) = T(M)$.

(ii) $B(u_0) = \emptyset$.

(iii) $\lim_{n\to\infty} u(a_n, T(M)) = \infty$.

Note that Theorem 1.1 (under the same hypotheses (1.7), (1.8)) was first proved in [17] for the equation $u_t = \Delta u + u^p$ $(p > 1)$. Giga and Umeda [7] then extended this result to the equation $u_t = \Delta u + f(u)$ under the assumption that $f(\sigma)/\sigma^p \to \infty$ as $\sigma \to \infty$ for some $p > 1$. Our assumptions (A1), (A2) relax this growth requirement by allowing such nonlinearities as $u(\log(1+u))^b$ $(b > 2)$. Note also that our proof is different from those of [6], [7], [17]; it relies genuinely on a simple sub- and supersolution argument.

Before stating our next theorems, which are the main focus of the present paper, we define $\varphi(s)$ as the unique solution of the following problem:

$$\varphi' = -f(\varphi) \ (s > 0), \quad \lim_{s\searrow 0} \varphi(s) = \infty. \tag{1.9}$$

Equivalently, φ is characterized by the identity

$$s = \int_{\varphi(s)}^{\infty} \frac{d\sigma}{f(\sigma)}.$$

Thus φ is a monotone decreasing function and it satisfies

$$\varphi(s) > 0 \quad (0 < s < s^*), \qquad \varphi(s) \to 0 \quad (s \nearrow s^*), \tag{1.10}$$

where

$$s^* = \int_0^{\infty} \frac{d\sigma}{f(\sigma)} \in (0, \infty].$$

As is easily seen, under the present hypotheses on f, we have

$$s^* = \infty \quad \text{if} \ \ f(0) = 0, \qquad s^* < \infty \quad \text{if} \ \ f(0) > 0.$$

Next we denote the inverse function of φ by ψ. More precisely, the function $\psi : (0, \infty) \to (0, s^*)$ is defined by the relation $\psi(\varphi(s)) = s$. Differentiating this relation gives

$$\psi'(\varphi(s))\, \varphi'(s) = 1.$$

Therefore ψ satisfies

$$\psi'(v) = -\frac{1}{f(v)}.$$

Note also that

$$\psi(v) \to s^* \quad \text{as } v \searrow 0, \qquad \psi(v) \to 0 \quad \text{as } v \to \infty. \tag{1.11}$$

For example, if $f(u) = u^p$ for some $p > 1$, then $s^* = \infty$ and

$$\varphi(s) = \kappa s^{-\frac{1}{p-1}} \quad (0 < s < \infty), \qquad \psi(v) = \frac{v^{-(p-1)}}{p-1} \quad (0 < v < \infty) \tag{1.12}$$

where $\kappa := (p-1)^{-1/(p-1)}$. If $f(u) = e^{\alpha u}$ for some $\alpha > 0$, then $s^* = 1/\alpha$ and

$$\varphi(s) = -\frac{1}{\alpha}\log(\alpha s) \quad (0 < s < 1/\alpha), \qquad \psi(v) = \frac{e^{-\alpha v}}{\alpha} \quad (0 < v < \infty). \tag{1.13}$$

We note that the solution of (1.6) is written as $U(t) = \varphi\big(T(M) - t\big)$, where $T(M)$ is the blow-up time for the initial data $U(0) = M$. Substituting $t = 0$ gives

$$M = \varphi(T(M)). \tag{1.14}$$

This implies

$$\psi(M) = T(M). \tag{1.15}$$

Now we state our main results on the global profile of solutions at the blow-up time. The first theorem gives a general upper bound on the profile. For this theorem we need the following additional condition:

(A3) There exists a constant $\gamma \in [1, 2)$ satisfying

$$f'(u)\Big(\int_u^\infty \frac{dw}{f(w)}\Big)^\gamma \le C$$

for some $C > 0$.

In the following, G is the fundamental solution of the heat equation in $\mathbb{R}^N$:

$$G(x,t) := \frac{1}{(4\pi t)^{N/2}} \exp\Big(-\frac{|x|^2}{4t}\Big).$$

Furthermore, we employ the notation

$$(e^{t\Delta} h)(x) := \int_{\mathbb{R}^N} G(x-y,t)h(y)\,dy.$$

Theorem 1.2 (General upper bound). *Assume* (A1), (A2) *and* (A3). *Then for any initial data* $u_0 \in C(\mathbb{R}^N)$ *satisfying* (1.7) *and* $T(u_0) = T(M)$, *there exist* $c_1 > 0$ *and* $a \in \mathbb{R}^N$ *such that*

$$u\big(x, T(u_0)\big) \le \varphi\big(\, c_1\, G\big(\, x - a, T(u_0)\,\big)\,\big), \tag{1.16}$$

where u *is the solution of the problem* (1.1) *with initial data* u_0.

Let $f(u) = u^p$ for some $p > 1$. Then from (1.12), the inequality (1.16) reduces to:

$$u(\,x, T(u_0)\,) \le C \exp\Big(\frac{|x-a|^2}{4(p-1)\,T(u_0)}\Big), \qquad x \in \mathbb{R}^N.$$

Similarly, if $f(u) = e^u$ for some $a > 0$, then (1.16) reduces to:

$$u(\,x, T(u_0)\,) \le \frac{|x-a|^2}{4\,T(u_0)} + C, \qquad x \in \mathbb{R}^N,$$

where the constant C depends on $a \in \mathbb{R}^N$ and u_0.

Remark 2. The following examples satisfy the conditions (A1) to (A3):

$$\begin{aligned} &f(u) = u^p && (p > 1),\\ &f(u) = e^{\alpha u} && (\alpha > 0),\\ &f(u) = u(\log(1+u))^\beta && (\beta > 2). \end{aligned}$$

On the other hand, the function $f(u) = u(\log(1+u))^2\{\log(\log(1+u))\}^\nu$ satisfy (A2) if and only if $\nu > 1$ but do not satisfy (A3)

Next result gives a lower bound for some special solution of the problem (1.1), which implies that the estimate of Theorem 1.2 is optimal.

Theorem 1.3 (Lower bound). *Assume* (A1), (A2). *For any* $\varepsilon > 0$ *there exists a solution* u *of* (1.1) *with an initial data* $u_0 \in C(\mathbb{R}^N)$ *satisfying* (1.7) *such that* $T(u_0) = T(M)$ *and that*

$$u(\,x, T(u_0)\,) \ge \varphi\Big(c_2\, G\big(\,x, T(u_0) + \varepsilon\,\big)\Big)$$

for some constant $c_2 > 0$.

The following result gives a estimate for some special solution of the problem (1.1). This theorem asserts that we can calculate the blow-up profile only by solving a linear heat equation.

Theorem 1.4 (Characterizing profile). *Suppose the function* f *satisfies* (A1), (A2), (A3) *and*

$$\liminf_{w\to\infty} \frac{f(\lambda w)}{\lambda f(w)} 1 \tag{1.17}$$

for any $\lambda > 1$. *Let* u *be a solution of the problem* (1.1) *that blows up at* $t = T$ *with an initial value* u_0 *satisfying* (1.7). *Assume the function* $\eta_0 := \psi(u_0) - T$ *satisfies*

(1.18)
$$\lim_{|x|\to\infty} \eta_0(x) = 0, \quad \limsup_{|x|\to\infty} |\nabla \eta_0(x)| < \infty, \quad \eta_0(x) \ge c_3 G(x,\delta), \quad x \in \mathbb{R}^N$$

for some $c_3 > 0$ *and* $\delta > 0$. *Then we have*

$$\frac{u(x,T)}{\varphi(e^{T\Delta}\eta_0)} \to 1 \quad as \quad |x| \to \infty. \tag{1.19}$$

Note that the condition (1.17) is satisfied if, for example, $f(u) = u^p$ or $f(u) = e^{\alpha u}$. The following theorem shows that any function $\rho(x)$ with a moderate growth rate at infinity can be a candidate for the global profile $u(x,T)$, at least asymptotically.

Theorem 1.5 (Prescribed profile)**.** *Assume the function* f *satisfies the conditions of Theorem* 1.4. *Let* $M > 0$ *be any constant and let* $\rho(x)$ *be any positive function on* $\mathbb{R}^N$ *with* $\lim_{|x|\to\infty} \rho(x) = \infty$ *and*

$$\lim_{|x|\to\infty} \frac{|\nabla\rho(x)|}{f(\rho(x))\,\psi(\rho(x))} = 0. \tag{1.20}$$

In the case where $s^* \neq \infty$, *assume further that* $\rho(x) > \varphi(s^* - T(M)\,)$ *for all* $x \in \mathbb{R}^N$. *Then there exists a solution* u *of* (1.1) *with the initial data* u_0 *satisfying* (1.7) *such that* $T(u_0) = T(M)$ *and that*

$$\frac{u(\,x, T(u_0)\,)}{\rho(x)} \to 1 \quad as \quad |x| \to \infty. \tag{1.21}$$

As a Corollary of Theorem 1.5, we have

Corollary 1.1. *Let* $f(u) = u^p$ *for some* $p > 1$. *Let* $M > 0$ *be any constant and let* $\rho(x)$ *be any positive function such that*

$$\lim_{|x|\to\infty} \rho(x) = \infty, \qquad \lim_{|x|\to\infty} |\nabla\rho(x)|/\rho(x) = 0.$$

Then there exists a solution u *of* (1.1) *with the initial data* u_0 *satisfying* (1.7), $T(u_0) = T(M)$ *and* (1.21).

Corollary 1.2. *Let* $f(u) = e^{\alpha u}$ *for some* $\alpha > 0$. *Let* $M > 0$ *be any constant and let* $\rho(x)$ *be any positive function such that*

$$\lim_{|x|\to\infty} \rho(x) = \infty, \qquad \lim_{|x|\to\infty} |\nabla\rho(x)| = 0.$$

Then there exists a solution u *of* (1.1) *with the initial data* u_0 *satisfying* (1.7), $T(u_0) = T(M)$ *and* (1.21).

Remark 3. It is clear from Theorem 1.5 that the growth rate of the global profile $u(x,T)$ as $|x| \to \infty$ can be made as slow as one wants. On the other hand, Theorem 1.2 shows that the growth rate cannot be arbitrarily large.

The proof of Theorems 1.2–1.5 will be done by using an appropriate pair of subsolutions and supersolutions.

Our next theorem is concerned with the question of whether a blow-up is complete or not. First let us recall the notion of complete blow-up due to Baras and Cohen [1]. Assume $T(u_0) < \infty$. Let $f_m(u) = \min\{f(u), m\}$. For each $m = 1, 2, 3, \cdots$, let u_m be the solution of the approximation problem

$$\begin{cases} (u_m)_t = \Delta u_m + f_m(u_m), & x \in \mathbb{R}^N,\ t > 0, \\ u_m(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.22}$$

The solutions u_m exist globally for $0 \le t < \infty$ and $u_m(x,t) \le u_{m+1}(x,t)$ $(m = 1, 2, 3, \cdots)$. Set

$$\bar{u}(x,t) := \lim_{m\to\infty} u_m(x,t) \in [0, \infty], \qquad x \in \mathbb{R}^N,\ t \in [0, \infty). \tag{1.23}$$

Then $\bar{u} = u$ for any $x \in \mathbb{R}^N$, $t \in [\,0, T(u_0)\,)$. We call the function $\bar{u}$ *the proper extension* (or *the minimal continuation*) of the solution u. Note that, by the boundedness of each $f_m(u)$ the function $u_m(x,t)$ is globally defined for $x \in \mathbb{R}^N$ and $t \ge 0$. Each u_m satisfies

$$u_m(x,t) = \int_{\mathbb{R}^N} G(x-y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s) f_m(u_m(y,s))\,ds$$

for $x \in \mathbb{R}^N$ and $t \ge 0$. By the positivity of $G(x,t)$ and the monotonicity of the sequences $\{u_m\}_{m=1}^{\infty}$ and $\{f_m(u_m)\}_{m=1}^{\infty}$, we can let $m \to \infty$ in the above integral identity to obtain

$$\bar{u}(x,t) = \int_{\mathbb{R}^N} G(x-y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s) f(\bar{u}(y,s))\,ds \tag{1.24}$$

for $x \in \mathbb{R}^N$ and $t \ge 0$. Here the identity (1.24) is understood in a generalized sense allowing the value ∞ for $\bar{u}$ and $f(\bar{u})$. Furthermore, $\bar{u}$ is the minimal element among all the functions that satisfy (1.24). See [1] for the detail.

Next we set

$$T^c = T^c(u_0) := \sup\{t \ge 0;\ \bar{u}(x,t) < \infty \quad \text{for a.e } x \in \mathbb{R}^N\}.$$

Clearly we have $T(u_0) \le T^c(u_0)$. Moreover one can easily see from the expression (1.24) that

$$\bar{u}(x,t) = \infty \quad \text{for a.e } x \in \mathbb{R}^N \quad \text{if} \quad t > T^c(u_0).$$

See [9] for the detail. We say that the blow-up is *complete* if $T(u_0) = T^c(u_0)$ and incomplete if $T(u_0) < T^c(u_0)$.

Theorem 1.6 (Completeness). *Let the hypotheses of Theorem* 1.1 *hold. Then the blow-up is complete.*

This paper is organized as follows. In Section 2, which is a preliminary section, we introduce useful sub- and supersolutions that will play a key role in the proof of our theorems. Section 3 is devoted to the proof of Theorem 1.1. The proof is based on the comparison argument and is much simpler than any of the previous methods in [6], [7], [17]. In Sections 4 and 5, we consider the global blow-up profiles and prove Theorems 1.2 to 1.5. In Section 6, we discuss the completeness of blow-up at space infinity and prove Theorem 1.6. The proof is based on our subsolution constructed in Section 2.

2. Preliminaries

In this section, we introduce subsolutions and a supersolution of the problem (1.1), which are useful in later arguments. We begin with the subsolutions. Note that the first subsolution has been used in [4], [10], [13], [14] and the second subsolution has been used in [8].

Lemma 2.1. *Suppose that f satisfies* (A1). *Let u be the solution of the problem* (1.1) *with initial data $u_0 \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Define $t^* \in (0, \infty]$ by*

$$t^* := \sup \big\{ t > 0 \,;\, e^{t\Delta} u_0(x) < \varphi(t) \quad \textit{for} \quad x \in \mathbb{R}^N \big\}.$$

Then $t^* \geq T(u_0)$, and the following inequality holds:

$$u(x,t) \geq \varphi\big(\psi(\, e^{t\Delta} u_0(x)\,) - t\big), \quad x \in \mathbb{R}^N,\, t \in [0, T(u_0)]. \tag{2.1}$$

Proof. Define

$$w(x,t) := \varphi\big(\psi(\, v(x,t)\,) - t\big), \qquad x \in \mathbb{R}^N,\, t \in [0, t^*),$$

where $v(x,t) := e^{t\Delta} u_0(x)$. Note that the function v satisfies

$$v_t = \Delta v, \quad v(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, t \geq 0. \tag{2.2}$$

By the definition of t^* we have $v(x,t) < \varphi(t)$ for $x \in \mathbb{R}^N$, $0 < t < t^*$, or equivalently $\psi(v(x,t)) - t > 0$. Therefore the function w is well-defined. First we prove that w is a subsolution of the problem (1.1). We can easily check that

$$\varphi''(s) = -f'(\varphi(s))\, \varphi'(s), \quad s > 0. \tag{2.3}$$

On the other hand,

$$\psi''(v) = -\frac{\varphi''(\,\psi(v)\,)|\,\psi'(v)\,|^2}{\varphi'(\,\psi(v)\,)} \tag{2.4}$$

It follows that

$$\begin{aligned}
w_t &= -\,\varphi'(\,\psi(v) - t\,) + \varphi'(\,\psi(v) - t\,)\psi'(v)\, v_t, \\
\nabla w &= \varphi'(\,\psi(v) - t\,)\, \psi'(v)\, \nabla v, \\
\Delta w &= \varphi''(\,\psi(v) - t\,)|\,\psi'(v)|^2|\,\nabla v\,|^2 + \varphi'(\,\psi(v) - t\,)\psi''(v)|\,\nabla v\,|^2 \\
&\qquad + \varphi'(\,\psi(v) - t\,)\psi'(v)\, \Delta v
\end{aligned}$$

Combining these and using (1.9), (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned}
&w_t - \Delta w - f(w)\\
&= -\,\varphi''(\,\psi(v) - t\,)|\,\psi'(v)|^2|\,\nabla v\,|^2 - \varphi'(\,\psi(v) - t\,)\psi''(v)|\,\nabla v\,|^2\\
&= -\,\varphi''(\,\psi(v) - t\,)|\,\psi'(v)|^2|\,\nabla v\,|^2 - \varphi'(\,\psi(v) - t\,)\frac{\varphi''(\,\psi(v)\,)|\,\psi'(v)\,|^2}{\varphi'(\,\psi(v)\,)}|\,\nabla v\,|^2\\
&= -\,\varphi'(\,\psi(v) - t\,)\Big\{\frac{\varphi''(\,\psi(v) - t\,)}{\varphi'(\,\psi(v) - t\,)} - \frac{\varphi''(\,\psi(v)\,)}{\varphi'(\,\psi(v)\,)}\Big\}|\,\psi'(v)\,|^2|\,\nabla v\,|^2\\
&= -\,\varphi'(\,\psi(v) - t)\,\Big\{f'\big(\varphi(\,\psi(v)\,)\big) - f'\big(\varphi(\,\psi(v) - t\,)\big)\Big\}|\,\psi'(v)\,|^2|\,\nabla v\,|^2 \le 0.
\end{aligned}$$

The last inequality follows from the convexity of f and the inequality $\varphi' < 0$. Therefore w is a subsolution of (1.1). Moreover, $w(x,0) = u_0(x) = u(x,0)$. Consequently,

$$u(x,t) \ge \varphi(\psi(v(x,t)) - t), \quad x \in \mathbb{R}^N,\ t \in (0, t_1), \tag{2.5}$$

where $t_1 := \min{(t^*, T(u_0))}$.

Next we prove $t^* \ge T(u_0)$. Suppose $T(u_0) > t^*$. Then the inequality (2.5) holds for $0 \le t < t^*$. On the other hand, from the definition of t^*, we have $\sup_{x\in\mathbb{R}^N} v(x,t^*) = \varphi(t^*)$. This implies

$$\lim_{t\to t^*} \sup_{x\in\mathbb{R}^N} w(x,t) = \infty.$$

Combining this and (2.5), we have

$$\lim_{t\to t^*} \sup_{x\in\mathbb{R}^N} u(x,t) = \infty,$$

which contradicts the assumption that $T(u_0) > t^*$. Finally, letting $t \nearrow T(u_0)$, we obtain the desired inequality for all $t \in [0, T(u_0)]$. □

Lemma 2.2. *Suppose* (A1). *Let u be a solution of the problem* (1.1) *with initial data $u_0 \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and let η be the solution of the problem*

$$\begin{cases} \eta_t = \Delta\eta, & x \in \mathbb{R}^N, t > 0,\\ \eta(x,0) = \psi(u_0) - T, & x \in \mathbb{R}^N, \end{cases} \tag{2.6}$$

where $T > 0$ is the blow-up time of u with the initial data u_0. Then the following inequality holds

$$u(x,t) \ge \varphi(T - t + \eta(x,t)), \qquad x \in \mathbb{R}^N, t \in [0,T].$$

Proof. Set $w^-(x,t) := \varphi(T - t + \eta(x,t))$. Then the function w^- is subsolution if and only if

$$\eta_t \ge \Delta\eta - f'(\varphi(T - t + \eta))\,|\nabla\eta|^2, \quad x \in \mathbb{R}^N,\ t \in (0,T).$$

This inequality holds if the function η satisfies $\eta_t = \Delta\eta$. On the other hand, $w^-(x,0) = u_0(x)$. Thus, by the comparison principle, we obtain the desired inequality for $t \in [0,T)$. Finally letting $t \nearrow T$, we obtain the inequality for all $t \in [0,T]$. The lemma is proved. $\square$

Next we introduce an supersolution of the problem (1.1). We obtain this supersolution by modifying the argument in [8].

Lemma 2.3. *Assume* (A1), (A2) *and* (A3). *Let u be a solution of the problem* (1.1) *with initial data $u_0 \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and let $T > 0$ be its blow-up time. Let η be the solution of the problem* (2.6). *Then there exists a positive continuous function $\mu(t)$ defined for $t \in [0,T]$ such that $\mu(0) = 1$, $\mu(T) > 0$ and*

$$u(x,t) \le \varphi(T - t + \mu(t)\,\eta(x,t)), \qquad x \in \mathbb{R}^N, t \in [0,T].$$

To prove Lemma 2.3, we need the following estimate.

Lemma 2.4. *Let η_0 be a bounded continuous function satisfying $\eta_0 \ge 0$, $\eta_0 \not\equiv 0$. Then, for any $\delta \in [0,1)$, there exists a constant $C_\delta > 0$ such that*

$$\frac{|\nabla e^{t\Delta}\eta_0|^2}{(e^{t\Delta}\eta_0)^{1+\delta}} \le \frac{C_\delta \|\eta_0\|^{1-\delta}_{L^\infty(\mathbb{R}^N)}}{t}, \quad x \in \mathbb{R}^N,\ t > 0.$$

Proof. By the Hölder inequality, we have

$$\begin{aligned}
|\nabla(e^{t\Delta}\eta_0)| &= \Big| \int_{\mathbb{R}^N} -\frac{x-y}{2t}\, G(x-y,t)\eta_0(y)\,dy \Big| \\
&\le \int_{\mathbb{R}^N} \frac{|x-y|}{2t}\, G^{\frac{1-\delta}{2}}(x-y,t)\eta_0^{\frac{1-\delta}{2}}(y)\; G^{\frac{1+\delta}{2}}(x-y,t)\eta_0^{\frac{1+\delta}{2}}(y)\,dy \\
&\le \Big(\int_{\mathbb{R}^N} \Big(\frac{|x-y|}{2t}\Big)^{\frac{2}{1-\delta}} G(x-y,t)\eta_0(y)\,dy \Big)^{\frac{1-\delta}{2}} \Big(\int_{\mathbb{R}^N} G(x-y,t)\eta_0(y)\,dy \Big)^{\frac{1+\delta}{2}}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\frac{|\nabla(e^{t\Delta}\eta_0)|^2}{(e^{t\Delta}\eta_0)^{1+\delta}} &\le \Big(\int_{\mathbb{R}^N} \Big(\frac{|x-y|}{2t}\Big)^{\frac{2}{1-\delta}} G(x-y,t)\eta_0(y)\,dy \Big)^{1-\delta} \\
&= \frac{\|\eta_0\|^{1-\delta}_{L^\infty(\mathbb{R}^N)}}{t} \Big(\int_{\mathbb{R}^N} \Big(\frac{|y|}{\sqrt{4t}}\Big)^{\frac{2}{1-\delta}} G(y,t)\,dy \Big)^{1-\delta} \\
&= \frac{\|\eta_0\|^{1-\delta}_{L^\infty(\mathbb{R}^N)}}{t} \Big(\frac{1}{\pi^{N/2}} \int_{\mathbb{R}^N} z^{\frac{2}{1-\delta}} e^{-z^2}\,dz \Big)^{1-\delta},
\end{aligned}$$

and the lemma follows. $\square$

Proof of Lemma 2.3. Set

$$w^+(x,t) := \varphi(T - t + h(x,t)), \quad h(x,t) := \mu(t)\eta,$$

where $\eta := e^{t\Delta}\eta_0$ with $\eta_0 := \psi(u_0) - T$. Then the function w^+ is supersolution if and only if

$$h_t \le \Delta h - f'(\varphi(T-t+h))\,|\nabla h|^2, \quad x \in \mathbb{R}^N,\ t \in (0,T). \tag{2.7}$$

This is equivalent to

$$-\frac{\dot\mu}{\mu^2} \ge f'(\varphi(T-t+\mu\eta))\,\frac{|\nabla\eta|^2}{\eta}, \quad x \in \mathbb{R}^N,\ t \in (0,T). \tag{2.8}$$

From an inequality $T - t + \mu\eta \ge 2\,(T-t)^{1/2}(\mu\eta)^{1/2}$, we find that the condition (2.8) follows from the following inequality:

$$-\frac{\dot\mu}{\mu^2} \ge f'\big(\varphi\big(2\,(T-t)^{1/2}(\mu\eta)^{1/2}\big)\big)\,\frac{|\nabla\eta|^2}{\eta}, \quad x \in \mathbb{R}^N,\ t \in (0,T). \tag{2.9}$$

We can easily check that (A3) is equivalent to $f'(\varphi(s)) \le \dfrac{C}{s^\gamma}$ for some $\gamma \in [1,2)$. Therefore, we have

$$\begin{aligned}
-\frac{\dot\mu}{\mu^2} &\ge \frac{C}{2^\gamma}\Big(\frac{1}{T-t}\Big)^{\gamma/2}\Big(\frac{1}{\mu\eta}\Big)^{\gamma/2}\frac{|\nabla\eta|^2}{\eta}\\
&= \frac{C}{2^\gamma}\Big(\frac{1}{T-t}\Big)^{\gamma/2}\Big(\frac{|\nabla\eta|^2}{\eta^{1+\gamma/2}}\Big)\Big(\frac{1}{\mu^{\gamma/2}}\Big), \quad x \in \mathbb{R}^N,\ t \in (0,T).
\end{aligned}$$

Consequently,

$$-\frac{\dot\mu}{\mu^{2-\gamma/2}} \ge \frac{C}{2^\gamma\,(T-t)^{\gamma/2}}\Big(\frac{|\nabla\eta|^2}{\eta^{1+\gamma/2}}\Big), \quad x \in \mathbb{R}^N,\ t \in (0,T). \tag{2.10}$$

By Lemma 2.4, we can see that (2.10) holds if the function $\mu(t)$ satisfies

$$-\frac{\dot\mu}{\mu^{2-\gamma/2}} = \frac{C_\delta\|\eta_0\|_{L^\infty(\mathbb{R}^N)}}{2^\gamma(T-t)^{\gamma/2}\,t}, \quad x \in \mathbb{R}^N,\ t \in (0,T).$$

Since $\gamma/2 \in [1/2, 1)$, we can integrate this equality from 0 to T. Hence we obtain

$$\frac{2^\gamma}{1-\gamma/2}\{\mu^{-1+\gamma/2}(t) - 1\} = C_\delta\|\eta_0\|_{L^\infty(\mathbb{R}^N)}\int_0^t \frac{ds}{(T-s)^{\gamma/2}\,s}\,dt, \quad t > 0.$$

Thus let us define

$$\mu(t) := \left(1 + \frac{(1-\gamma/2)C_\delta\|\eta_0\|_{L^\infty(\mathbb{R}^N)}}{2^\gamma}\int_0^t \frac{ds}{(T-s)^{\gamma/2}\,s}\,dt\right)^{-\frac{1}{1-\gamma/2}}.$$

Immediately, we have

$$\mu(T) = \left(1 + \frac{(1-\gamma/2)C_\delta\|\eta_0\|_{L^\infty(\mathbb{R}^N)}}{2^\gamma}\int_0^T \frac{ds}{(T-s)^{\gamma/2}\,s}\,dt\right)^{-\frac{1}{1-\gamma/2}} > 0 \tag{2.11}$$

and $\mu(0) = 1$. Thus $w^+(x, 0) = u_0(x)$ for $x \in \mathbb{R}^N$. Finally, by the comparison principle and the limiting argument, we obtain the desired inequality. $\square$

3. The blow-up time and the blow-up set

Now let us prove Theorem 1.1. In this section $v(x, t)$ will stand for the solution of the problem

$$\begin{cases} v_t = \Delta v, & x \in \mathbb{R}^N, t > 0, \\ v(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where the function u_0 is determined from (1.1).

Lemma 3.1. *Assume the same conditions as in Theorem* 1.1*, then*

$$\lim_{n\to\infty} v(a_n, T(M)) = M. \tag{3.2}$$

Proof. The comparison principle yields

$$0 \le v(x, t) \le M, \quad (x, t) \in \mathbb{R}^N \times (0, \infty).$$

In particular, $v(a_n, T(M)) \le M$ for $n = 1, 2, 3 \cdots$. On the other hand,

$$\begin{aligned} 0 \le M - v(\, a_n, T(M)\,) &= \int_{B_{R_n}(a_n)} G(\, a_n - y, T(M)\,)\,(M - u_0(y))\,dy \\ &\quad + \int_{\mathbb{R}^N \setminus B_{R_n}(a_n)} G(\, a_n - y, T(M)\,)\,(M - u_0(y))\,dy \\ &\le \|G(\,\cdot, T(M)\,)\|_{L^\infty(\mathbb{R}^N)} \|M - u_0\|_{L^1(B_{R_n}(a_n))} \\ &\quad + M \|G(\cdot, T(M)\,)\|_{L^1(\mathbb{R}^N \setminus B_{R_n}(0))}. \end{aligned}$$

From (1.8) and $\lim_{n\to\infty} \|G(\cdot, T(M)\,)\|_{L^1(\mathbb{R}^N \setminus B_{\frac{R_n}{2}}(0))} = 0$, we obtain (3.2). $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First we show (i). By the comparison principle, we have $0 \le u \le U$ in $\mathbb{R}^N \times [0, T(M))$, where U is the solution of (1.6). Hence $T(u_0) \ge T(M)$. To obtain a contradiction, suppose that $T(u_0) > T(M)$. If we substitute $(x, t) = (a_n, T(M))$ in (2.1) and use (1.15), we get

$$\|u(\cdot, T(M))\|_{L^\infty(\mathbb{R}^N)} \ge u(a_n, T(M)) \ge \varphi\Big(\psi\big(v(\, a_n, T(M)\,)\big) - \psi(M)\Big)$$

By Lemma 3.1, the right-hand side tends to infinity as $n \to \infty$, which contradicts $T(u_0) > T(M)$. Therefore $T(u_0) = T(M)$ must holds..

Next we prove (ii). Since u_0 is continuous and satisfying (1.7), we can find a point $b \in \mathbb{R}^N$ and a continuous monotone function $W_0(r)$ $(0 \le r < \infty)$ such that

$$u_0(x) \le W_0(|x - b|) \le M \quad (x \in \mathbb{R}^N), \qquad \lim_{r\to\infty} W_0(r) = M, \qquad W_0 \not\equiv M.$$

Let w be the solution of (1.1) with initial data $w(x,0) = W_0(|x-b|)$. By the comparison principle, we have

$$u(x,t) \le w(x,t) \le U(t), \qquad x \in \mathbb{R}^N,\ 0 \le t < T(M).$$

This implies that $T(w_0) = T(u_0) = T(M)$ and $B(u_0) \subset B(w_0)$. Moreover, it is easily seen from the maximum principle that $w(x,t)$ is radially symmetric with respect to $x = b$ for every $0 \le t < T(M)$, and that

$$\frac{\partial}{\partial r} w > 0, \qquad x \in \mathbb{R}^N, 0 < t < T(M),$$

where $r = |x-b|$. In view of this and the assumptions (A1) and (A2), and applying the result of Friedman and McLeod [2], we see that $B(w_0) = \emptyset$, hence $B(u_0) = \emptyset$.

Finally we prove (iii). By Lemma 2.1, we obtain

$$u(a_n,t) \ge \varphi\Big(\psi\big(v(\,a_n,t\,)\big) - t\Big) \tag{3.3}$$

for all $n \in \mathbb{N}$ and $t \in (0, T(u_0))$. By the result of (i), we have $T(u_0) = T(M)$. Moreover since (ii) holds, the global blow-up profile $u(x,T(M)) := \lim_{t \to T(M)} u(x,t)$ exists and is finite for every $x \in \mathbb{R}^N$. Thus, letting $t \to T(M) = \psi(M)$ in (3.3), we see that

$$u(a_n, T(M)) \ge \varphi\Big(\psi\big(v(\,a_n, T(M)\,)\big) - \psi(M)\Big).$$

for all $n \in \mathbb{N}$. Combining this with Lemma 3.1 and (1.9), we conclude that $\lim_{n\to\infty} u(\,a_n, T(M)\,) = \infty$. $\square$

4. The global profile

We begin with proving Theorem 1.2. In the following, we will write $T(u_0)$ simply T when no confusion can arise.

Proof of Theorem 1.2. Since (1.7) holds, there exist constants $r > 0$, $\delta > 0$ and $a \in \mathbb{R}^N$ such that

$$u_0(x) \le \varphi(\,T + \delta\chi(x)\,) =: w_0(x), \quad x \in \mathbb{R}^N.$$

where χ is the characteristic function for the ball $B_r(a)$. Let w be the solution of the problem (1.1) with initial value w_0. Then, from the comparison principle and Lemma 2.3, we have

$$u(x,t) \le w(x,t) \le \varphi(T - t + \mu(t)\,\delta e^{t\Delta}\chi(x)\,), \quad x \in \mathbb{R}^N,\ t \in [0,T]. \tag{4.1}$$

Furthermore, there exists a constant $\varepsilon > 0$ such that

$$\delta e^{t\Delta}\chi(x) \ge \varepsilon G(x-a,t), \qquad x \in \mathbb{R}^N\ t \in (0,T]. \tag{4.2}$$

Since φ is monotone decreasing, (4.1) and (4.2) imply

$$u(x,t) \le \varphi(T - t + \varepsilon\mu(t)\, G(x-a,t)), \quad x \in \mathbb{R}^N,\ t \in (0,T].$$

Substituting $t = T$, we conclude that

$$u(x,T) \le \varphi\big(\varepsilon\mu(T)G(x-a,T)\big), \quad x \in \mathbb{R}^N.$$

Hence the theorem is proved. □

Next we shall prove Theorem 1.3.

Proof of Theorem 1.3. Define $u_0(x) := \varphi(\psi(M) + \delta G\big(x,\varepsilon)\big)$ for some $\delta > 0$. We can easily check that u_0 satisfies (1.7) and that $\lim_{|x|\to\infty} u_0(x) = M$. Thus, by Theorem 1.1, $T = T(u_0) = T(M)$. From Lemma 2.2, we have the following inequality:

$$\varphi(T - t + \delta G(x, t+\varepsilon)) \le u(x,t), \quad x \in \mathbb{R}^N,\ t \in [0,T].$$

Substituting $t = T$, we obtain the result. □

Finally, we prove Theorem 1.4. We need the next lemma to prove it.

Lemma 4.1. *Assume that* (1.17) *holds for any* $\lambda > 1$. *Then* $\lim_{A\to\infty} \psi(B)/\psi(A) = 1$ *implies* $\lim_{A\to\infty} B/A = 1$.

Proof. First we prove that

$$\limsup_{A\to\infty} \frac{\psi(\lambda A)}{\psi(A)} < 1 \tag{4.3}$$

for any $\lambda > 1$. By (1.17), there exists some constant $\theta \in (0,1)$ such that $\lambda/f(\lambda w) \le \theta/f(w)$ for sufficiently large $w > 0$. Hence

$$\psi(\lambda A) = \int_{\lambda A}^{\infty} \frac{dw}{f(w)} = \int_{A}^{\infty} \frac{\lambda\, dw}{f(\lambda w)} \le \theta \int_{A}^{\infty} \frac{dw}{f(w)} = \theta\psi(A)$$

for sufficiently large $A > 0$. This gives (4.3).

Without loss of generality, we assume that $B \ge A$. Suppose there exists a sequence A_n, B_n such that $\lim_{A_n\to\infty} \psi(B_n)/\psi(A_n) = 1$ but $\lim_{A_n\to\infty} B_n/A_n \ne 1$. Since this implies

$$A_n \le (1+\delta)A_n \le B_n$$

for some $\delta > 0$, we have

$$1 = \lim_{A_n\to\infty} \frac{\psi(A_n)}{\psi(A_n)} \ge \lim_{A_n\to\infty} \frac{\psi((1+\delta)A_n)}{\psi(A_n)} \ge \lim_{A_n\to\infty} \frac{\psi(B_n)}{\psi(A_n)} = 1,$$

which contradicts (4.3). This contradiction proves the lemma. □

Proof of Theorem 1.4. Let $\eta := e^{t\Delta}\eta_0$ with $\eta_0 =: \psi(u_0) - T$ and let $\{\eta_n\}_{n=1}^{\infty}$ be solutions of $\eta_t = \Delta\eta$ with initial values

$$\eta_{n,0}(x) := \min\Big\{\eta_0(x), \frac{1}{n}\Big\}, \quad x \in \mathbb{R}^N.$$

First we prove that

$$\lim_{|x|\to\infty} \frac{\eta_n(x,T)}{\eta(x,T)} = 1 \tag{4.4}$$

for all $n \in \mathbb{N}$. Note that $\eta_{n,0} \leq \eta_0$ and that there exists a constant $r = r(n)$ such that $\eta_{n,0}(x) = \eta_0(x)$ for $|x| \geq r$. Define $w(x,t) := \eta(x,t) - \eta_n(x,t)$ and $w_0(x) = w(x,0)$. Since the support of w_0 is compact, $w(x,t) \leq c_4 G(x, \delta/2)$ for some $c_4 > 0$. By the comparison principle and (1.18), we have

$$\eta(x,t) \geq c_3 G(x, t+\delta), \quad w(x,t) \leq c_4 G\Big(x, t + \frac{\delta}{2}\Big), \quad x \in \mathbb{R}^N,\ t \geq 0.$$

This yields

$$0 \leq \frac{w(x,t)}{\eta(x,t)} \leq \frac{c_4 G\big(x, t+\delta/2\big)}{c_3 G(x, t+\delta)} \to 0 \quad \text{as} \quad |x| \to \infty.$$

Consequently,

$$\frac{\eta_n(x,t)}{\eta(x,t)} = 1 - \frac{w(x,t)}{\eta(x,t)} \to 1 \quad \text{as} \quad |x| \to \infty$$

and we obtain (4.4).

Furthermore, $\lim_{n\to\infty} \mu_n(T) = 1$, where $\mu_n(t)$ are functions corresponding to $\eta_{n,0}$ introduced in Lemma 2.3. This is a result of the inequality (2.11) in the proof of Lemma 2.3 and $\lim_{n\to\infty} \|\eta_{n,0}(x)\|_{L^\infty(\mathbb{R}^N)} = 0$. Therefore, for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \delta \leq \mu_n(T) \tag{4.5}$$

for all $n \geq n_0$. From the assumption (1.18) along with Lemma 2.2 and Lemma 2.3, we have

$$\mu_n(T)\eta_n(x,T) \leq \psi(u(x,T)) \leq \eta(x,T), \qquad x \in \mathbb{R}^N. \tag{4.6}$$

Since (4.4), (4.5) and (4.6) imply

$$1 - \delta \leq \mu_n(T) = \lim_{|x|\to\infty} \frac{\mu_n(T)\eta_n(x,T)}{\eta(x,T)} \leq \lim_{|x|\to\infty} \frac{\psi(u(x,T))}{\eta(x,T)} \leq 1$$

for arbitrary $\delta > 0$, we get

$$\lim_{|x|\to\infty} \frac{\psi(u(x,T))}{\eta(x,T)} = 1.$$

Combining this with Lemma 4.1, we obtain

$$\lim_{|x|\to\infty} \frac{u(x,T)}{\varphi(\eta(x,T))} = 1.$$

This completes the proof of the theorem. □

5. The prescribed profile

To prove Theorem 1.5, we use the following lemma.

Lemma 5.1. *Let $\rho(x)$ be a positive function on $\mathbb{R}^N$ that satisfies*

$$\lim_{|x|\to\infty} \frac{|\nabla\rho(x)|}{\rho(x)} = 0. \tag{5.1}$$

Then, for any $T \in (0,\infty)$,

$$\lim_{|x|\to\infty} \frac{e^{T\Delta}\rho(x)}{\rho(x)} = 1. \tag{5.2}$$

Proof. First we shall prove

$$e^{-K|y|} \le \frac{\rho(x+y)}{\rho(x)} \le e^{K|y|}, \quad y \in \mathbb{R}^N \tag{5.3}$$

for some constant $K > 0$. By (5.1), there exists a positive constant $K > 0$ such that

$$|\nabla\rho(x)| \le K\rho(x), \quad x \in \mathbb{R}^N. \tag{5.4}$$

Define $h(t) := \dfrac{\rho(x+ty)}{\rho(x)}$. Then, by the assumption (5.4),

$$|h'(t)| = \left| y \cdot \frac{\nabla\rho(x+ty)}{\rho(x)} \right| \le K|y| \frac{\rho(x+ty)}{\rho(x)} = K|y|\, h(t).$$

Since $h(0) = 1$, we obtain $e^{-K|y|t} \le h(t) \le e^{K|y|t}$. Substituting $t = 1$, we have (5.3). On the other hand, for any $\varepsilon_1 > 0$, there is some constant $R_0 > 0$ such that the following inequality holds:

$$\frac{1}{(4\pi T)^{N/2}} \int_{|y|\ge R_0} e^{-\frac{|y|^2}{4T}+K|y|}\, dy \le \varepsilon_1. \tag{5.5}$$

Again by the inequality (5.1), for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$|\nabla\rho(x)| \le \varepsilon\rho(x) \quad \text{for} \quad |x| > R_\varepsilon.$$

Thus

$$|\nabla\rho(x+ty)| \le \varepsilon\rho(x+ty) \quad \text{for} \quad |x| \ge R_0 + R_\varepsilon,\ |y| \le R_0$$

for any $t \in (0,1)$. Hence we obtain

$$e^{-\varepsilon|y|} \le \frac{\rho(x+y)}{\rho(x)} \le e^{\varepsilon|y|} \quad \text{for} \quad |x| \ge R_0 + R_\varepsilon,\ |y| \le R_0. \tag{5.6}$$

Combining (5.3), (5.5) and (5.6), we have

$$\begin{aligned}
|e^{T\Delta}\rho(x) - \rho(x)| &= \Big| \frac{1}{(4\pi T)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4T}} (\rho(x+y) - \rho(x))\, dy \Big| \\
&\le \Big(\frac{1}{(4\pi T)^{N/2}} \int_{|y|\ge R_0} (e^{K|y|} - 1) e^{-\frac{|y|^2}{4T}}\, dy \Big) \rho(x) \\
&\quad + \Big(\frac{1}{(4\pi T)^{N/2}} \int_{|y|\le R_0} (e^{\varepsilon|y|} - 1) e^{-\frac{|y|^2}{4T}}\, dy \Big) \rho(x) \\
&\le \varepsilon_1 \rho(x) + (e^{\varepsilon R_0} - 1) \Big(\frac{1}{(4\pi T)^{N/2}} \int_{|y|\le R_0} e^{-\frac{|y|^2}{4T}}\, dy \Big) \rho(x) \\
&\le \varepsilon_1 \rho(x) + (e^{\varepsilon R_0} - 1) \rho(x)
\end{aligned}$$

for any $|x| \ge R_0 + R_\varepsilon$. This implies

$$\frac{|e^{T\Delta}\rho(x) - \rho(x)|}{\rho(x)} \le \varepsilon_1 + (e^{\varepsilon R_0} - 1) \quad \text{for} \quad |x| > R_0 + R_\varepsilon.$$

Letting $|x| \to \infty$, we obtain

$$\limsup_{|x|\to\infty} \frac{|e^{T\Delta}\rho(x) - \rho(x)|}{\rho(x)} \le \varepsilon_1 + (e^{\varepsilon R_0} - 1).$$

Since $\varepsilon, \varepsilon_1 > 0$ are arbitrary, we have (5.2) as desired. The lemma is proved. □

Proof of Theorem 1.5. Define $u_0 := \varphi(T + \eta_0(x))$, where $\eta_0(x) := \psi(\rho(x))$ and $T := T(M)$. In the following, we write $\eta(x,t) = e^{t\Delta}\eta_0$ for simplicity. From (1.11) and $\lim_{|x|\to\infty} \rho(x) = \infty$, we obtain $\lim_{|x|\to\infty} \eta_0(x) = 0$. Therefore,

$$\lim_{|x|\to\infty} u_0(x) = \lim_{|x|\to\infty} \varphi(T(M) + \eta_0(x)) = M.$$

Since φ is monotone decreasing, $\eta_0(x) = \psi(\rho(x)) > 0$ and (1.14) yield

$$u_0(x) \le \varphi(T(M)) = M, \quad x \in \mathbb{R}^N.$$

On the other hand, $\rho(x) > \varphi(s^* - T)$ and (1.10) imply

$$u_0(x) \ge \varphi(\psi(\rho(x)) + T) \ge \varphi(s^*) = 0, \quad x \in \mathbb{R}^N.$$

Hence (1.7) holds. Thus, from Theorem 1.1, we obtain $T = T(M) = T(u_0)$.

By (1.20), we have

$$\begin{aligned}
\lim_{|x|\to\infty} \frac{|\nabla \eta_0(x)|}{\eta_0(x)} &= \lim_{|x|\to\infty} \frac{|\nabla \psi(\rho(x))|}{\psi(\rho(x))} = \lim_{|x|\to\infty} \frac{|\psi'(\rho)||\nabla\rho(x)|}{\psi(\rho(x))} \\
&= \lim_{|x|\to\infty} \frac{|\nabla\rho(x)|}{f(\rho(x))\psi(\rho(x))} = 0.
\end{aligned} \tag{5.7}$$

This immediately yields $\lim_{|x|\to\infty} |\nabla\eta_0(x)| = 0$. Furthermore, by (5.3), we obtain $\eta_0(x) \geq e^{-K|x|}\eta_0(0)$ for some $K > 0$. In particular, we get

$$\eta_0(x) \geq c_4\, G(x,\delta), \quad x \in \mathbb{R}^N$$

for some $c_4 > 0$ and $\delta > 0$. Then we have verified all the conditions in (1.18). Therefore, by Theorem 1.4, we obtain

$$\lim_{|x|\to\infty} \frac{u(x,T)}{\varphi(e^{T\Delta}\eta_0(x))} = 1. \tag{5.8}$$

On the other hand, (5.7) and Lemma 5.1 imply $\lim\limits_{|x|\to\infty} \dfrac{e^{T\Delta}\eta_0(x)}{\eta_0(x)} = 1$. Combining this with Lemma 4.1, we have

$$\lim_{|x|\to\infty} \frac{\varphi(e^{T\Delta}\eta_0(x))}{\varphi(\eta_0(x))} = 1.$$

From this and (5.8), we conclude that

$$\lim_{|x|\to\infty} \frac{u(x,T)}{\rho(x)} = \lim_{|x|\to\infty} \frac{u(x,T)}{\varphi(\eta_0(x))} = 1.$$

□

6. Completeness

In this section we prove Theorem 1.6, which states that the blow-up is complete. What we have to show is $\bar{u}(x,t) = \infty$ for all $x \in \mathbb{R}^N$ and $t > T(M)$, where $\bar{u}$ denotes the minimal continuation of the solution u as defined in (1.23).

Lemma 6.1. *Let w be a solution of the problem* (1.1) *with the initial value w_0 and let $\bar{w}$ be the minimal continuation of the solution w. Assume $w_0(x) \leq u_0(x)$ for all $x \in \mathbb{R}^N$. Then $\bar{w}(x,t) \leq \bar{u}(x,t)$ for all $x \in \mathbb{R}^N$ and $t > 0$.*

Proof. Let w_m be a solution of the approximating problem (1.22). Then by the comparison principle, we have $w_m(x,t) \leq u_m(x,t)$ for all $x \in \mathbb{R}^N$, $t > 0$ and $m \in \mathbb{N}$. Letting $m \to \infty$, we obtain the result. □

First we prove Theorem 1.6 under the condition

$$\lim_{n\to\infty} \|u_0 - M\|_{L^\infty(B_{R_n}(a_n))} = 0 \tag{6.1}$$

instead of (1.8). Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence with $\varepsilon_n \to 0$ $(n \to \infty)$ such that

$$M - \varepsilon_n \leq u_0(x), \qquad x \in B_{R_n}(a_n), \quad n = 1, 2, 3, \cdots$$

In the following, we define $v_n(x,t) := (e^{t\Delta}v_{n,0})(x)$, where functions $v_{n,0}$ satisfy $0 \leq v_{n,0} \leq u_0$ in $\mathbb{R}^N$ and

$$v_{n,0}(x) = \begin{cases} M - \varepsilon_n, & x \in B_{\frac{R_n}{2}-1}(a_n), \\ 0, & x \in \mathbb{R}^N \setminus B_{\frac{R_n}{2}}(a_n) \end{cases}$$

for $n = 1, 2, 3, \cdots$. Next we introduce functions of the form:

$$w_n^{(z)}(x,t) := \varphi\big(\psi(\, v_n(x-z,t)\,) - t\big), \qquad x \in \mathbb{R}^N, \quad t > 0, \quad z \in B_{\frac{R_n}{2}}(0).$$

Let $\tilde{T}(w_n^{(z)})$ be the time such that

$$\lim_{t \to \tilde{T}(w_n^{(z)})} \|w_n^{(z)}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \lim_{t \to \tilde{T}(w_n^{(z)})} w_n^{(z)}(a_n + z, t) = \infty. \tag{6.2}$$

Since $\tilde{T}(w_n^{(z)})$ are independent of $z \in B_{\frac{R_n}{2}}(0)$, we denote these time by $\tilde{T}(w_n)$. By Lemma 2.1, we have $0 \le w_n^{(z)} \le u$ in $\mathbb{R}^N \times [0, T(u_0))$. Hence $T(u_0) \le \tilde{T}(w_n)$ for $n = 1, 2, 3, \cdots$. This immediately yields $T(u_0) \le \lim_{n\to\infty} \tilde{T}(w_n)$.

Lemma 6.2. *Assume* (1.7) *and* (1.8). *Then* $\lim_{n\to\infty} \tilde{T}(w_n) = T(u_0)$.

Proof. Seeking a contradiction, we assume $T(u_0) < \limsup_{n\to\infty} \tilde{T}(w_n)$. Then there exists a sequence $n_1 < n_2 < n_3 < \cdots$ such that

$$T(u_0) < \lim_{j\to\infty} \tilde{T}(w_{n_j}). \tag{6.3}$$

Note that $w_{n_j}^{(z)}$ can be defined for all $t \in [0, T(u_0)]$ and $j \in \mathbb{N}$. Since $\tilde{T}(w_n)$ is independent of $z \in B_{\frac{R_n}{2}}(0)$, we assume that $z = 0$ for simplicity. By the same argument as in the proof of Lemma 3.1, we obtain $\lim_{n\to\infty} v_n(a_n, T(M)) = M$. Hence, by (1.15) , we conclude that

$$\begin{aligned} w_{n_j}^{(0)}(a_n, T(M)) &= \varphi\big(\psi(\, v_{n_j}(a_{n_j}, T(M))\,) - T(M)\big) \\ &\to \varphi\big(\psi(M) - T(M)\big) = \infty \quad \text{as} \quad j \to \infty. \end{aligned}$$

Since $T(M) = T(u_0)$, this contradicts (6.3). The Lemma is proved. □

Now we are ready to prove Theorem 1.6 under the condition (6.1).

Proof of Theorem 1.6 *under* (6.1). First we prove

$$\lim_{t\to\tilde{T}(w_n)} \bar{u}(x,t) = \infty, \quad x \in B_{\frac{R_n}{2}}(a_n) \quad (n = 1, 2, 3, \cdots). \tag{6.4}$$

This is equivalent to the following:

$$\lim_{t\to\tilde{T}(w_n)} \bar{u}(a_n + z, t) = \infty, \qquad z \in B_{\frac{R_n}{2}}(0). \tag{6.5}$$

To obtain a contradiction, we suppose $\lim_{t\to\tilde{T}(w_n)} \bar{u}(a_n + z, t) < \infty$ for some $z_0 \in B_{\frac{R_n}{2}}(0)$. By Lemma 2.1 and Lemma 6.1, we obtain

$$\bar{u}(x,t) \ge \bar{w}_n^{(z)}(x,t) \ge w_n^{(z)}(x,t),\ x \in \mathbb{R}^N,\ t \in [0, \tilde{T}(w_n)),\ \text{for all}\ z \in B_{\frac{R_n}{2}}(0),$$

where $\bar{w}_n^{(z)}$ be the minimal continuation of the solution to the problem (1.1) with the initial value $v_{n,0}(x-z)$. Hence

$$\lim_{t\to\tilde{T}(w_n)} w_n^{(z)}(a_n+z_0,t) \le \lim_{t\to\tilde{T}(w_n)} \bar{u}(a_n+z_0,t) < \infty,$$

contradicting (6.2). Therefore, (6.5) must hold.

Lemma 6.2 implies that for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\tilde{T}(w_n) \le T(u_0)+\delta$ for all $n > N$. Thus (1.24) yields

$$\bar{u}(x, T(u_0)+\delta) \ge \int_{B_{\frac{R_n}{2}}(a_n)} G\big(x-y, T(u_0)+\delta-t\big)\, \bar{u}(y,t)\, dy$$

for all $x \in \mathbb{R}^N$ and $t \in (0, \tilde{T}(w_n))$. Using (6.4) and Fatou's lemma, we conclude that $\bar{u}(x, T(u_0)+\delta) = \infty$ for all $x \in \mathbb{R}^N$ by letting $t \to \tilde{T}(w_n)$. The theorem is proved. $\square$

Next we prove Theorem 1.6 under the condition (1.8).

Proof of Theorem 1.6. It is sufficient to prove that the condition (1.8) yields

$$\lim_{n\to\infty} \|u(\cdot,\tau) - U(\tau)\|_{L^\infty(B_{R_n/2}(a_n))} = 0 \tag{6.6}$$

for any small $\tau > 0$, where U is the solution of (1.6). Because we conclude Theorem 1.6 by replacing u_0 by $u(\cdot,\tau)$ and repeating the above argument.

Since $U(x,t) \ge u(x,t)$ for all $x \in \mathbb{R}^N$ and $t \in (0, T(M))$, for any $x \in \mathbb{R}^N$ and $t \in [0,\tau]$, there exists $L = L_\tau > 0$ such that

$$(U-u)_t \le \Delta(U-u) + L(U-u).$$

Hence, by the comparison principle

$$\begin{aligned}(U-u)(x,t) &\le e^{-Lt}\int_{\mathbb{R}^N} G(x-y,t)\,(M-u_0)(y)\,dy \\ &= e^{-Lt}\int_{B_{\frac{R_n}{2}}(x)} G(x-y,t)\,(M-u_0)(y)\,dy \\ &\quad + e^{-Lt}\int_{\mathbb{R}^N\setminus B_{\frac{R_n}{2}}(x)} G(x-y,t)\,(M-u_0)(y)\,dy\end{aligned}$$

for any $x \in B_{\frac{R_n}{2}}(a_n), t \in (0,\tau)$. Since $|x-y| \le R_n/2$ and $|x-a_n| \le R_n/2$ imply $|y-a_n| \le R_n$, we obtain

$$\begin{aligned}(U-u)(x,\tau) \le e^{-L\tau}\|G(\cdot,\tau)\|_{L^\infty(B_{R_n/2}(0))}\, &\|M-u_0\|_{L^1(B_{R_n}(a_n))} \\ &+ Me^{-L\tau}\|G(\cdot,\tau)\|_{L^1(\mathbb{R}^N\setminus B_{R_n}(0))}\end{aligned}$$

for any $x \in B_{\frac{R_n}{2}}(a_n)$. Therefore,

$$\|(U-u)(\cdot,\tau)\|_{L^\infty(B_{R_n/2}(a_n))} \le e^{-L\tau}\|G(\cdot,\tau)\|_{L^\infty(B_{R_n/2}(0))}\,\|M-u_0\|_{L^1(B_{R_n}(a_n))} + Me^{-L\tau}\|G(\cdot,\tau)\|_{L^1(\mathbb{R}^N\setminus B_{R_n}(0))}.$$

Letting $n \to \infty$, and using the condition (1.8), we obtain (6.6). □

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