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IMMERSIONS IN A MANIFOLD WITH A PAIR OF SYMPLECTIC FORMS

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Let N be a manifold with a pair of symplectic forms σ_1, σ_2, and M a manifold with a pair of closed two-forms ω_1 and ω_2. For certain pairs of symplectic forms on N, we prove the existence of smooth immersions $f : M \to N$ such that $f^*\sigma_i = \omega_i$ for $i = 1, 2$.

1. Introduction

Let (N, σ) be a symplectic manifold with the symplectic form σ and M a manifold with a closed two-form ω. An immersion $f : M \to N$ is said to be a symplectic immersion if f pulls back the form σ onto ω. All manifolds and maps in this article are assumed to be smooth. The symplectic immersion theorem of Gromov states that the symplectic immersions $f : M \to N$ satisfy the C^0 dense h-principle near the continuous maps $f_0 : M \to N$ which pull back the deRham cohomology class of σ onto that of ω [**8**, 3.4.2(A)]. The aim of this paper is to generalize this theorem when the manifold N comes equipped with a pair of symplectic forms σ_1 and σ_2 and M has a pair of closed two-forms ω_1 and ω_2. An immersion $f : M \to N$ will be called a *bisymplectic* immersion if it satisfies the relations $f^*\sigma_1 = \omega_1$ and $f^*\sigma_2 = \omega_2$. The bisymplectic immersions are solutions to a system of first-order partial differential equations (PDEs) on a manifold. In fact, we can associate a first-order partial differential operator $\mathcal{D}$ defined on the space of C^∞ maps from M to N such that the bisymplectic maps are solutions to the equation $\mathcal{D} = (\omega_1, \omega_2)$ for a given pair of closed two-forms ω_1, ω_2 on M. This takes us into the theory of C^∞ operators.

Generally, to solve a PDE we need to prove an appropriate implicit function theorem so as to obtain a local inversion of the operator $\mathcal{D}$. The implicit function theorem in the present case should ensure the C^∞-smoothness (regularity) of the inversions. Gromov proves in [**8**, 2.3] that if an rth-order C^∞ operator $\mathcal{D}$ is infintesimally invertible on an open subset $\mathcal{U}$ in the space of admissible maps defined by a dth-order differential relation for $d \geq r$, then the operator $\mathcal{D}$ restricted to $\mathcal{U}$ is an open map relative to the fine C^∞-topologies on the function spaces and there is a smooth local inversion.

In this case, the associated differential operator takes values in closed forms only, so it cannot admit local inversion. However, we observe that there is a first-order differential operator $\bar{\mathcal{D}}$ such that the solutions of the associated PDE give rise to the solutions of the original equation (see Section 4). Moreover, the operator $\bar{\mathcal{D}}$ is infinitesimally invertible on a set of maps which are solutions to some open differential relation. Such maps will be referred as (σ_1, σ_2)-regular maps in this paper (see Definition 2.3). We observe in Section 2 that a generic map is (σ_1, σ_2)-regular under mild dimension restriction. Applying the implicit function theorem of Gromov we then derive the following result in Section 3.

Theorem A. *Let* $\sigma_1 = \sum_{k=1}^{2q} dx_k \wedge dy_k$ *and* $\sigma_2 = \sum_{k=1}^{q}(dx_{2k-1} \wedge dy_{2k} - dx_{2k} \wedge dy_{2k-1})$ *be two linear symplectic forms on* $\mathbb{R}^{4q}$. *Let* M *be a closed manifold with two exact two-forms* ω_1 *and* ω_2. *If* $2q \geq 3 \dim M$ *and* q *is even, then there exists a* (σ_1, σ_2)*-regular bisymplectic immersion* $f : M \longrightarrow \mathbb{R}^{4q}$.

We also partially answer a question of Gromov pertaining to inducing square four-forms from a small perturbation of square symplectic forms. Note that a symplectic immersion $f : M \to N$ (i.e., $f^*\sigma = \omega$) satisfies $f^*(\sigma^2) = \omega^2$. Suppose we break the symmetry of σ^2 by perturbing it a little to Ω. Will it still be possible to induce the form ω^2 by an immersion from the perturbed four-form Ω?[1] We take a linear symplectic form σ_1 on $\mathbb{R}^{2q}$ and some specific perturbation σ_2 of σ_1 which allow (σ_1, σ_2)-regular immersions. If we set the wedge of two such forms as Ω then it is possible to induce the square forms ω^2 from Ω by means of immersions $f : M \to \mathbb{R}^{2q}$.

Theorem B. *Let* $\sigma_1 = \sum_{k=1}^{2q} dx_k \wedge dy_k$ *and* $\sigma_2 = \sum_{k=1}^{q} \lambda_k(dx_{2k-1} \wedge dy_{2k-1} + dx_{2k} \wedge dy_{2k})$ *be two linear symplectic forms on* $\mathbb{R}^{4q}$, *where* λ_k*'s are distinct real numbers. Then given any exact two-form* ω *on a closed manifold* M, *there exists an immersion* $f : M \longrightarrow \mathbb{R}^{4q}$ *such that* $f^*(\sigma_1 \wedge \sigma_2) = \omega^2$ *for* $2q \geq 3 \dim M$.

We also prove the following h-principle in Section 4.

Theorem C. *Suppose* N *is a smooth manifold with closed two-forms* σ_1 *and* σ_2, *and* M *is an open manifold with a closed two-form* ω. *Then in the following two cases (a) and (b), the* (σ_1, σ_2)*-regular immersions* $f : M \to N$ *which pull back both the forms* σ_1 *and* σ_2 *onto* ω, *satisfy the h-principle in the space of continuous maps* $f_0 : M \to N$ *such that* $f_0^*[\sigma_i] = [\omega]$ *for* $i = 1, 2$.

(a) ω *is the zero form on* M.

(b) M *is a symplectic manifold and* ω *is a symplectic form on* M.

[1] The question was posed by Gromov in discussions with the author at the IHES in the year 2005.

The problem of inducing a pair of structures by maps were first considered in [**1**] and later in [**2–4**]. In all these articles, one of the two structures was a Riemannian metric, while the other structure was a contact form (in [**1**]), a Riemannian metric (in [**2**, **3**]) or a symplectic form (in [**4**]), and except in [**3**], the authors exhibit the existence of C^1-immersions which induce the given pair of structures by adapting Nash's technique [**11**]. In [**3**], on the other hand, the authors prove the existence of Lipschitz solutions to the given problem by the convex integration technique [**8**, 2.4]. In this paper, we consider a pair of symplectic forms and prove the existence of smooth immersions which induce a given pair of closed two-forms. We employ, in contrast with earlier works, the analytic technique and the sheaf technique in the theory of h-principle which we discuss in Appendix A.

2. (σ_1, σ_2)-Regular maps

In this section, we introduce the notion of (σ_1, σ_2)-regular maps into a manifold N which comes with a pair of closed two-forms σ_1 and σ_2. The main result of this section gives a sufficient condition for the existence of such regular maps when σ_1, σ_2 is a symplectic pair.

Definition 2.1. Let σ_1 and σ_2 be a pair of linear two-forms on a vector space W. A subspace V of W is said to be (σ_1, σ_2)-*regular* (or simply *regular*) if the linear map

$$(\tilde{\sigma}_1, \tilde{\sigma}_2) : W \longrightarrow \Lambda^1(V) \times \Lambda^1(V)$$

defined by

$$\partial \mapsto (\partial.\sigma_1|_V, \partial.\sigma_2|_V)$$

is an epimorphism.

A necessary condition for the existence of regular subspaces is that $\dim W \geq 2 \dim V$. If V is a regular subspace of (W, σ_1, σ_2) then any subspace of V is also regular.

Proposition 2.1. *V is (σ_1, σ_2)-regular if and only if $W = \ker \tilde{\sigma}_1 + \ker \tilde{\sigma}_2$; in other words, $\ker \tilde{\sigma}_1$ is transversal to $\ker \tilde{\sigma}_2$.*

Proof. Let

$$\bar{\sigma}_1 = \tilde{\sigma}_1|_{\ker \tilde{\sigma}_2} : \ker \tilde{\sigma_2} \longrightarrow \Lambda^1(V) \text{ and } \bar{\sigma}_2 = \tilde{\sigma}_2|_{\ker \tilde{\sigma_1}} : \ker \tilde{\sigma_1} \longrightarrow \Lambda^1(V).$$

Observe that $\ker \bar{\sigma}_1 = \ker \bar{\sigma}_2 = \ker(\tilde{\sigma}_1, \tilde{\sigma}_2) = \ker \tilde{\sigma}_1 \cap \ker \tilde{\sigma}_2$. If V is regular then both $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are surjective. The converse is also true. To see this let $(\alpha_1, \alpha_2) \in \Lambda^1(V) \times \Lambda^1(V)$. Then there exist vectors $\partial_1 \in \ker \tilde{\sigma_1}$ and $\partial_2 \in \ker \tilde{\sigma_2}$ such that

$$\bar{\sigma}_2(\partial_1) = \alpha_2 \text{ and } \bar{\sigma}_1(\partial_2) = \alpha_1.$$

Hence, $\tilde{\sigma}_1(\partial_1+\partial_2)=\alpha_1$ and $\tilde{\sigma}_2(\partial_1+\partial_2)=\alpha_2$ which proves (σ_1,σ_2)-regularity of V. Consequently, V is regular if and only if the following equalities hold:

$$\dim\ker\tilde{\sigma}_1=\dim V+\dim[\ker\tilde{\sigma}_1\cap\ker\tilde{\sigma}_2]=\dim\ker\tilde{\sigma}_2,$$

which implies that

$$\begin{aligned}\dim(\ker\tilde{\sigma}_1+\ker\tilde{\sigma}_2)&=\dim\ker\tilde{\sigma}_1+\dim\ker\tilde{\sigma}_2-\dim[\ker\tilde{\sigma}_1\cap\ker\tilde{\sigma}_2]\\&=2\dim V+\dim\ker(\tilde{\sigma}_1,\tilde{\sigma}_2)\\&=\dim W.\end{aligned}$$

□

If σ_1 and σ_2 are two linear symplectic forms on W then we can characterize the regularity condition as follows:

Corollary 2.1. *Suppose σ_1 and σ_2 are two linear symplectic forms on W. A subspace V is (σ_1,σ_2)-regular if and only if*

$$V^{\sigma_1}\text{ is transversal to }V^{\sigma_2},$$

where $V^{\sigma_i}=\{w\in W:\sigma_i(v,w)=0$ for all $v\in V\}$ is the symplectic complement of V relative to σ_i, $i=1,2$.

Proof. We may identify $\ker\tilde{\sigma}_1$ as the symplectic complement of V relative to σ_1. Similarly $\ker\tilde{\sigma}_2$ is the symplectic complement of V relative to σ_2. □

Proposition 2.2. *Let σ_1 and σ_2 be two linear symplectic forms on W. Let A denote the unique vector space isomorphism $W\longrightarrow W$ determined by the relation $\sigma_2(v,w)=\sigma_1(v,Aw)$ for all $v,w\in W$. A subspace V of W is (σ_1,σ_2)-regular if and only if $V+A(V)$ has the maximum possible dimension.*

Proof. We observe that V^{σ_1} is transversal to V^{σ_2} if and only if V^{σ_1} is transversal to $A(V)^{\sigma_1}$. Since $(V+A(V))^{\sigma_1}=V^{\sigma_1}\cap A(V)^{\sigma_1}$, we obtain $\operatorname{codim}(V+A(V))^{\sigma_1}=2n$. Consequently, $V+A(V)$ has the maximum possible dimension, as σ_1 is a symplectic form. □

Definition 2.2. Let W be a vector space with two linear two-forms σ_1,σ_2. A linear map $\ell:V\to W$ from a vector space V to W will be called (σ_1,σ_2)-*regular* if $\ell(V)$ is a regular subspace of W. Equivalently, the linear map

$$\begin{aligned}W&\to\Lambda^1(V)\times\Lambda^1(V),\\ \partial&\mapsto(\ell^*(\partial.\sigma_1),\ell^*(\partial.\sigma_2))\end{aligned}$$

is an epimorphism.

Observation. Let p_1 and p_2 denote the projection maps of the product vector space $V\times W$ onto V and W, respectively. Suppose that σ_1, σ_2 are two linear two-forms on W and ω_1,ω_2 are two linear two-forms on V. Let $\bar{\sigma}_1=p_1^*\omega_1-p_2^*\sigma_1$ and $\bar{\sigma}_2=p_1^*\omega_2-p_2^*\sigma_2$. If $\ell:V\to W$ is (σ_1,σ_2) regular then the graph map of ℓ, $\bar{\ell}:V\to V\times W$, is $(\bar{\sigma}_1,\bar{\sigma}_2)$-regular. This follows from the relation that $\bar{\ell}^*(\partial.\bar{\sigma}_i)=\ell^*(\partial.\sigma_i)$ for all $\partial\in W$, $i=1,2$.

Definition 2.3. Let σ_1, σ_2 be two closed two-forms on a manifold N. A smooth immersion $f : M \longrightarrow N$ is (σ_1, σ_2)-*regular* if the derivative map $df_x : T_xM \to T_{f(x)}N$ is $(\sigma_1(f(x)), \sigma_2(f(x)))$-regular for each $x \in M$. We shall often refer the (σ_1, σ_2)-regular maps as regular maps.

Theorem 2.1. *Consider two linear symplectic forms σ_1 and σ_2 on $\mathbb{R}^{2q}$ which are related by $\sigma_2(v, w) = \sigma_1(v, Aw)$ for some linear isomorphism A of $\mathbb{R}^{2q}$. Suppose that k is the maximum of the geometric multiplicities of real eigenvalues of A. Then generic maps $f : M \to \mathbb{R}^{2q}$ are (σ_1, σ_2)-regular immersions for $2q \geq \max\{3 \dim M, 2 \dim M + k\}$.*

In particular, if A has no real eigenvalues then generic maps $f : M \to \mathbb{R}^{2q}$ are (σ_1, σ_2)-regular immersions for $2q \geq 3 \dim M$.

Proof. Let $\dim M = n$. Let Σ be the subset of the Grassmannian manifold $Gr_n(\mathbb{R}^{2q})$ which consists of all n-planes T in $\mathbb{R}^q$ such that $T \cap A(T) \neq 0$. Then Σ is the union of two sets Σ' and Σ'' where

(1) Σ' consists of all n-planes T in $\mathbb{R}^{2q}$ which contains an eigenvector of A, where the eigenspaces of A are at most k-dimensional,
(2) Σ'' consists of all n-planes T in $\mathbb{R}^{2q}$ which contains a two-dimensional subspace spanned by a pair $\{v, Av\}$ for some $v \in T$.

Since the geometric multiplicities of the eigenvalues of A are at the most k, the dimension of Σ' is less than or equal to $k-1+(n-1)(2q-n)$. On the other hand, the dimension of Σ'' is less than or equal to $(2q-1)+(n-2)(2q-n)$. Therefore, $\dim \Sigma = \max(\dim \Sigma', \dim \Sigma'')$.

Let $\mathcal{R}$ denote the open subset of $J^1(M, \mathbb{R}^{2q})$ consisting of one-jet of germs of immersions from M to $\mathbb{R}^{2q}$ and let $p : \mathcal{R} \longrightarrow Gr_n(\mathbb{R}^{2q})$ be the canonical projection which maps an one-jet $j^1_f(x)$, $x \in M$, onto the n-dimensional subspace $\operatorname{Im} df_x$ in $\mathbb{R}^{2q}$. A smooth map $f : M \longrightarrow \mathbb{R}^{2q}$ is a regular immersion if and only if $p \circ j^1_f$ misses the set Σ, if and only if j^1_f misses the set $p^{-1}(\Sigma)$.

Now, observe that if $2q \geq \max\{3n, 2n+k\}$ then $\operatorname{codim} \Sigma > n$, and hence the same is true for the codimension of $p^{-1}(\Sigma)$, since p is a submersion. Therefore, by an application of the Thom Transversality Theorem [**7**] j^1_f misses $p^{-1}\Sigma$ for a generic f. Thus, a generic map $f : M \longrightarrow \mathbb{R}^{2q}$ is regular if $2q \geq \max\{3n, 2n+k\}$. □

Remark 2.1. If σ_1 and σ_2 are two symplectic forms on a manifold N, then there is a bundle isomorphism $A : TN \to TN$ satisfying the following relation: $\sigma_2(v, w) = \sigma_1(v, Aw)$ for all $v, w \in T_xN$ and $x \in N$. Suppose that

$$k = \max_{x \in M}\{\text{geometric multiplicities of the real eigenvalues of } A_x\}.$$

Then, we can obtain an exact analogue of Theorem 2.1 for maps $f : M \to (N, \sigma_1, \sigma_2)$ with this k.

Let σ_1, σ_2 be two linear symplectic forms on W and let A satisfy $\sigma_2(v,w) = \sigma_1(v, Aw)$ for all $v, w \in W$. If $\{u_1, \dots, u_q, v_1, \dots, v_q\}$ is a canonical symplectic basis for σ_1 then relative to this basis A can be represented by a matrix of the following form:

$$M(A) = \begin{pmatrix} B & C \\ D & B^t \end{pmatrix},$$

where B, C, D are $q \times q$ square matrices of which C and D are skew-symmetric. Indeed, writing $\sigma_1 = \sum_{k=1}^{q} u_k^* \wedge v_k^*$ we have $B = (\sigma_2(u_j, v_i))_{i,j}$, $C = (\sigma_2(v_i, v_j))_{i,j}$ and $D = (\sigma_2(u_i, u_j))_{i,j}$.

If $M(A)$ is symmetric then all eigenvalues are real and if $M(A)$ is skew-symmetric then all eigenvalues are purely imaginary. We consider two special cases under the above criteria. The first, when B is symmetric and $C = D = 0$, and the second, when B is skew-symmetric and $C = D = 0$.

Example 2.1. Let $\sigma_1 = u_k^* \wedge v_k^*$. If $\sigma_2 = \sum_{k=1}^{q} \lambda_k \, u_k^* \wedge v_k^*$, then $M(A)$ is symmetric. If $q = 2n$ is even and $\sigma_2 = \sum_{k=1}^{n} (u_{2k-1}^* \wedge v_{2k}^* - u_{2k}^* \wedge v_{2k-1}^*)$, then $M(A)$ is skew-symmetric.

We can now easily deduce the following two corollaries from Theorem 2.1.

Corollary 2.2. *Let $\sigma_1 = \sum_{i=1}^{q} dx_i \wedge dy_i$ and $\sigma_2 = \sum_{i=1}^{q} \lambda_i dx_i \wedge dy_i$ be two symplectic forms on $\mathbb{R}^{2q}$, where the multiplicities of λ_i's are less than equal to k. Then a generic map $f : M \to \mathbb{R}^{2q}$ is a (σ_1, σ_2)-regular immersion for $2q \geq \max\{3 \dim M, 2 \dim M + k\}$.*

Corollary 2.3. *If q is even, say $q = 2n$ and $\sigma_1 = \sum_{i=1}^{2n} dx_i \wedge dy_i$ and $\sigma_2 = \sum_{k=1}^{n} (dx_{2k-1} \wedge dy_{2k} - dx_{2k} \wedge dy_{2k-1})$, then generic maps $f : M \to \mathbb{R}^{2q}$ are (σ_1, σ_2)-regular immersions for $2q \geq 3 \dim M$.*

We end this section by formulating equivalent criteria for the symmetry and the skew-symmetry conditions on $M(A)$. The following is a standard fact from symplectic geometry [**9**].

Lemma 2.1. *Let W be a vector space with a linear symplectic form σ which is invariant under an almost complex structure J. Define a bilinear form g on W by*

$$g(u, v) = \sigma(u, Jv) \text{ for all } u, v \in W.$$

Then

(1) *$\sigma(u, v) = g(Ju, v)$;*
(2) *g is a non-degenerate symmetric form;*
(3) *g is J-invariant.*

The triple (g, J, σ) is such that given any two of these structures the third structure is obtained uniquely by the relation $g(u, v) = \sigma(u, Jv)$.

Lemma 2.2. *Let σ_1 and σ_2 be two linear symplectic forms on W. Let A denote the unique vector space isomorphism $W \longrightarrow W$ determined by the relation $\sigma_2(v,w) = \sigma_1(v, Aw)$. Let J be an almost complex structure on W such that σ_1 is invariant under J. Define g_1 by $g_1(v,w) = \sigma_1(v, Jw)$ for $v, w \in W$. Then the following are equivalent:*

(1) *σ_2 is invariant under J.*
(2) *A commutes with J.*
(3) *A is symmetric relative to the symmetric form g_1.*

Therefore, under any of the above conditions, eigenvalues of A are all real and they have even multiplicities.

Proof. (1) $\Longrightarrow$ (2): Since both σ_1 and σ_2 are J invariant, $\sigma_1(Ju, JAv) = \sigma_1(u, Av) = \sigma_2(u,v) = \sigma_2(Ju, Jv) = \sigma_1(Ju, AJv)$ for all v, w. Now, the non-degeneracy of σ_1 implies that $AJ = JA$. Consequently the eigenvalues occur in pairs.

(1) $\Longrightarrow$ (3): $g_1(Au, v) = \sigma_1(Au, Jv) = -\sigma_1(Jv, Au) = -\sigma_2(Jv, u) = \sigma_2(u, Jv)$. Since σ_2 is J-invariant, $\sigma_2(u, Jv) = \sigma_2(v, Ju)$ and hence we have $g_1(Au, v) = g_1(u, Av)$.

(2) $\Longrightarrow$ (1): Since A commutes with J we obtain $\sigma_2(Jv, Jw) = \sigma_1(Jv, AJw) = \sigma_1(Jv, JAw)$. Further, since σ_1 is J-invariant, $\sigma_1(Jv, JAw) = \sigma_1(v, Aw) = \sigma_2(v, w)$. Thus σ_2 is J-invariant.

(3) $\Longrightarrow$ (1): For any $v, w \in W$, $\sigma_2(Jv, Jw) = \sigma_1(Jv, AJw) = -g_1(v, AJw) = -g_1(Av, Jw)$ (since A is symmetric with respect to g_1) $= -g_1(Jw, Av) = -\sigma_1(w, Av) = \sigma_2(v, w)$.

If A is symmetric with respect to g_1 then there is a g_1-orthonormal basis $u_1, u_2, \ldots, u_{2n}$ consisting of eigenvectors of A. Suppose $Au_i = \lambda_i u_i$ for $i = 1, 2, \ldots, 2n$, where λ_i are real numbers. Consider $u_1 \in W$. There exists at least one u_{n_1}, $n_1 \neq 1$, such that $\sigma_2(u_1, u_{n_1}) \neq 0$. Using the relation between σ_1 and σ_2 we obtain that $\lambda_1 = \lambda_{n_1}$. Now, the g_1-orthogonal complement W_1 of the span of u_1 and u_{n_1} is spanned by the set $\{u_i | i \neq 1, i \neq n_1\}$. Hence, we can repeat the above argument for the pair $(W_1, A|_{W_1})$, where $\dim W_1 < \dim W$. Consequently, an induction on $\dim W$ proves that the eigenvalues are real and repeated even number of times. □

Analogously, we can prove:

Lemma 2.3. *Let σ_1 and σ_2 be two linear symplectic forms on W and A, J and g_1 be defined as in Lemma 2.2. Then the following are equivalent:*

(1) *$\sigma_2(v, Jw) = \sigma_2(Jv, w)$ for all $v, w \in W$; in other words, $J^*\sigma_2 = -\sigma_2$.*
(2) *A anticommutes with J, that is, $AJ = -JA$.*
(3) *A is skew-symmetric relative to the symmetric bilinear form g_1.*

Therefore, under any of the above conditions, eigenvalues of A are purely imaginary.

3. Existence of immersions inducing a given pair of forms

In this section we study the existence of bisymplectic immersions in a manifold (N, σ_1, σ_2), where $\sigma_1 = d\tau_1$ and $\sigma_2 = d\tau_2$ are two exact two-forms on N. Let M be a manifold with a pair of exact two-forms $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$. Consider the differential operator

$$\mathcal{D} : C^\infty(M, N) \to \Omega^2(M) \times \Omega^2(M)$$

which takes an $f \in C^\infty(M, N)$ onto the pair $(f^*\sigma_1, f^*\sigma_2)$, where $C^\infty(M, N)$ denotes the space of smooth maps from M to N and $\Omega^2(M)$ denotes the space of two-forms on M. The bisymplectic immersions $f : M \to N$ are solutions to the differential equation $\mathcal{D}f = (\omega_1, \omega_2)$ for the given pair of two-forms on M. If $f : M \to N$ is a bisymplectic immersion, then $f^*\tau_1 - \alpha_1$ and $f^*\tau_2 - \alpha_2$ are closed one-form, and conversely.

Let us now consider the following first-order differential operator:

$$\tilde{\mathcal{D}} : C^\infty(M, N) \times C^\infty(M) \times C^\infty(M) \longrightarrow \Omega^1(M) \times \Omega^1(M)$$

defined by

$$(f, \phi_1, \phi_2) \mapsto (f^*\tau_1 + d\phi_1, f^*\tau_2 + d\phi_2),$$

where $C^\infty(M)$ is the space of smooth real valued functions on M and $\Omega^1(M)$ is the space of one-forms on M. This operator is closely associated with the operator $\mathcal{D}$; indeed, if (f, ϕ_1, ϕ_2) is a solution of the equation $\tilde{\mathcal{D}} = (\alpha_1, \alpha_2)$, then clearly f satisfies the equations $f^*\sigma_1 = \omega_1$ and $f^*\sigma_2 = \omega_2$.

The linearization of $\tilde{\mathcal{D}}$ at (f, ϕ_1, ϕ_2) is an operator

$$L : \Gamma^\infty(f^*TN) \times C^\infty(M) \times C^\infty(M) \longrightarrow \Omega^1(M) \times \Omega^1(M),$$

which is given by

$$(\partial, \psi_1, \psi_2) \mapsto \left(f^*(\partial.\sigma_1 + d(\partial.\tau_1)) + d\psi_1, f^*(\partial.\sigma_2 + d(\partial.\tau_2)) + d\psi_2\right),$$

where ∂ is a vector field on N along f, and ψ_1 and ψ_2 are smooth functions on M. L is right invertible if we can solve the following system of equations in ∂, ψ_1 and ψ_2 for arbitrary one-forms g_1 and g_2 on M:

$$\begin{aligned} f^*(\partial.\sigma_1) = g_1, \quad & f^*(\partial.\sigma_2) = g_2, \\ f^*(\partial.\tau_1) + \psi_1 = 0, \quad & f^*(\partial.\tau_2) + \psi_2 = 0. \end{aligned}$$

If f is (σ_1, σ_2)-regular (see Definition 2.3), then the first two equations can be solved for ∂, the value of which is then inserted in the second set of equations to obtain ψ_1 and ψ_2. Thus, L is right invertible by a zeroth-order operator $L^{-1} : (g_1, g_2) \mapsto (\partial, \psi_1, \psi_2)$ when f is (σ_1, σ_2)-regular. Hence, $\tilde{\mathcal{D}}$ is infinitesimally invertible on (σ_1, σ_2)-regular immersions. Now, the (σ_1, σ_2)-regular immersions are solutions to an open differential relation $A \subset J^1(M, N)$. Consequently, the set of regular C^∞ immersions, $\mathcal{A}$, form an open subspace of $C^\infty(M,N)$ in the fine C^∞-topology. Hence, we obtain the following result by an application of Theorem A.2 (see Section 5).

Proposition 3.1. *The restriction of $\tilde{\mathcal{D}}$ to the space of regular immersions is an open map relative to the fine C^∞-topologies on the function spaces.*

We are now in a position to prove Theorem A.

Proof of Theorem A. Denote the coordinates on $\mathbb{R}^{2q}$ by $x_1, y_1, \dots, x_q, y_q$. Suppose that $q = 2n$ and let

$$\bar\sigma_1 = \sum_{k=1}^{2n} dx_k \wedge dy_k \text{ and } \bar\sigma_2 = \sum_{k=1}^{n} (dx_{2k-1} \wedge dy_{2k} - dx_{2k} \wedge dy_{2k-1}),$$

so that $\sigma_1 = \bar\sigma_1 \oplus \bar\sigma_1$ and $\sigma_2 = \bar\sigma_2 \oplus \bar\sigma_2$. Take a $(\bar\sigma_1, \bar\sigma_2)$-regular immersion $h : M \longrightarrow \mathbb{R}^{2q}$; such an h is guaranteed by Corollary 2.3 since $2q \geq 3 \dim M$. Define $h' : M \longrightarrow \mathbb{R}^{2q}$ by $h'(x) = (h_1(x), -h_2(x), \dots, h_{2q-1}(x), -h_{2q}(x))$, and set $\bar h = (h, h') : M \to \mathbb{R}^{4q}$. Clearly, $\bar h$ is (σ_1, σ_2)-regular and it pulls back both σ_1 and σ_2 onto the zero form on M, that is, $\bar h^* \sigma_i = 0$ for $i = 1, 2$. Since both σ_1 and σ_2 are exact, we can write $\sigma_1 = d\tau_1$ and $\sigma_2 = d\tau_2$ for some one-forms τ_1 and τ_2 on $\mathbb{R}^{2q}$, and this implies that $\bar h^*\tau_1$ and $\bar h^*\tau_2$ are closed one-forms on M. Therefore, if we define $\tilde{\mathcal{D}}$ as above then its image contains an ordered pair (c_1, c_2) of closed one-forms on M, where $c_i = \bar h^* \tau_i$ for $i = 1, 2$. Since M is a closed manifold and $\tilde{\mathcal{D}}$ is an open map (by Proposition 3.1), for every pair of one-forms (α_1, α_2) on M there exists a scalar $\lambda > 0$ such that $(c_1 + \lambda\alpha_1, c_2 + \lambda\alpha_2)$ also belongs to the image of $\tilde{\mathcal{D}}$. This implies that, we have a triple $(\bar f, \phi_1, \phi_2)$ such that $\bar f^*\tau_1 + d\phi_1 = c_1 + \lambda\alpha_1$ and $\bar f^*\tau_2 + d\phi_2 = c_2 + \lambda\alpha_2$, where $\bar f : M \to \mathbb{R}^{4q}$ is a regular immersion and ϕ_1 and ϕ_2 are smooth functions on M. Consequently, $\bar f^*(\sigma_1) = \lambda d\alpha_1$ and $\bar f^*(\sigma_2) = \lambda d\alpha_2$. The required map f is then defined as $f = \lambda^{-\frac{1}{2}} \bar f$.

Remark 3.1. Let (N, σ_1, σ_2) and (M, ω_1, ω_2) be as described in the beginning of this section. Now, suppose that

- the pair (σ_1, σ_2) admits regular immersions $M \to N$ and
- there exists a diffeomorphism ϕ of N such that $\phi^*\sigma_i = -\sigma_i$ for $i = 1, 2$.

If M is a closed manifold, then by setting $h' = \phi \circ h$ it may be seen as in the proof of Theorem A that there exists a regular immersion $f : M \to N \times N$ which satisfies $f^*(\sigma_1 \oplus \sigma_1) = \omega_1$ and $f^*(\sigma_2 \oplus \sigma_2) = \omega_2$.

Also, if we have

- a pair of closed two-forms σ_1 and σ_2 on N and
- a (σ_1, σ_2)-regular embedding $f : M \to N$ such that $f^*\sigma_1 = 0 = f^*\sigma_2$

then both σ_1 and σ_2 are exact on a tubular neighbourhood of image f in N. Therefore, as in the above theorem, an arbitrary pair of exact forms on M can be induced from the pair (σ_1, σ_2) by a regular immersion $\bar f$. Moreover, we can choose $\bar f$ sufficiently C^0 close to f.

The next result follows from the above remark, together with Corollary 2.2.

Corollary 3.1. *Let*

$$\sigma_1 = \sum_{k=1}^{2q} dx_k \wedge dy_k \text{ and } \sigma_2 = \sum_{k=1}^{q} \lambda_k \left(dx_{2k-1} \wedge dy_{2k-1} + dx_{2k} \wedge dy_{2k}\right)$$

be two symplectic forms on $\mathbb{R}^{4q}$, where the multiplicity of each λ_k is less than or equal to k. If M is a closed manifold, then for $2q \geq \max\{3\dim M, 2\dim M + k\}$, there exists a smooth regular immersion $f : M \longrightarrow \mathbb{R}^{4q}$ such that $f^*(\sigma_1) = \omega_1$ and $f^*(\sigma_2) = \omega_2$, where ω_1 and ω_2 are given exact two-forms on M.

Theorem B is now immediate from the above corollary if we take $\omega_1 = \omega_2 = \omega$.

We end this section with the following result in h-principle.

Theorem 3.1. *Let σ_1, σ_2 be two exact two-forms on a manifold N, and ω_1 and ω_2 two exact two-forms on M. Then (σ_1, σ_2)-regular maps $f : M \times \mathbb{R} \longrightarrow N$ which pull back the forms σ_1 and σ_2 onto $p^*\omega_1$ and $p^*\omega_2$, respectively, satisfy the h-principle.*

We postpone the proof of this theorem as of now.

4. h-Principle of immersions inducing given pair of forms

In this section, we assume that σ_1 and σ_2 are arbitrary closed two-forms on N and ω_1, ω_2 are two closed two-forms on M. We aim to see if the regular maps $f : M \to N$ satisfying $f^*\sigma_i = \omega_i$, $i = 1, 2$ follow the h-principle. We first note that such an f pulls back the deRham cohomology classes of σ_1 and σ_2, respectively, onto those of ω_1 and ω_2. Therefore, the h-principle can at most be C^0-dense (Definition A.2)in the space of continuous maps $f_0 : M \to N$ such that $f_0^*[\sigma_i] = [\omega_i]$ for $i = 1, 2$, in which case the solution space, if non-empty, is dense in the space of such continuous maps f_0. In view of this we start with a smooth map $f_0 : M \to N$ which satisfies these cohomology conditions. Let p_1 and p_2 denote the projection maps of the product manifold $M \times N$ onto M and N, respectively. Consider the two product forms on $M \times N$:

$$\bar{\sigma}_1 =: p_1^*\omega_1 - p_2^*\sigma_1 \text{ and } \bar{\sigma}_2 =: p_1^*\omega_2 - p_2^*\sigma_2.$$

Since the graph of f_0 is an embedded submanifold of the product manifold $M \times N$, the cohomology condition on f_0 implies that both $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are exact in a tubular neighbourhood Y of graph f_0. Suppose, $\bar{\sigma}_i = d\tau_i$ for some one-forms τ_i on Y, $i = 1, 2$.

If $\bar{f} : M \to M \times N$ is a section of p_1 then we will denote the underlying map of $\bar{f}$, namely $p_2 \circ \bar{f}$, by f. If f is (σ_1, σ_2)-regular then $\bar{f}$ is $(\bar{\sigma}_1, \bar{\sigma}_2)$-regular (see Observation in Section 2).

Let $\Gamma^\infty(Y)$ denote the sheaf of C^∞ sections of the product bundle $M \times N \to M$ whose images lie in Y. Define a differential operator $\bar{\mathcal{D}}$ as follows:

$$\begin{aligned}\Gamma^\infty(Y) \times C^\infty(M) \times C^\infty(M) &\xrightarrow{\bar{\mathcal{D}}} \Omega^1(M) \times \Omega^1(M), \\ (\bar{f}, \phi_1, \phi_2) &\mapsto (\bar{f}^*\tau_1 + d\phi_1, \bar{f}^*\tau_2 + d\phi_2).\end{aligned}$$

If $\bar{\mathcal{D}}(\bar{f}, \phi_1, \phi_2) = 0$ then $f^*\sigma_1 = \omega_1$ and $f^*\sigma_2 = \omega_2$.

Now, as we observed in the previous section, $\bar{\mathcal{D}}$ is infinitesimally invertible at all triples $(\bar{f}, \phi_1, \phi_2)$ for which $p_2 \circ \bar{f}$ is (σ_1, σ_2)-regular. Consequently, $\bar{\mathcal{D}}$ is a first-order differential operator which admits a zeroth-order inversion of defect 1 (see Section 5).

Let E denote the fibre bundle over M whose total space is $Y \times \mathbb{R} \times \mathbb{R}$ and the projection map $\pi : E \to M$ is defined by $\pi(y, t, s) = p_1(y)$, where $(y, t, s) \in Y \times \mathbb{R} \times \mathbb{R}$. The sections of E are in one-to-one correspondence with triples $(\bar{f}, \phi_1, \phi_2)$, where $\bar{f} \in \Gamma^\infty(Y)$ and $\phi_1, \phi_2 \in C^\infty(M)$. Let G denote the vector bundle $\Lambda^1(M) \oplus \Lambda^1(M)$, where $\Lambda^1(M)$ is the cotangent bundle of M. The operator $\bar{\mathcal{D}}$ induces a sequence of bundle maps $\bar{\Delta}_\alpha : E^{(\alpha+1)} \to G^{(\alpha)}$, $\alpha \geq 0$, satisfying the relations $\bar{\Delta}_\alpha \circ j^{\alpha+1}_{(\bar{f},\phi_1,\phi_2)} = j^\alpha_{\bar{\mathcal{D}}(\bar{f},\phi_1,\phi_2)}$ (see Section 5). For each non-negative integer α, we now define a differential relation $\bar{\mathcal{R}}^\alpha = \bar{\Delta}_\alpha^{-1}(0)$. All these relations have the same C^∞ solutions as the equation $\bar{\mathcal{D}} = 0$.

Let $\bar{\mathcal{R}}_2$ be the subset of $\bar{\mathcal{R}}^2$ consisting of all three-jets at x, $x \in M$, which can be represented by a local section $(\bar{f}, \phi_1, \phi_2)$ of E such that $p_2 \circ \bar{f}$ is regular at x. Therefore, $(\bar{f}, \phi_1, \phi_2)$ is a solution of $\bar{\mathcal{R}}_2$ if

(1) $f = p_2 \circ \bar{f}$ is (σ_1, σ_2)-regular, and
(2) $\bar{f}^*\tau_1 + d\phi_1 = 0$ and $\bar{f}^*\tau_2 + d\phi_2 = 0$, that is $\bar{\mathcal{D}}(\bar{f}, \phi_1, \phi_2) = 0$.

Let $\bar{\Phi}$ denote the solution sheaf of $\bar{\mathcal{R}}_2$ and $\bar{\Psi}_2$ the sheaf of sections of $\bar{\mathcal{R}}_2$.

The next result follows from Theorem A.2 and Proposition A.1 in Section 5.

Proposition 4.1. *$\bar{\Phi}$ is a microflexible sheaf. Moreover, the three-jet map $j^3 : \bar{\Phi} \to \bar{\Psi}_2$ is a local weak homotopy equivalence; in other words, $\bar{\mathcal{R}}_2$ satisfies the local parametric h-principle.*

Remark 4.1. Let Φ denote the sheaf of regular solutions of the original differential equation, namely, $\mathcal{D}f \equiv (f^*\sigma_1, f^*\sigma_2) = (\omega_1, \omega_2)$. Let $\mathcal{R}_1$ be the subset of $J^2(M, N)$ consisting of two-jets of infinitesimal regular solutions of order 1 of the differential equation $\mathcal{D} = (\omega_1, \omega_2)$ and Ψ_1 the sheaf of sections of $\mathcal{R}_1$. There is a canonical map $p' : \bar{\Phi} \to \Phi$ which takes the triple $(\bar{f}, \phi_1, \phi_2)$ onto $p_2 \circ \bar{f}$. Then p' induces a map $p : \bar{\mathcal{R}}_2 \to \mathcal{R}_1$ defined by $(j^3_{\bar{f}}, j^3_{\phi_1}, j^3_{\phi_2})(x) \mapsto j^2_{p_2 \circ \bar{f}}(x)$. To see this, we note that the exterior differential operator d determines, for each $k \geq 1$, a sequence of bundle maps $d_\alpha : (\Lambda^k(M))^{(\alpha+1)} \longrightarrow (\Lambda^{k+1}(M))^{(\alpha)}$, $\alpha = 0, 1, 2, \dots$ such that $d_\alpha \circ j_\tau^{\alpha+1} = j_{d\tau}^\alpha$, where τ is a k-form. Therefore, if $j^2_{(\bar{f}^*\tau_1 + d\phi_1)} = 0$ and $j^2_{(\bar{f}^*\tau_2 + d\phi_2)} = 0$ at x, then applying d_1 on both sides we get $j^1_{f^*\sigma_1}(x) = j^1_{\omega_1}(x)$ and $j^1_{f^*\sigma_2}(x) = j^1_{\omega_2}(x)$, where $f = p_2 \circ \bar{f}$. Thus, f is an infinitesimal solution of order 1 of the equation $\mathcal{D} = (\omega_1, \omega_2)$.

Further, we have the following commutative diagram which relates the solution sheaves of the two differential equations:

$$\begin{array}{ccc} \bar{\Phi} & \xrightarrow{j^3} & \bar{\Psi}_2 \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p_*} \\ \Phi & \xrightarrow{j^2} & \Psi_1 \end{array}$$

where p_* is the map induced by p. It can be proved following [**5**] that $p : \bar{\mathcal{R}}_2 \to \mathcal{R}_1$ is a surjective submersion and the fibres of p are affine subspaces; hence p has a section. These are, in fact, consequences of the following sequence of vector bundles and maps which is exact by the formal Poincaré Lemma:

$$\cdots \longrightarrow \Big(\Lambda^{k-2}(M)\Big)^{(3)} \xrightarrow{d_2} \Big(\Lambda^{k-1}(M)\Big)^{(2)} \xrightarrow{d_1} \Big(\Lambda^{k}(M)\Big)^{(1)} \longrightarrow \cdots .$$

Since p has a section, p_* is onto. It is now easy to see from the above commutative square, that if $\bar{\mathcal{R}}_2$ satisfies the h-principle, then $\mathcal{R}_1$ also satisfies the h-principle.

We recall the following definitions from [**8**, 3.4.1(B)].

Definition 4.1. Let M be a smooth manifold with a closed two-form ω. A vector field ∂ on M is said to be ω-*isometric* if the Lie derivative $L_\partial \omega = 0$, in other words, $\partial.\omega$ is a closed form. The vector field ∂ is said to be ω-*exact* if there exists a zero-form α (i.e., a function on M) such that $\partial.\omega = d\alpha$.

A (local) isotopy $\delta_t : U \to M$ is called *exact* if $\delta'_t = \frac{d\delta_t}{dt}$ is an exact vector field on $\delta_t(U)$ for all $t \in [0,1]$ and there exists a homotopy of zero-forms α_t defined on $\delta_t(U)$ such that $\delta'_t.\omega = d\alpha_t$.

Observation. The isotopy defined by a ω-isometric vector field consists of diffeomorphisms which preserve the form ω. If δ_t is a ω exact diffeotopy which fixes an open set U_0 pointwise then the exact one-forms $\delta'_t.\omega$ vanish on U_0. This means that any primitive of $\delta'_t.\omega$ takes a constant real value on U_0. Hence, we can choose a primitive ϕ_t which also vanishes on the set U_0. We would like to remark here that diffeotopies with this property are referred as *strictly exact* diffeotopy in [**8**, 3.4.1].

Lemma 4.1. *Suppose that $\omega_1 = \omega_2 = \omega$. Then the ω-exact diffeotopies of M act on the sheaf $\bar{\Phi}$.*

Proof. We follow [**8**, 3.4.1(B)] to define an action of strictly ω-exact diffeotopies on $\bar{\Phi}$. First note that each diffeotopy $\delta_t : U \to V$ of open subsets of M lifts to a diffeotopy $\bar{\delta}_t : U \times N \to V \times N$ by $\bar{\delta}_t(u,x) = (\delta_t(u), x)$.

Suppose that, $Y' \subset Y \cap (U \times N)$ and $\bar{\delta}_t(Y') \subset Y$ for all t. Then δ_t has a natural action on sections $\bar{f} : V \to V \times N$ whose images lie in Y. The action is given by $\delta_t.\bar{f} = \bar{\delta}_t^{-1}\bar{f}\delta_t$ (see Example A.1 in Section 5).

Differentiating the homotopy of one-forms $\bar{\delta}_t^*\tau_1$ with respect to t we obtain

$$L_{\bar{\delta}'_t}\tau_1 = \bar{\delta}'_t.\bar{\sigma}_1 + d(\bar{\delta}'_t.\tau_1) = p_1^*(\delta'_t.\omega) + d(\bar{\delta}'_t.\tau_1).$$

If δ_t is strictly ω-exact, then there exists a homotopy of C^∞ functions α_t along $\delta_t(U)$ such that $\delta'_t.\omega = d\alpha_t$. Hence

$$\bar{\delta}_t^*\tau_1 = \tau_1 + d\phi_t, \text{ where } \phi_t = \int_0^t (p_1^*\alpha_t + \bar{\delta}'_t.\tau_1)dt. \tag{4.1}$$

Further, if δ_t is constant on U_0 for $t \leq t_0$, we can and we do choose $\phi_t = 0$ for $t \leq t_0$.

Consider a triple $(\bar{f}, \phi, \psi)$ in $\bar{\Phi}$ so that $\bar{f} : V \to V \times N$ has its image in Y and $\bar{f}^*\tau_1 + d\phi = 0$, $\bar{f}^*\tau_2 + d\psi = 0$. If we define $\delta_t.\bar{f} = \bar{\delta}_t^{-1}\bar{f}\delta_t$, then using equation (4.1) we obtain

$$\begin{aligned}(\delta_t.\bar{f})^*\tau_1 &= \delta_t^*\bar{f}^*(\bar{\delta}_t^{-1})^*\tau_1 \\ &= \delta_t^*\bar{f}^*[\tau_1 - d(\bar{\delta}_t^{-1})^*\phi_t] \\ &= -\delta_t^* d\phi + d(\delta_t.\bar{f})^*\phi_t \\ &= -d[\delta_t^*\phi - (\delta_t.\bar{f})^*\phi_t].\end{aligned}$$

Finally, since δ_t is strictly ω-exact diffeotopy, $\phi \mapsto \delta_t^*\phi - (\delta_t.\bar{f})^*\phi_t$ defines an action on the space of C^∞ functions on M (for a fixed $\bar{f}$). Indeed, if δ_t is constant in t on a maximal open set U_0 then we can choose $\phi_t = 0$ on U_0 and then $\delta_t^*\phi - (\delta_t.\bar{f})^*\phi_t$ is constant on U_0.

Therefore, we can define the action of a strictly exact diffeotopy δ_t satisfying $\bar{\delta}_t(Y') \subset Y$ on $\bar{\Phi}$ by

$$\delta_t.(\bar{f}, \phi, \psi) = (\delta_t.\bar{f}, \delta_t^*\phi - (\delta_t.\bar{f})^*\phi_t, \delta_t^*\psi - (\delta_t.\bar{f})^*\psi_t),$$

where $\delta_t.\bar{f} = \bar{\delta}_t^{-1}\bar{f}\delta_t$ and ϕ_t, ψ_t satisfy the relations $\bar{\delta}_t^*\tau_1 = \tau_1 + d\phi_t$, $\bar{\delta}_t^*\tau_2 = \tau_2 + d\psi_t$. □

Proposition 4.2. *Suppose that $\omega_1 = \omega_2 = \omega$, where ω is the zero-form or a symplectic form on M. If M_0 is a submanifold of M of positive codimension then $j^3 : \bar{\Phi}|_{M_0} \to \bar{\Psi}_2|_{M_0}$ is a weak homotopy equivalence.*

Proof. In view of Theorem A.3 in Section 5 and Proposition 4.1 we need to show that there is an appropriate class of (local) diffeotopies which act on the sheaf $\bar{\Phi}$ and have the desired sharply moving property. We first consider the case when ω is a symplectic form. Recall that ω defines a canonical isomorphism $I_\omega : TM \to T^*M$ which in turn defines a one-to-one correspondence between vector fields and one-forms on M. Indeed, if ∂ is a vector field on M then $\partial.\omega$ is a global one-form on M. The ω-exact (local) diffeotopies are obtained by integrating the vector fields which correspond to the exact one-forms under this correspondence. As we observed in Lemma 4.1, these diffeotopies act on the sheaf $\bar{\Phi}$. It is also known that these diffeotopies sharply move any submanifold of M of positive codimension (see [**8**, 3.4.2]). This proves the proposition for the case when ω is symplectic.

If ω is the zero form, it is enough to observe that any local diffeotopy is ω-exact. □

We are now in a position to prove Theorem C.

Proof of Theorem C. Since M is an open manifold, it admits a Morse function without any critical point of index equal to the dimension of M [**10**]. It then follows from Morse theory that, there is a simplicial complex K in M of positive codimension which is a strong deformation retract of M. In fact, M is isotopic to an arbitrarily small open neighbourhood of K. Now, by the above proposition, the h-principle for $\bar{\Phi}$ localizes near K, and hence the h-principle for the sheaf Φ also localizes near K (see Remark 4.1). This means that a section of $\mathcal{R}_1$ can be homotoped to a solution f_1 of the differential equation $\mathcal{D} = (\omega, \omega)$ near K, where ω is either the zero-form or a symplectic form on M.

To prove (a), take an isotopy ϕ_t such that ϕ_1 brings M into the domain of f_1. Then the composition map $f_1 \circ \phi_1$ is a global solution of the equation $\mathcal{D} = (0, 0)$.

To obtain global h-principle stated in (b) we observe that there is a homotopy of symplectic immersions $\phi_t : (M, \omega) \to (M, \omega)$ such that $\phi_0 = \mathrm{id}$ and ϕ_1 maps M into K ([**5**, **6**]). Composing f_1 with ϕ_1 we obtain a global solution of the equation $\mathcal{D} = (\omega, \omega)$. □

Remark 4.2. If (M_0, ω_0) is a symplectic manifold then as a direct consequence of the above theorem we obtain the h-principle with $M = M_0 \times \mathbb{R}^2$ and $\omega = \omega_0 \oplus dx \wedge dy$.

Proof of Theorem 3.1. Let $\omega_1 = d\alpha_1$ and $\omega_2 = d\alpha_2$ for some one-forms α_1, α_2 on M. Let $\tilde{\Phi}$ denote the sheaf of solutions of the differential equation $\tilde{\mathcal{D}} = (p^*\alpha_1, p^*\alpha_2)$, where $\tilde{\mathcal{D}}$ is defined as in Section 3. A diffeomorphism $\lambda : M \times \mathbb{R} \to M \times \mathbb{R}$ is said to be *fibre-preserving* if $p \circ \lambda = \lambda$, where $p : M \times \mathbb{R} \to M$ is the projection onto the first factor. Hence $\lambda^* p^* \alpha_i = p^* \alpha_i, i = 1, 2$, for such a λ. This allows us to define an action of fibre-preserving diffeomorphisms on the sheaf $\tilde{\Phi}$. Indeed, if $\tilde{\mathcal{D}}(f, \phi_1, \phi_2) = (p^*\alpha_1, p^*\alpha_2)$ and f is regular, then $f^*\tau_i + d\phi_i = p^*\alpha_i$, where $\sigma_i = d\tau_i$ for $i = 1, 2$. If λ is a fibre-preserving diffeomorphism then we define an action by the following simple rule:

$$\lambda.(f, \phi_1, \phi_2) = (f \circ \lambda, \phi_1 \circ \lambda, \phi_2 \circ \lambda).$$

(Note that if λ_t is a fibre-preserving diffeotopy, then the vector fields λ'_t has no component along M_0. Hence, λ'_t is ω-exact for any two-form ω of the form $p_1^*\omega_0$, where ω_0 is a two-form on M.) Since we observed in Proposition 3.1 that $\tilde{\mathcal{D}}$ is infinitesimally invertible on regular maps, the hypothesis of Theorem A.4 is satisfied (by Theorem A.2 and Proposition A.1 in Section 5). This proves that $\tilde{\Phi}$ satisfies the h-principle. This h-principle then descends to the desired h-principle for Φ by an argument similar to that in Remark 4.1. □

5. Appendix A. Preliminaries of h-principle

Here we briefly discuss the sheaf technique and the analytic technique in the theory of h-principle following [**8**].

Let $p : E \longrightarrow M$ be a C^∞-fibration, and let $E^{(r)}$ denote the r-jet space of C^∞-sections of E for $r \geq 1$. Then the canonical projection $p^{(r)} : E^{(r)} \longrightarrow M$ is also a fibration. We endow the space of sections of p and $p^{(r)}$ with the C^∞ and C^0-compact open topologies, respectively. The canonical projection maps $E^{(r)} \to E^{(i)}$ are denoted by p^r_i.

A *partial differential relation* of order r for sections of E is a subset $\mathcal{R}$ of $E^{(r)}$. A section $f : M \longrightarrow E$ is said to be a *solution* of $\mathcal{R}$ if the r-jet map j^r_f (which is a section of $p^{(r)}$) maps M into $\mathcal{R}$. A section of $p^{(r)}$ is called *holonomic* if it is the r-jet map of a solution of $\mathcal{R}$.

We denote the space of solutions of $\mathcal{R}$ by $\operatorname{Sol}\mathcal{R}$, while $\Gamma(\mathcal{R})$ denotes the space of sections of $E^{(r)} \longrightarrow M$ whose images lie in $\mathcal{R}$.

Definition A.1. A relation $\mathcal{R}$ is said to satisfy the *h-principle* if a section of $\mathcal{R}$ can be homotoped within $\Gamma(\mathcal{R})$ to a holonomic section.

$\mathcal{R}$ satisfies the *parametric h-principle* if the r-jet map $j^r : \operatorname{Sol}\mathcal{R} \longrightarrow \Gamma(\mathcal{R})$ is a weak homotopy equivalence.

Definition A.2. Let $\mathcal{S}$ be a subspace of the space of continuous sections of E. A relation $\mathcal{R} \subset E^{(r)}$ is said to satisfy the h-principle *C^0-dense in $\mathcal{S}$* if for every $f_0 \in \mathcal{S}$, for every neighbourhood U of graph f_0 and for every section $\phi_0 : M \to \mathcal{R}$ satisfying $p^r_0 \circ \phi_0 = j^0_{f_0}$, there exists a homotopy of sections $\phi_t : M \to \mathcal{R}$ such that the image of $p^r_0 \circ \phi_t$ is contained in U and ϕ_1 is holonomic.

Let Φ denote the sheaf of solutions of a given relation $\mathcal{R}$, and Ψ the sheaf of sections of $\mathcal{R}$. The topologies on $\Phi(U)$ and $\Psi(U)$ are, respectively, the C^∞ and C^0 compact open topologies. The r-jet map j^r defines a sheaf homomorphism from Φ to Ψ. This takes us into the realm of topological sheaves.

Sometimes, by an abuse of language, we say that the sheaf Φ satisfies the h-principle without giving any reference to a partial differential relation. But one should be careful at this point, since two subsets $\mathcal{R}$ and $\mathcal{R}'$ possibly in different jet spaces $E^{(r)}$ and $E^{(s)}$ may have the same set of solutions, but one of the relations may satisfy the h-principle while the other may not. In fact, given an rth-order relation $\mathcal{R}$, we can form an sth-order relation $\mathcal{R}'$ (for any $s > r$) by taking s-jets of C^s solutions of $\mathcal{R}$.

We recall some general definitions and terminology from [**8**].

Definition A.3. Let $\mathcal{F}$ be a topological sheaf over M and A a compact set in M. Then $\mathcal{F}(A)$ will denote the direct limit of the sets $\mathcal{F}(U)$, where U runs over all open sets containing A.

However, these sets, $\mathcal{F}(A)$, will have only quasi-topological structures [**8**, 1.4.1]. A map $f : P \longrightarrow \mathcal{F}(A)$ on a polyhedron P is called *continuous* (in the quasi-topological sense) if there exists an open set $U \supset A$ such that each f_p is defined over U and the resulting map $P \longrightarrow \mathcal{F}(U)$ is continuous with respect to the given topology on $\mathcal{F}(U)$.

Definition A.4. $\mathcal{R}$ satisfies the *local parametric h-principle* if for each $x \in M$, $j^r : \Phi(x) \longrightarrow \Psi(x)$ is a weak homotopy equivalence.

Definition A.5. A topological sheaf $\mathcal{F}$ over M is *flexible* if the restriction maps $\mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ are Serre fibrations for every pair of compact sets (A, B), $A \supset B$. The restriction map $\mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ is called a *microfibration* if given a continuous map $f'_0 : P \longrightarrow \mathcal{F}(A)$ on a polyhedron P and a homotopy $f_t, 0 \le t \le 1$, of $f'_0|_B$ there exists an $\varepsilon > 0$ and a homotopy f'_t of f'_0 such that $f'_t|\operatorname{Op} B = f_t$ for $0 \le t \le \varepsilon$. If for every pair of compact sets the restriction morphism is a microfibration, then the sheaf $\mathcal{F}$ is called *microflexible*.

The following topological result provides a sufficient condition for a sheaf homomorphism to be a weak homotopy equivalence.

Theorem A.1 ([8, 2.2.1(B)]). *Let $\mathcal{F}$ and $\mathcal{G}$ be two flexible sheaves over M, and let $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ be a continuous sheaf homomorphism such that $\alpha(x) : \mathcal{F}(x) \longrightarrow \mathcal{G}(x)$ is a weak homotopy equivalence for each $x \in M$. Then α is a weak homotopy equivalence.*

Thus, if the solution sheaf Φ is flexible and if $\mathcal{R}$ satisfies the local parametric h-principle, then $\mathcal{R}$ satisfies the parametric h-principle (because Ψ is always flexible [**8**, 1.4.2(A$'$)]).

The solution sheaf turns out to be non-flexible in many important problems, though microflexibility is a much more common property. For example, when $\mathcal{R}$ is open the solution sheaf is easily seen to be microflexible; however, many of the relations that are of special interest are fibrewise closed in the jet space. This is the case, when the solutions of $\mathcal{R}$ also arise as solutions to some PDEs $\mathcal{D}(f) = g$, where $\mathcal{D}$ is defined on sections of the fibre bundle E taking values in the space of sections of a vector bundle. Gromov proves that Φ (or possibly a subsheaf of Φ) is microflexible when the operator $\mathcal{D}$ is infinitesimally invertible over an open subset of $\Gamma^\infty(E)$ in the fine C^∞ topology. On the other hand, he observes that there are higher order relations $\mathcal{R}^\alpha \subset J^{(r+\alpha)}$, $\alpha = 0, 1, 2, \ldots$, that have the same solution space as $\mathcal{R}$ and which satisfy the local parametric h-principle for $\alpha > \alpha_0$, where α_0 is some positive integer.

To elaborate this let $E \longrightarrow M$ be a C^∞-fibration and $G \longrightarrow M$ be a C^∞ vector bundle over a manifold M. We denote by $\mathcal{E}^\alpha$ and $\mathcal{G}^\alpha$, respectively, the spaces of C^α sections of E and G with the *fine* C^α topologies, for $\alpha = 1, 2, \ldots, \infty$. Let $\mathcal{D} : \mathcal{E}^r \longrightarrow \mathcal{G}^0$ be a C^∞ differential operator of order r, which means that $\mathcal{D}$ is given by a C^∞ bundle map $\Delta : E^{(r)} \to G$ such that $\mathcal{D}(f) = \Delta \circ j^r_f$. As a consequence, we obtain a sequence of bundle maps $\Delta_\alpha : E^{(r+\alpha)} \to G^{(\alpha)}$ such that $j^\alpha_{\mathcal{D}(f)} = \Delta_\alpha \circ j^{r+\alpha}_f$, where α is any non-negative integer.

Let V denote the subbundle of TE consisting of all vectors which are tangent to the fibres of E over points of M. We shall refer V as the vertical tangent bundle of E. For any section f of E, the vector space of C^β sections of the pullback bundle f^*V will be denoted by $\mathcal{E}^\beta_f$. The space $\mathcal{E}^\beta_f$ is defined as the *infinite-dimensional tangent space* of $\mathcal{E}$ at f. It is not difficult to see that when E is a vector bundle, f^*V is canonically isomorphic to E and therefore $\mathcal{E}^\beta_f$ is isomorphic to $\mathcal{E}^\beta$.

The linearization L_f of $\mathcal{D}$ at f is a map $L_f : \mathcal{E}^r_f \longrightarrow \mathcal{G}^0$ which is defined as follows:

$$L_f(y) = \lim_{t\to\infty} \frac{\partial}{\partial t}\mathcal{D}(f_t)|_{t=0},$$

where f_t is a differentiable curve in $\mathcal{E}^r$ such that $f_0 = f$ and the tangent to f_t at $t = 0$ is $y \in \mathcal{E}^r_f$. Clearly, L_f is a linear differential operator of order r in y and $L(f, y) = L_f(y)$ is a differential operator of order r in both f and y.

Let $A \subset E^{(d)}$ be an open relation of order d for some $d \geq r$, and $\mathcal{A}$ denote the space of solutions of the relation A. Clearly, $\mathcal{A}$ is contained in $\mathcal{E}^d$, and $\mathcal{A}^{\alpha+d} = \mathcal{A} \cap \mathcal{E}^{\alpha+d}$ is an open subset of $\mathcal{E}^{\alpha+d}$ in the fine $C^{\alpha+d}$ topology. A solution of A will be referred as an A-*regular* section of E.

$\mathcal{D}$ is said to be *infinitesimally invertible* over the subset $\mathcal{A} \subset \mathcal{E}^d$ if for every $f \in \mathcal{A}$ there is a linear differential operator $M_f : \mathcal{G}^s \longrightarrow \mathcal{E}^0_f$ of a certain order s (independent of f) such that the following properties are satisfied:

(1) The global operator

$$M : \mathcal{A}^d \times \mathcal{G}^s \longrightarrow T(\mathcal{E}^0)$$

is a differential operator that is given by a C^∞ map $A \oplus G^{(s)} \longrightarrow V$.

(2) $L(f, M(f,g)) = g$ for all $f \in \mathcal{A}^{d+r}$ and $g \in \mathcal{G}^{r+s}$, where $M(f,g) = M_f(g)$. In other words, M_f is a right inverse of L_f.

The integer d is called the *defect* of the infinitesimal inversion M [**8**, 2.3.1].

We now quote two results from [**8**] which are consequences of an Implicit Function Theorem (due to Gromov) in the context of differential operators.

Theorem A.2 ([**8**, 2.3.2(B),(D″)])**.** *Suppose that $\mathcal{D}$ is a C^∞ differential operator of order r and it admits an infinitesimal inversion of defect d on $\mathcal{A}$.*

(i) The operator $\mathcal{D} : \mathcal{A}^\infty \longrightarrow \mathcal{G}^\infty$ is an open map in the respective fine C^∞ topologies.

(ii) The sheaf of A-regular solutions of the differential equation $\mathcal{D} = g$ is microflexible, where g is a smooth section of G.

Definition A.6. A local section f of E, defined on a neighbourhood of some $x \in M$, is said to be an *infinitesimal solution of $\mathcal{D} = g$ of order α* if the α-jet of $\mathcal{D}(f) - g$ is zero at x.

Let $\mathcal{R}^\alpha \subset E^{(\alpha+r)}$ consist of $(\alpha+r)$- jets of infinitesimal solutions of $\mathcal{D} = g$ of order α and let $\mathcal{R}^0$ be denoted as $\mathcal{R}$. Since $j^\alpha_{\mathcal{D}(f)} = \Delta_\alpha \circ j^{r+\alpha}_f$, therefore, $\mathcal{R}^\alpha = (\Delta_\alpha)^{-1}(j^\alpha_g)$.

Define

$$\mathcal{R}_\alpha = \mathcal{R}^\alpha \cap (p^{\alpha+r}_d)^{-1}(A),$$

where $p^{\alpha+r}_d : E^{(\alpha+r)} \longrightarrow E^{(d)}$ is the canonical projection map for $\alpha \geq d - r$. The relations $\mathcal{R}_\alpha$ have the same C^∞ solutions for all $\alpha \geq d - r$, namely the C^∞ solutions of the equation $\mathcal{D}(x) = g$ in $\mathcal{A}$.

Let Φ_{reg} denote the sheaf of A-regular solutions of the equation $\mathcal{D} = g$ with the C^∞ compact open topology and let Ψ_α be the sheaf of sections of $\mathcal{R}_\alpha$ with C^0 compact open topology.

Proposition A.1 ([**8**, 2.3.2(D′),(D″)])**.** *If $\mathcal{D}$ admits an infinitesimal inversion of order s and defect d on $\mathcal{A}$ then the map $J : \Phi_{\text{reg}} \longrightarrow \Psi_\alpha$, defined by $J(\phi) = j^{r+\alpha}_\phi$, is a local weak homotopy equivalence for each $\alpha \geq \max(d+s, 2r+2s)$. In other words, $\mathcal{R}_\alpha$ satisfies the local parametric h-principle.*

Thus we see that there is a large class of relations $\mathcal{R}$ which satisfy the local h-principle and for which the solution sheaves are microflexible. The following result of Gromov in [**8**, 2.2.3(C′)] is the central result in the theory of h-principle as far as the sheaf technique is concerned:

Theorem A.3. *If Φ is a microflexible sheaf on a manifold M and N is an embedded submanifold of positive codimension, then $\Phi|N$ is flexible, provided there is a class of 'acting diffeotopies' $\mathcal{D}_0$ which 'sharply moves N'.*

We now explain the notion of 'acting diffeotopies' and 'sharply moving diffeotopies'.
Action of diffeotopies: Let Φ be a topological sheaf over a manifold M and U' an open subset of M. Consider a diffeotopy $\delta_t : U \to U'$ that moves an open subset $U \subset U'$ inside U', where $\delta_0 = \text{id}$. Let Φ' be a subset of $\Phi(U')$, and the diffeotopy δ_t act on Φ' by assigning a homotopy of sections $\delta_t^*\phi$ in $\Phi(U)$ to every $\phi \in \Phi'$ such that $\delta_0^*\phi = \phi|_U$ and the following conditions are satisfied:

(1) If two sections ϕ_1 and ϕ_2 in Φ' are such that $\phi_1(u') = \phi_2(u')$ for some $u' \in U'$ and if $\delta_{t_0}(u) = u'$ for some $u \in U$, then $\delta_{t_0}^*\phi_1(u) = \delta_{t_0}^*\phi_1(u)$. In particular, if the two sections ϕ_1 and ϕ_2 restrict to the same section on U, then $(\delta_t|_U)^*\phi_1 = (\delta_t|_U)^*\phi_2$.
(2) If U_0 is a maximal open subset where δ_t is constant, (that is, $\delta_t(x) = x$ for all $x \in U_0$,) then $\delta_t^*\phi$ is also constant on U_0 (that is $\delta_t^*\phi = \phi$ on U_0).
(3) If the diffeotopy δ_t is constant for $t \geq t_0$, then $\delta_t^*\phi$ is also constant for $t \geq t_0$ for some $t_0 \in [0, 1]$.
(4) If $\phi_p \in \Phi'$, $p \in P$, is a continuous family of sections then the family $\delta_t^*\phi_p$ is jointly continuous in p and t.

Conditions (2) and (3) are natural in the sense that they make the action compatible with the presheaf structure.

One must note that this is a partial action as δ_t need not in general act on $\Phi(U)$. Further, there is no condition on the subset Φ'.

Example A.1. Let Φ denote the sheaf of sections of the product bundle $M \times N$ over M. Then Diff(M), the pseudogroup of local diffeomorphisms of M, has a natural action on Φ given by $\delta.\bar{f} = \bar{\delta}^{-1}\bar{f}\delta$, where $\bar{f} \in \Phi$ and $\bar{\delta} : U \times N \to V \times N$ is given by $\bar{\delta}(x, y) = (\delta(x), y)$. We can extend this action to an action by diffeotopies of M. However, if we consider the subsheaf Φ_Y of sections of $M \times N$ whose images lie in an open subset Y then Φ_Y is not invariant under this action. In this case we get only a partial action by diffeotopies: indeed, if δ_t is sufficiently C^0-close to the identity map then it acts on $\Phi_Y(U)$. More generally, if δ_t is a diffeotopy that moves U in M and if there is an open subset $Y' \subset (U \times N)$ such that $\bar{\delta}_t(Y') \subset Y$ for all $t \in [0, 1]$, then δ_t acts on the sheaf Φ_Y.

Definition A.7. We fix a metric d on M. Let M_0 be a submanifold of M of positive codimension which lies in an open subset U' of M. A class of diffeotopies $\mathcal{D}$ on M is said to *sharply move* M_0 in M if given any hypersurface S in M_0 and any positive numbers ε, we can obtain a diffeotopy $\delta_t : \mathrm{Op}\, M_0 \to U'$ in $\mathcal{D}$ which satisfies the following conditions:

(1) δ_0 is the identity map;
(2) $\delta_t|_{Op\, v}$ is identity for all $v \in M_0$ for which $d(v, S) \geq \varepsilon$;
(3) $d(\delta_1(S), M_0) > r$ for some positive number r.

We end this section with the following result of h-principle.

Theorem A.4. *Let $M = M_0 \times \mathbb{R}$. Suppose that $\mathcal{R}$ satisfies the local parametric h-principle and the solution sheaf Φ of $\mathcal{R}$ is microflexible. If the fibre-preserving diffeotopies of M act on Φ then $\mathcal{R}$ satisfies the h-principle.*

Proof. Since the fibre-preserving diffeotopies of M sharply move the submanifold M_0, it follows from Theorem A.3 that a section of $\mathcal{R}$ can be homotoped to a holonomic section j^r_f over an open neighbourhood U of $M_0 \times \{0\}$ in M. Since M is split as $M_0 \times \mathbb{R}$, we can deform M into U by a smooth one-parameter family of embeddings $F_t : M_0 \times \mathbb{R} \longrightarrow M_0 \times \mathbb{R}, 0 \leq t \leq 1$, such that F_t is fibre-preserving and F_1 takes M into U. Since F_t acts on Φ, $F_1^* f$ is a global solution of $\mathcal{R}$. This proves the theorem. $\square$

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