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On left-orderable fundamental groups and Dehn surgeries on knots

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Abstract. We show that the resulting manifold by r-surgery on a large class of two-bridge knots has left-orderable fundamental group if the slope r satisfies certain conditions. This result gives a supporting evidence to a conjecture of Boyer, Gordon and Watson that relates L-spaces and the left-orderability of their fundamental groups.

Introduction.

The motivation of this paper is a conjecture of Boyer, Gordon and Watson that relates L-spaces and the left-orderability of their fundamental groups. Let Y be a closed, connected, oriented 3-manifold, and denote by $\widehat{HF}(Y)$ the 'hat' version of Heegaard Floer homology of Y. We are interested in a class of manifolds with minimal Heegaard Floer homology which was introduced in [**OS**]. A rational homology sphere Y is called an L-space if $\widehat{HF}(Y)$ is a free abelian group whose rank coincides with the number of elements in $H_1(Y;\mathbb{Z})$. Examples of L-spaces include lens spaces as well as all spaces with elliptic geometry [**OS**]. It is natural to ask if there are characterizations of L-spaces which do not refer to Heegaard Floer homology.

A non-trivial group G is called left-orderable if there exists a strict total ordering $<$ on its elements such that $g < h$ implies $fg < fh$ for all elements $f, g, h \in G$. It is known that the fundamental group of an irreducible 3-manifold with positive first Betti number is left-orderable [**HSt**], [**BRW**]. There is a conjectured connection between L-spaces and the left-orderability of their fundamental groups. Precisely, a conjecture of Boyer, Gordon and Watson [**BGW**] states that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable. The conjecture was confirmed for Seifert fibered manifolds, Sol manifolds, double branched covers of non-splitting alternating links [**BGW**].

In a related direction, it was shown that if $-4 \le r \le 4$ then r-surgery on the figure-eight knot yields a manifold whose fundamental group is left-orderable [**BGW**], [**CLW**]. Recently, Hakamata and Teragaito have generalized this result to all hyperbolic twist knots. They show that if $0 \le r \le 4$ then r-surgery on any hyperbolic twist knot yields a manifold whose fundamental group is left-orderable [**HT1**], [**HT2**]. In this paper, we study the left-orderability of the fundamental group of manifolds obtained by Dehn surgeries on a large class of two-bridge knots that includes all twist knots. Let $J(k,l)$

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be the knot in Figure 1. Note that $J(k,l)$ is a knot if and only if kl is even, and is the trivial knot if $kl=0$. Furthermore, $J(k,l) \cong J(l,k)$ and $J(-k,-l)$ is the mirror image of $J(k,l)$. Hence, without loss of generality, we consider $J(k,2n)$ for $k>0$ and $|n|>0$ only. When $k=2$, $J(2,2n)$ is the twist knot. Note that the twist knot K_n in [**HT2**] is $J(-2,2n)$, which is the mirror image of $J(2,-2n)$.

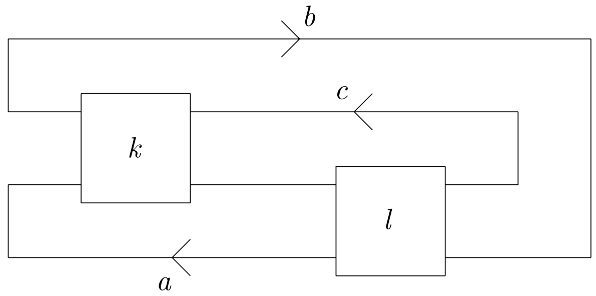

Figure 1. The knot $K = J(k,l)$. Here k and l denote the numbers of half twists in the boxes. Positive numbers correspond to right-handed twists and negative numbers correspond to left-handed twists.

The main result of the paper is as follows.

THEOREM 1. *Let m and n be integers such that $m \geq 1$. Suppose $r \in \mathbb{Q}$ satisfies*

$$r \in \begin{cases} (-\max\{4m,4n\},0], & n \geq 2 \text{ and } m \geq 2, \\ \Big(-(4n+2), -\Big(\dfrac{4(2n-1)}{\omega_n}+4\Big)\Big) \cup [-4,0], & n \geq 2 \text{ and } m = 1, \\ \Big(-(4m+2), -\Big(\dfrac{4(2m-1)}{\omega_m}+4\Big)\Big) \cup [-4,0], & n = 1 \text{ and } m \geq 2, \\ (-4m,-4n), & n \leq -1, \end{cases}$$

where ω_m (resp. ω_n) is the unique real solution of the equation $te^t = 4(2m-1)$ (resp. $te^t = 4(2n-1)$). Then the resulting manifold by r-surgery on the hyperbolic knot $J(2m,2n)$ has left-orderable fundamental group.

REMARK 0.1. a) It is known that $J(k,l)$ is a hyperbolic knot if and only if $|k|,|l| \geq 2$ and $J(k,l)$ is not the trefoil knot. We exclude $J(2,2)$ from Theorem 1 since it is the trefoil knot.

b) Since $J(-2m,-2n)$ is the mirror image of $J(2m,2n)$, the following follows from Theorem 1. Let m and n be integers such that $m \geq 1$. Suppose $r \in \mathbb{Q}$ satisfies

$$r \in \begin{cases} [0, \max\{4m, 4n\}), & n \geq 2 \text{ and } m \geq 2, \\ [0,4] \cup \left(\dfrac{4(2n-1)}{\omega_n} + 4, 4n+2\right), & n \geq 2 \text{ and } m = 1, \\ [0,4] \cup \left(\dfrac{4(2m-1)}{\omega_m} + 4, 4m+2\right), & n = 1 \text{ and } m \geq 2, \\ (4n, 4m), & n \leq -1. \end{cases}$$

Then the resulting manifold by r-surgery on the hyperbolic knot $J(-2m, -2n)$ has left-orderable fundamental group.

c) Since $J(2m, 2n)$ does not yield an L-space by any non-trivial Dehn surgery [**OS**], Theorem 1 gives a supporting evidence to the conjecture of Boyer, Gordon and Watson.

Plan of the paper. In Sections 1, 2 and 3, we respectively study the knot group, the non-abelian $SL_2(\mathbb{C})$-representation space and the canonical longitude of the knot $J(2m, 2n)$. Sections 4 and 5 contain crucial calculations involving the meridian and the canonical longitude of $J(2m, 2n)$ which will be needed in the proof of the main theorem in the last section. Section 6 is devoted to the proof of Theorem 1.

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1. Knot groups.

Let X be the closure of S^3 minus a tubular neighborhood of a knot K. The fundamental group of X is called the knot group of K and is denoted by $\pi_1(K)$. By [**HSn**, Section 4], the knot group of $K = J(2m, 2n)$ has a presentation

$$\pi_1(K) = \langle a, b \mid aw^n = w^n b \rangle,$$

where $w = (ab^{-1})^m (a^{-1}b)^m$ and a, b are meridians of K depicted in Figure 1.

In the case of $m = 1$ (twist knots), the following presentation is more useful. Let c be the meridian of $J(2, 2n)$ depicted in Figure 1.

Lemma 1.1. *One has*

$$\pi_1(J(2, 2n)) = \langle b, c \mid bu = uc \rangle$$

where $u = (b^{-1}c)^n c (b^{-1}c)^{-n}$.

Proof. Let $b_1, \ldots, b_{|n|+1}$ and $c_1, \ldots, c_{|n|+1}$ be meridians of $K = J(2, 2n)$ depicted in Figures 2 and 3, where $b_1 = b$ and $c_1 = c$.

Case 1: $n < 0$. From the Wirtinger relations corresponding to the bottom $2|n|$

(positive) crossings of K, it follows that $b_{j+1} = c_j^{-1} b_j c_j$ and $c_{j+1} = b_{j+1} c_j b_{j+1}^{-1}$. Then, by induction on j, we have $b_{j+1} = (c^{-1}b)^j b (c^{-1}b)^{-j}$ and $c_{j+1} = (c^{-1}b)^j c (c^{-1}b)^{-j}$.

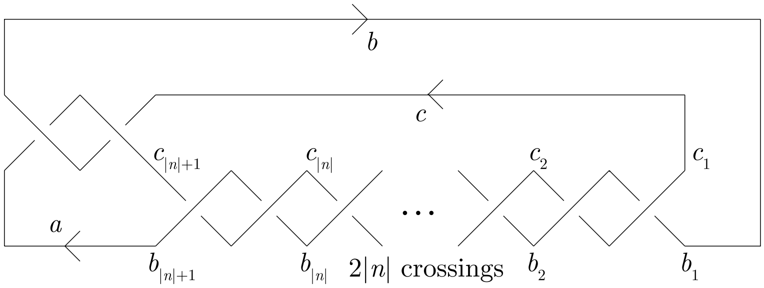

Figure 2. $J(2, 2n),\ n < 0$.

Case 2: $n > 0$. From the Wirtinger relations corresponding to the bottom $2|n|$ (negative) crossings of K, it follows that $c_{j+1} = b_j^{-1} c_j b_j$ and $b_{j+1} = c_{j+1} b_j c_{j+1}^{-1}$. Then, by induction on j, we have $c_{j+1} = (b^{-1}c)^j c (b^{-1}c)^{-j}$ and $b_{j+1} = (b^{-1}c)^j b (b^{-1}c)^{-j}$.

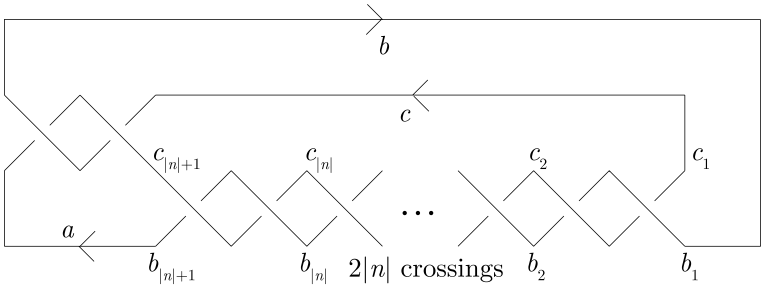

Figure 3. $J(2, 2n),\ n > 0$.

In both cases, we have $b_{|n|+1} = (b^{-1}c)^n b (b^{-1}c)^{-n}$ and $c_{|n|+1} = (b^{-1}c)^n c (b^{-1}c)^{-n}$. The Wirtinger relations corresponding to the top 2 (negative) crossings of K are equivalent to the same relation $c = c_{|n|+1}^{-1} b c_{|n|+1}$. The lemma follows by letting $u = c_{|n|+1}$. $\square$

REMARK 1.2. The above presentation of the knot group of $J(2, 2n)$ follows from the choice of generators of its Kauffman bracket skein algebra in [**GN**] and is very useful for understanding the character variety of $J(2, 2n)$, see [**NT**].

2. Non-abelian $SL_2(\mathbb{C})$-representations.

Recall that $K = J(2m, 2n)$. A representation $\rho : \pi_1(K) \to SL_2(\mathbb{C})$ is called non-abelian if $\rho(\pi_1(K))$ is a non-abelian subgroup of $SL_2(\mathbb{C})$. Taking conjugation if necessary, we can assume that ρ has the form

$$\rho(a) = A = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = B = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \tag{2.1}$$

where $(M, y) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the matrix equation $AW^n - W^nB = O$. Here $W = \rho(w)$. It can be easily checked that $y = \operatorname{tr} AB^{-1}$. Let $x = \operatorname{tr} A = \operatorname{tr} B = M + M^{-1}$.

Let $\{S_j(t)\}_j$ be the sequence of Chebyshev polynomials defined by $S_0(t) = 1$, $S_1(t) = t$, and $S_{j+1}(t) = tS_j(t) - S_{j-1}(t)$ for all integers j. Note that $S_{-j}(t) = -S_{j-2}(t)$. Moreover if $t = s + s^{-1}$, where $s \neq \pm 1$, then $S_j(t) = (s^{j+1} - s^{-j-1})/(s - s^{-1})$.

By [**MT**, Section 2], the assignment (2.1) gives a non-abelian representation $\rho : \pi_1(K) \to SL_2(\mathbb{C})$ if and only if $(M, y) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the equation

$$\phi_K(x, y) := \alpha_m S_{n-1}(\beta_m) - S_{n-2}(\beta_m) = 0,$$

where

$$\begin{aligned}\beta_m &= 2 + (y-2)(y+2-x^2)S_{m-1}^2(y),\\ \alpha_m &= 1 - (y+2-x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y)).\end{aligned}$$

The polynomial $\phi_K(x, y)$ is also known as the Riley polynomial [**Ri**], [**Le**] of K. Certain roots of $\phi_K(x, y)$ can be described as follows.

Lemma 2.1. *Suppose $|n| \geq 2$. There are $0 < \delta_1 < \delta_2 < 4$ (depending on n) such that for every real $y > 2$, there exists*

$$x \in \left(\sqrt{y + 2 + \frac{\delta_1}{(y-2)S_{m-1}^2(y)}}, \sqrt{y + 2 + \frac{\delta_2}{(y-2)S_{m-1}^2(y)}} \right)$$

such that $\phi_K(x, y) = 0$.

Proof. Fix $y > 2$. We consider the following 3 cases.

Case 1: $n = 2$. We have $\phi_K(x, y) = \alpha_m \beta_m - 1$. If $x = \sqrt{y + 2 + 2/((y-2)S_{m-1}^2(y))}$ then $\beta_m = 0$, and $\phi_K(x, y) = -1 < 0$. If $x = \sqrt{y + 2 + 1/((y-2)S_{m-1}^2(y))}$ then $\beta_m = 1$ and $\alpha_m > 1$, which implies that $\phi_K(x, y) = \alpha_m - 1 > 0$. Hence there exists

$$x \in \left(\sqrt{y + 2 + \frac{1}{(y-2)S_{m-1}^2(y)}}, \sqrt{y + 2 + \frac{2}{(y-2)S_{m-1}^2(y)}} \right)$$

such that $\phi_K(x, y) = 0$.

Case 2: $n > 2$. It is known that the polynomial $S_{n-1}(t) - S_{n-2}(t)$ has exactly $n - 1$ roots given by $t = 2\cos((2j-1)\pi/(2n-1))$, where $1 \leq j \leq n-1$.

Let $x_j = \sqrt{y + 2 + \frac{2 - 2\cos((2j-1)\pi/(2n-1))}{(y-2)S_{m-1}^2(y)}}$. Note that if $x = x_j$ then $\beta_m = 2\cos((2j-1)\pi/(2n-1))$, which implies that $S_{n-1}(\beta_m) = S_{n-1}(\beta_m)$ and $\phi_K(x_j, y) = (\alpha_m - 1)S_{n-1}(2\cos((2j-1)\pi/(2n-1)))$. In particular, we have $\phi_K(x_1, y) > 0 > \phi_K(x_2, y)$, since $S_{n-1}(2\cos(\pi/(2n-1))) > 0 > S_{n-1}(2\cos(3\pi/(2n-1)))$ (see e.g. [**HT2**, Lemma 3.1]). Hence there exists $x \in (x_1, x_2)$ such that $\phi_K(x, y) = 0$.

Case 3: $n \le -2$. Let $l = -n \ge 2$. We have

$$\phi_K(x,y) := \alpha_m S_{n-1}(\beta_m) - S_{n-2}(\beta_m) = S_l(\beta_m) - \alpha_m S_{l-1}(\beta_m).$$

Let $x'_j = \sqrt{y + 2 + \frac{2-2\cos((2j-1)\pi/(2l+1))}{(y-2)S^2_{m-1}(y)}}$, where $1 \le j \le l$. By a similar argument as in the previous case, we can show that $\phi_K(x'_1, y) < 0 < \phi_K(x'_2, y)$. Hence there exists $x \in (x'_1, x'_2)$ such that $\phi_K(x,y) = 0$. $\square$

In the case of $m = 1$ (twist knots), by using the presentation in Lemma 1.1 we can also describe non-abelian $SL_2(\mathbb{C})$-representations of $K = J(2, 2n)$ as follows. Suppose $\rho : \pi_1(K) \to SL_2(\mathbb{C})$ is a non-abelian representation. Taking conjugation if necessary, we can assume that ρ has the form

$$\rho(b) = B = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(c) = C = \begin{bmatrix} M & 0 \\ 2-z & M^{-1} \end{bmatrix} \tag{2.2}$$

where $(M, z) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the matrix equation $BU - UC = O$. Here $U = \rho(u)$.

It can be easily checked that $z = \operatorname{tr} BC^{-1}$. The following lemma is standard.

LEMMA 2.2. *Suppose the sequence $\{D_j\}_j$ of 2×2 matrices satisfies the recurrence relation $D_{j+1} = tD_j - D_{j-1}$ for all integers j. Then*

$$D_j = S_{j-1}(t)D_1 - S_{j-2}(t)D_0. \tag{2.3}$$

PROPOSITION 2.3. *One has*

$$BU - UC = \begin{bmatrix} (2-z)\gamma_n(x,z) & M^{-1}\gamma_n(x,z) \\ (z-2)M\gamma_n(x,z) & 0 \end{bmatrix}$$

where

$$\begin{aligned} \gamma_n(x,z) = &-(z+1)S^2_{n-1}(z) + S^2_{n-2}(z) + 2S_{n-1}(z)S_{n-2}(z) \\ &+ x^2 S_{n-1}(z)(S_{n-1}(z) - S_{n-2}(z)). \end{aligned}$$

PROOF. We first note that, by the Cayley-Hamilton theorem, $D^{j+1} = (\operatorname{tr} D)D^j - D^{j-1}$ for all matrices $D \in SL_2(\mathbb{C})$ and all integers j. By applying (2.3) twice, we have

$$\begin{aligned} BU &= B(B^{-1}C)^n C(C^{-1}B)^n \\ &= S^2_{n-1}(z)B(B^{-1}C)C(C^{-1}B) + S^2_{n-2}(z)BC \\ &\quad - S_{n-1}(z)S_{n-2}(z)(B(B^{-1}C)C + BC(C^{-1}B)) \\ &= S^2_{n-1}(z)CB + S^2_{n-2}(z)BC - S_{n-1}(z)S_{n-2}(z)(C^2 + B^2). \end{aligned}$$

Similarly,

$$\begin{aligned}
UC &= (B^{-1}C)^n C(C^{-1}B)^n C \\
&= S_{n-1}^2(z)(B^{-1}C)C(C^{-1}B)C + S_{n-2}^2(z)CC \\
&\quad - S_{n-1}(z)S_{n-2}(z)((B^{-1}C)CC + C(C^{-1}B)C) \\
&= S_{n-1}^2(z)B^{-1}CBC + S_{n-2}^2(z)C^2 - S_{n-1}(z)S_{n-2}(z)(B^{-1}C^3 + BC).
\end{aligned}$$

Hence, by direct calculations using (2.2), we obtain

$$\begin{aligned}
BU - UC &= S_{n-1}^2(z)(CB - B^{-1}CBC) + S_{n-2}^2(z)(BC - C^2) \\
&\quad - S_{n-1}(z)S_{n-2}(z)(C^2 - B^{-1}C^3 + B^2 - BC) \\
&= \begin{bmatrix} (2-z)\gamma_n(x,z) & M^{-1}\gamma_n(x,z) \\ (z-2)M\gamma_n(x,z) & 0 \end{bmatrix}
\end{aligned}$$

where

$$\gamma_n(x,z) = (M^2 + M^{-2} + 1 - z)S_{n-1}^2(z) - (M^2 + M^{-2})S_{n-1}(z)S_{n-2}(z) + S_{n-2}^2(z).$$

The proposition follows since $M^2 + M^{-2} = x^2 - 2$. □

Proposition 2.3 implies that the assignment (2.2) gives a non-abelian representation $\rho : \pi_1(J(2,2n)) \to SL_2(\mathbb{C})$ if and only if $\gamma_n(x,z) = 0$.

3. Canonical longitudes.

Recall that X is the closure of S^3 minus a tubular neighborhood of a knot K. The boundary of X is a torus $\mathbb{T}^2$. There is a standard choice of a meridian μ and a longitude λ on $\mathbb{T}^2$ such that the linking number between the longitude and the knot is 0. We call λ the canonical longitude of K corresponding to the meridian μ.

Let $\mu = b$ be the meridian of $K = J(2m, 2n)$ and λ the canonical longitude corresponding to μ. Suppose $\rho : \pi_1(K) \to SL_2(\mathbb{C})$ is a non-abelian representation. By taking conjugation if necessary, we can assume that ρ has the form

$$\rho(a) = A = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = B = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}$$

where $y = \operatorname{tr} AB^{-1}$. Recall that $x = \operatorname{tr} A = \operatorname{tr} B = M + M^{-1}$.

By [**HSn**, Section 4], we have $\rho(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1} \end{smallmatrix}\right]$ where $L = -\widetilde{W}_{12}/W_{12}$. Here W_{ij} is the ij-entry of $W = \rho(w)$ and $\widetilde{W}_{ij}$ is obtained from W_{ij} by replacing M by M^{-1}.

Lemma 3.1. *One has*

$$W_{12} = S_{m-1}(y)\big[xS_{m-1}(y) - (M - M^{-1})S_{m-2}(y) - yM^{-1}S_{m-1}(y)\big].$$

Proof. The proof is similar to that of [**MT**, Lemma 2.3], so we omit the details. □

In the case of $m = 1$ (twist knots), by Lemma 3.1 we have $\rho(\lambda) = \left[\begin{smallmatrix} L & * \\ 0 & L^{-1} \end{smallmatrix}\right]$ where

$$L = \frac{1-(y-1)M^2}{y-1-M^2}. \tag{3.1}$$

By Lemma 1.1, the knot group of $J(2, 2n)$ also has the following presentation

$$\pi_1(J(2,2n)) = \langle b, c \mid bu = uc \rangle$$

where $u = (b^{-1}c)^n c(b^{-1}c)^{-n}$. Recall from the previous section that $C = \rho(c)$ and $z = \operatorname{tr} BC^{-1}$. We can express $y = \operatorname{tr} AB^{-1}$ in terms of x and z as follows.

LEMMA 3.2. *One has*

$$y = (z^2-2)S_{n-1}^2(z) + 2S_{n-2}^2(z) - 2zS_{n-1}(z)S_{n-2}(z) - x^2(z-2)S_{n-1}^2(z).$$

PROOF. From the proof of Lemma 1.1, we have $a = b_{|n|+1} = (b^{-1}c)^n b(b^{-1}c)^{-n}$, see Figures 2 and 3. By applying (2.3) twice, we have

$$\begin{aligned} AB^{-1} &= (B^{-1}C)^n B(C^{-1}B)^n B^{-1} \\ &= S_{n-1}^2(z)(B^{-1}C)B(C^{-1}B)B^{-1} + S_{n-2}^2(z)BB^{-1} \\ &\quad - S_{n-1}(z)S_{n-2}(z)\big((B^{-1}C)BB^{-1} + B(C^{-1}B)B^{-1}\big) \\ &= S_n^2(z)B^{-1}CBC^{-1} + S_{n-1}^2(z)I - S_{n-1}(z)S_{n-2}(z)(B^{-1}C + BC^{-1}), \end{aligned}$$

where I is the 2×2 identity matrix. Taking traces, we obtain

$$\begin{aligned} \operatorname{tr} AB^{-1} &= S_{n-1}^2(z)\operatorname{tr}(B^{-1}CBC^{-1}) + 2S_{n-2}^2(z) - 2zS_{n-1}(z)S_{n-2}(z) \\ &= (z^2 - zx^2 + 2x^2 - 2)S_{n-1}^2(z) + 2S_{n-2}^2(z) - 2zS_{n-1}(z)S_{n-2}(z), \end{aligned}$$

since $\operatorname{tr}(B^{-1}CBC^{-1}) = z^2 - zx^2 + 2x^2 - 2$. The lemma follows. $\square$

In Sections 4 and 5 below we will perform crucial calculations involving the meridian and the canonical longitude of the knot $J(2m, 2n)$ which will be needed in the proof of Theorem 1 in the last section.

4. Calculations: The case of $|n| \geq 2$.

Recall that $K = J(2m, 2n)$. Let $s > 1$ and $y = s + s^{-1}$. By Lemma 2.1, there exists

$$x \in \left(\sqrt{y+2+\frac{\delta_1}{(y-2)S_{m-1}^2(y)}}, \sqrt{y+2+\frac{\delta_2}{(y-2)S_{m-1}^2(y)}}\right)$$

such that $\phi_K(x,y) = 0$, where $0 < \delta_1 < \delta_2 < 4$ depending on n only. Since $x > \sqrt{y+2} > 2$, there exists $M_s > 1$ such that $x = M_s + M_s^{-1}$. Because $\phi_K(x,y) = 0$, there exists a non-abelian representation $\rho_s : \pi_1(K) \to SL_2(\mathbb{R})$ of the form

$$\rho_s(a) = A = \begin{bmatrix} M_s & 0 \\ 2-y & M_s^{-1} \end{bmatrix} \quad \text{and} \quad \rho_s(b) = B = \begin{bmatrix} M_s & 1 \\ 0 & M_s^{-1} \end{bmatrix}.$$

Recall from the previous section that $\mu = b$ is the meridian of K and λ is the canonical longitude corresponding to μ. We have $\rho_s(\lambda) = \left[\begin{smallmatrix} L_s & * \\ 0 & L_s^{-1} \end{smallmatrix}\right]$ where

$$\begin{aligned} L_s &= -\frac{\widetilde{W}_{12}}{W_{12}} = -\frac{xS_{m-1}(y) + (M-M^{-1})S_{m-2}(y) - yMS_{m-1}(y)}{xS_{m-1}(y) - (M-M^{-1})S_{m-2}(y) - yM^{-1}S_{m-1}(y)} \\ &= \frac{M^2 - s - s^{2m} + M^2 s^{1+2m}}{-1 + M^2 s + M^2 s^{2m} - s^{1+2m}} \end{aligned}$$

by Lemma 3.1.

Lemma 4.1. *One has $M_s^2 > s > 1$. Hence $L_s > 1$.*

Proof. We have $x^2 > y+2$, or equivalently $M_s^2 + M_s^{-2} + 2 > s + s^{-1} + 2$. It follows that $M_s^2 > s > 1$, and hence $L_s > 1$. □

Lemma 4.2. *One has* $\lim_{s\to 1^+}(\log L_s/\log M_s) = 0$ *and* $\lim_{s\to\infty}(\log L_s/\log M_s) = 4m$.

Proof. Let $s \to \infty$. Since $x^2 \in (y + 2 + \delta_1/((y-2)S_{m-1}^2(y)),\ y + 2 + \delta_2/((y-2)S_{m-1}^2(y)))$, we have $x^2 - (y+2) \to 0$, or equivalently $(M_s^2 - s)(1 - 1/(sM_s^2)) \to 0$. It follows that $M^2 - s \to 0$, and

$$L - s^{2m} = \frac{M^2 - s - s^{2m} + M^2 s^{1+2m}}{-1 + M^2 s + M^2 s^{2m} - s^{1+2m}} - s^{2m} \to 0.$$

Hence $\lim_{s\to\infty}(\log L_s/\log M_s) = 4m$.

Let $s \to 1^+$, $y \to 2^+$. Since $x^2 \in (y + 2 + \delta_1/((y-2)S_{m-1}^2(y)),\ y + 2 + \delta_2/((y-2)S_{m-1}^2(y)))$, we have $x^2 \to \infty$. It follows that $M_s \to \infty$ and

$$L_s = \frac{M^2 - s - s^{2m} + M^2 s^{1+2m}}{-1 + M^2 s + M^2 s^{2m} - s^{1+2m}} \to 1.$$

Hence $\lim_{s\to 1^+}(\log L_s/\log M_s) = 0$. □

Let $f_0 : (1,\infty) \to \mathbb{R}$ be the function defined by $f_0(s) = -\log L_s/\log M_s$. Lemmas 4.1 and 4.2 imply the following.

Proposition 4.3. *The image of f_0 contains the interval $(-4m, 0)$.*

5. Calculations: The case of $m = 1$.

Let $K = J(2, 2n)$. Recall from Proposition 2.3 and Lemma 3.2 that

$$\begin{aligned}\gamma_n(x,z) &= -(z+1)S_{n-1}^2(z) + S_{n-2}^2(z) + 2S_{n-1}(z)S_{n-2}(z)\\ &\quad + x^2 S_{n-1}(z)(S_{n-1}(z) - S_{n-2}(z))\\ y &= (z^2-2)S_{n-1}^2(z) + 2S_{n-2}^2(z) - 2zS_{n-1}(z)S_{n-2}(z) - x^2(z-2)S_{n-1}^2(z).\end{aligned}$$

Let $s \in \mathbb{C} \setminus \{-1, 0, 1\}$ and $z = s + s^{-1}$. Note that $S_j(z) = (s^{j+1} - s^{-j-1})/(s - s^{-1})$ for all integers j.

Lemma 5.1. *Suppose $(s^{2n}-1)(s^{2n-1}+1)s \neq 0$ and $x^2 = (2+s+s^{-1})((s^{4n-1}-1)/((s^{2n}-1)(s^{2n-1}+1)))$. Then $\gamma_n(x,z) = 0$ and $y - 1 = (s^{2n+1}+1)/(s^{2n}+s)$.*

Proof. Since $z = s + s^{-1}$, by direct calculations, we have

$$\begin{aligned}-(z+1)S_{n-1}^2(z) + S_{n-2}^2(z) + 2S_{n-1}(z)S_{n-2}(z) &= -\frac{s^{4n-1}-1}{s^{2n-1}(s-1)},\\ S_{n-1}(z)(S_{n-1}(z) - S_{n-2}(z)) &= \frac{(s^{2n-1}+1)(s^{2n}-1)}{s^{2n-2}(s-1)(s+1)^2}.\end{aligned}$$

By assumption, $x^2 = (2+s+s^{-1})((s^{4n-1}-1)/((s^{2n}-1)(s^{2n-1}+1)))$. It follows that $\gamma_n(x,z) = 0$.

Similarly, $y - 1 = (s^{2n+1}+1)/(s^{2n}+s)$ by direct calculations. $\square$

5.1. The case of $n > 0$.

Lemma 5.2. *On the real interval $(1, \infty)$, the equation $(2+s+s^{-1})((s^{4n-1}-1)/((s^{2n}-1)(s^{2n-1}+1))) = 4$ has a unique solution s_0.*

Proof. Suppose s is a real number > 1. Then the equation is equivalent to $((s^{2n}-1)(s^{2n-1}+1))/(s^{4n-1}-1) = (s+1)^2/(4s)$, i.e. $(s^{2n} - s^{2n-1})/(s^{4n-1}-1) = (s-1)^2/(4s)$, or equivalently $(s^{2n-1} - s^{-2n})(s-1) = 4$. The $LHS = (s^{2n-1} - s^{-2n})(s-1)$ is a strictly increasing function in $s > 1$. Hence the lemma follows since $\lim_{s\to 1^+} LHS = 0 < 4 < \infty = \lim_{s\to\infty} LHS$. $\square$

5.1.1. The case of $s > s_0$.

Suppose $s > s_0$. Since

$$(2+s+s^{-1})\frac{s^{4n-1}-1}{(s^{2n}-1)(s^{2n-1}+1)} > 4$$

by Lemma 5.2, there exists $x > 2$ such that $x^2 = (2+s+s^{-1})((s^{4n-1}-1)/((s^{2n}-1)(s^{2n-1}+1)))$. By Lemma 5.1, $\gamma_n(x,z) = 0$.

Choose $M_s > 1$ such that $x = M_s + M_s^{-1}$. Since $\gamma_n(x,z) = 0$, Proposition 2.3 implies that there exists a non-abelian representation $\rho_s : \pi_1(K) \to SL_2(\mathbb{R})$ satisfying

$$\rho_s(a) = A = \begin{bmatrix} M_s & 0 \\ 2-y & M_s^{-1} \end{bmatrix} \quad \text{and} \quad \rho_s(b) = B = \begin{bmatrix} M_s & 1 \\ 0 & M_s^{-1} \end{bmatrix}$$

where $y = \operatorname{tr} AB^{-1} = 1 + (s^{2n+1}+1)/(s^{2n}+s)$ by Lemmas 3.2 and 5.1.

By (3.1), we have $\lambda = \left[\begin{smallmatrix} L_s & * \\ 0 & L_s^{-1} \end{smallmatrix}\right]$ where $L_s = (1-(y-1)M_s^2)/(y-1-M_s^2)$.

LEMMA 5.3. *One has*

$$(2+s+s^{-1})\frac{s^{4n-1}-1}{(s^{2n}-1)(s^{2n-1}+1)} < \frac{s^{2n+1}+1}{s^{2n}+s} + \frac{s^{2n}+s}{s^{2n+1}+1} + 2. \tag{5.1}$$

PROOF. Since

$$LHS - RHS = \frac{-(s+1)^2(s^{2n}-s)}{(s^{2n+1}+1)(s^{2n}-1)} < 0,$$

the lemma follows. □

LEMMA 5.4. *One has $y-1 > M_s^2 > 1$. Hence $L_s < -1$.*

PROOF. We have $y-1 = (s^{2n+1}+1)/(s^{2n}+s) > 1$. The inequality (5.1) is equivalent to $M_s^2 + M_s^{-2} < y-1+1/(y-1)$. It follows that $y-1 > M_s^2 > 1$ and $L_s = (1-(y-1)M_s^2)/(y-1-M_s^2) < -1$. □

LEMMA 5.5. *One has* $\lim_{s\to\infty}(\log|L_s|/\log M_s^2) = 2n+1$.

PROOF. We have

$$M_s^2 + M_s^{-2} = x^2 - 2 = s + s^{-1} - (2+s+s^{-1})\frac{s^{2n-1}(s-1)}{(s^{2n}-1)(s^{2n-1}+1)}.$$

It follows that

$$\begin{aligned} M_s^2 = {} & \frac{1}{2}\left(s+s^{-1}-(2+s+s^{-1})\frac{s^{2n-1}(s-1)}{(s^{2n}-1)(s^{2n-1}+1)}\right) \\ & + \frac{1}{2}\sqrt{\left(s+s^{-1}-(2+s+s^{-1})\frac{s^{2n-1}(s-1)}{(s^{2n}-1)(s^{2n-1}+1)}\right)^2 - 4}. \end{aligned}$$

It is easy to show that

$$\lim_{s\to\infty}(s+s^{-1}-s^{2-2n}-s^{1-2n})^{-1}\left(s+s^{-1}-(2+s+s^{-1})\frac{s^{2n-1}(s-1)}{(s^{2n}-1)(s^{2n-1}+1)}\right) = 1,$$

$$\begin{aligned} & \lim_{s\to\infty}(s-s^{-1}-s^{2-2n}-s^{1-2n})^{-1} \\ & \qquad \cdot \sqrt{\left(s+s^{-1}-(2+s+s^{-1})\frac{s^{2n-1}(s-1)}{(s^{2n}-1)(s^{2n-1}+1)}\right)^2 - 4} = 1. \end{aligned}$$

Hence

$$\lim_{s\to\infty}(s - s^{2-2n} - s^{1-2n})^{-1}M_s^2 = 1 \quad \text{and} \quad \lim_{s\to\infty}\left(M_s^2 - \frac{s^{2n+1}+1}{s^{2n}+s}\right)\Big/ s^{1-2n} = -1.$$

Since

$$L_s = \left(\frac{s^{2n+1}+1}{s^{2n}+s}M_s^2 - 1\right)\Big/\left(M_s^2 - \frac{s^{2n+1}+1}{s^{2n}+s}\right),$$

we have $\lim_{s\to\infty} s^{-2n-1}L_s = -1$. The lemma follows. $\square$

Let $\omega > 1$ be the unique real solution of the equation $se^s = 4(2n-1)$ satisfying $s > 1$.

Lemma 5.6. *One has* $\lim_{s\to s_0^+}(\log|L_s|/\log M_s^2) < 2(2n-1)/\omega + 2$.

Proof. From the proof of Lemma 5.2, it follows that $s_0 > 1$ is the solution of $(s^{4n-1}-1)(s-1) = 4s^{2n}$, or equivalently $(s^{2n}-1)^2 = s(s^{2n-1}+1)^2$. Hence $(s_0^{2n}-1)/(s_0^{2n-1}+1) = \sqrt{s_0}$ and

$$\lim_{s\to s_0^+} y - 1 = \lim_{s\to s_0^+}\frac{s^{2n+1}+1}{s^{2n}+s} = \lim_{s\to s_0^+} 1 + \frac{(s-1)(s^{2n}-1)}{s(s^{2n-1}+1)} = 1 + \frac{s_0-1}{\sqrt{s_0}}.$$

Let $\gamma = 1 + (s_0-1)/\sqrt{s_0}$. By L'Hospital's rule, we have

$$\lim_{s\to s_0^+}\left(\frac{\log|L_s|}{\log M_s^2}\right) = \lim_{t=M_s^2\to 1^+}\frac{\log(\gamma t - 1) - \log(\gamma - t)}{\log t} = \frac{\gamma+1}{\gamma-1} = 1 + \frac{2}{\gamma-1}.$$

We claim that $s_0 > 1 + \omega/(2n-1)$. Indeed, assume that $s_0 \le 1 + \omega/(2n-1)$. Then

$$\begin{aligned} 4 &= \left(s_0^{2n-1} - s_0^{-2n}\right)(s_0-1) < s_0^{2n-1}(s_0-1) \\ &\le \left(1 + \frac{\omega}{2n-1}\right)^{2n-1}\frac{\omega}{2n-1} < e^{\omega}\frac{\omega}{2n-1} = 4, \end{aligned}$$

a contradiction. Hence $s_0 > 1 + \omega/(2n-1)$ and

$$\gamma - 1 = \frac{s_0-1}{\sqrt{s_0}} > \frac{\omega/(2n-1)}{\sqrt{1+\omega/(2n-1)}} = \frac{\omega}{\sqrt{(2n-1)(2n-1+\omega)}} > \frac{2\omega}{4n-2+\omega}.$$

Therefore $\lim_{s\to s_0^+}(\log|L_s|/\log M_s^2) = 1 + 2/(\gamma-1) < 1 + (4n-2+\omega)/\omega = 2(2n-1)/\omega + 2$. $\square$

Let $f_1 : (s_0, \infty) \to \mathbb{R}$ be the function defined by $f_1(s) = -\log|L_s|/\log M_s$. Lemmas 5.4, 5.5 and 5.6 imply the following.

Proposition 5.7. *The image of* f_1 *contains the interval* $(-(4n+2), -(4(2n-1)/\omega+4))$.

5.1.2. The case of $s = e^{2\theta i}$.
Then $z = 2\cos 2\theta$ and

$$(2+s+s^{-1})\frac{s^{4n-1}-1}{(s^{2n}-1)(s^{2n-1}+1)} = \frac{4\cos^2\theta\sin(4n-1)\theta}{2\sin(2n)\theta\cos(2n-1)\theta},$$
$$\frac{s^{2n+1}+1}{s^{2n}+s} = \frac{\cos(2n+1)\theta}{\cos(2n-1)\theta}.$$

Suppose $n > 1$. Consider $\pi/(2(2n-1)) < \theta < \pi/(2n)$.

Lemma 5.8. *One has*

$$\frac{4\cos^2\theta\sin(4n-1)\theta}{2\cos(2n-1)\theta\sin(2n)\theta} > \frac{\cos(2n-1)\theta}{\cos(2n+1)\theta} + \frac{\cos(2n+1)\theta}{\cos(2n-1)\theta} + 2. \tag{5.2}$$

Proof. We have

$$LHS - RHS = \frac{2\cos^2\theta}{\cos(2n-1)\theta}\left(\frac{\sin(4n-1)\theta}{\sin(2n\theta)} - \frac{2\cos^2(2n\theta)}{\cos(2n+1)\theta}\right)$$
$$= \frac{-2\cos^2\theta\sin\theta}{\sin(2n\theta)\cos(2n+1)\theta} > 0.$$

The lemma follows. □

We have $\cos(2n-1)\theta - \cos(2n+1)\theta = 2\sin\theta\sin(2n\theta) > 0$. It follows that $\cos(2n+1)\theta < \cos(2n-1)\theta < 0$ and $\cos(2n+1)\theta/\cos(2n-1)\theta > 1$. Lemma 5.8 implies that

$$\frac{4\cos^2\theta\sin(4n-1)\theta}{2\cos(2n-1)\theta\sin(2n)\theta} > \frac{\cos(2n-1)\theta}{\cos(2n+1)\theta} + \frac{\cos(2n+1)\theta}{\cos(2n-1)\theta} + 2 > 4.$$

Hence there exists $x > 2$ such that

$$x^2 = \frac{4\cos^2\theta\sin(4n-1)\theta}{2\sin(2n)\theta\cos(2n-1)\theta} = (2+s+s^{-1})\frac{s^{4n-1}-1}{(s^{2n}-1)(s^{2n-1}+1)}.$$

By Lemma 5.1, $\gamma_n(x,z) = 0$.

Choose $M_\theta > 1$ such that $x = M_\theta + M_\theta^{-1}$. Since $\gamma_n(x,z) = 0$, Proposition 2.3 implies that there exists a non-abelian representation $\rho_\theta : \pi_1(K) \to SL_2(\mathbb{R})$ satisfying

$$\rho_\theta(a) = A = \begin{bmatrix} M_\theta & 0 \\ 2-y & M_\theta^{-1} \end{bmatrix} \quad \text{and} \quad \rho_\theta(b) = B = \begin{bmatrix} M_\theta & 1 \\ 0 & M_\theta^{-1} \end{bmatrix}$$

where $y = \operatorname{tr} AB^{-1} = 1 + (s^{2n+1}+1)/(s^{2n}+s) = 1 + \cos(2n+1)\theta/\cos(2n-1)\theta$ by

Lemmas 3.2 and 5.1.

By (3.1), we have $\lambda = \left[\begin{smallmatrix} L_\theta & * \\ 0 & L_\theta^{-1} \end{smallmatrix}\right]$ where $L_\theta = (1-(y-1)M_\theta^2)/(y-1-M_\theta^2)$.

LEMMA 5.9. *One has* $M_\theta^2 > y-1 > 1$. *Hence* $L_\theta > 1$.

PROOF. We have $y-1 = \cos(2n+1)\theta/\cos(2n-1)\theta > 1$. The inequality (5.2) is equivalent to $M_\theta^2 + M_\theta^{-2} + 2 > y-1+(1/(y-1))+2$. It follows that $M_\theta^2 > y-1 > 1$ and $L_\theta = (1-(y-1)M_\theta^2)/(y-1-M_\theta^2) > 1$. □

LEMMA 5.10. *One has*

$$\lim_{\theta\to(\pi/(2(2n-1)))^+}\left(\frac{\log L_\theta}{\log M_\theta^2}\right) = 2 \quad \textit{and} \quad \lim_{\theta\to(\pi/(2n))^-}\left(\frac{\log L_\theta}{\log M_\theta^2}\right) = 0.$$

PROOF. For the first limit, let $\theta_1 = \pi/(2(2n-1))$. Since

$$\lim_{\theta\to\theta_1^+}\left(\frac{-2\cos^2\theta\sin\theta}{\sin(2n\theta)\cos(2n+1)\theta}\right) = \frac{-2\cos^2\theta_1\sin\theta_1}{\cos\theta_1(-\sin 2\theta_1)} = 1,$$

the proof of Lemma 5.9 implies that $\lim_{\theta\to\theta_1^+}(M_\theta^2 + M_\theta^{-2}) - (y-1+1/(y-1)) = 1$. Hence $\lim_{\theta\to\theta_1^+} M_\theta^2 - (y-1) = 1$ and

$$\lim_{\theta\to\theta_1^+}\left(\frac{\log L_\theta}{\log M_\theta^2}\right) = \lim_{\theta\to\theta_1^+}\frac{\log((y-1)M_\theta^2-1)-\log(M_\theta^2-(y-1))}{\log M_\theta^2} = 2.$$

The second limit is clear, since $M_\theta^2 \to \infty$ and $L_\theta \to 1$ as $\theta \to (\pi/(2n))^-$. □

Let $f_2 : (\pi/(2(2n-1)), \pi/(2n)) \to \mathbb{R}$ be the function defined by $f_2(\theta) = -\log L_\theta/\log M_\theta$. Lemmas 5.9 and 5.10 imply the following.

PROPOSITION 5.11. *The image of* f_2 *contains the interval* $(-4,0)$.

5.2. The case of $n < 0$.

Let $l = -n > 0$. From Lemma 5.1, we have

LEMMA 5.12. *Suppose* $(s^{2l+1}+1)(s^{2l}-1)s \neq 0$ *and* $x^2 = (2+s+s^{-1}) \cdot((s^{4l+1}-1)/((s^{2l+1}+1)(s^{2l}-1)))$. *Then* $\gamma_n(x,z) = 0$ *and* $y-1 = (s^{2l}+s)/(s^{2l+1}+1)$.

5.2.1. The case of $s > 1$.

Suppose $s > 1$. Since

$$(2+s+s^{-1})\frac{s^{4l+1}-1}{(s^{2l+1}+1)(s^{2l}-1)} = (2+s+s^{-1})\left(1+\frac{s^{2l}(s-1)}{(s^{2l+1}+1)(s^{2l}-1)}\right) > 4,$$

there exists $x > 2$ such that $x^2 = (2+s+s^{-1})((s^{4l+1}-1)/((s^{2l+1}+1)(s^{2l}-1)))$. By Lemma 5.12, $\gamma_n(x,z) = 0$.

Choose $M_s > 1$ such that $x = M_s + M_s^{-1}$. Since $\gamma_n(x,z) = 0$, Proposition 2.3 implies that there exists a non-abelian representation $\rho_s : \pi_1(K) \to SL_2(\mathbb{R})$ satisfying

$$\rho_s(a) = A = \begin{bmatrix} M_s & 0 \\ 2-y & M_s^{-1} \end{bmatrix} \quad \text{and} \quad \rho_s(b) = B = \begin{bmatrix} M_s & 1 \\ 0 & M_s^{-1} \end{bmatrix}$$

where $y = \operatorname{tr} AB^{-1} = 1 + (s^{2l}+s)/(s^{2l+1}+1)$ by Lemmas 3.2 and 5.12.

By (3.1), we have $\lambda = \left[\begin{smallmatrix} L_s & * \\ 0 & L_s^{-1} \end{smallmatrix}\right]$ where

$$L_s = \frac{1-(y-1)M_s^2}{y-1-M_s^2} = \left(\frac{s^{2l}+s}{s^{2l+1}+1}M_s^2 - 1\right) \Big/ \left(M_s^2 - \frac{s^{2l}+s}{s^{2l+1}+1}\right).$$

Lemma 5.13. *One has $M_s^2 > s$. Hence $0 < L_s < 1$.*

Proof. We have

$$M_s^2 + M_s^{-2} = x^2 - 2 = s + s^{-1} + (2 + s + s^{-1})\frac{s^{2l}(s-1)}{(s^{2l+1}+1)(s^{2l}-1)}.$$

It follows that

$$\begin{aligned} M_s^2 &= \frac{1}{2}\left(s + s^{-1} + (2+s+s^{-1})\frac{s^{2l}(s-1)}{(s^{2l+1}+1)(s^{2l}-1)}\right) \\ &\quad + \frac{1}{2}\sqrt{\left(s + s^{-1} + (2+s+s^{-1})\frac{s^{2l}(s-1)}{(s^{2l+1}+1)(s^{2l}-1)}\right)^2 - 4} \\ &> \frac{1}{2}(s+s^{-1}) + \frac{1}{2}\sqrt{(s+s^{-1})^2 - 4} = s > 1. \end{aligned}$$

Since $M_s^2 > s > (s^{2l+1}+1)/(s^{2l}+s) > 1 > (s^{2l}+s)/(s^{2l+1}+1)$, we obtain $0 < L_s < 1$. □

The following lemma is easy to check.

Lemma 5.14. *One has $\lim_{s\to 1^+} M_s^2 = 1 + (1+\sqrt{4l+1})/(2l)$ and $\lim_{s\to 1^+} L_s = 1$.*

Lemma 5.15. *One has $\lim_{s\to\infty}(M_s^2/(s+s^{1-2l})) = 1$ and $\lim_{s\to\infty} s^{2l}L_s = 1$.*

Proof. It is easy to show that

$$\lim_{s\to\infty}(s+s^{-1}+s^{1-2l})^{-1}\left(s+s^{-1}+(2+s+s^{-1})\frac{s^{2l}(s-1)}{(s^{2l+1}+1)(s^{2l}-1)}\right) = 1,$$

$$\lim_{s\to\infty}(s-s^{-1}+s^{1-2l})^{-1}\sqrt{\left(s+s^{-1}+(2+s+s^{-1})\frac{s^{2l}(s-1)}{(s^{2l+1}+1)(s^{2l}-1)}\right)^2 - 4} = 1.$$

Hence

$$\lim_{s\to\infty}(s+s^{1-2l})^{-1}M_s^2 = 1 \quad \text{and} \quad \lim_{s\to\infty}\left(M_s^2 - \frac{s^{2l+1}+1}{s^{2l}+s}\right)\Big/ s^{2-2l} = 1.$$

Then, from

$$L_s = \left(\frac{s^{2l}+s}{s^{2l+1}+1}M_s^2 - 1\right)\Big/\left(M_s^2 - \frac{s^{2l}+s}{s^{2l+1}+1}\right)$$

we obtain $\lim_{s\to\infty} s^{2l}L_s = 1$. □

Let $f_3 : (1,\infty) \to \mathbb{R}$ be the function defined by $f_3(s) = -\log L_s/\log M_s$. Lemmas 5.13, 5.14 and 5.15 imply the following.

Proposition 5.16. *The image of f_3 contains the interval $(0, -4n)$.*

5.2.2. The case of $s = e^{2\theta i}$.

Suppose $s = e^{2\theta i}$. Then $z = s + s^{-1} = 2\cos 2\theta$. By direct calculations, we have

$$(2+s+s^{-1})\frac{s^{4l+1}-1}{(s^{2l+1}+1)(s^{2l}-1)} = \frac{4\cos^2\theta\sin(4l+1)\theta}{2\cos(2l+1)\theta\sin(2l)\theta},$$
$$\frac{s^{2l}+s}{s^{2l+1}+1} = \frac{\cos(2l-1)\theta}{\cos(2l+1)\theta}.$$

Let $\theta_2 = \pi/(2(2l+1))$. Consider $0 < \theta < \theta_2$.

Lemma 5.17. *One has*

$$\frac{4\cos^2\theta\sin(4l+1)\theta}{2\cos(2l+1)\theta\sin(2l)\theta} > \frac{\cos(2l-1)\theta}{\cos(2l+1)\theta} + \frac{\cos(2l+1)\theta}{\cos(2l-1)\theta} + 2. \tag{5.3}$$

Proof. We have

$$RHS = \frac{(\cos(2l-1)\theta + \cos(2l+1)\theta)^2}{\cos(2l-1)\theta\cos(2l+1)\theta} = \frac{4\cos^2\theta\cos^2(2l\theta)}{\cos(2l-1)\theta\cos(2l+1)\theta}.$$

It follows that

$$LHS - RHS = \frac{2\cos^2\theta}{\cos(2l+1)\theta}\left(\frac{\sin(4l+1)\theta}{\sin(2l\theta)} - \frac{2\cos^2(2l\theta)}{\cos(2l-1)\theta}\right)$$
$$= \frac{2\cos^2\theta\sin\theta}{\sin(2l\theta)\cos(2l-1)\theta} > 0.$$

The lemma follows. □

Since $0 < (2l-1)\theta < (2l+1)\theta < \pi/2$, we have $\cos(2l-1)\theta > \cos(2l+1)\theta > 0$. Lemma 5.17 implies that

$$\frac{4\cos^2\theta\sin(4l+1)\theta}{2\cos(2l+1)\theta\sin(2l)\theta} > \frac{\cos(2l-1)\theta}{\cos(2l+1)\theta} + \frac{\cos(2l+1)\theta}{\cos(2l-1)\theta} + 2 > 4.$$

Hence there exists $x > 2$ such that

$$x^2 = \frac{4\cos^2\theta\sin(4l+1)\theta}{2\cos(2l+1)\theta\sin(2l)\theta} = (2+s+s^{-1})\frac{s^{4l+1}-1}{(s^{2l+1}+1)(s^{2l}-1)}.$$

By Lemma 5.12, $\gamma_n(x,z) = 0$.

Choose $M_\theta > 1$ such that $x = M_\theta + M_\theta^{-1}$. Since $\gamma_n(x,z) = 0$, Proposition 2.3 implies that there exists a non-abelian representation $\rho_\theta : \pi_1(K) \to SL_2(\mathbb{R})$ satisfying

$$\rho_\theta(a) = A = \begin{bmatrix} M_\theta & 0 \\ 2-y & M_\theta^{-1} \end{bmatrix} \quad \text{and} \quad \rho_\theta(b) = B = \begin{bmatrix} M_\theta & 1 \\ 0 & M_\theta^{-1} \end{bmatrix}$$

where $y = \operatorname{tr} AB^{-1} = 1 + (s^{2l}+s)/(s^{2l+1}+1) = 1 + \cos(2l-1)\theta/\cos(2l+1)\theta$ by Lemmas 3.2 and 5.12.

By (3.1), we have $\lambda = \left[\begin{smallmatrix} L_\theta & * \\ 0 & L_\theta^{-1} \end{smallmatrix}\right]$ where $L_\theta = (1-(y-1)M_\theta^2)/(y-1-M_\theta^2)$.

LEMMA 5.18. *One has $M_\theta^2 > y-1 > 1$. Hence $L_\theta > 1$.*

PROOF. We have $y-1 = \cos(2l-1)\theta/\cos(2l+1)\theta > 1$. The inequality (5.3) is equivalent to $M_\theta^2 + M_\theta^{-2} + 2 > y-1+(1/(y-1))+2$. Hence $M_\theta^2 > y-1 > 1$ and $L_\theta = (1-(y-1)M_\theta^2)/(y-1-M_\theta^2) > 1$. □

LEMMA 5.19. *One has $\lim_{\theta\to\theta_2^-}(\log L_\theta/\log M_\theta^2) = 2$ and $\lim_{\theta\to 0^+}(\log L_\theta/\log M_\theta^2) = 0$.*

PROOF. For the first limit, we have

$$\lim_{\theta\to\theta_2^-}\frac{2\cos^2\theta\sin\theta}{\sin(2l\theta)\cos(2l-1)\theta} = \frac{2\cos^2\theta_2\sin\theta_2}{\cos\theta_2\sin 2\theta_2} = 1.$$

The proof of Lemma 5.17 then implies that $\lim_{\theta\to\theta_2^-}(M_\theta^2+M_\theta^{-2})-(y-1+1/(y-1)) = 1$. Hence $\lim_{\theta\to\theta_2^-} M_\theta^2-(y-1) = 1$ and

$$\begin{aligned}\lim_{\theta\to\theta_2^-}\left(\frac{\log L_\theta}{\log M_\theta^2}\right) &= \lim_{\theta\to\theta_2^-}\frac{\log((y-1)M_\theta^2-1)-\log(M_\theta^2-(y-1))}{\log M_\theta^2} \\ &= \lim_{t=M_\theta^2\to\infty}\frac{\log((t-1)t-1)}{\log t} = 2.\end{aligned}$$

The second limit follows from Lemma 5.14. □

Let $f_4 : (0,\pi/(2(2l+1))) \to \mathbb{R}$ be the function defined by $f_4(\theta) = -\log L_\theta/\log M_\theta$. Lemmas 5.18 and 5.19 imply the following.

PROPOSITION 5.20. *The image of f_4 contains the interval $(-4, 0)$.*

6. Proof of Theorem 1.

Let $X_{m,n}$ be the closure of S^3 minus a tubular neighborhood of the knot $J(2m, 2n)$. Here $m > 0$ and $|n| > 0$. Let μ and λ be the pair of the meridian and the canonical longitude of $J(2m, 2n)$ as defined in Section 3.

For $r \in \mathbb{Q}$, let $M_{m,n}(r)$ denote the resulting manifold by r-surgery on the hyperbolic knot $J(2m, 2n)$. For $r = 0$, $M_{m,n}(0)$ is irreducible and has positive first Betti number, so $\pi_1(M_{m,n}(0))$ is left-orderable.

LEMMA 6.1. *Suppose there are a continuous family of non-abelian representations $\rho_t : \pi_1(X_{m,n}) \to PSL_2(\mathbb{R})$, $t \in (t_0, t_1)$, and a continuous function $g : (t_0, t_1) \to \mathbb{R}$ such that the image of g contains some interval (r_0, r_1) and $g(t) = r \in \mathbb{Q}$ if and only if $\rho_t(\mu^p\lambda^q) = \pm I$ where $r = p/q$ is a reduced fraction. Then $M_{m,n}(r)$ has left-orderable fundamental group if $r \in \mathbb{Q} \cap (r_0, r_1)$.*

PROOF. The proof is similar to that of [**BGW**, Section 7] and [**HT2**, Section 7]. The crucial point here is that the knot $J(2m, 2n)$ has genus one.

Suppose $r = p/q$ is a reduced fraction in $\mathbb{Q} \cap (r_0, r_1)$. By assumption, there exists $t \in (t_0, t_1)$ such that $g(t) = r$ and $\rho_t(\mu^p\lambda^q) = \pm I$.

Let $\widetilde{SL_2}$ be the universal covering of $PSL_2(\mathbb{R})$ and $\varphi : \widetilde{SL_2} \to PSL_2(\mathbb{R})$ the covering map. It is known that there is an identification $\widetilde{SL_2} \cong \Delta \times \mathbb{R}$, where $\Delta = \{z \in \mathbb{C} : |z| = 1\}$, and $\ker\varphi = \{(0, j\pi) \mid j \in \mathbb{Z}\}$, see e.g. [**Kh**].

There is a lift of $\rho_t : \pi_1(X_{m,n}) \to PSL_2(\mathbb{R})$ to a homomorphism $\widetilde{\rho}_t : \pi_1(X_{m,n}) \to \widetilde{SL_2}$ since the obstruction to its existence is the Euler class $e(\rho_t) \in H^2(X_{m,n}; \mathbb{Z}) \cong 0$, see [**Gh**]. Since the knot $J(2m, 2n)$ has genus one, without loss of generality we can assume that $\widetilde{\rho}_t(\pi_1(\partial X_{m,n}))$ is contained in the subgroup $(-1, 1) \times \{0\}$ of $\widetilde{SL_2}$, by [**HT2**, Lemma 7.1]. Because $\rho_t(\mu^p\lambda^q) = \pm I$, we have $\varphi(\widetilde{\rho}_t(\mu^p\lambda^q)) = I$. This means that $\widetilde{\rho}_t(\mu^p\lambda^q)$ lies in $\ker\varphi = \{(0, j\pi) \mid j \in \mathbb{Z}\}$. Hence $\widetilde{\rho}_t(\mu^p\lambda^q) = (0, 0)$, the identity of $\widetilde{SL_2}$, and so $\widetilde{\rho}_t$ induces a homomorphism $\pi_1(M_{m,n}(r)) \to \widetilde{SL_2}$ with non-abelian image. Since $\widetilde{SL_2}$ is left-orderable [**Be**], any non-trivial subgroup of $\widetilde{SL_2}$ is left-orderable. Because $M_{m,n}(r)$ is irreducible [**HT**], $\pi_1(M_{m,n}(r))$ is left-orderable by [**BRW**, Theorem 1.1]. $\square$

We are ready to prove Theorem 1. Let $r = p/q$ be a reduced fraction. Suppose $\rho : \pi_1(X_{m,n}) \to PSL_2(\mathbb{R})$ is a representation such that

$$\rho(\mu) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix}$$

where $M, L \in \mathbb{R} \setminus \{0, \pm 1\}$. Since μ and λ commute, it is easy to see that $\rho(\mu^p\lambda^q) = \pm I$ if and only if $M^pL^q = \pm I$, or equivalently

$$-\frac{\log|L|}{\log|M|} = \frac{p}{q}.$$

We first consider $m = 1$. Propositions 5.7, 5.11, 5.16, 5.20 and Lemma 6.1 imply that $M_{m,n}(r)$ has left-orderable fundamental group if the slope r satisfies the condition

$$r \in \begin{cases} \left(-(4n+2), -\left(\dfrac{4(2n-1)}{\omega_n}+4\right)\right) \cup (-4, 0], & n \geq 2, \\ (-4, -4n), & n \leq -1. \end{cases}$$

(Note that $\pi_1(M_{m,n}(0))$ is left-orderable.) Since $\pi_1(M_{1,n}(-4))$ is left-orderable by [**Te**], Theorem 1 follows.

Suppose now $m \geq 2$. We consider the following cases.

Case 1: $n = 1$. Since $J(2m, 2) \cong J(2, 2m)$, $M_{m,1}(r)$ has left-orderable fundamental group if $r \in (-(4m+2), -(4(2m-1)/\omega_m + 4)) \cup [-4, 0]$.

Case 2: $n = -1$. Since $J(2m, -2) \cong J(-2, 2m)$ is the mirror image of $J(2, -2m)$, $M_{m,-1}(r)$ has left-orderable fundamental group if $r \in (-4m, 4]$.

Case 3: $|n| \geq 2$. Proposition 4.3 and Lemma 6.1 imply that $M_{m,n}(r)$ has left-orderable fundamental group if the slope r satisfies the condition $r \in (-4m, 0]$.

If $n \geq 2$, then since $J(2m, 2n) \cong J(2n, 2m)$, $M_{m,n}(r)$ also has left-orderable fundamental group if $r \in (-4n, 0]$. Hence we conclude that $M_{m,n}(r)$ has left-orderable fundamental group $r \in (-\max\{4m, 4n\}, 0]$.

If $n \leq -2$, then since $J(2m, 2n) \cong J(2n, 2m)$ is the mirror image of $J(-2n, -2m)$, $M_{m,n}(r)$ also has left-orderable fundamental group if $r \in [0, -4n)$. Hence we conclude that $M_{m,n}(r)$ has left-orderable fundamental group if $r \in (-4m, -4n)$.

This completes the proof of Theorem 1.

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