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Asymptotic expansion of resolvent kernels and behavior of spectral functions for symmetric stable processes

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Abstract. We give a precise behavior of spectral functions for symmetric stable processes applying the asymptotic expansion of resolvent kernels.

1. Introduction.

Let $\{X_t\}_{t\geq 0}$ be the *rotationally invariant α-stable* process on $\mathbb{R}^d$, the Hunt process with generator $H = (-\Delta)^{\alpha/2}$ $(0 < \alpha < 2)$. We denote by $(\mathcal{E}, \mathcal{F})$ the associated Dirichlet form on $L^2(\mathbb{R}^d, m)$, i.e. $\mathcal{E}(u, v) = \langle \sqrt{H}u, \sqrt{H}v\rangle$ for $u, v \in \mathcal{F} = \mathcal{D}(\sqrt{H})$. Here m is the Lebesgue measure on $\mathbb{R}^d$ and $\langle \cdot, \cdot\rangle$ is the inner product of $L^2(\mathbb{R}^d, m)$.

Let $V(x)$ be a non-negative continuous function on $\mathbb{R}^d$ with compact support and define the Schrödinger-type operator $H^\lambda = H - \lambda V$ for $\lambda \geq 0$. Then the equation $\partial u/\partial t = -H^\lambda u$ admits the fundamental solution $p^\lambda(t, x, y)$, in particular, $p^0(t, x, y)$ is the transition density function of $\{X_t\}_{t\geq 0}$. We write simply $p(t, x, y)$ for $p^0(t, x, y)$. In [**12**], we established a necessary and sufficient condition for $p^\lambda(t, x, y)$ to satisfy

$$c_1 p(t, x, y) \leq p^\lambda(t, x, y) \leq c_2 p(t, x, y)$$

for some positive constants c_1 and c_2; we call this property *stability of fundamental solution*. To show this, we define the bottom of the spectrum of the operator $(1/V)H$ by

$$\lambda_V = \inf\left\{\mathcal{E}(u, u) \;\middle|\; u \in \mathcal{F}, \int_{\mathbb{R}^d} u^2(x)V(x)dx = 1\right\}.$$

Note that λ_V describes the smallness of $V(x)$; if $V_1 \leq V_2$, $\lambda_{V_1} \geq \lambda_{V_2}$. The stability of the fundamental solution is equivalent to $\lambda < \lambda_V$, and then $\lambda V(x)$ is said to be *subcritical*. Takeda and Uemura [**11**] established other conditions equivalent to the subcriticality. More specifically, they defined the Feynman–Kac expectation $e_{\lambda V}(t, x)$ by

$$e_{\lambda V}(t, x) = \mathbb{E}_x\left[\exp\left(\lambda\int_0^t V(X_s)ds\right)\right]$$

and showed that the subcriticality of $\lambda V(x)$ is equivalent to the *gaugeability*, i.e. $\sup_{x\in\mathbb{R}^d} e_{\lambda V}(\infty, x) < \infty$. In particular, $e_{\lambda V}(t, x)$ converges to $e_{\lambda V}(\infty, x)$ as $t \to \infty$. If $\lambda = \lambda_V$(resp.$\lambda > \lambda_V$), $\lambda V(x)$ is said to be *critical* (resp. *supercritical*). In these

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cases, the Feynman–Kac expectation diverges as $t \to \infty$. To know the growth order of the Feynman–Kac expectation, we consider the *logarithmic moment generating function* $\Lambda(x)$ defined by

$$\Lambda(x) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp \left(\lambda \int_0^t V(X_s) ds \right) \right].$$

Takeda [**9**] showed that $\Lambda(x)$ is equal to the bottom of the spectrum for H^λ, that is,

$$C(\lambda) = -\inf \left\{ \mathcal{E}^\lambda(u,u) \;\middle|\; u \in \mathcal{F}, \int_{\mathbb{R}^d} u^2(x)dx = 1 \right\},$$

where $\mathcal{E}^\lambda$ is the corresponding form to the operator H^λ given by

$$\mathcal{E}^\lambda(u,u) = \mathcal{E}(u,u) - \lambda \int_{\mathbb{R}^d} u^2(x)V(x)dx. \tag{1.1}$$

We see that $C(\lambda_V) = 0$ by the definition of λ_V. Moreover, Takeda and Tsuchida [**10**] proved that $C(\lambda) > 0$ if and only if $\lambda > \lambda_V$. For $\lambda = \lambda_V$, there exists a positive function $h_0(x)(\notin \mathcal{F}$ in general) which attains the minimum of (1.1), i.e. $\mathcal{E}^{\lambda_V}(h_0, h_0) = 0$. $h_0(x)$ is uniquely determined up to multiple constant and is called *ground state* of $H - \lambda_V V$.

It was proved in [**10**] that for a transient $\{X_t\}_{t\geq 0}$ with $1 < d/\alpha \leq 2$, the spectral function $C(\lambda)$ is differentiable on $\mathbb{R}$, in particular, at $\lambda = \lambda_V$. The purpose of this paper is to give more precise asymptotic behavior of the spectral function as $\lambda \downarrow \lambda_V$. Moreover, we also treat recurrent processes. Note that $\lambda_V = 0$ for recurrent processes, while $\lambda_V > 0$ for transient ones. We then prove the following main theorem:

THEOREM 1.1. *If $\{X_t\}$ is recurrent, the spectral function $C(\lambda)$ satisfies the following asymptotics as $\lambda \downarrow 0$:*

$$C(\lambda) \sim \left(\frac{C_V \lambda}{\alpha \sin(\pi/\alpha)} \right)^{\alpha/(\alpha-1)}, \quad C_V = \int_{\mathbb{R}^d} V(x)dx \qquad (d = 1 < \alpha < 2),$$
$$C(\lambda) \asymp \exp\left(-\frac{\pi}{C_V \lambda} \right) \quad (d = \alpha = 1),$$

where $A \sim B$ means $B/A \to 1$ and $A \asymp B$ means $c_1 \leq B/A \leq c_2$ for some positive constants c_1 and c_2.
If $\{X_t\}$ is transient, the spectral function $C(\lambda)$ satisfies the following asymptotics as $\lambda \downarrow \lambda_V$:

$$C(\lambda) \sim \left\{ \frac{\alpha\Gamma(d/2)\sin(((d/\alpha)-1)\pi)\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}{2^{1-d}\pi^{1-(d/2)}\langle \lambda_V\sqrt{V}, \sqrt{V}h_0\rangle^2}(\lambda - \lambda_V) \right\}^{\alpha/(d-\alpha)} \qquad (1 < d/\alpha < 2),$$
$$C(\lambda) \sim \frac{\Gamma(\alpha+1)\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}{2^{1-d}\pi^{-d/2}\langle \lambda_V\sqrt{V}, \sqrt{V}h_0\rangle^2} \cdot \frac{\lambda - \lambda_V}{\log(\lambda-\lambda_V)^{-1}} \qquad (d/\alpha = 2),$$
$$C(\lambda) \sim \frac{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}{\langle h_0, h_0\rangle}(\lambda - \lambda_V) \qquad (d/\alpha > 2).$$

Note that this result is an extension of Cranston et al [**2**, Theorem 6.1] where the same problem was studied for the Brownian motion on $\mathbb{R}^d$, i.e. the Hunt process with generator $(-\Delta)$. Indeed, substituting $\alpha = 2$, we can obtain their preceding result. In [**2**], they first considered the asymptotic expansion of β-order resolvent as $\beta \downarrow 0$ to obtain the asymptotic behavior of the spectral function. This is easy for the Brownian motion, since the resolvent is expressed through the Hankel functions. For the rotationally invariant α-stable processes, we cannot express the resolvent through a special function. We first express the heat kernel by means of the function with respect to $|x-y|/t^{1/\alpha}$. Using this expression, we obtain the asymptotic expansion of β-order resolvent as $\beta \downarrow 0$.

This paper is organized as follows: In Section 2, we give the asymptotic behavior of the β-order resolvent. In Section 3, we define a compact operator on $L^2(\mathbb{R}^d)$ from the resolvent kernel and the function $V(x)$, following the method of Klaus and Simon [**5**]. We also give a relation between the principal eigenvalue of the compact operator and the spectral function. In Section 4, we give the asymptotic expansion of the principal eigenvalue applying the first order perturbation theory in Kato [**4**], and prove Theorem 1.1. c_i's are unimportant positive constants varying from line to line.

2. The asymptotic behavior of the resolvent.

Let $\{X_t\}_{t\geq 0}$ be the rotationally invariant α-stable process $(0<\alpha<2)$ on $\mathbb{R}^d$. Then the associated Dirichlet form is given by

$$\mathcal{E}(u,v) = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x))(v(y)-v(x))\frac{A_{d,\alpha}}{|x-y|^{d+\alpha}}dxdy,\;\; \mathcal{F} = H^{\alpha/2}(\mathbb{R}^d).$$

Here $H^{\alpha/2}(\mathbb{R}^d)$ is the Sobolev space with order $\alpha/2$ and $A_{d,\alpha}$ is a positive constant

$$A_{d,\alpha} = \frac{\alpha\cdot 2^{\alpha-1}\Gamma((\alpha+d)/2)}{\pi^{d/2}\Gamma(1-(\alpha/2))}, \qquad\qquad \Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx.$$

The characteristic function of $\{X_t\}_{t\geq 0}$ is

$$\mathbb{E}_x[\exp(i\xi\cdot(X_t-x))] = \int_{\mathbb{R}^d}\exp(i\xi\cdot(y-x))p(t,x,y)dy = e^{-t|\xi|^\alpha}, \quad \xi\in\mathbb{R}^d.$$

Here $p(t,x,y)$ is the transition density function. Applying the Fourier inverse transformation, we have

$$p(t,x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp(-t|\xi|^\alpha + i\xi\cdot(x-y))d\xi. \tag{2.1}$$

The following lemma is a precise version of Kolokoltsov [**6**, p. 314] or Blumenthal and Getoor [**1**, (2,1)].

LEMMA 2.1. *The transition density function of $\{X_t\}_{t\geq 0}$ is expressed by*

$$p(t,x,y) = C_{d,\alpha}t^{-d/\alpha}g\left(\frac{|x-y|}{t^{1/\alpha}}\right), \tag{2.2}$$

where

$$C_{d,\alpha} = \begin{cases} (\pi\alpha)^{-1} & (d=1) \\ \left(2^{d-1}\pi^{(d+1)/2}\Gamma\left(\dfrac{d-1}{2}\right)\alpha\right)^{-1} & (d \geq 2) \end{cases}$$

and g *is the function on* $[0,\infty)$ *defined by*

$$g(w) = \int_0^\infty s^{(1/\alpha)-1}e^{-s}\cos\left(ws^{1/\alpha}\right)ds \qquad (d=1), \tag{2.3}$$

$$g(w) = \int_0^\pi \int_0^\infty s^{(d/\alpha)-1}e^{-s}\cos\left(ws^{1/\alpha}\cos\theta\right)\sin^{d-2}\theta ds d\theta \qquad (d \geq 2). \tag{2.4}$$

Proof. For $d=1$, (2.1) implies

$$\begin{aligned} p(t,x,y) &= \frac{1}{\pi}\int_0^\infty \exp(-tr^\alpha)\cos(|x-y|r)dr \\ &= \frac{t^{-1/\alpha}}{\pi\alpha}\int_0^\infty s^{(1/\alpha)-1}e^{-s}\cos\left(\frac{|x-y|}{t^{1/\alpha}}s^{(1/\alpha)}\right)ds. \end{aligned}$$

For $d \geq 2$, (2.1) implies

$$\begin{aligned} p(t,x,y) &= C_d\int_0^\pi\int_0^\infty \exp(-tr^\alpha + ir|x-y|\cos\theta)\cdot r^{d-1}\sin^{d-2}\theta dr d\theta \\ &= C_d\int_0^\pi\int_0^\infty \exp(-tr^\alpha)\cos(r|x-y|\cos\theta)\cdot r^{d-1}\sin^{d-2}\theta dr d\theta, \end{aligned} \tag{2.5}$$

where

$$C_d = \frac{2}{(2\pi)^d}\prod_{n=0}^{d-3}\int_0^\pi \sin^n\theta d\theta = \frac{1}{2^{d-1}\pi^{(d+1)/2}\Gamma\left((d-1)/2\right)}.$$

Substituting $tr^\alpha = s$ in (2.5), we have

$$\frac{t^{-d/\alpha}}{2^{d-1}\pi^{(d+1)/2}\Gamma\left((d-1)/2\right)\alpha}\int_0^\pi\int_0^\infty s^{(d/\alpha)-1}e^{-s}\cos\left(\frac{|x-y|}{t^{1/\alpha}}s^{1/\alpha}\cos\theta\right)\sin^{d-2}\theta ds d\theta.$$

$\square$

We next consider the properties of the function g in (2.3)–(2.4).

Lemma 2.2. *The function* g *in* (2.3)–(2.4) *satisfies the following properties*:

(i) *It follows that*

$$g(0) = \begin{cases} \Gamma\left(\dfrac{1}{\alpha}\right) & (d=1) \\ \sqrt{\pi}\Gamma\left(\dfrac{d}{\alpha}\right)\Gamma\left(\dfrac{d-1}{2}\right)\Big/\Gamma\left(\dfrac{d}{2}\right) & (d \geq 2). \end{cases} \tag{2.6}$$

(ii) *There exists positive constants* c_1, c_2 *satisfying*

$$c_1(1 \wedge w^{-d-\alpha}) \leq g(w) \leq c_2(1 \wedge w^{-d-\alpha}). \tag{2.7}$$

(iii) *There exists a positive constant* c_3 *depending on* d *and* α *such that*

$$g(0) - g(w) \leq c_3 w^2. \tag{2.8}$$

PROOF. We obtain (i) by simple calculation. Blumenthal and Getoor [**1**] showed that $g(w)$ is a positive continuous function and satisfies

$$\lim_{w \to \infty} w^{d+\alpha} g(w) = c_1.$$

Hence, we have (ii). For $d = 1$,

$$\begin{aligned} g(0) - g(w) &= \int_0^\infty s^{(1/\alpha)-1} e^{-s} (1 - \cos(ws^{1/\alpha})) ds \\ &= 2 \int_0^\infty s^{(1/\alpha)-1} e^{-s} \sin^2 \left(\frac{ws^{1/\alpha}}{2} \right) ds \leq \frac{w^2}{2} \int_0^\infty s^{(3/\alpha)-1} e^{-s} ds = c_1 w^2. \end{aligned}$$

For $d \geq 2$,

$$\begin{aligned} g(0) - g(w) &= \int_0^\pi \int_0^\infty s^{(d/\alpha)-1} e^{-s} \left(1 - \cos\left(ws^{1/\alpha} \cos\theta\right)\right) \sin^{d-2}\theta ds d\theta \\ &= 2 \int_0^\pi \int_0^\infty s^{(d/\alpha)-1} e^{-s} \sin^2 \left(\frac{ws^{1/\alpha} \cos\theta}{2} \right) \sin^{d-2}\theta ds d\theta \\ &\leq \frac{w^2}{2} \int_0^\pi \int_0^\infty s^{(d+2)/\alpha-1} e^{-s} \cos^2\theta \sin^{d-2}\theta ds d\theta = c_1 w^2. \end{aligned}$$

Hence, we have (iii). □

For $\beta > 0$, let $G_\beta(x, y)$ be the β-resolvent kernel,

$$G_\beta(x, y) := \int_0^\infty p(t, x, y) e^{-\beta t} dt. \tag{2.9}$$

We next consider the asymptotic behavior of the β-resolvent when β tends to 0. If $\{X_t\}_{t \geq 0}$ is recurrent, $G_\beta(x, y)$ diverges as $\beta \downarrow 0$. The following theorem describes an exact behavior of this divergence.

THEOREM 2.3. *Suppose* $d = 1$ *and* $1 \leq \alpha < 2$, *that is,* $\{X_t\}_{t \geq 0}$ *is recurrent. Then the resolvent kernel* $G_\beta(x, y)$ *satisfies the following asymptotics as* β *tends to* 0.

$$G_\beta(x, y) = \frac{\beta^{(1/\alpha)-1}}{\alpha \sin \pi/\alpha} + E_\beta(x, y) \quad (1 < \alpha < 2), \tag{2.10}$$

$$G_\beta(x, y) = \frac{\log \beta^{-1}}{\pi} + E_\beta(x, y) \quad (\alpha = 1), \tag{2.11}$$

where $E_\beta(x,y)$ *satisfies*

$$|E_\beta(x,y)| \le \begin{cases} c_1|x-y|^{\alpha-1} & (1<\alpha<2) \\ c_2(1+|\log|x-y||+\beta|x-y|) & (\alpha=1). \end{cases}$$

Proof. Suppose $1<\alpha<2$. Using (2.2), we have

$$\begin{aligned} G_\beta(x,y) &= \int_0^\infty \frac{1}{\pi\alpha} t^{-1/\alpha} g\left(\frac{|x-y|}{t^{1/\alpha}}\right) e^{-\beta t} dt \\ &= \frac{\beta^{(1/\alpha)-1}}{\pi\alpha} \int_0^\infty s^{-1/\alpha} g\left(\frac{\beta^{1/\alpha}|x-y|}{s^{1/\alpha}}\right) e^{-s} ds. \end{aligned}$$

Note that

$$\frac{1}{\pi\alpha}\int_0^\infty s^{-1/\alpha} g\left(\frac{\beta^{1/\alpha}|x-y|}{s^{1/\alpha}}\right) e^{-s} ds \to \frac{1}{\pi\alpha} g(0)\Gamma\left(1-\frac{1}{\alpha}\right) = \frac{1}{\alpha\sin\pi/\alpha}$$

as β tends to 0. Setting the function $E_\beta(x,y)$ by

$$E_\beta(x,y) = \frac{-1}{\pi\alpha}\int_0^\infty t^{-1/\alpha}\left(g(0)-g\left(\frac{|x-y|}{t^{1/\alpha}}\right)\right) e^{-\beta t} dt,$$

we obtain (2.10). Moreover, (2.7) and (2.8) imply $g(0)-g(w)\le c_1(1\wedge w^2)$, and thus

$$|E_\beta(x,y)| \le \int_0^{|x-y|^\alpha} c_1 t^{-1/\alpha} dt + \int_{|x-y|^\alpha}^\infty c_2 t^{-3/\alpha}|x-y|^2 dt \le c_3|x-y|^{\alpha-1}.$$

Suppose $\alpha=1$. First we express $G_\beta(x,y)$ as follows:

$$G_\beta(x,y) = \int_0^{|x-y|} p(t,x,y)e^{-\beta t}dt + \int_{|x-y|}^\infty p(t,x,y)e^{-\beta t}dt =: I_1+I_2.$$

By the upper bound of the heat kernel, the first term satisfies

$$I_1 \le c_1\int_0^{|x-y|}\frac{t}{|x-y|^2}dt = c_2.$$

For the second term, we have

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_{|x-y|}^\infty g\left(\frac{|x-y|}{t}\right)\cdot\frac{e^{-\beta t}}{t}dt = \frac{1}{\pi}\int_{\beta|x-y|}^\infty g\left(\frac{\beta|x-y|}{s}\right)\cdot\frac{e^{-s}}{s}ds \\ &= \frac{g(0)}{\pi}\int_{\beta|x-y|}^\infty \frac{e^{-s}}{s}ds - \frac{1}{\pi}\int_{\beta|x-y|}^\infty \left(g(0)-g\left(\frac{\beta|x-y|}{s}\right)\right)\frac{e^{-s}}{s}ds =: J_1-J_2. \end{aligned}$$

To obtain the asymptotic expansion for J_1, we first note

$$\int_{\beta|x-y|}^\infty \frac{e^{-s}}{s}ds = \lim_{N\to\infty}\int_{\beta|x-y|}^N \left(1+\sum_{n=1}^\infty\frac{(-s)^n}{n!}\right)\frac{1}{s}ds$$

$$= -\log(\beta|x-y|) - \sum_{n=1}^{\infty} \frac{(-\beta|x-y|)^n}{n \cdot n!} + \lim_{N\to\infty} \left\{ \log N + \sum_{n=1}^{\infty} \frac{(-N)^n}{n \cdot n!} \right\}$$

$$= -\log(\beta|x-y|) - \sum_{n=1}^{\infty} \frac{(-\beta|x-y|)^n}{n \cdot n!} - \gamma + \lim_{N\to\infty} \left\{ \sum_{k=1}^{N} \frac{1}{k} - \int_0^N \frac{1-e^{-w}}{w} dw \right\}, \tag{2.12}$$

where

$$\gamma = \lim_{N\to\infty} \left\{ \sum_{k=1}^{N} \frac{1}{k} - \log N \right\}.$$

We see that the last term of (2.12) converges to 0 as $N \to \infty$. Indeed, we have

$$\sum_{k=1}^{N} \frac{1}{k} - \int_0^N \frac{1-e^{-w}}{w} dw = \sum_{k=1}^{N} \frac{1}{k} - (1-e^{-N}) \log N + \int_0^N e^{-w} \log w dw$$
$$\to \gamma + \int_0^\infty e^{-w} \log w dw \quad (N \to \infty). \tag{2.13}$$

Since the Gamma function has the representation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right\},$$

we obtain

$$\int_0^\infty e^{-w} \log w dw = \Gamma'(1) = \frac{\Gamma'(1)}{\Gamma(1)} = \frac{d \log \Gamma(z)}{dz} \Big|_{z=1} = -\gamma. \tag{2.14}$$

Hence, J_1 has the asymptotic expansion

$$J_1 = \frac{1}{\pi} \left(-\log(\beta|x-y|) - \gamma - \sum_{n=1}^{\infty} \frac{(-\beta|x-y|)^n}{n \cdot n!} \right) =: \frac{\log \beta^{-1}}{\pi} + \tilde{E}_\beta(x,y).$$

Furthermore, noting that for $z \le 0$,

$$0 \le -\sum_{n=1}^{\infty} \frac{z^n}{n \cdot n!} = -\int_0^z \sum_{n=1}^{\infty} \frac{w^{n-1}}{n!} dw = \int_z^0 \frac{e^w - 1}{w} dw \le -z,$$

we have

$$|\tilde{E}_\beta(x,y)| \le c_1 (1 + |\log|x-y|| + \beta|x-y|).$$

For J_2, we obtain from (2.8)

$$J_2 \le c_1 \int_{\beta|x-y|}^{\infty} \frac{\beta^2 |x-y|^2}{s^2} \cdot \frac{e^{-s}}{s} ds \le c_1 \int_{\beta|x-y|}^{\infty} \frac{\beta^2|x-y|^2}{s^3} ds \le c_2.$$

Define $E_\beta(x,y)$ by $E_\beta(x,y) = I_1 + \tilde{E}_\beta(x,y) - J_2$. Then we have (2.11) and

$$|E_\beta(x,y)| \le I_1 + |\tilde{E}_\beta(x,y)| + J_2 \le c_1(1 + |\log|x-y|| + \beta|x-y|). \qquad \square$$

If $\{X_t\}_{t\ge 0}$ is transient, (2.9) makes a sense for $\beta = 0$, i.e. $G_0(x,y) < \infty$ for $x \ne y$. $G_0(x,y)$ is called *Green kernel* and in the sequel we write $G(x,y)$ for $G_0(x,y)$ simply. The next theorem gives us the asymptotic expansion of $G_\beta(x,y)$ as $\beta \downarrow 0$.

THEOREM 2.4. *Suppose $d/\alpha > 1$, that is, $\{X_t\}_{t\ge 0}$ is transient.*

(i) *For $1 < d/\alpha < 2$,*

$$\begin{aligned} G_\beta(x,y) &= G(x,y) - \frac{2^{1-d}\pi^{1-(d/2)}}{\alpha\Gamma(d/2)\sin(((d/\alpha)-1)\pi)}\beta^{(d/\alpha)-1} + E_\beta(x,y), \qquad (2.15)\\ E_\beta(x,y) &\le c_1\beta|x-y|^{2\alpha-d}. \end{aligned}$$

(ii) *For $d/\alpha = 2$,*

$$\begin{aligned} G_\beta(x,y) &= G(x,y) - \frac{2^{1-d}\pi^{-d/2}}{\Gamma(\alpha+1)}\beta\log\beta^{-1} + E_\beta(x,y), \qquad (2.16)\\ |E_\beta(x,y)| &\le c_1\beta(1 + |\log|x-y|| + \beta|x-y|^\alpha). \end{aligned}$$

(iii) *For $d/\alpha > 2$,*

$$G_\beta(x,y) = G(x,y) - \tilde{G}(x,y)\beta + E_\beta(x,y), \qquad (2.17)$$

where $\tilde{G}(x,y)$ is defined by

$$\tilde{G}(x,y) = \int_0^\infty tp(t,x,y)dt = c_1|x-y|^{2\alpha-d}$$

and $E_\beta(x,y)$ satisfies

$$E_\beta(x,y) \le \begin{cases} c_2\beta^{(d/\alpha)-1} & (2 < d/\alpha < 3)\\ c_2\beta^2\log\beta^{-1} + c_3\beta^2(1 + |\log|x-y|| + \beta|x-y|^\alpha) & (d/\alpha = 3)\\ c_2\beta^2|x-y|^{3\alpha-d} & (d/\alpha > 3). \end{cases}$$

PROOF. Suppose $1 < d/\alpha < 2$. Applying (2.2), we have

$$\begin{aligned} G(x,y) - G_\beta(x,y) &= C_{d,\alpha}\int_0^\infty t^{-d/\alpha} g\left(\frac{|x-y|}{t^{1/\alpha}}\right)(1 - e^{-\beta t})dt\\ &= C_{d,\alpha}\beta^{(d/\alpha)-1}\int_0^\infty s^{-d/\alpha} g\left(\frac{\beta^{1/\alpha}|x-y|}{s^{1/\alpha}}\right)(1 - e^{-s})ds. \end{aligned}$$

Note that

$$\int_0^\infty s^{-d/\alpha}(1-e^{-s})ds = -\left(1-\frac{d}{\alpha}\right)^{-1}\int_0^\infty s^{1-(d/\alpha)}e^{-s}ds = \frac{\alpha}{d-\alpha}\Gamma\left(2-\frac{d}{\alpha}\right).$$

Thus, we obtain

$$\begin{aligned}G_\beta(x,y) &= G(x,y) - C_{d,\alpha}g(0)\cdot\frac{\alpha}{d-\alpha}\Gamma\left(2-\frac{d}{\alpha}\right)\beta^{(d/\alpha)-1} + E_\beta(x,y)\\ &= G(x,y) - \frac{2^{1-d}\pi^{1-(d/2)}}{\alpha\Gamma(d/2)\sin(((d/\alpha)-1)\pi)}\beta^{d/\alpha-1} + E_\beta(x,y),\end{aligned}$$

where $E_\beta(x,y)$ is defined by

$$E_\beta(x,y) = C_{d,\alpha}\int_0^\infty t^{-d/\alpha}(1-e^{-\beta t})\left(g(0)-g\left(\frac{|x-y|}{t^{1/\alpha}}\right)\right)dt.$$

Since $g(0)-g(w)\le c_1(1\wedge|w|^2)$ from (2.7)–(2.8),

$$\begin{aligned}E_\beta(x,y) &\le c_2\beta\int_0^\infty t^{1-(d/\alpha)}\left(g(0)-g\left(\frac{|x-y|}{t^{1/\alpha}}\right)\right)dt\\ &\le c_3\beta\left(\int_0^{|x-y|^\alpha} t^{1-(d/\alpha)}dt + \int_{|x-y|^\alpha}^\infty t^{1-((2+d)/\alpha)}|x-y|^2dt\right) \le c_4\beta|x-y|^{2\alpha-d}\end{aligned}$$

and we obtain (2.15).

Suppose $d/\alpha = 2$. Similarly to the case $d=\alpha=1$, we first have

$$\begin{aligned}G(x,y)-G_\beta(x,y) &= \int_0^{|x-y|^\alpha} p(t,x,y)(1-e^{-\beta t})dt\\ &\quad + \int_{|x-y|^\alpha}^\infty p(t,x,y)(1-e^{-\beta t})dt =: I_1+I_2. \end{aligned}\tag{2.18}$$

Using the upper bound of the heat kernel, we have

$$I_1 \le c_3\int_0^{|x-y|^\alpha}\frac{t}{|x-y|^{d+\alpha}}\cdot\beta t dt = c_3\beta\int_0^{|x-y|^\alpha}\frac{t^2}{|x-y|^{3\alpha}}dt = c_4\beta. \tag{2.19}$$

For I_2, we have

$$I_2 = C_{d,\alpha}\beta\int_{\beta|x-y|^\alpha}^\infty s^{-2}g\left(\frac{\beta^{1/\alpha}|x-y|}{s^{1/\alpha}}\right)(1-e^{-s})ds.$$

Since $s^{-2}(1-e^{-s})$ is integrable on $[\beta|x-y|^\alpha,\infty)$,

$$\begin{aligned}I_2 &= C_{d,\alpha}g(0)\beta\int_{\beta|x-y|^\alpha}^\infty s^{-2}(1-e^{-s})ds\\ &\quad - C_{d,\alpha}\beta\int_{\beta|x-y|^\alpha}^\infty s^{-2}\left(g(0)-g\left(\frac{\beta^{1/\alpha}|x-y|}{s^{1/\alpha}}\right)\right)(1-e^{-s})ds\\ &=: J_1-J_2.\end{aligned}\tag{2.20}$$

Using integration by parts, we have

$$J_1 = C_{d,\alpha} g(0) \beta \left(\frac{1 - e^{-\beta|x-y|^\alpha}}{\beta|x-y|^\alpha} + \int_{\beta|x-y|^\alpha}^\infty \frac{e^{-s}}{s} ds \right).$$

Substituting $\beta|x-y|^\alpha$ for $\beta|x-y|$ in (2.12) and combining with (2.13) and (2.14), we have

$$\int_{\beta|x-y|^\alpha}^\infty \frac{e^{-s}}{s} ds = -\log(\beta|x-y|^\alpha) - \sum_{n=1}^\infty \frac{(-\beta|x-y|^\alpha)^n}{n \cdot n!} - \gamma.$$

Thus, J_1 admits the asymptotic expansion

$$\begin{aligned} J_1 &= C_{d,\alpha} g(0) \beta \left(\frac{1 - e^{-\beta|x-y|^\alpha}}{\beta|x-y|^\alpha} - \log(\beta|x-y|^\alpha) - \gamma - \sum_{n=1}^\infty \frac{(-\beta|x-y|^\alpha)^n}{n \cdot n!} \right) \\ &=: \frac{\beta \log \beta^{-1}}{2^{d-1} \pi^{d/2} \Gamma(\alpha+1)} + \tilde{E}_\beta(x,y). \end{aligned} \tag{2.21}$$

Similarly to the case $d = \alpha = 1$,

$$|\tilde{E}_\beta(x,y)| \le c_2 \beta (1 + |\log|x-y|| + \beta|x-y|^\alpha). \tag{2.22}$$

Moreover, (2.8) implies

$$\begin{aligned} J_2 &\le c_1 \beta \int_{\beta|x-y|^\alpha}^\infty s^{-1} \cdot \left(\frac{\beta^{1/\alpha}|x-y|}{s^{1/\alpha}} \right)^2 ds \\ &= c_1 \beta^{1+(2/\alpha)} |x-y|^2 \int_{\beta|x-y|^\alpha}^\infty s^{-1-(2/\alpha)} ds \le c_2 \beta. \end{aligned} \tag{2.23}$$

Combining the formulae (2.18), (2.20) and (2.21), we have

$$\begin{aligned} G_\beta(x,y) &= G(x,y) - (I_1 + I_2) = G(x,y) - I_1 - J_1 + J_2 \\ &= G(x,y) - \frac{\beta \log \beta^{-1}}{2^{d-1} \pi^{d/2} \Gamma(\alpha+1)} - \tilde{E}_\beta(x,y) - I_1 + J_2. \end{aligned}$$

Set $E_\beta(x,y) = -\tilde{E}_\beta(x,y) - I_1 + J_2$. Then we see from (2.19), (2.22) and (2.23) that

$$|E_\beta(x,y)| \le c_3 \beta (1 + |\log|x-y|| + \beta|x-y|^\alpha)$$

and obtain (2.16).

It follows that for $d/\alpha > 2$,

$$\int_0^\infty t p(t,x,y) dt = \int_0^\infty C_{d,\alpha} t^{1-(d/\alpha)} g\left(\frac{|x-y|}{t^{1/\alpha}} \right) dt = c_1 |x-y|^{2\alpha - d}.$$

Thus, we have

$$\begin{aligned}G(x,y) - G_\beta(x,y) &= \int_0^\infty p(t,x,y)(1-e^{-\beta t})dt \\ &= \tilde{G}(x,y)\beta + \int_0^\infty p(t,x,y)(1-\beta t - e^{-\beta t})dt.\end{aligned}$$

Setting

$$E_\beta(x,y) = \int_0^\infty p(t,x,y)(\beta t - 1 + e^{-\beta t})dt,$$

we obtain (2.17). For the estimate of $E_\beta(x,y)$, first note that

$$0 \le \beta t - 1 + e^{-\beta t} \le \frac{(\beta t)^2}{2}. \tag{2.24}$$

For $d/\alpha > 3$,

$$\begin{aligned}E_\beta(x,y) &\le \int_0^\infty \frac{(\beta t)^2}{2} p(t,x,y)dt \\ &\le c_1\beta^2 \int_0^\infty \left(t^{2-(d/\alpha)} \wedge \frac{t^3}{|x-y|^{d+\alpha}}\right) dt \le c_2\beta^2|x-y|^{3\alpha-d}\end{aligned}$$

and the desired result follows.

For $d/\alpha < 3$,

$$\begin{aligned}E_\beta(x,y) &= \int_0^\infty p(t,x,y)(\beta t - 1 + e^{-\beta t})dt \\ &= c_1\beta^{(d/\alpha)-1} \int_0^\infty s^{-d/\alpha} g\left(\frac{\beta^{1/\alpha}|x-y|}{s^{1/\alpha}}\right)(s-1+e^{-s})ds \\ &\le c_2\beta^{(d/\alpha)-1} \int_0^\infty s^{-d/\alpha}(s-1+e^{-s})ds \le c_3\beta^{(d/\alpha)-1}\end{aligned}$$

on account of the integrability of $s^{-d/\alpha}(s-1+e^{-s})$ and the inequality $g(w) \le g(0)$.

For $d/\alpha = 3$, we have

$$\begin{aligned}E_\beta(x,y) = &\int_0^{|x-y|^\alpha} p(t,x,y)(\beta t - 1 + e^{-\beta t})dt \\ &+ \int_{|x-y|^\alpha}^\infty C_{d,\alpha} t^{-3} g\left(\frac{|x-y|}{t^{1/\alpha}}\right)(\beta t - 1 + e^{-\beta t})dt =: I_1 + I_2.\end{aligned}$$

By the upper bound of the heat kernel and (2.24),

$$I_1 \le c_1 \int_0^{|x-y|^\alpha} \frac{t}{|x-y|^{4\alpha}} \cdot \beta^2 t^2 dt \le c_2\beta^2.$$

For I_2, we have

$$I_2 \le C_{d,\alpha} g(0) \int_{|x-y|^\alpha}^\infty t^{-3}(\beta t - 1 + e^{-\beta t}) dt = c_1 \beta^2 \int_{\beta|x-y|^\alpha}^\infty s^{-3}(s - 1 + e^{-s}) ds$$

$$= c_2 \beta^2 \left(\frac{\beta|x-y|^\alpha - 1 + e^{-\beta|x-y|^\alpha}}{2(\beta|x-y|^\alpha)^2} + \int_{\beta|x-y|^\alpha}^\infty \frac{1}{2s^2}(1 - e^{-s}) ds \right).$$

Thus, similarly to the case $d/\alpha = 2$, we have

$$I_2 \le c_3 \beta^2 \log \beta^{-1} + c_4 \beta^2 (1 + |\log|x-y|| + \beta|x-y|^\alpha)$$

and we can conclude

$$E_\beta(x,y) \le c_1 \beta^2 \log \beta^{-1} + c_2 \beta^2 (1 + |\log|x-y|| + \beta|x-y|^\alpha). \qquad \square$$

3. Construction of compact operators and their principal eigenvalues.

Let V be a positive continuous function with support contained in $\{x : |x| \le R\}$ for some $R > 0$. We define the bottom of the spectrum of the operator $(1/V)H$ by

$$\lambda_V = \inf \left\{ \mathcal{E}(u,u) \;\middle|\; u \in \mathcal{F}_e, \quad \int_{\mathbb{R}^d} u^2(x) V(x) dx = 1 \right\}, \tag{3.1}$$

where $\mathcal{F}_e$ is an extended Dirichlet space, i.e., the family of m-measurable functions u such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. For $\lambda \ge 0$, we define the Schrödinger form $\mathcal{E}^\lambda$ and the spectral function $C(\lambda)$ by

$$\begin{aligned} \mathcal{E}^\lambda(u,v) &= \mathcal{E}(u,v) - \lambda \int_{\mathbb{R}^d} u(x) v(x) V(x) dx, \\ C(\lambda) &= -\inf \left\{ \mathcal{E}^\lambda(u,u) \;\middle|\; u \in \mathcal{F}, \int_{\mathbb{R}^d} u^2(x) dx = 1 \right\}. \end{aligned} \tag{3.2}$$

LEMMA 3.1. *The function $C(\lambda)$ is positive if and only if $\lambda > \lambda_V$.*

PROOF. Suppose $\{X_t\}_{t \ge 0}$ is recurrent. Since $1 \in \mathcal{F}_e$ and $\mathcal{E}(1,1) = 0$, we have $\lambda_V = 0$. Let $\{u_n\} \subset \mathcal{F}$ be an approximating sequence for $1 \in \mathcal{F}_e$. For $\lambda > 0$,

$$\mathcal{E}^\lambda(u_n,u_n) = \mathcal{E}(u_n,u_n) - \lambda \int_{\mathbb{R}^d} u_n^2(x) V(x) dx.$$

By Fatou's lemma, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^d} u_n^2(x) V(x) dx \ge \int_{\mathbb{R}^d} V(x) dx > 0.$$

Thus, for large $n \in \mathbb{N}$, $\mathcal{E}^\lambda(u_n,u_n) < 0$ and it follows that $C(\lambda) > 0$.

Let $\{v_n\}$ be the normalization of $\{u_n\}$ in $L^2(\mathbb{R}^d)$. Noting that $\|u_n\|_2 \to \infty$ as $n \to \infty$, we have

$$\mathcal{E}(v_n, v_n) \to 0 \qquad (n \to \infty).$$

Hence, $C(0) = 0$ and the assertion follows. For transient $\{X_t\}_{t\geq 0}$, the assertion follows from [**10**, Lemma 2.2]. □

For $\beta \geq 0$, we define λ_β by

$$\lambda_\beta = \inf\left\{\mathcal{E}_\beta(u, u) \;\middle|\; u \in \mathcal{F}_e, \int_{\mathbb{R}^d} u^2(x)V(x)dx = 1\right\}. \tag{3.3}$$

where $\mathcal{E}_\beta(u, u) = \mathcal{E}(u, u) + \beta\langle u, u\rangle$. In particular, $\lambda_0 = \lambda_V$. The next lemma follows from Rellich's theorem [**7**, Theorem 8.6].

LEMMA 3.2. *For $\beta > 0$, $(\mathcal{F}, \mathcal{E}_\beta)$ is compactly embedded into $L^2(V \cdot m)$. If $\{X_t\}_{t\geq 0}$ is transient, $(\mathcal{F}_e, \mathcal{E})$ is also compactly embedded into $L^2(V \cdot m)$.*

The next lemma gives a relation between (3.2) and (3.3).

LEMMA 3.3. *$C(\lambda)$ is the inverse function of λ_β, i.e. $C(\lambda_\beta) = \beta$ for $\beta \geq 0$.*

PROOF. We first assume that $\beta > 0$. By the definition of λ_β, we have for $h \in \mathcal{F}$

$$\mathcal{E}_\beta(h, h) \geq \lambda_\beta \int_{\mathbb{R}^d} h^2(x)V(x)dx. \tag{3.4}$$

By Lemma 3.2, there exists an element h_β in $\mathcal{F}$ which attains the infimum of the right hand side of (3.3). Thus h_β satisfies the equality in (3.4) and we have

$$\mathcal{E}^{\lambda_\beta}(h_\beta, h_\beta) = -\beta \int_{\mathbb{R}^d} h_\beta^2(x)dx. \tag{3.5}$$

$h_\beta/\|h_\beta\|_2$ attains the infimum of (3.2) and thus $C(\lambda_\beta) = \beta$.

If $\{X_t\}_{t\geq 0}$ is recurrent and $\beta = 0$, we have the assertion from $C(0) = \lambda_0 = 0$. If $\{X_t\}_{t\geq 0}$ is transient and $\beta = 0$, we can prove that there exists an element h_0 in $\mathcal{F}_e$ such that

$$\mathcal{E}^{\lambda_0}(h_0, h_0) = 0$$

by the same argument as in $\beta > 0$. If $h_0 \in L^2(\mathbb{R}^d)$, $h_0/\|h_0\|_2$ attains the infimum of (3.2) and $C(\lambda_0) = 0$. If $h_0 \notin L^2(\mathbb{R}^d)$, let $\{u_n\} \subset \mathcal{F}$ be an approximating sequence for h_0. By the definition of λ_0, we have

$$\liminf_{n\to\infty} \mathcal{E}^{\lambda_0}(u_n, u_n) \geq 0.$$

Moreover, Fatou's lemma implies

$$\limsup_{n\to\infty} \mathcal{E}^{\lambda_0}(u_n, u_n) \leq \mathcal{E}^{\lambda_0}(h_0, h_0) = 0$$

and thus, $\mathcal{E}^{\lambda_0}(u_n, u_n) \to 0$ as $n \to \infty$. Noting that $\|u_n\|_2 \to \infty$ as $n \to \infty$ by Fatou's lemma, we see that

$$\lim_{n\to\infty} \mathcal{E}^{\lambda_0}(v_n, v_n) = 0,$$

where $\{v_n\}$ is the normalization of $\{u_n\}$ in $L^2(\mathbb{R}^d)$. Thus, we have $C(\lambda_0) = 0$. □

REMARK 3.4. We call the function $h_0(x)$ *ground state* of the Schrödinger operator $H^{\lambda_V} = H - \lambda_V V$. Takeda and Tsuchida [**10**] showed

$$c_1(1 \wedge |x|^{\alpha-d}) \le h_0(x) \le c_2(1 \wedge |x|^{\alpha-d}).$$

In particular, $h_0(x) \in L^2(\mathbb{R}^d)$ if and only if $d/\alpha > 2$.

Define the operator $K_\beta : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$K_\beta f(x) = \sqrt{V(x)} \int_{\mathbb{R}^d} G_\beta(x, y)\sqrt{V(y)} f(y) dy.$$

THEOREM 3.5. *For $\beta > 0$, the operator K_β is a compact operator. If $\{X_t\}$ is transient, K_0 is also compact.*

PROOF. Let f be in $L^2(\mathbb{R}^d)$. For $\beta > 0$, $G_\beta(\sqrt{V} f)$ belongs to $\mathcal{F}$. Since $\mathcal{F}$ is compactly embedded into $L^2(V \cdot m)$ by Lemma 3.2, K_β is a compact operator. If $\{X_t\}_{t\ge 0}$ is transient and $\beta = 0$, it is sufficient to prove that $G_\beta(\sqrt{V} f)$ belongs to the class $\mathcal{F}_e$ similarly to the case $\beta > 0$. We first show that K_0 is a bounded operator. Since $\mathrm{supp}[V] \subset \{x \mid |x| \le R\}$ for some $R > 0$, we have

$$|K_0 f(x)| \le c_1 \sqrt{V(x)} \int_{\mathbb{R}^d} U(x - y)\sqrt{V(y)}|f(y)|dy,$$

where $U(x) = G(0, x) \cdot 1_{\{|x|\le 2R\}}$. Applying the Young inequality and the Hölder inequality, we have

$$\left\| \sqrt{V(\cdot)} \int_{\mathbb{R}^d} U(\cdot - y)\sqrt{V(y)}|f(y)|dy \right\|_2 \le \|\sqrt{V}\|_\infty \|U\|_1 \|\sqrt{V}\|_\infty \|f\|_2 \le c_2 \|f\|_2.$$

Hence $\|K_0 f\|_2 \le c_3 \|f\|_2$ and K_0 is a bounded operator. If f is non-negative, $G(\sqrt{V} f)$ satisfies

$$\int_{\mathbb{R}^d} \sqrt{V} f(x) \cdot G(\sqrt{V} f)(x) dx = \int_{\mathbb{R}^d} f(x) \cdot K_0 f(x) dx < \infty.$$

Hence, [**3**, Theorem 1.5.4] implies $G(\sqrt{V} f) \in \mathcal{F}_e$. For general $f \in L^2(\mathbb{R}^d)$, we have $f = f_+ - f_-$ for non-negative functions $f_+ := f \vee 0$ and $f_- := -(f \wedge 0)$, and thus the assertion holds. □

The following theorem describes a relation between the eigenvalue of K_β and the bottom of the spectrum λ_β.

THEOREM 3.6. *The compact operator K_β admits the principal eigenvalue λ_β^{-1}.*

PROOF. Note that

$$\mathcal{E}_\beta(h_\beta, h_\beta) = \lambda_\beta \int_{\mathbb{R}^d} h_\beta^2(x)V(x)dx = \lambda_\beta \mathcal{E}_\beta(G_\beta(Vh_\beta), h_\beta). \tag{3.6}$$

Setting $\gamma_\beta = \lambda_\beta^{-1}$, we have

$$\mathcal{E}_\beta(G_\beta(Vh_\beta) - \gamma_\beta h_\beta, h_\beta) = 0.$$

Let k_β be the function satisfying $G_\beta(Vh_\beta) = \gamma_\beta h_\beta + k_\beta$. Then we see that $k_\beta = 0$. Indeed, (3.3) implies

$$\mathcal{E}_\beta(\gamma_\beta h_\beta + k_\beta, \gamma_\beta h_\beta + k_\beta) \geq \gamma_\beta^{-1} \int_{\mathbb{R}^d} (\gamma_\beta h_\beta(x) + k_\beta(x))^2 V(x)dx.$$

Noting that $\mathcal{E}_\beta(h_\beta, k_\beta) = 0$ and h_β satisfies (3.6), we have

$$\begin{aligned}\mathcal{E}_\beta(k_\beta, k_\beta) &\geq 2\int_{\mathbb{R}^d} h_\beta(x)k_\beta(x)V(x)dx + \gamma_\beta^{-1}\int_{\mathbb{R}^d} k_\beta^2(x)V(x)dx \\ &= 2\mathcal{E}_\beta(G_\beta(Vh_\beta), k_\beta) + \gamma_\beta^{-1}\int_{\mathbb{R}^d} k_\beta^2(x)V(x)dx \\ &= 2\mathcal{E}_\beta(k_\beta, k_\beta) + \gamma_\beta^{-1}\int_{\mathbb{R}^d} k_\beta^2(x)V(x)dx.\end{aligned}$$

For $\beta > 0$ (resp. $\beta = 0$), $\mathcal{F}$ (resp. $\mathcal{F}_e$) is a Hilbert space with inner product $\mathcal{E}_\beta$ (resp. $\mathcal{E}$). Thus, we have $k_\beta = 0$ and $G_\beta(Vh_\beta) = \gamma_\beta h_\beta$. Substituting $g_\beta = \sqrt{V}h_\beta$, we see that $K_\beta g_\beta = \gamma_\beta g_\beta$ and γ_β is the eigenvalue of K_β. If γ_β were not maximum, there would be $\hat{g}_\beta \in L^2(\mathbb{R}^d)$ and $\hat{\gamma}_\beta > \gamma_\beta$ such that $K_\beta \hat{g}_\beta = \hat{\gamma}_\beta \hat{g}_\beta$. Set $\hat{h}_\beta = G_\beta(\sqrt{V}\hat{g}_\beta)$. Then we see that $\hat{h}_\beta \in \mathcal{F}_e$ from Theorem 3.5 and thus

$$\begin{aligned}\mathcal{E}_\beta(\hat{h}_\beta, \hat{h}_\beta) &= \hat{\gamma_\beta}^{-1}\mathcal{E}_\beta(G_\beta(V\hat{h}_\beta), \hat{h}_\beta) \\ &= \hat{\gamma}_\beta^{-1}\int_{\mathbb{R}^d} \hat{h}_\beta^2(x)V(x)dx < \gamma_\beta^{-1}\int_{\mathbb{R}^d} \hat{h}_\beta^2(x)V(x)dx.\end{aligned}$$

This is a contradiction to the minimum property of $\lambda_\beta = \gamma_\beta^{-1}$. □

4. Asymptotics of spectral functions.

To know the behavior of the spectral function, we first consider the behavior of γ_β, the principal eigenvalue of K_β. We give the asymptotic behavior of γ_β as $\beta \downarrow 0$ via the asymptotic expansion for K_β obtained from the asymptotic expansion for $G_\beta(x, y)$ in Section 2.

LEMMA 4.1. *Suppose $d = 1$ and $1 \leq \alpha < 2$, namely $\{X_t\}_{t\geq 0}$ is recurrent. For $\beta \downarrow 0$, the principal eigenvalue γ_β satisfies*

$$\gamma_\beta = \frac{C_V}{\alpha \sin \pi/\alpha} \beta^{(1/\alpha)-1} + O(1), \qquad C_V = \int_{\mathbb{R}} V(x)dx \quad (1 < \alpha < 2),$$
$$\gamma_\beta = \frac{C_V \log \beta^{-1}}{\pi} + O(1) \qquad (\alpha = 1).$$

Proof. For $1 < \alpha < 2$, the operator K_β satisfies

$$K_\beta = \frac{\beta^{(1/\alpha)-1}}{\alpha \sin \pi/\alpha} D_1 + D_2,$$

where the operators D_1 and D_2 are defined by

$$D_1 f(x) = \sqrt{V(x)} \int_{\mathbb{R}} \sqrt{V(y)} f(y) dy, \tag{4.1}$$
$$D_2 f(x) = \sqrt{V(x)} \int_{\mathbb{R}} E_\beta(x,y) \sqrt{V(y)} f(y) dy. \tag{4.2}$$

Since D_1 and D_2 are bounded operators, this formula gives an asymptotic expansion for K_β. Indeed, for $f \in L^2(\mathbb{R})$ it follows that

$$\begin{aligned} \|D_1 f\|_2^2 &= \int_{\mathbb{R}} V(x) \left| \int_{\mathbb{R}} \sqrt{V(y)} f(y) dy \right|^2 dx \\ &\le \|V\|_1 \left(\int_{\mathbb{R}} \sqrt{V(y)} |f(y)| dy \right)^2 \le \|V\|_1^2 \|f\|_2^2 \le c_1 \|f\|_2^2 \end{aligned}$$

and thus D_1 is a bounded operator. Since $|E_\beta(x,y)| \le c_1 |x-y|^{\alpha-1}$, we have

$$|D_2 f(x)| \le c_1 \sqrt{V(x)} \int_{\mathbb{R}} U(x-y) \sqrt{V(y)} |f(y)| dy,$$

where $U(x) = 1_{\{|x| \le 2R\}} |x|^{\alpha-1}$. Applying the Young inequality and the Hölder inequality, we have

$$\left\| \sqrt{V(\cdot)} \int_{\mathbb{R}} U(\cdot - y) \sqrt{V(y)} |f(y)| dy \right\|_2 \le \|\sqrt{V}\|_\infty \|U\|_1 \|\sqrt{V}\|_\infty \|f\|_2 \le c_2 \|f\|_2.$$

Hence D_2 is a bounded operator.

Let γ_β be the principal eigenvalue of K_β and g_β be the corresponding eigenfunction. Note that $g_\beta = \sqrt{V} h_\beta$ for some $h_\beta \in \mathcal{F}$. Using the Schwartz inequality, we have for $h \in \mathcal{F}$

$$\langle \sqrt{V} h, D_1(\sqrt{V} h) \rangle = \left(\int_{\mathbb{R}} V(y) h(y) dy \right)^2 \le \int_{\mathbb{R}} V(y) dy \int_{\mathbb{R}} V(y) h^2(y) dy. \tag{4.3}$$

Thus, we see that

$$\gamma_\beta \le \frac{C_V}{\alpha \sin \pi/\alpha} \beta^{(1/\alpha)-1} + O(1). \tag{4.4}$$

Furthermore, there exists $h_0 \in \mathcal{F}$ such that $h_0(x) = 1$ on the support of $V(x)$, since $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{F} = H^{\alpha/2}(\mathbb{R})$. Here $C_0^\infty(\mathbb{R})$ is the family of infinitely differentiable functions with compact support. Then, h_0 is an element which attains the equality in (4.3) and we have

$$\gamma_\beta \geq \frac{C_V}{\alpha \sin \pi/\alpha} \beta^{(1/\alpha)-1} + O(1). \tag{4.5}$$

Combining (4.4) and (4.5), we have the desired result.

Suppose $\alpha = 1$. K_β satisfies

$$K_\beta = \frac{\log \beta^{-1}}{\pi} D_1 + D_2$$

for D_1 and D_2 defined by (4.1) and (4.2). Similarly to the case $1 < \alpha < 2$, we obtain the desired assertion. $\square$

Setting $\gamma_0^{-1} = 0$ for convention, we see that γ_β^{-1} is the inverse function of the spectral function $C(\lambda)$ for $\beta \geq 0$.

If $\{X_t\}_{t\geq 0}$ is transient, we obtain the asymptotic expansion for γ_β by the first order perturbation theory of compact operators in Kato [**4**].

LEMMA 4.2. *Suppose $d/\alpha > 1$, namely $\{X_t\}_{t\geq 0}$ is transient. For $\beta \downarrow 0$, the principal eigenvalue γ_β satisfies*

$$\gamma_\beta = \gamma_0 - \frac{2^{1-d}\pi^{1-(d/2)}\langle \sqrt{V}, \sqrt{V}h_0\rangle^2}{\alpha\Gamma(d/2)\sin(((d/\alpha)-1)\pi)\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}\beta^{(d/\alpha)-1} + o(\beta^{(d/\alpha)-1}) \ (1 < d/\alpha < 2),$$

$$\gamma_\beta = \gamma_0 - \frac{2^{1-d}\pi^{-d/2}\langle \sqrt{V}, \sqrt{V}h_0\rangle^2}{\Gamma(\alpha+1)\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}\beta \log \beta^{-1} + o(\beta \log \beta^{-1}) \quad (d/\alpha = 2),$$

$$\gamma_\beta = \gamma_0 - \frac{\langle h_0, h_0\rangle}{\langle \lambda_V \sqrt{V}h_0, \lambda_V \sqrt{V}h_0\rangle}\beta + o(\beta) \quad (d/\alpha > 2).$$

PROOF. Suppose $1 < d/\alpha < 2$. By (2.15), the operator K_β admits the asymptotic expansion as follows:

$$K_\beta = K_0 - \kappa_1 \beta^{(d/\alpha)-1} D_1 + D_2, \qquad \kappa_1 = \frac{2^{1-d}\pi^{1-(d/2)}}{\alpha\Gamma\left(d/2\right)\sin(((d/\alpha)-1)\pi)},$$

where the operators D_1 and D_2 are defined by

$$D_1 f(x) = \sqrt{V(x)} \int_{\mathbb{R}^d} \sqrt{V(y)} f(y) dy, \tag{4.6}$$

$$D_2 f(x) = \sqrt{V(x)} \int_{\mathbb{R}^d} E_\beta(x,y) \sqrt{V(y)} f(y) dy. \tag{4.7}$$

We first consider the principal eigenvalue of the operator $K_0 - \kappa_1 \beta^{(d/\alpha)-1} D_1$. Since K_0 is a compact operator and D_1 is a bounded operator, we can apply [**4**, Theorem VIII.2.6]. Recall that γ_0 is the principal eigenvalue of K_0 and $\sqrt{V}h_0$ is the corresponding eigenfunction, where h_0 is the ground state for the Schrödinger operator $H - \lambda_V V$. Let P be the projection operator defined by

$$Pf(x) = \frac{\langle f, \sqrt{V}h_0\rangle}{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}\sqrt{V}h_0(x).$$

Then the eigenvalue of the operator PD_1P considered in the principal eigenspace is calculated by

$$\frac{\langle PD_1P(\sqrt{V}h_0), \sqrt{V}h_0\rangle}{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle} = \frac{\langle \sqrt{V}, \sqrt{V}h_0\rangle^2}{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}.$$

Thus, the principal eigenvalue of $K_0 - \kappa_1\beta^{d/\alpha-1}D_1$ admits the asymptotic expansion as follows:

$$\gamma_0 - \kappa_1\frac{\langle \sqrt{V}, \sqrt{V}h_0\rangle^2}{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}\beta^{(d/\alpha)-1} + o(\beta^{(d/\alpha)-1}).$$

Furthermore, similarly to Lemma 4.1, $E_\beta(x,y) \le c_1\beta|x-y|^{2\alpha-d}$ implies that the operator norm of D_2 is dominated by $c_2\beta$. Hence, γ_β satisfies

$$\gamma_\beta = \gamma_0 - \kappa_1\frac{\langle \sqrt{V}, \sqrt{V}h_0\rangle^2}{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}\beta^{(d/\alpha)-1} + o(\beta^{(d/\alpha)-1}).$$

Suppose $d/\alpha = 2$. By (2.16),

$$K_\beta = K_0 - \kappa_2\beta\log\beta^{-1}D_1 + D_2, \qquad \kappa_2 = \frac{2^{1-d}\pi^{-d/2}}{\Gamma(\alpha+1)},$$

where D_1 and D_2 are defined by (4.6) and (4.7). Thus, the principal eigenvalue of $K_0 - \kappa_2\beta\log\beta^{-1}D_1$ admits the asymptotic expansion

$$\gamma_0 - \kappa_2\frac{\langle \sqrt{V}, \sqrt{V}h_0\rangle^2}{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}\beta\log\beta^{-1} + o(\beta\log\beta^{-1}).$$

Furthermore, noting that the operator norm of D_2 is dominated by $c_1\beta$ from the estimate of $E_\beta(x,y)$ in Theorem 2.4, we obtain the desired formula.

Suppose $d/\alpha > 2$. By (2.17),

$$K_\beta = K_0 - \beta D_1 + D_2,$$

where D_1 is defined by

$$D_1f(x) = \sqrt{V(x)}\int_{\mathbb{R}^d}\tilde{G}(x,y)\sqrt{V(y)}f(y)dy, \quad \tilde{G}(x,y) = \int_0^\infty tp(t,x,y)dt$$

and D_2 is defined by (4.7). We see that D_1 is a bounded operator. Indeed, $\tilde{G}(x,y) = c_1|x-y|^{2\alpha-d}$ and $\mathrm{supp}[V] \subset \{x \mid |x| \le R\}$ imply $\|D_1f\|_2 \le c_2\|K_0f\|_2$. Thus, the principal eigenvalue of $K_0 - \beta D_1$ admits the asymptotic expansion

$$\gamma_0 - \frac{\langle PD_1P(\sqrt{V}h_0), \sqrt{V}h_0\rangle}{\langle \sqrt{V}h_0, \sqrt{V}h_0\rangle}\beta + o(\beta).$$

Note that the operator $H^{-2} = G^2$ has the integral kernel $\tilde{G}(x, y)$ since the operator e^{-tH} has the integral kernel $p(t, x, y)$. Hence we have

$$\begin{aligned}\langle PD_1P(\sqrt{V}h_0), \sqrt{V}h_0\rangle &= \langle \sqrt{V}G^2(Vh_0), \sqrt{V}h_0\rangle \\ &= \langle G(Vh_0), G(Vh_0)\rangle = \lambda_V^{-2}\langle h_0, h_0\rangle.\end{aligned}$$

Furthermore, we see that the operator norm of D_2 is dominated by $c_1\beta^{((d/\alpha)-1)\wedge 3/2}$ from the estimate of $E_\beta(x, y)$ in Theorem 2.4. Hence γ_β satisfies

$$\gamma_\beta = \gamma_0 - \frac{\langle h_0, h_0\rangle}{\langle \lambda_V\sqrt{V}h_0, \lambda_V\sqrt{V}h_0\rangle}\beta + o(\beta). \qquad \square$$

(Proof of Theorem 1.1)
The asymptotic behavior of γ_β is given in Lemmas 4.1 and 4.2. The spectral function $C(\lambda)$ is the inverse function of $\gamma_\beta^{-1} = \lambda_\beta$ and we have the desired result.

Remark 4.3. Let δ_0 be the Dirac measure at the origin. In [**8**] the principal eigenvalue of $(-\Delta)^{\alpha/2}/2 - \lambda\delta_0$ is calculated for $d = 1$ and $1 < \alpha < 2$, which is consistent with Theorem 1.1.

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