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Topological aspect of Wulff shapes

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Abstract. In this paper we investigate Wulff shapes in $\mathbb{R}^{n+1}$ ($n \geq 0$) from the topological viewpoint. A topological characterization of the limit of Wulff shapes and the dual Wulff shape of the given Wulff shape are provided. Moreover, we show that the given Wulff shape is never a polytope if its support function is of class C^1. Several characterizations of the given Wulff shape from the viewpoint of pedals are also provided. One of such characterizations may be regarded as a bridge between the mathematical aspect of crystals at equilibrium and the mathematical aspect of perspective projections.

1. Introduction.

In 1901 Wulff gave the simple geometric construction for the shape of a crystal at equilibrium ([**22**], see also [**16**], [**20**], [**21**]). In this paper, we study Wulff shapes, which are the sets obtained by Wulff's geometric construction, from the topological viewpoint.

We first review Wulff's construction. For any non-negative integer n we let S^n be the unit sphere in $\mathbb{R}^{n+1}$. Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function where $\mathbb{R}_+ = \{\lambda \in \mathbb{R} \mid \lambda > 0\}$. For any $\theta \in S^n \subset \mathbb{R}^{n+1}$ put

$$\Gamma_{\gamma,\theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\},$$

where the dot in the center stands for the scalar product of $x, \theta \in \mathbb{R}^{n+1}$. Then, the *Wulff shape* associated with the support function γ is the following set $\mathcal{W}_\gamma$:

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma,\theta}.$$

Wulff showed in [**22**] that for any crystal at equilibrium the shape of it can be constructed as the Wulff shape $\mathcal{W}_\gamma$ by an appropriate support function γ. It is clearly seen that any Wulff shape $\mathcal{W}_\gamma$ is compact, convex and the origin of $\mathbb{R}^{n+1}$ is contained in $\mathcal{W}_\gamma$ as an interior point. It is known that its converse, too, holds as follows (see page 573 of [**20**]).

Proposition 1.1. *Let W be a subset of $\mathbb{R}^{n+1}$. Then, there exists a parallel translation $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $T(W)$ is the Wulff shape associated with an appropriate support function if and only if W is compact, convex and has an interior point.*

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Key Words and Phrases. Wulff shape, spherical polar set, Maehara's lemma, central projection, pedal, Legendrian map.

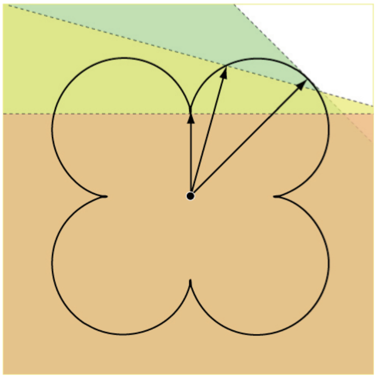

Figure 1. Wulff's construction.

In this paper, we first study dissolution of Wulff shapes. Let $\mathcal{H}(\mathbb{R}^{n+1})$ be the set of non-empty compact subsets of $\mathbb{R}^{n+1}$. Let $d_H : \mathcal{H}(\mathbb{R}^{n+1}) \times \mathcal{H}(\mathbb{R}^{n+1}) \to \mathbb{R}_+ \cup \{0\}$ be the Hausdorff metric (for the Hausdorff metric, see for instance [**4**], [**5**]). Then, it is well-known that the metric space $(\mathcal{H}(\mathbb{R}^{n+1}), d_H)$ is a complete metric space (for the complete metric space $(\mathcal{H}(\mathbb{R}^{n+1}), d_H)$, see for instance [**4**], [**5**]). Let $\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1})$ be the subset of $\mathcal{H}(\mathbb{R}^{n+1})$ consisting of non-empty compact convex subsets:

$$\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1}) = \left\{ W \in \mathcal{H}(\mathbb{R}^{n+1}) \mid W \text{ is convex} \right\}.$$

Any Wulff shape $\mathcal{W}_\gamma$ belongs to $\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1})$ since it is compact, convex and having an interior point. Any Cauchy sequence of Wulff shapes with respect to the Hausdorff metric converges in $\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1})$ since the following Lemma 1.1 holds.

Lemma 1.1. *The metric space $(\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1}), d_H)$ is complete.*

Proof of Lemma 1.1. Let $\{W_i\}_{i=1,2,\ldots} \subset \mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1})$ be a Cauchy sequence with respect to the Hausdorff metric d_H. Put

$$W = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \exists x_i \in W_i \ (i \in \mathbb{N}); \lim_{i\to\infty} x_i = x \right\}.$$

Then, it is known that $\{W_i\}_{i=1,2,\ldots}$ is convergent to W in $(\mathcal{H}(\mathbb{R}^{n+1}), d_H)$ (see for instance [**4**]). Thus, it is sufficient to show that W is convex.

Let x, y be two points of W and let $\{x_i \in W_i\}_{i=1,2,\ldots}$ (resp. $\{y_i \in W_i\}_{i=1,2,\ldots}$) be a sequence such that $\lim_{i\to\infty} x_i = x$ (resp. $\lim_{i\to\infty} y_i = y$). Then, since $W_i \in \mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1})$, it follows that $(1-t)x_i + ty_i \in W_i$ for any $t \in [0,1]$ and any $i \in \mathbb{N}$. On the other hand, it is clear that

$$(1-t)x + ty = \lim_{i\to\infty} ((1-t)x_i + ty_i)$$

for any $t \in [0,1]$. Thus, by definition of W, we have that $(1-t)x + ty \in W$ for any $t \in [0,1]$. Therefore, W is convex. $\square$

The zero dimensional Euclidean space $\mathbb{R}^0 = \{0\}$ itself may be regarded as the Wulff shape in $\mathbb{R}^0$ associated with a support function $S^{-1} \to \mathbb{R}_+$ where $S^{-1} = \{x \in \mathbb{R}^0 \mid \|x\| = 1\} = \emptyset$; since $\mathbb{R}^0$ is compact, convex and has an interior point. Then, we have the following:

THEOREM 1.1. *Let $\{\mathcal{W}_{\gamma_i}\}_{i=1,2,\dots}$ be a Cauchy sequence of Wulff shapes in $\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^{n+1})$ with respect to the Hausdorff metric d_H. Then, there exist an integer k $(0 \le k \le n+1)$, a rotation $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ around the origin of $\mathbb{R}^{n+1}$ and a parallel translation $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $T \circ R(\lim_{i\to\infty} \mathcal{W}_{\gamma_i})$ is a Wulff shape in $(\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^k \times \{(0,\dots,0)\}), d_H)$.*

Since the definitions of $\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^k \times \{(0,\dots,0)\})$ and Wulff shapes in it are clear, we omit to state them.

Secondly, we study the dual Wulff shape for the given Wulff shape $\mathcal{W}_\gamma$ of a given support function $\gamma : S^n \to \mathbb{R}_+$. Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. For any $\theta \in S^n$ put

$$\widetilde{\Gamma}_{\gamma,\theta} = \big\{(x,1) \in \mathbb{R}^{n+1} \times \{1\} \mid (x,1)\cdot(\theta,0) \le \gamma(\theta)\big\},$$

where the dot in the center stands for the scalar product of $(x,1), (\theta,0)$ of $\mathbb{R}^{n+2}$. Consider the following set:

$$\widetilde{\mathcal{W}}_\gamma = \bigcap_{\theta \in S^n} \widetilde{\Gamma}_{\gamma,\theta}.$$

It is clear that $\mathcal{W}_\gamma$ and $\widetilde{\mathcal{W}}_\gamma$ are congruent. Thus, $\widetilde{\mathcal{W}}_\gamma$ may be regarded as the Wulff shape. Our result is stated in terms of $\widetilde{\mathcal{W}}_\gamma$, the following spherical polar set X° of a set $X \subset S^{n+1}$ and the following central projection α_N. For any point P of S^{n+1}, we let $H(P)$ be the following set:

$$H(P) = \{Q \in S^{n+1} \mid P \cdot Q \ge 0\}.$$

Here, the dot in the center stands for the scalar product of $P, Q \in \mathbb{R}^{n+2}$.

DEFINITION 1.1. Let X be a subset of S^{n+1}. Then, the set

$$\bigcap_{P \in X} H(P)$$

is called the *spherical polar set* of X and is denoted by X°.

Let N be the point $(0,\dots,0,1) \in S^{n+1}$ where N stands for the north pole of S^n and let $S^{n+1}_{N,+}$ be the upper hemisphere $\{P \in S^{n+1} \mid N \cdot P > 0\}$. Thus, $S^{n+1}_{N,+} = S^{n+1} - H(-N)$. We let $\alpha_N : S^{n+1}_{N,+} \to \mathbb{R}^{n+1} \times \{1\}$ be the map defined by

$$\alpha_N(P_1, \ldots, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \ldots, \frac{P_{n+1}}{P_{n+2}}, 1\right)$$

for any $P = (P_1, \ldots, P_{n+2}) \in S^{n+1}_{N,+}$. The map α_N is called the *central projection* relative to N (see Figure 2).

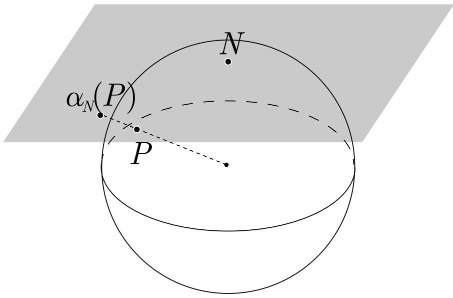

Figure 2. Central projection relative to N.

Definition 1.2. Let $\widetilde{X}$ be a subset of $\mathbb{R}^{n+1} \times \{1\}$. Then the following set is called the *convex hull* of $\widetilde{X}$ and is denoted by $\mathrm{conv}(\widetilde{X})$.

$$\mathrm{conv}(\widetilde{X}) = \left\{ \sum_{i=1}^{k} t_i(x_i, 1) \,\middle|\, (x_i, 1) \in \widetilde{X},\ \sum_{i=1}^{k} t_i = 1,\ t_i \geq 0, k \in \mathbb{N} \right\}.$$

In Definition 1.2 we may assume $k \leq n+2$ by Carathéodory's theorem (for Carathéodory's theorem, see for instance [**10**]).

Definition 1.3. Let $\{(x_1, 1), \ldots, (x_k, 1)\}$ be a finite subset of $\mathbb{R}^{n+1} \times \{1\}$. Suppose that $\mathrm{conv}(\{(x_1, 1), \ldots, (x_k, 1)\})$ has an interior point. Then, we call $\mathrm{conv}(\{(x_1, 1), \ldots, (x_k, 1)\})$ the *polytope* generated by $(x_1, 1), \ldots, (x_k, 1)$.

Theorem 1.2. *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Then, for the Wulff shape $\widetilde{\mathcal{W}}_\gamma \subset \mathbb{R}^{n+1} \times \{1\}$ the following hold*:

1. *The set $\alpha_N((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ)$ is the Wulff shape associated with an appropriate support function.*
2. *The given Wulff shape $\widetilde{\mathcal{W}}_\gamma$ is a polytope if and only if $\alpha_N((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ)$ is a polytope.*

By Theorem 1.2 it is reasonable to call the Wulff shape $\alpha_N((\alpha_N^{-1}(\mathcal{W}_\gamma))^\circ)$ the *dual Wulff shape* of $\mathcal{W}_\gamma$. In Section 5 it turns out that the dual Wulff shape of $\mathcal{W}_\gamma$ is exactly the convex hull of $1/\gamma$ polar plot. Thus, the dual Wulff shape of $\mathcal{W}_\gamma$ may be regarded as a generalization of Frank-Meijering construction (for details, see Section 5).

Thirdly, as an application of Theorem 1.2, we show the following:

Theorem 1.3. *Let $\gamma : S^n \to \mathbb{R}_+$ be a function of class C^1. Then the Wulff shape $\mathcal{W}_\gamma$ is never a polytope.*

This paper is organized as follows. In Section 2, we prepare several properties of spherical polar sets for proofs of Theorems 1.2 and 1.3. Theorems 1.1, 1.2 and 1.3 are proved in Sections 3, 4 and 5 respectively. Finally, in Section 6, we investigate Wulff shapes from the viewpoint of pedals.

2. Spherical polar sets.

In this section we investigate properties of spherical polar sets in S^{n+1}. The notion of spherical polar sets seems to be less common. Since Theorem 1.3 is proved by using spherical polar sets and Theorem 1.2 is stated in terms of spherical polar sets, we emphasize that the notion of spherical polar sets is significant.

It is clear that $X^\circ = \cap_{P\in X} H(P)$ is closed for any $X \subset S^{n+1}$.

LEMMA 2.1. *Let X, Y be subsets of S^{n+1}. Suppose that the inclusion $X \subset Y$ holds. Then, the inclusion $Y^\circ \subset X^\circ$ holds.*

PROOF OF LEMMA 2.1. Let Q be an element of Y°. Then, by definition we have that $P \cdot Q \geq 0$ for any $P \in Y$. Thus by the assumption we have that $\widetilde{P} \cdot Q \geq 0$ for any $\widetilde{P} \in X$ and therefore by definition Lemma 2.1 follows. □

LEMMA 2.2. *For any subset X of S^{n+1}, the inclusion $X \subset X^{\circ\circ}$ holds.*

PROOF OF LEMMA 2.2. For any point P of X the inclusion $X^\circ \subset \{P\}^\circ = H(P)$ holds by Lemma 2.1. Hence the inequality $P\cdot Q \geq 0$ holds for any $Q \in X^\circ$ by definition. Therefore, again by definition we have that $P \in X^{\circ\circ}$. □

DEFINITION 2.1. A subset $X \subset S^{n+1}$ is said to be *hemispherical* if there exists a point $P \in S^{n+1}$ such that $H(P) \cap X = \emptyset$.

DEFINITION 2.2. A hemispherical subset $X \subset S^{n+1}$ is said to be *spherical convex* if $PQ \subset X$ for any $P, Q \in X$.

Here, PQ stands for the following arc:

$$PQ = \left\{ \frac{(1-t)P + tQ}{\|(1-t)P + tQ\|} \in S^{n+1} \;\middle|\; 0 \leq t \leq 1 \right\}.$$

Note that $\|(1-t)P + tQ\| \neq 0$ for any $P, Q \in X$ and any $t \in [0,1]$ if $X \subset S^{n+1}$ is hemispherical. Note further that X° is spherical convex if X is hemispherical and has an interior point. However, in general, X° is not necessarily spherical convex even if X is hemispherical (for instance if $X = \{P\}$ then $X^\circ = H(P)$ is not spherical convex).

LEMMA 2.3. *Let $X_\lambda \subset S^{n+1}$ be a spherical convex subset for any $\lambda \in \Lambda$. Then, the intersection $\cap_{\lambda\in\Lambda} X_\lambda$ is spherical convex.*

PROOF OF LEMMA 2.3. Let P, Q be two points of $\cap_{\lambda\in\Lambda} X_\lambda$. Since P, Q belong to X_λ and X_λ is spherical convex for any $\lambda \in \Lambda$ we have that $PQ \subset X_\lambda$ for any $\lambda \in \Lambda$. Therefore $\cap_{\lambda\in\Lambda} X_\lambda$ contains PQ and thus it is spherical convex. □

Definition 2.3. Let X be a hemispherical subset of S^{n+1}. Then, the following set is called the *spherical convex hull of* X and is denoted by s-conv(X).

$$\text{s-conv}(X) = \left\{ \frac{\sum_{i=1}^k t_i P_i}{\| \sum_{i=1}^k t_i P_i \|} \,\middle|\, P_i \in X,\ \sum_{i=1}^k t_i = 1,\ t_i \geq 0, k \in \mathbb{N} \right\}.$$

It is clear that s-conv$(X) = X$ if X is spherical convex. More generally, we have the following:

Lemma 2.4. *For any hemispherical subset X, the spherical convex hull of X is the smallest spherical convex set containing X.*

Proof of Lemma 2.4. Let Y be a spherical convex set such that $X \subset Y$. Let $\sum_{i=1}^k t_i P_i / \| \sum_{i=1}^k t_i P_i \|$ be an element of s-conv(X). Then, since $P_i \in X \subset Y$ for any i $(1 \leq i \leq k)$ and Y is spherical convex, $P_{i_1} P_{i_2}$ is contained in Y for any i_1, i_2 $(1 \leq i_1, i_2 \leq k)$. Let t_{i_1}, t_{i_2} be two non-negative real numbers such that $t_{i_1} + t_{i_2} = 1$. Then, since $(t_{i_1} P_{i_1} + t_{i_2} P_{i_2}) / \| t_{i_1} P_{i_1} + t_{i_2} P_{i_2} \|$ and P_{i_3} are contained in Y and Y is spherical convex, the set

$$\left\{ \frac{(1 - t_{i_3}) t_{i_1} P_{i_1} + (1 - t_{i_3}) t_{i_2} P_{i_2} + t_{i_3} P_{i_3}}{\| (1 - t_{i_3}) t_{i_1} P_{i_1} + (1 - t_{i_3}) t_{i_2} P_{i_2} + t_{i_3} P_{i_3} \|} \,\middle|\, 0 \leq t_{i_3} \leq 1 \right\}$$

is contained in Y. In this way, it is seen that the given point $\sum_{i=1}^k t_i P_i / \| \sum_{i=1}^k t_i P_i \|$ is contained in Y. $\square$

Definition 2.4. Let $\{P_1, \ldots, P_k\}$ be a hemispherical finite subset of S^{n+1}. Suppose that s-conv$(\{P_1, \ldots, P_k\})$ has an interior point. Then, we call s-conv$(\{P_1, \ldots, P_k\})$ the *spherical polytope* generated by $P_1, \ldots, P_k$.

Proposition 2.1. *For any closed hemispherical subset $X \subset S^{n+1}$, the following hold*:

1. *The equality* s-conv$(X) = (\text{s-conv}(X))^{\circ\circ}$ *holds.*[1]
2. *The set* s-conv(X) *is a spherical polytope if and only if* $(\text{s-conv}(X))^{\circ}$ *is a spherical polytope.*

Note that for any closed hemispherical subset $X \subset S^{n+1}$, s-conv(X), too, is closed and hemispherical. Note also that for any subset $X \subset S^{n+1}$, the inclusion $X \subset X^{\circ\circ}$ holds always by Lemma 2.2. However, even if X is closed and hemispherical, the inverse inclusion $X \supset X^{\circ\circ}$ does not hold in general as Figure 3 shows.

For the proof of Proposition 2.1, we need the following Maehara's lemma.

Lemma 2.5 (Maehara's lemma ([**9**])). *For any hemispherical finite subset X =*

[1] The assertion 1 of Proposition 2.1 has been already known (see [**7**]). However, since no proofs of this fact have been given in [**7**], we give a proof of the assertion 1 of Proposition 2.1 for the sake of readers' convenience.

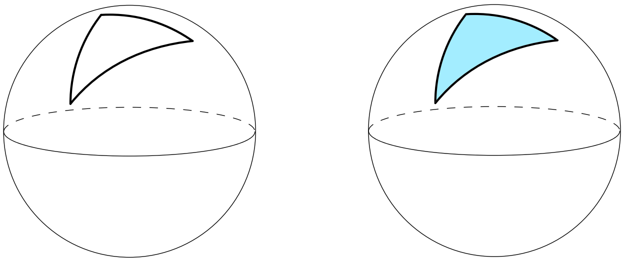

Figure 3. Left:X. Right: $X^{\circ\circ}$.

$\{P_1, \ldots, P_k\} \subset S^{n+1}$, *the following holds*:

$$\left\{ \frac{\sum_{i=1}^k t_i P_i}{\| \sum_{i=1}^k t_i P_i \|} \;\middle|\; P_i \in X,\ \sum_{i=1}^k t_i = 1,\ t_i \geq 0 \right\}^\circ = H(P_1) \cap \cdots \cap H(P_k).$$

Since the reference [**9**] is written in Japanese, we give a proof of Lemma 2.5 here for the sake of readers' convenience.

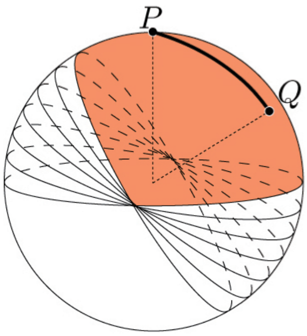

Figure 4. $(PQ)^\circ = H(P) \cap H(Q)$.

Proof of Lemma 2.5. Let Q be a point of S^{n+1}. Then, we see that the inequality $Q \cdot \left(\sum_{i=1}^k t_i P_i\right) \geq 0$ holds for any $t_1, \ldots, t_k$ such that $\sum_{i=1}^k t_i = 1, t_i \geq 0 \quad (1 \leq i \leq k)$ if and only if $Q \cdot P_i \geq 0$ for any i $(1 \leq i \leq k)$. Therefore, Lemma 2.5 follows. $\square$

Proof of the assertion 1 of Proposition 2.1. By Lemma 2.2, we have the inclusion s-conv$(X) \subset$ (s-conv$(X))^{\circ\circ}$. Conversely, suppose that there exists a point $P \in$ (s-conv$(X))^{\circ\circ}$ such that $P \notin$ s-conv(X). Since s-conv(X) is hemispherical closed and $P \notin$ s-conv(X), there exists a point $Q \in S^{n+1}$ such that s-conv$(X) \subset H(Q)$ and $P \notin H(Q)$ by the separation theorem (for the separation theorem, see for instance [**10**]). Since s-conv$(X) \subset H(Q)$ we have that $Q \cdot R \geq 0$ for any $R \in$ s-conv(X), which implies that $Q \in$ (s-conv$(X))^\circ$. Hence and since $P \in$ (s-conv$(X))^{\circ\circ}$ we have that $P \cdot Q \geq 0$, which contradicts that $P \notin H(Q)$. $\square$

Proof of the assertion 2 of Proposition 2.1. Suppose that s-conv(X) is a spherical polytope. Let $F_1, \ldots, F_\ell$ be n-dimensional cells of s-conv(X) (that is, $F_1, \ldots, F_\ell$

are facets of s-conv(X)). Then, since s-conv(X) is a spherical polytope, we have that $\ell \geq n+2$. Let A_i be the point of S^{n+1} such that s-conv$(X) = H(A_1) \cap \cdots \cap H(A_\ell)$. By Maehara's lemma (Lemma 2.5) we have that $(\text{s-conv}(\{A_1, \ldots, A_\ell\}))^\circ = H(A_1) \cap \cdots \cap H(A_\ell)$. Thus, by the assertion 1 of Proposition 2.1, we have that $(\text{s-conv}(X))^\circ =$ s-conv$(\{A_1, \ldots, A_\ell\})$. On the other hand, since s-conv(X) has an interior point, it follows that there exists a subset $\{i_1, \ldots, i_{n+2}\} \subset \{1, \ldots, \ell\}$ such that $A_{i_1}, \ldots, A_{i_{n+2}}$ are linearly independent. Hence, s-conv$(\{A_1, \ldots, A_\ell\})$ has an interior point. Therefore, $(\text{s-conv}(X))^\circ$ is a spherical polytope.

Conversely, suppose that $(\text{s-conv}(X))^\circ$ is a spherical polytope. Then, by the argument so far, $(\text{s-conv}(X))^{\circ\circ}$ is a spherical polytope. Therefore, by the assertion 1 of Proposition 2.1, s-conv(X) is a spherical polytope. $\square$

3. Proof of Theorem 1.1.

Since $\{\mathcal{W}_{\gamma_i}\}_{i=1,2,\ldots}$ is a Cauchy sequence in $(\mathcal{H}_{\text{conv}}(\mathbb{R}^{n+1}), d_H)$, $\lim_{i\to\infty} \mathcal{W}_{\gamma_i}$ exists in $\mathcal{H}_{\text{conv}}(\mathbb{R}^{n+1})$ by Lemma 1.1. Hence, $\lim_{i\to\infty} \mathcal{W}_{\gamma_i}$ is a non-empty, compact and convex subset of $\mathbb{R}^{n+1}$. Then, since $\lim_{i\to\infty} \mathcal{W}_{\gamma_i}$ is convex, there exists the unique integer k $(0 \leq k \leq n+1)$ and the unique k-dimensional linear subspace V^k of $\mathbb{R}^{n+1}$ such that $\lim_{i\to\infty} \mathcal{W}_{\gamma_i} \subset V^k$ and $\lim_{i\to\infty} \mathcal{W}_{\gamma_i}$ has an interior point in V^k. Therefore, there exist a rotation $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ around the origin of $\mathbb{R}^{n+1}$ such that $R(V^k) = \mathbb{R}^k \times \{(0, \ldots, 0)\}$ and $R(\lim_{i\to\infty} \mathcal{W}_{\gamma_i})$ is compact, convex and has an interior point in $R(V^k)$. Hence, by Proposition 1.1, there exists a parallel translation $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $T \circ R(\lim_{i\to\infty} \mathcal{W}_{\gamma_i})$ is the Wulff shape of an appropriate support function $\gamma : S^{k-1} \to \mathbb{R}_+$ in $\mathbb{R}^k \times \{(0, \ldots, 0)\}$. $\square$

4. Proof of Theorem 1.2.

As defined in Section 1, N is the point $(0, \ldots, 0, 1)$ of S^{n+1}, $S^{n+1}_{N,+}$ is the upper hemisphere $\{P \in S^{n+1} \mid N \cdot P > 0\}$ and $\alpha_N : S^{n+1}_{N,+} \to \mathbb{R}^{n+1} \times \{1\}$ is the central projection relative to N.

Lemma 4.1. 1. *For any spherical convex $X \subset S^{n+1}_{N,+}$, $\alpha_N(X)$ is convex.*
2. *For any convex $\widetilde{X} \subset \mathbb{R}^{n+1} \times \{1\}$, $\alpha_N^{-1}(\widetilde{X})$ is spherical convex.*

Proof of Lemma 4.1. Let P, Q be two points of $\alpha_N(X)$. Suppose that there exists $t \in [0,1]$ such that $(1-t)P + tQ \notin \alpha_N(X)$. Let ℓ be the linear line of $\mathbb{R}^{n+2}$ spanned by the $(n+2)$-dimensional vector $(1-t)P + tQ$. Since X is spherical convex, the intersection of ℓ and $S^{n+1}_{N,+}$ belongs to X. Thus, the point $(1-t)P + tQ$, which is the image of the intersection by α_N belongs to $\alpha_N(X)$. The contradiction shows that $\alpha_N(X)$ must be convex.

Next, let P, Q be two points of $\alpha_N^{-1}(\widetilde{X})$. Suppose that there exists $t \in [0,1]$ such that $((1-t)P + tQ)/\|(1-t)P + tQ\| \notin \alpha_N^{-1}(\widetilde{X})$. Let ℓ be the linear line of $\mathbb{R}^{n+2}$ spanned by the $(n+2)$-dimensional vector $(1-t)P + tQ$. Since $\widetilde{X}$ is convex, the intersection of ℓ and $\mathbb{R}^{n+1} \times \{1\}$ belongs to $\widetilde{X}$. Thus, the point $((1-t)P + tQ)/\|(1-t)P + tQ\|$, which is the inverse image of the intersection by α_N, belongs to $\alpha_N^{-1}(\widetilde{X})$. The contradiction shows that $\alpha_N^{-1}(\widetilde{X})$ must be spherical convex. □

Let the cylinder $\{(\theta,\rho) \mid \theta \in S^n, \rho \in \mathbb{R}\}$ be denoted by C_N and let $\beta_N : S^{n+1} - \{\pm N\} \to C_N$ be the map defined by

$$\beta_N(P) = \left(\frac{P_1}{\sqrt{P_1^2 + \cdots + P_{n+1}^2}}, \ldots, \frac{P_{n+2}}{\sqrt{P_1^2 + \cdots + P_{n+1}^2}} \right)$$

for any $P = (P_1, \ldots, P_{n+2}) \in S^{n+1} - \{\pm N\}$. The map β_N is called the *central cylindrical projection relative to* N (see Figure 5).

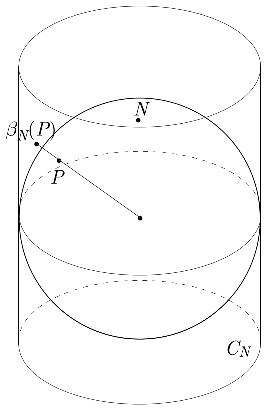

Figure 5. Central cylindrical projection relative to N.

Lemma 4.2. *Let $X \subset S^{n+1}$ be a closed and spherical convex subset. Suppose that $N = (0, \ldots, 0, 1) \in S^{n+1}$ is an interior point of X and $X \subset S^{n+1}_{N,+}$. Define the function $\gamma : S^n \to \mathbb{R}$ by*

$$\beta_N(X - \{N\}) \cap (\{-\theta\} \times \mathbb{R}) = \{-\theta\} \times [\gamma(\theta), \infty) \quad (\forall \theta \in S^n).$$

Then, γ is well-defined, continuous, $\gamma(\theta) > 0$ for any $\theta \in S^n$ and the following equality holds:

$$\widetilde{\mathcal{W}_\gamma} = \alpha_N(X^\circ).$$

Proof of Lemma 4.2. Put $\Pi_\theta = \mathbb{R}(\theta, 0) + \mathbb{R}N$ for any $\theta \in S^n$. Then, since X is closed and spherical convex and N is an interior point of X, for any $\theta \in S^n$ we have two points $P(\theta), P(-\theta) \in X$ such that $P(\theta) \cdot (\theta, 0) > 0$, $P(-\theta) \cdot (-\theta, 0) > 0$ and the intersection $X \cap \Pi_\theta$ is exactly the arc $P(\theta)P(-\theta)$.

Let $\{\theta_i\}_{i=1,2,\ldots}$ be a sequence of S^n satisfying

$$\lim_{i\to\infty} \theta_i = \theta_0 \quad \text{and} \quad \lim_{i\to\infty} P(\theta_i) = P_0.$$

Then, since X is closed, $P_0 \in X$. Thus, by the definition of $P(\theta_0)$, the scalar product $N \cdot P_0$ must be greater than or equal to $N \cdot P(\theta_0)$. Suppose that $N \cdot P_0 > N \cdot P(\theta_0)$. Then, by the definition of P_0, we may assume that there exists a sufficiently small $\varepsilon > 0$ such that $P(\theta_i) \notin D^{n+2}_{\varepsilon}(P(\theta_0))$ for any $i \in \mathbb{N}$, where $D^{n+2}_{\varepsilon}(P(\theta_0))$ is the $(n+2)$-dimensional disk with radius ε centered at $P(\theta_0)$. However, since X is spherical convex, the arc $P(\theta_i)P(\theta_0)$ belongs to X for any $i \in \mathbb{N}$. Thus, there must exist a point in $X \cap \Pi_{\theta_i} \cap D^{n+2}_{\varepsilon}(P(\theta_0))$ for any sufficiently large i. This contradicts the definition of $P(\theta_i)$ for any sufficiently large i. Hence, we have that $N \cdot P_0 = N \cdot P(\theta_0)$ which implies that the map $P : S^n \to S^{n+1}$ is continuous. Since N is an interior point of X, it is clearly seen that $P(\theta) \neq N$ for any $\theta \in S^n$. Furthermore, since $X \cap H(-N) = \emptyset$, it is trivial that $P(S^n) \cap H(-N) = \emptyset$. Since it is clear that $\beta_N : S^{n+1} - \{\pm N\} \to C_N$ is a C^∞ diffeomorphism and $\beta_N(P(-\theta)) = (-\theta, \gamma(\theta))$, $\gamma : S^n \to \mathbb{R}_+$ is a well-defined continuous function.

Let $\Psi_N : S^{n+1} - \{\pm N\} \to S^{n+1}$ be the map defined by

$$\Psi_N(P) = \frac{1}{\sqrt{1-(N \cdot P)^2}}(N - (N \cdot P)P).$$

The map Ψ_N, which has been introduced in [**12**] and has been used in [**12**], [**13**] for the study of singularities of spherical pedal curves, in [**14**] for the study of pedal unfoldings of pedal curves and in [**15**] for the study of hedgehogs (see also [**8**] where the hyperbolic version of Ψ_N has been introduced and studied), has the following interesting properties:

1. For any $P \in S^{n+1} - \{\pm N\}$, the equality $P \cdot \Psi_N(P) = 0$ holds,
2. for any $P \in S^{n+1} - \{\pm N\}$, the property $\Psi_N(P) \in \mathbb{R}N + \mathbb{R}P$ holds,
3. for any $P \in S^{n+1} - \{\pm N\}$, the property $N \cdot \Psi_N(P) > 0$ holds,
4. the restriction $\Psi_N|_{S^{n+1}_{N,+} - \{N\}} : S^{n+1}_{N,+} - \{N\} \to S^{n+1}_{N,+} - \{N\}$ is a C^∞ diffeomorphism.

By the property 3, $\alpha_N \circ \Psi_N \circ P(\theta)$ is well-defined for any $\theta \in S^n$. Properties 1 and 2 yield the following by elementary geometry:

$$\gamma(\theta) = (\alpha_N \circ \Psi_N \circ P(-\theta)) \cdot (\theta, 0) \quad (\forall \theta \in S^n). \tag{a}$$

By using of Maehara's lemma (Lemma 2.5) and the equality (a), we have the following:

$$\begin{aligned}
&(x,1) \in \alpha_N(X^\circ) \\
&\quad \Leftrightarrow \alpha_N^{-1}(x,1) \in X^\circ \\
&\quad \Leftrightarrow \alpha_N^{-1}(x,1) \cdot P \geq 0 \quad (\forall P \in X) \\
&\quad \Leftrightarrow \alpha_N^{-1}(x,1) \cdot P(-\theta) \geq 0 \quad (\forall \theta \in S^n) \\
&\quad \Leftrightarrow (x,1) \in \Gamma_{\gamma,\theta} \quad (\forall \theta \in S^n).
\end{aligned}$$

Here, the equivalence of the third line and the fourth line (resp., the fourth line and the fifth line) is obtained by Maehara's lemma (resp., the above equality (a)). Therefore, the following holds:

$$\alpha_N(X^\circ) = \bigcap_{\theta \in S^n} \Gamma_{\gamma,\theta} = \widetilde{\mathcal{W}}_\gamma. \qquad \square$$

Definition 4.1. Let $\{(p_1, 1), \dots, (p_k, 1)\}$ be a subset of $\mathbb{R}^{n+1} \times \{1\}$. Suppose that the convex hull of $\{(p_1, 1), \dots, (p_k, 1)\}$ has an interior point. Then, the convex hull of $\{(p_1, 1), \dots, (p_k, 1)\}$ is called the *polytope* generated by $(p_1, 1), \dots, (p_k, 1)$.

Lemma 4.3. 1. *Let $X \subset S^{n+1}_{N,+}$ be the spherical polytope generated by $P_1, \dots, P_k$. Then, $\alpha_N(X)$ is the polytope generated by $\alpha_N(P_1), \dots, \alpha_N(P_k)$.*

2. *Let $\widetilde{X} \subset \mathbb{R}^{n+1} \times \{1\}$ be the polytope generated by $(p_1, 1), \dots, (p_k, 1)$. Then, $\alpha_N^{-1}(\widetilde{X})$ is the spherical polytope generated by $\alpha_N^{-1}((p_1, 1)), \dots, \alpha_N^{-1}((p_k, 1))$.*

Proof of Lemma 4.3. Since α_N is a C^∞ diffeomorphism, $\alpha_N(X)$ has an interior point if X has an interior point and $\alpha_N^{-1}(\widetilde{X})$ has an interior point if $\widetilde{X}$ has an interior point. Hence, Lemma 4.3 follows. $\square$

Proof of the assertion 1 of Theorem 1.2. We put $C = \beta_N((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma)) \setminus \{N\})$. Then, since $\widetilde{\mathcal{W}}_\gamma$ is compact, C is a closed subset of $S^n \times \mathbb{R}$ and $C \cap (S^n \times \{0\}) = \emptyset$. Let $\widetilde{\gamma} : S^n \to \mathbb{R}$ be the function defined by $C \cap (\{-\theta\} \times \mathbb{R}) = \{-\theta\} \times [\widetilde{\gamma}(\theta), \infty)$ for any $\theta \in S^n$. Then, as in the proof of Lemma 4.2, $\widetilde{\gamma}(\theta) > 0$ holds for any $\theta \in S^n$ and $\widetilde{\gamma}$ is continuous. Thus, by Proposition 2.1 and Lemma 4.2, we have that $\alpha_N((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ) = \widetilde{\mathcal{W}}_{\widetilde{\gamma}}$. $\square$

Proof of the assertion 2 of Theorem 1.2. Suppose that $\widetilde{\mathcal{W}}_\gamma$ is a polytope. Then, by Lemma 4.3, $\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma)$ is a spherical polytope. Thus, $(\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ$ is a spherical polytope by Proposition 2.1. Hence, $\alpha_N((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ)$ is a polytope by Lemma 4.3.

Conversely, suppose that $\alpha_N((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ)$ is a polytope. Then, by Lemma 4.3, the following set is a spherical polytope:

$$\alpha_N^{-1}\big(\alpha_N\big((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ\big)\big) = \big(\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma)\big)^\circ.$$

Thus, the following set is a spherical polytope by Proposition 2.1.

$$\big(\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma)\big)^{\circ\circ} = \big(\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma)\big).$$

Hence, $\alpha_N(\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma)) = \widetilde{\mathcal{W}}_\gamma$ is a polytope by Lemma 4.3. $\square$

5. Proof of Theorem 1.3.

Let $\widetilde{f}_\gamma : S^n \to \mathbb{R}^{n+1} \times \{1\} - \{N\}$ be the C^1 embedding defined by $\widetilde{f}_\gamma(\theta) = (\theta, \gamma(\theta), 1)$, where N is the point $(0, \dots, 0, 1) \in \mathbb{R}^{n+1} \times \{1\}$ and $(\theta, \gamma(\theta), 1)$ is the polar coordinate expression of the point of $\mathbb{R}^{n+1} \times \{1\} - \{N\}$. Put $f_\gamma = \alpha_N^{-1} \circ \widetilde{f}_\gamma$. Then, $f_\gamma : S^n \to S^{n+1}$ is a C^1 embedding. Then, by Maehara's lemma, we have the following:

$$
\begin{aligned}
&Q \in (\Psi_N \circ f_\gamma(S^n))^\circ \\
&\quad \Leftrightarrow P \cdot Q \geq 0 \quad (\forall P \in \Psi_N \circ f_\gamma(S^n)) \\
&\quad \Leftrightarrow \frac{\sum_{i=1}^k t_i P_i}{\|\sum_{i=1}^k t_i P_i\|} \cdot Q \geq 0 \\
&\qquad\qquad (\forall P_i \in \Psi_N \circ f_\gamma(S^n),\ \forall t_i \geq 0 \text{ such that } \sum_{i=1}^k t_i = 1,\ \forall k \in \mathbb{N}),
\end{aligned}
$$

where $\Psi_N : S^{n+1} - \{\pm N\} \to S^{n+1}$ is the map defined in Section 4. Thus, the following holds:

$$(\Psi_N \circ f_\gamma(S^n))^\circ = (\text{s-conv}\,(\Psi_N \circ f_\gamma(S^n)))^\circ.$$

On the other hand, as in the proof of Lemma 4.2 the following holds:

$$\widetilde{\mathcal{W}}_\gamma = \alpha_N((\Psi_N \circ f_\gamma(S^n))^\circ).$$

Therefore, the following holds:

$$\widetilde{\mathcal{W}}_\gamma = \alpha_N((\text{s-conv}(\Psi_N \circ f_\gamma(S^n)))^\circ).$$

Hence, by Proposition 2.1 we have the following:

$$\alpha_N\big((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ\big) = \alpha_N(\text{s-conv}(\Psi_N \circ f_\gamma(S^n))).$$

Since γ is of class C^1 and the property 4 of Ψ_N in Section 4, the boundary of $\alpha_N(\text{s-conv}(\Psi_N \circ f_\gamma(S^n)))$ is a C^1 manifold (for instance, see [**19**], [**23**]). Hence, $\alpha_N\big((\alpha_N^{-1}(\widetilde{\mathcal{W}}_\gamma))^\circ\big)$ is not a polytope. Therefore, $\widetilde{\mathcal{W}}_\gamma$ is not a polytope by Theorem 1.2. $\square$

As a by-product of the above proof, we have the following:

Theorem 5.1. *Let $\gamma_1, \gamma_2 : S^n \to \mathbb{R}_+$ be two continuous functions. Furthermore, we let $\widetilde{f}_{\gamma_i} : S^n \to \mathbb{R}^{n+1} \times \{1\}$ be the topological embedding defined by $\widetilde{f}_{\gamma_i}(\theta) = (\theta, \gamma_i(\theta), 1)$ and let f_{γ_i} be the composition $\alpha_N^{-1} \circ \widetilde{f}_{\gamma_i}$ for any i $(i = 1, 2)$. Then, $\widetilde{\mathcal{W}}_{\gamma_1} = \widetilde{\mathcal{W}}_{\gamma_2}$ if and only if* $\text{s-conv}(\Psi_N \circ f_{\gamma_1}(S^n)) = \text{s-conv}(\Psi_N \circ f_{\gamma_2}(S^n))$.

Furthermore, we can characterize the dual Wulff shape of $\mathcal{W}_\gamma$ for a given continuous function $\gamma : S^n \to \mathbb{R}_+$ as follows. For any continuous function $\gamma : S^n \to \mathbb{R}_+$, let $\widetilde{f}_{(1/\gamma,-)} : S^n \to \mathbb{R}^{n+1} \times \{1\}$ be the map defined by $\widetilde{f}_{(1/\gamma,-)}(\theta) = (\theta, 1/\gamma(-\theta), 1)$ and put $f_{(1/\gamma,-)} = \alpha_N^{-1} \circ \widetilde{f}_{(1/\gamma,-)}$. The image of $\widetilde{f}_{(1/\gamma,-)}$ is called the $1/\gamma$ *polar plot*. Put

$$D^{n+1}(\widetilde{f}_{(1/\gamma,-)}) = \left\{(1-t)\left(\theta, \frac{1}{\gamma(-\theta)}, 1\right) + t\left(-\theta, \frac{1}{\gamma(\theta)}, 1\right) \,\middle|\, \theta \in S^n,\ 0 \leq t \leq 1\right\}.$$

Note that the boundary of $D^{n+1}(\widetilde{f}_{(1/\gamma,-)})$ is exactly the $1/\gamma$ polar plot and $D^{n+1}(\widetilde{f}_{(1/\gamma,-)})$ is not convex in general. Then, since $\alpha_N\big((\alpha_N^{-1}(\widetilde{\mathcal{W}_\gamma}))^\circ\big) = \alpha_N(\text{s-conv}(\Psi_N \circ f_\gamma(S^n)))$ and $\widetilde{f}_{(1/\gamma,-)}(\theta) = \alpha_N \circ \Psi_N \circ f_\gamma(-\theta)$, by Maehara's lemma and Theorem 1.2 we have the following:

THEOREM 5.2. *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Then, the following hold*:

1. *The Wulff shape $\mathcal{W}_\gamma$ is exactly $\alpha_N\big((f_{(1/\gamma,-)}(S^n))^\circ\big)$.*
2. *The dual Wulff shape $\alpha_N\big((\alpha_N^{-1}(\widetilde{\mathcal{W}_\gamma}))^\circ\big)$ is exactly the convex hull of the $1/\gamma$ polar plot.*
3. *Suppose that $D^{n+1}(\widetilde{f}_{(1/\gamma,-)})$ is a polytope. Then, $\mathcal{W}_\gamma$ is a polytope.*

By Theorem 5.2, the dual Wulff shape of $\mathcal{W}_\gamma$ may be regarded as a generalization of Frank-Meijering construction ([**6**], [**11**]).

6. Wulff shapes from the viewpoint of pedals.

Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function, $\widetilde{f}_\gamma : S^n \to \mathbb{R}^{n+1} \times \{1\}$ be the topological embedding defined by $\widetilde{f}_\gamma(\theta) = (\theta, \gamma(\theta), 1)$ and $f_\gamma : S^n \to S^{n+1}$ be the composition $\alpha_N^{-1} \circ \widetilde{f}_\gamma$ respectively. Then, as in Section 5, we have that

$$\widetilde{W}_\gamma = \alpha_N\big((\text{s-conv}(\Psi_N \circ f_\gamma(S^n)))^\circ\big),$$
$$\alpha_N\big(\big(\alpha_N^{-1}(\widetilde{W}_\gamma)\big)^\circ\big) = \alpha_N(\text{s-conv}(\Psi_N \circ f_\gamma(S^n))).$$

In this section, we investigate $\mathcal{W}_\gamma$ in the case that there exists a Legendrian map $\boldsymbol{r} : S^n \to S^{n+1}_{N,+}$ such that the spherical convex hull of the image of the dual of $\boldsymbol{r} : S^n \to S^{n+1}_{N,+}$ is exactly the spherical convex hull of $\Psi_N \circ f_\gamma(S^n)$. In this case, $\mathcal{W}_\gamma$ can be expressed in three ways.

DEFINITION 6.1.
1. A tangent oriented hyperplane field K on a $(2m+1)$-dimensional oriented C^∞ manifold M is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^m \neq 0$ at any point of M where α is a 1-form defining K locally.
2. For a $(2m+1)$-dimensional oriented C^∞ manifold M and a tangent oriented hyperplane field K on M, (M, K) is said to be a *contact manifold* if K is a non-degenerate hyperplane field.
3. A C^∞ submanifold of a contact manifold (M, K) is said to be *integral* if its tangent plane at every point belongs to K.
4. Integral manifolds of the greatest possible dimension are said to be *Legendrian* submanifolds of the contact manifold.
5. A C^∞ bundle $\pi : E^{2m+1} \to B^{m+1}$ is said to be *Legendrian* if its space E furnished with a contact structure and its fibers are Legendrian submanifolds. The projective cotangent bundle $(PT^*(M), K)$ furnished with the canonical contact structure is a Legendrian bundle.
6. Let $i : L \to PT^*(M)$ be a C^∞ embedding of a Legendrian submanifold L to the space of the projective cotangent bundle $(PT^*(M), K)$ of a C^∞ oriented manifold M

furnished with the canonical contact structure. Then, the composition $\pi \circ i$ is said to be a *Legendrian map*.

7. For a Legendrian map $\pi \circ i : L \to B$, its image $\pi \circ i(L)$ is said to be a *front*.

For details on these definitions, see for instance [**3**]. Note that any C^∞ immersion $S^n \to S^{n+1}$ is a Legendrian map. For a Legendrian map $\boldsymbol{r} : S^n \to S^{n+1}$, as in [**1**], [**17**], [**18**], [**12**], [**13**], [**15**], we can define the spherical dual of $\boldsymbol{r}$ as follows. For any $\theta \in S^n$ let $GH_{\boldsymbol{r}(\theta)}$ be the co-oriented great hypersphere tangent to $\boldsymbol{r}(S^n)$ at $\boldsymbol{r}(\theta)$. Let $\boldsymbol{n} : S^n \to S^{n+1}$ be the map which maps $\theta \in S^n$ to the unique point $\boldsymbol{n}(\theta)$ satisfying

$$\boldsymbol{n}(\theta) \cdot P = 0 \ (\forall P \in GH_{\boldsymbol{r}(\theta)}) \quad \text{and} \quad \boldsymbol{n}(\theta) \cdot N \geq 0.$$

The map $\boldsymbol{n} : S^n \to S^{n+1}$ is called the *dual of* $\boldsymbol{r}$. Note that $\boldsymbol{n}$ is also a Legendrian map and singularities of $\boldsymbol{n}$ belongs to the class of Legendrian singularities which are relatively well-investigated (for instance, see [**1**], [**2**], [**3**]).

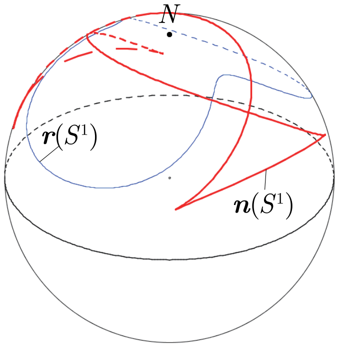

Figure 6. Images of $\boldsymbol{r}$ and its dual.

Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Hereafter until Theorem 6.3, we assume that there exists a Legendrian map $\boldsymbol{r}_\gamma : S^n \to S^{n+1}_{N,+}$ such that the following (b) is satisfied; where $\boldsymbol{n}_\gamma : S^n \to S^{n+1}$ is the dual of $\boldsymbol{r}_\gamma$, f_γ is given by $f_\gamma(\theta) = \alpha_N^{-1}(\theta, \gamma(\theta), 1)$ and $(\theta, \gamma(\theta), 1)$ is the polar coordinate expression of the point of $\mathbb{R}^{n+1} \times \{1\} - \{N\}$:

$$\text{s-conv}(\Psi_N \circ f_\gamma(S^n)) = \text{s-conv}(\boldsymbol{n}_\gamma(S^n)). \tag{b}$$

Our assumption is not strong, or rather, reasonable for studying Wulff shapes from the viewpoint of Legendrian singularity theory. Actually, we can show the following Theorem 6.1 which asserts that the condition (b) is equivalent to the following condition (c):

$$\widetilde{\mathcal{W}_\gamma} = \alpha_N\big((\boldsymbol{n}_\gamma(S^n))^\circ\big). \tag{c}$$

Theorem 6.1. *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function.*

1. *Suppose that there exists a Legendrian map $\boldsymbol{r}_\gamma : S^n \to S^{n+1}_{N,+}$ such that the condition (b) is satisfied, where $\boldsymbol{n}_\gamma : S^n \to S^{n+1}$ is the dual of $\boldsymbol{r}_\gamma$, f_γ is given by $f_\gamma(\theta) =$*

$\alpha_N^{-1}(\theta, \gamma(\theta), 1)$ and $(\theta, \gamma(\theta), 1)$ is the polar coordinate expression of the point of $\mathbb{R}^{n+1} \times \{1\} - \{N\}$. Then, the condition (c) is satisfied.

2. *Suppose that there exists a Legendrian map $\boldsymbol{r}_\gamma : S^n \to S^{n+1}_{N,+}$ such that the condition (c) is satisfied, where $\boldsymbol{n}_\gamma : S^n \to S^{n+1}$ is the dual of $\boldsymbol{r}_\gamma$. Then, the condition (b) is satisfied, where f_γ is given by $f_\gamma(\theta) = \alpha_N^{-1}(\theta, \gamma(\theta), 1)$ and $(\theta, \gamma(\theta), 1)$ is the polar coordinate expression of the point of $\mathbb{R}^{n+1} \times \{1\} - \{N\}$.*

Proof of the assertion 1 of Theorem 6.1. Note that the condition (b) implies that $\boldsymbol{n}_\gamma(S^n) \subset S^{n+1}_{N,+}$. In particular, we have that $N \notin \bigcup_{\theta \in S^n} GH_{\boldsymbol{r}_\gamma(\theta)}$. As in Section 5, the following holds:

$$\widetilde{\mathcal{W}_\gamma} = \alpha_N\big((\text{s-conv}(\Psi_N \circ f_\gamma(S^n)))^\circ\big).$$

On the other hand, the following holds by Maehara's lemma:

$$\alpha_N\big((\boldsymbol{n}_\gamma(S^n))^\circ\big) = \alpha_N\big((\text{s-conv}(\boldsymbol{n}_\gamma(S^n)))^\circ\big).$$

Therefore, the assertion 1 of Theorem 6.1 follows. $\square$

Proof of the assertion 2 of Theorem 6.1. Note that the condition (c) implies that $\boldsymbol{n}_\gamma(S^n) \subset S^{n+1}_{N,+}$. In particular, we have that $N \notin \bigcup_{\theta \in S^n} GH_{\boldsymbol{r}_\gamma(\theta)}$. As in Section 5, the following holds:

$$(\text{s-conv}(\Psi_N \circ f_\gamma(S^n)))^\circ = \alpha_N^{-1}\big(\widetilde{\mathcal{W}_\gamma}\big).$$

Thus, by the assertion 1 of Proposition 2.1, the following holds:

$$\text{s-conv}(\Psi_N \circ f_\gamma(S^n)) = \big(\alpha_N^{-1}(\widetilde{\mathcal{W}_\gamma})\big)^\circ.$$

On the other hand, the following holds by Maehara's lemma:

$$(\text{s-conv}(\boldsymbol{n}_\gamma(S^n)))^\circ = (\boldsymbol{n}_\gamma(S^n))^\circ.$$

Thus, again by the assertion 1 of Proposition 2.1, the following holds:

$$\text{s-conv}(\boldsymbol{n}_\gamma(S^n)) = (\boldsymbol{n}_\gamma(S^n))^{\circ\circ}.$$

Therefore, the assertion 2 of Theorem 6.1 follows. $\square$

Define the map $\textit{s-ped}_{\boldsymbol{r}_\gamma,N} : S^n \to S^{n+1}_{N,+}$ as $\textit{s-ped}_{\boldsymbol{r}_\gamma,N}(\theta)$ is the unique nearest point in $GH_{\boldsymbol{r}_\gamma(\theta)}$ from N. The map $\textit{s-ped}_{\boldsymbol{r}_\gamma,N}$ is called the *spherical pedal* relative to the *pedal point* N for $\boldsymbol{r}_\gamma$. Note that $\textit{s-ped}_{\boldsymbol{r}_\gamma,N}$ is well-defined since $\boldsymbol{r}_\gamma(S^n) \subset S^{n+1}_{N,+}$. It is easy to show that the spherical pedal relative to the pedal point N for $\boldsymbol{r}_\gamma$ can be characterized as follows (see [**12**]).

Lemma 6.1. *s-$ped_{\boldsymbol{r}_\gamma,N} = \Psi_N \circ \boldsymbol{n}_\gamma$.*

Put $\widetilde{\boldsymbol{r}}_\gamma = \alpha_N \circ \boldsymbol{r}_\gamma$. Note that $\widetilde{\boldsymbol{r}}_\gamma$ is a Legendrian map since $\boldsymbol{r}_\gamma$ is a Legendrian map and $\alpha_N : S^{n+1}_{N,+} \to \mathbb{R}^{n+1}\times\{1\}$ is a C^∞ diffeomorphism. For any $\theta \in S^n$ let $HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}$ be the hyperplane tangent to $\widetilde{\boldsymbol{r}}_\gamma(S^n)$ at $\widetilde{\boldsymbol{r}}_\gamma(\theta)$. Then, we have that $N \notin \bigcup_{\theta\in S^n} HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}$ since $N \notin \bigcup_{\theta\in S^n} GH_{\boldsymbol{r}_\gamma(\theta)}$. Define the map $ped_{\widetilde{\boldsymbol{r}}_\gamma,N} : S^n \to \mathbb{R}^{n+1}\times\{1\}$ as $ped_{\widetilde{\boldsymbol{r}}_\gamma,N}(\theta)$ is the unique nearest point in $HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}$ from N. The map $ped_{\widetilde{\boldsymbol{r}}_\gamma,N}$ is called the *pedal* relative to the *pedal point* N for $\widetilde{\boldsymbol{r}}_\gamma$. Then, since the nearest point in $GH_{\boldsymbol{r}_\gamma}(\theta)$ from N is mapped to the nearest point in $HP_{\widetilde{\boldsymbol{r}}_\gamma}(\theta)$ from N by the central projection α_N, the following clearly holds:

Lemma 6.2. *$ped_{\widetilde{\boldsymbol{r}}_\gamma,N} = \alpha_N \circ s$-$ped_{\boldsymbol{r}_\gamma,N}$.*

For the central cylindrical projection $\beta_N : S^{n+1} - \{\pm N\} \to C_N$, we put $\beta_N(P) = (\beta_{N,S^n}(P), \beta_{N,\mathbb{R}}(P))$ where $\beta_{N,S^n}(P) \in S^n$ and $\beta_{N,\mathbb{R}}(P)) \in \mathbb{R}$. Then, the following equality holds by elementary geometry:

$$ped_{\widetilde{\boldsymbol{r}}_\gamma,N}(\theta) = (-\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta)), \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta)), 1)$$

Here, $(-\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta)), \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta)), 1)$ is the polar coordinate expression of the point of $\mathbb{R}^{n+1}\times\{1\} - \{N\}$. Furthermore, put

$$\Delta_{ped,\theta} = \left\{(x,1) \in \mathbb{R}^{n+1}\times\{1\} \mid (x,1)\cdot(-\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta)),0) \le \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta))\right\}.$$

Note that the boundary of $\Delta_{ped,\theta}$ is exactly $HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}$. By Theorem 6.1, we have the following characterization of the Wulff shape associated with the support function γ by using the pedal relative to N for $\widetilde{\boldsymbol{r}}_\gamma$:

Theorem 6.2. *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Suppose that there exists a Legendrian map $\boldsymbol{r}_\gamma : S^n \to S^{n+1}$ such that* s-conv$(\Psi_N\circ f_\gamma(S^n))$ = s-conv$(\boldsymbol{n}_\gamma(S^n))$ *is satisfied; where $\boldsymbol{n}_\gamma : S^n \to S^{n+1}$ is the dual of $\boldsymbol{r}_\gamma$, f_γ is given by $f_\gamma(\theta) = \alpha_N^{-1}(\theta,\gamma(\theta),1)$ and $(\theta,\gamma(\theta),1)$ is the polar coordinate expression of the point of $\mathbb{R}^{n+1}\times\{1\}-\{N\}$. Then, the following holds:*

$$\widetilde{\mathcal{W}}_\gamma = \bigcap_{\theta\in S^n} \Delta_{ped,\theta}.$$

Moreover, we can show the following:

Theorem 6.3. *Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Suppose that there exists a Legendrian map $\boldsymbol{r}_\gamma : S^n \to S^{n+1}$ such that* s-conv$(\Psi_N\circ f_\gamma(S^n))$ = s-conv$(\boldsymbol{n}_\gamma(S^n))$ *is satisfied. Then, the following holds*:

$$\widetilde{\mathcal{W}}_\gamma = \overline{\mathbb{R}^{n+1}\times\{1\} - \bigcup_{\theta\in S^n} HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}}.$$

Here, $\boldsymbol{n}_\gamma : S^n \to S^{n+1}$ is the spherical dual of $\boldsymbol{r}_\gamma$, f_γ is given by $f_\gamma(\theta) = \alpha_N^{-1}(\theta, \gamma(\theta), 1)$ and $(\theta, \gamma(\theta), 1)$ is the polar coordinate expression of the point of $\mathbb{R}^{n+1} \times \{1\} - \{N\}$, $\widetilde{\boldsymbol{r}}_\gamma = \alpha_N \circ \boldsymbol{r}_\gamma$ and $\overline{X}$ stands for the topological closure of $X \subset \mathbb{R}^{n+1} \times \{1\}$.

Proof of Theorem 6.3. Let $(x, 1)$ be an element of $\mathbb{R}^{n+1} \times \{1\} - \bigcup_{\theta \in S^n} HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}$. Then, for any $\theta \in S^n$ the following holds:

$$(x, 1) \cdot (-\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta)), 0) \neq \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta)). \tag{d}$$

For the x suppose that there exists an element $\theta_0 \in S^n$ such that

$$(x, 1) \cdot (-\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta_0)), 0) > \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta_0)).$$

Then, since both $\beta_{N,S^n} : S^{n+1} - \{\pm N\} \to S^n$ and $\beta_{N,\mathbb{R}} : S^{n+1} - \{\pm N\} \to \mathbb{R}$ are continuous, for the x and any $\theta \in S^n$ the following (e) must hold by (d):

$$(x, 1) \cdot (-\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta)), 0) > \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta)). \tag{e}$$

On the other hand, by Theorem 6.2, we have that for any $\xi \in S^n$ there exist $\theta_1, \theta_2 \in S^n$ such that

$$\begin{aligned} \xi &= -\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta_1)), \\ -\xi &= -\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta_2)). \end{aligned}$$

Thus, by (e) we have the following:

$$\begin{aligned} (x, 1) \cdot (\xi, 0) &> \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta_1)) > 0, \\ -(x, 1) \cdot (\xi, 0) = (x, 1) \cdot (-\xi, 0) &> \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta_2)) > 0. \end{aligned}$$

By this contradiction we have that for any $(x, 1) \in \mathbb{R}^{n+1} \times \{1\} - \bigcup_{\theta \in S^n} HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}$ and any $\theta \in S^n$ the following holds:

$$(x, 1) \cdot (-\beta_{N,S^n}(\boldsymbol{n}_\gamma(\theta)), 0) < \beta_{N,\mathbb{R}}(\boldsymbol{n}_\gamma(\theta)).$$

Hence we have the inclusion $\widetilde{\mathcal{W}}_\gamma \supset \overline{\mathbb{R}^{n+1} \times \{1\} - \bigcup_{\theta \in S^n} HP_{\widetilde{\boldsymbol{r}}_\gamma(\theta)}}$. Since it is clear that the converse holds, Theorem 6.3 follows. $\square$

Theorem 6.3 may be regarded as a bridge between the mathematical aspect of crystals and the mathematical aspect of computer vision as follows.

Let $f(S^n) \subset \mathbb{R}^{n+1} \times \{1\}$ be a given front of a Legendrian map $f : S^n \to \mathbb{R}^{n+1} \times \{1\}$. For any point $p = (p_1, \ldots, p_{n+1})$ of $\mathbb{R}^{n+1}$, the parallel translation $T_p : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ defined by $T_p(x_1, \ldots, x_{n+2}) = (x_1 - p_1, \ldots, x_{n+1} - p_{n+1}, x_{n+2})$ maps the point $(p, 1) \in \mathbb{R}^{n+1} \times \{1\}$ to the point $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \times \{1\}$. Put $\widetilde{\boldsymbol{r}}_p = T_p \circ f$. Furthermore, put $E_N = \{P \in S^{n+1} \mid N \cdot P = 0\}$ and define the map $\pi_N : S^{n+1} - \{\pm N\} \to E_N$ as

$\pi_N(P)$ is the unique point $Q \in E_N$ such that $Q \in E_N \cap (\mathbb{R}N + \mathbb{R}P)$ and $P \cdot Q > 0$ for any $P \in S^{n+1}$. Then, we call the restricted map $\pi_N \circ \alpha_N^{-1} \circ T_p|_{f(S^n)} : f(S^n) \to E_N$ the *perspective projection* of $f(S^n)$ from the *perspective point* p ([**15**]). The perspective projection of the front $f(S^n)$ from the perspective point p is said to have *no silhouette* if $N \notin \cup_{\theta \in S^n} HP_{\widetilde{\boldsymbol{r}}_p(\theta)}$. Put $\boldsymbol{r}_p = \alpha_N^{-1} \circ \widetilde{\boldsymbol{r}}_p$ and let $\boldsymbol{n}_p : S^n \to S^{n+1}$ be the dual of $\boldsymbol{r}_p$. Put

$$\begin{aligned}\mathcal{NS}_f &= \left\{ (p,1) \in \mathbb{R}^{n+1} \times \{1\} \,\middle|\, N \notin \bigcup_{\theta \in S^n} HP_{\widetilde{\boldsymbol{r}}_p(\theta)} \right\} \\ &= \left\{ (p,1) \in \mathbb{R}^{n+1} \times \{1\} \,\middle|\, (p,1) \notin \bigcup_{\theta \in S^n} HP_{f(\theta)} \right\}.\end{aligned}$$

Here, $\mathcal{NS}$ stands for "No Silhouette". Figure 7 is a set of examples of $\mathcal{NS}_f$. In Figure 7, the thick curves are the given fronts and the blank region is $\mathcal{NS}_f$ for each front $f(S^1)$. The following Lemma 6.3 is known for $\mathcal{NS}_f$.

Lemma 6.3 ([**15**]).

$$\begin{aligned}&(p,1) \in \mathcal{NS}_f \\ &\qquad \Leftrightarrow \boldsymbol{n}_p(S^n) \subset S^{n+1} - E_N.\end{aligned}$$

Hence, by changing the given orientation of the canonical hyperplane field K of the projective cotangent bundle $PT^*(S^{n+1})$ if necessary, we may assume

$$\begin{aligned}&(p,1) \in \mathcal{NS}_f \\ &\qquad \Leftrightarrow \boldsymbol{n}_p(S^n) \subset S^{n+1}_{N,+}.\end{aligned}$$

Then, note that $N \notin \boldsymbol{n}_p(S^n)$ for any $p \in \mathbb{R}^{n+1}$ such that $(p,1) \in \mathcal{NS}_f$ since $\boldsymbol{r}_p \subset S^{n+1}_{N,+}$. Thus, for any $p \in \mathbb{R}^{n+1}$ such that $(p,1) \in \mathcal{NS}_f$, the function $\gamma_p : S^n \to \mathbb{R}_+$ given by $\beta_N(\text{s-conv}(\boldsymbol{n}_p(S^n))) \cap (\{-\theta\} \times \mathbb{R}) = \{-\theta\} \times [\gamma_p(\theta), \infty)$ $(\theta \in S^n)$ is well-defined. Therefore, by Theorem 6.3, we have the following equality if $\mathcal{NS}_f$ is not empty:

$$\overline{\mathcal{NS}_f} = T_{-p}(\mathcal{W}_{\gamma_p}) \quad \text{for any } p \in \mathcal{NS}_f.$$

Thus, $\overline{\mathcal{NS}_f}$ is an equilibrium form of crystal if $\mathcal{NS}_f$ is not empty. Perspective projections having no silhouette themselves seem to be meaningless because we can obtain no information about the object $f(S^n)$ by the perspective projections. However, such meaningless perspective points themselves, if exist, create the morphology $\overline{\mathcal{NS}_f}$.

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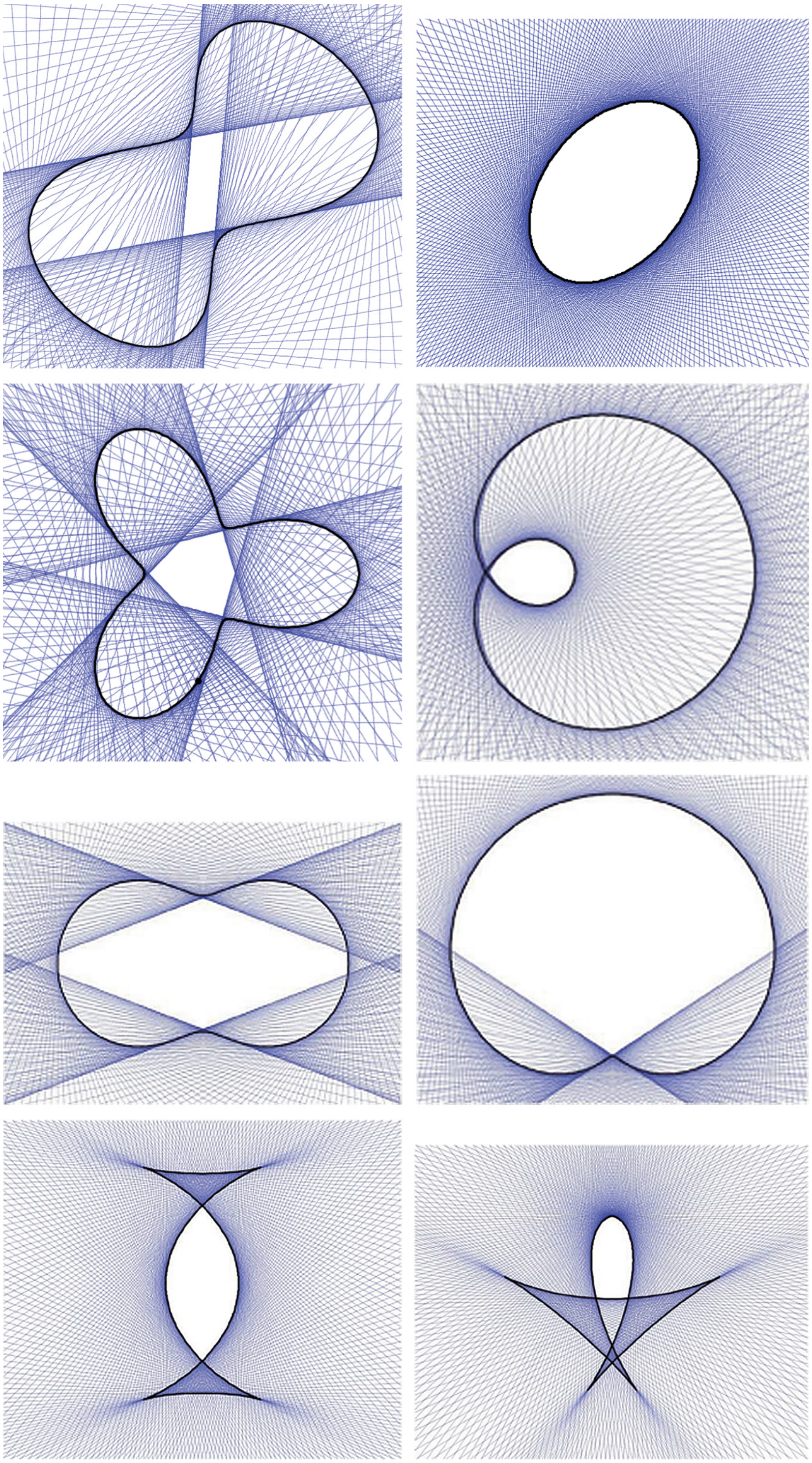

Figure 7. Various Wulff shapes constructed by tangent lines to fronts.

References

[1] V. I. Arnol'd, The geometry of spherical curves and the algebra of quaternions, Russian Math. Surveys, **50** (1995), 1–68.

[2] V. I. Arnol'd, V. V. Goryunov, O. V. Lyashko and V. A. Vasil'ev, Dynamical Systems. VIII, Encyclopaedia Math. Sci., **39**, Springer-Verlag, Berlin, 1993.

[3] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps. I, Monogr. Math., **82**, Birkhäuser, Boston Inc., Boston, MA, 1985.

[4] M. F. Barnsley, Fractals Everywhere. 2nd edition, Academic Press Professional, Boston, MA, 1993.

[5] K. Falconer, Fractal Geometry –Mathematical Foundations and Applications. 2nd edition, John Wiley & Sons, Inc., Hoboken, NJ, 2003.

[6] F. C. Frank, Metal Surfaces, ASM, Cleveland, OH, 1963.

[7] F. Gao, D. Hug and R. Schneider, Intrinsic volumes and polar sets in spherical space, Math. Notae, **41** (2001/02), 159–176 (2003).

[8] S. Izumiya and F. Tari, Projections of hypersurfaces in the hyperbolic space to hyperhorospheres and hyperplanes, Rev. Mat. Iberoam., **24** (2008), 895–920.

[9] H. Maehara, Geometry of Circles and Spheres, Asakura Publishing, 1998 (in Japanese).

[10] J. Matoušek, Lectures on Discrete Geometry, Grad. Texts in Math., **212**, Springer-Verlag, New York, 2002.

[11] J. L. Meijering, Usefulness of a $1/\gamma$ plot in the theory of thermal etching, Acta Metallurgica, **11** (1963), 847–849.

[12] T. Nishimura, Normal forms for singularities of pedal curves produced by non-singular dual curve germs in S^n, Geom. Dedicata, **133** (2008), 59–66.

[13] T. Nishimura, Singularities of pedal curves produced by singular dual curve germs in S^n, Demonstratio Math., **43** (2010), 447–459.

[14] T. Nishimura, Singularities of one-parameter pedal unfoldings of spherical pedal curves, J. Singul., **2** (2010), 160–169.

[15] T. Nishimura and Y. Sakemi, View from inside, Hokkaido Math. J., **40** (2011), 361–373.

[16] A. Pimpinelli and J. Villain, Physics of Crystal Growth, Monographs and Texts in Statistical Physics, Cambridge University Press, Cambridge, 1998.

[17] I. R. Porteous, Some remarks on duality in S^3, In: Geometry and Topology of Caustics–CAUSTICS '98, Warsaw, 1998, (eds. S. Janeczko and V. M. Zakalyukin), Banach Center Publ., **50**, Polish Acad. Sci., 1999, pp. 217–226.

[18] I. R. Porteous, Geometric Differentiation. second edition, Cambridge University Press, Cambridge, 2001.

[19] S. A. Robertson and M. C. Romero-Fuster, The convex hull of a hypersurface, Proc. London Math. Soc. (3), **50** (1985), 370–384.

[20] J. E. Taylor, Crystalline variational problems, Bull. Amer. Math. Soc., **84** (1978), 568–588.

[21] J. E. Taylor, J. W. Cahn and C. A. Handwerker, Geometric models of crystal growth, Acta Metallurgica et Materialia, **40** (1992), 1443–1474.

[22] G. Wulff, Zur frage der geschwindigkeit des wachstums und der auflösung der krystallflachen, Z. Kristallographine und Mineralogie, **34** (1901), 449–530.

[23] V. M. Zakalyukin, Singularities of convex hulls of smooth manifolds, Funct. Anal. Appl., **11** (1977), 225–227.

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