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Whittaker functions associated to newforms for $GL(n)$ over p-adic fields

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Abstract. Let F be a non-Archimedean local field of characteristic zero. Jacquet, Piatetski-Shapiro and Shalika introduced the notion of newforms for irreducible generic representations of $GL_n(F)$. In this paper, we give an explicit formula for Whittaker functions associated to newforms on the diagonal matrices in $GL_n(F)$.

1. Introduction.

Let F be a non-Archimedean local field of characteristic zero. Shintani [**12**] gave an explicit formula for spherical Whittaker functions of unramified principal series representations of $GL_n(F)$. His formula is a key to the unramified computation of Rankin-Selberg type zeta integrals (see for example [**2**] and [**3**]). In this paper, we extend Shintani's result to Whittaker functions associated to newforms for $GL_n(F)$.

Jacquet, Piatetski-Shapiro and Shalika [**7**] introduced the notion of newforms for irreducible generic representations of $GL_n(F)$, which is an extension of that for $GL_2(F)$ by Casselman [**4**]. Newforms for $GL_n(F)$ are defined by using a certain family of open compact subgroups $\{K_n\}_{n\geq 0}$. Given an irreducible generic representation π of $GL_n(F)$, the smallest integer $c(\pi)$ among those n such that π has K_n-fixed vectors is called the conductor of π. We say that a vector in π is a newform if it is fixed by $K_{c(\pi)}$. When the conductor of π is zero, its newforms are just $GL_n(\mathfrak{o})$-fixed vectors, where $\mathfrak{o}$ is the ring of integers in F. In this paper, we give an explicit formula for Whittaker functions associated to newforms on the diagonal matrices in $GL_n(F)$.

Originally, Shintani's explicit formula for spherical Whittaker functions is written in terms of Hecke eigenvalues. We will follow his method. For an irreducible generic representation π, the Hecke algebra associated to $K_{c(\pi)}$ acts on the space of its newforms. Since this space is one-dimensional, the actions of elements in this Hecke algebra are given by scalar multiplication. Suppose that the conductor of π is positive. Then, similar to the unramified case, we get a formula for the Whittaker function W associated to a newform on $T_1 = \{\mathrm{diag}(a_1, \ldots, a_{n-1}, 1) \mid a_i \in F^\times\}$ in terms of Hecke eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ (see Section 3 for precise definition). Kondo and Yasuda [**9**] showed the relation between these Hecke eigenvalues and the L-factor of π. We therefore obtain an explicit formula for W on T_1 in terms of the L-factor of π (Theorem 4.1).

As a corollary, we show that Whittaker functions associated to newforms attain L-

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factors when they are integrated on $GL_1(F)$ which is embedded into the upper left side in $GL_n(F)$ (Theorem 5.1).

We note that our formula determines Whittaker functions associated to newforms for π on $BK_{c(\pi)}$, where B denotes the upper triangular Borel subgroup of $GL_n(F)$. But the set $BK_{c(\pi)}$ is smaller than $GL_n(F)$ when the conductor of π is positive. However it seems that our formula is enough to compute several kinds of zeta integrals when all data in them are related to newforms.

Recently, Matringe [**11**] gave a constructive proof of the existence of newforms for generic representations π of $GL_n(F)$, by investigating derivatives of π. One can find the same formula for Whittaker functions associated to newforms in *loc. cit.*

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2. Local newforms.

In this section, we recall from [**7**] the notion and basic properties of local newforms for $GL(n)$. Let F be a non-Archimedean local field of characteristic zero, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal in $\mathfrak{o}$, and ϖ a generator of $\mathfrak{p}$. We write $|\cdot|$ for the absolute value of F normalized so that $|\varpi| = q^{-1}$, where q denotes the cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$. We fix a non-trivial additive character ψ of F whose conductor is $\mathfrak{o}$.

We set $G = GL_n(F)$. Let B denote the Borel subgroup of G consisting of the upper triangular matrices, and U its unipotent radical. We use the same letter ψ for the following character of U induced from ψ:

$$\psi(u) = \psi\biggl(\sum_{i=1}^{n-1} u_{i,i+1}\biggr), \quad \text{for } u = (u_{i,j}) \in U.$$

Let (π, V) be an irreducible generic representation of G. Then there exists a unique element l in $\mathrm{Hom}_U(\pi, \psi)$ up to constant. For $v \in V$, we define the Whittaker function associated to v by

$$W_v(g) = l(\pi(g)v), \quad g \in G.$$

We call $\mathcal{W}(\pi, \psi) = \{W_v \mid v \in V\}$ *the Whittaker model of* π with respect to ψ.

Put $K_0 = GL_n(\mathfrak{o})$. For each positive integer m, let K_m be the subgroup of K_0 consisting of the elements $k = (k_{ij})$ in K_0 which satisfy

$$(k_{n1}, k_{n2}, \dots, k_{nn}) \equiv (0, 0, \dots, 0, 1) \pmod{\mathfrak{p}^m}.$$

We write $V(m)$ for the space of K_m-fixed vectors in V. Due to [**7**, (5.1) Théorème (ii)], there exists a non-negative integer m such that $V(m) \neq \{0\}$. We denote by $c(\pi)$ the smallest integer with this property. We call $c(\pi)$ *the conductor of* π, and elements in $V(c(\pi))$ *newforms for* π. By [**7**, (5.1) Théorème (ii)], we have

$$\dim V(c(\pi)) = 1. \tag{2.1}$$

For simplicity, we say that an element W in $\mathcal{W}(\pi, \psi)$ is a newform if W is the Whittaker function associated to a newform for π. Suppose that W is a non-zero newform in $\mathcal{W}(\pi, \psi)$. Then by the existence of Kirillov model for π (see [**1**, Theorem 5.20]), there exists an element $g \in GL_{n-1}(F)$ such that

$$W\begin{pmatrix} g & \\ & 1 \end{pmatrix} \neq 0.$$

For any element $f = (f_1, \ldots, f_{n-1})$ in $\boldsymbol{Z}^{n-1}$, we set

$$\varpi^f = \mathrm{diag}(\varpi^{f_1}, \ldots, \varpi^{f_{n-1}}, 1) \in G.$$

By using the Iwasawa decomposition of $GL_{n-1}(F)$, we see that there exists $f \in \boldsymbol{Z}^{n-1}$ such that $W(\varpi^f) \neq 0$.

Proposition 2.2. *Let W be a newform in $\mathcal{W}(\pi, \psi)$. For $f \in \boldsymbol{Z}^{n-1}$, we have $W(\varpi^f) = 0$ unless $f_1 \geq \cdots \geq f_{n-1} \geq 0$.*

Proof. The proposition follows since W is $U \cap M_n(\mathfrak{o})$-invariant. □

3. Hecke operators.

Let (π, V) be an irreducible generic representation of G with conductor $c = c(\pi)$. Throughout this section, we suppose that c is positive. For each $g \in G$, we define the Hecke operator T_g on $V(c)$ by

$$T_g v = \frac{1}{\mathrm{vol}(K_c)} \int_{K_c g K_c} \pi(k) v dk = \sum_{k \in K_c / K_c \cap g K_c g^{-1}} \pi(kg) v,$$

for $v \in V(c)$. By (2.1), there exists a complex number λ_g such that $T_g = \lambda_g 1_{V(c)}$. We call λ_g the Hecke eigenvalue of T_g.

For $1 \leq i \leq n-1$, we denote by λ_i the Hecke eigenvalue of $T_i = T_{\varpi^{f^i}}$, where

$$f^i = (\overbrace{1, \ldots, 1}^{i}, 0, \ldots, 0) \in \boldsymbol{Z}^{n-1}.$$

To describe T_i, we give a complete system of representatives for $K_m / K_m \cap \varpi^{f^i} K_m \varpi^{-f^i}$ ($m > 0$, $1 \leq i \leq n-1$). We write $A \in M_n(F)$ as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in M_{n-1}(F)$, $A_{12} \in M_{n-1,1}(F)$, $A_{21} \in M_{1,n-1}(F)$ and $A_{22} \in F$. We embed the group $GL_{n-1}(F)$ into G by

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}, \quad g \in GL_{n-1}(F).$$

Then we may regard ϖ^f, $f \in \boldsymbol{Z}^{n-1}$ as an element in $GL_{n-1}(F)$. Set $H = GL_{n-1}(\mathfrak{o})$. For $0 \leq i \leq n-1$, we define an $\mathfrak{o}$-lattice L_i in $M_{n-1,1}(F)$ by

$$L_i = {}^t(\overbrace{\mathfrak{p} \oplus \cdots \oplus \mathfrak{p}}^{i} \oplus \mathfrak{o} \oplus \cdots \oplus \mathfrak{o}).$$

Then the groups H and $H \cap \varpi^{f^i} H \varpi^{-f^i}$ fix L_0 and L_i respectively.

Lemma 3.1. *Let m be a positive integer. For $1 \leq i \leq n-1$, we can take*

$$\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}, \quad a \in H/H \cap \varpi^{f^i} H \varpi^{-f^i}, \quad x \in L_0/aL_i$$

as a complete system of representatives for $K_m/K_m \cap \varpi^{f^i} K_m \varpi^{-f^i}$.

Proof. We use the block notation. An element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_m$ lies in $K_m \cap \varpi^{f^i} K_m \varpi^{-f^i}$ if and only if $a \in H \cap \varpi^{f^i} H \varpi^{-f^i}$ and $b \in L_i$. Thus, one can observe that the elements in the lemma belong to pairwise disjoint cosets in $K_m/K_m \cap \varpi^{f^i} K_m \varpi^{-f^i}$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_m$, we see that g is equivalent to $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ modulo $K_m \cap \varpi^{f^i} K_m \varpi^{-f^i}$. This completes the proof of the lemma. □

Let W be a newform in $\mathcal{W}(\pi, \psi)$. Set $w(f) = W(\varpi^f)$, for $f \in \boldsymbol{Z}^{n-1}$. Then we obtain the following lemma:

Lemma 3.2. *Suppose that $f \in \boldsymbol{Z}^{n-1}$ satisfies $f_1 \geq \cdots \geq f_{n-1} \geq 0$. Then we have*

$$q^{-i} \lambda_i w(f) = q^{i(n-1)-i(i-1)/2} \sum_{\varepsilon \in I_i} q^{-\sum_{j=1}^{n-1} \varepsilon_j j} w(f + \varepsilon), \tag{3.3}$$

where $I_i = \{\varepsilon \in \boldsymbol{Z}^{n-1} \mid \varepsilon_j \in \{0,1\},\ \sum_{j=1}^{n-1} \varepsilon_j = i\}$.

Proof. By Lemma 3.1, we get

$$\begin{aligned} \lambda_i W(\varpi^f) &= \sum_{k \in K_c/K_c \cap \varpi^{f^i} K_c \varpi^{-f^i}} W(\varpi^f k \varpi^{f^i}) \\ &= \sum_{a \in H/H \cap \varpi^{f^i} H \varpi^{-f^i},\ x \in L_0/aL_i} W\left(\varpi^f \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \varpi^{f^i}\right) \end{aligned}$$

$$= q^i \sum_{a \in H/H \cap \varpi^{f^i} H \varpi^{-f^i}} W\left(\varpi^f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \varpi^{f^i}\right).$$

In the last equality, we use the equation $[L_0 : aL_i] = [aL_0 : aL_i] = [L_0 : L_i] = q^i$. Now the proof is quite similar to that of the theorem in [**12**, p. 181] because $W|_{GL_{n-1}(F)}$ is $GL_{n-1}(\mathfrak{o})$-invariant. We note that in [**12**] one should take the set I_i as $I_i = \{\varepsilon \in \boldsymbol{Z}^n \mid \varepsilon_j \in \{0,1\},\ \sum_{j=1}^n \varepsilon_j = i\}$. $\square$

4. An explicit formula for Whittaker functions.

We prepare some notation to state our main theorem. For an irreducible generic representation π of G, let $L(s,\pi)$ denote its L-factor defined in [**5**]. It follows from [**6**, Section 3] that the degree of $L(s,\pi)$ is equal to or less than n. So we can write it as

$$L(s,\pi) = \prod_{i=1}^n (1 - \alpha_i q^{-s})^{-1}, \quad \alpha_i \in \boldsymbol{C}.$$

By [**6**, Section 3] again, $L(s,\pi)$ is of degree n if and only if π is unramified, that is, $c(\pi)$ equals to zero. We take α_n to be zero if the degree of $L(s,\pi)$ is less than n. Let $X = (X_1, \ldots, X_n)$ be n indeterminates. If $f \in \boldsymbol{Z}^n$ satisfies $f_1 \geq \cdots \geq f_n \geq 0$, then we denote by $s_f(X)$ the Schur polynomial in $X_1, \ldots, X_n$ associated to f, that is,

$$s_f(X) = \frac{|(X_j^{f_i+n-i})_{1 \leq i,j \leq n}|}{\prod_{1 \leq i < j \leq n}(X_i - X_j)}$$

(see [**10**, Chapter I, Section 3]). Since $s_f(X)$ is a symmetric polynomial in $X_1, \ldots, X_n$, the number $s_f(\alpha) = s_f(\alpha_1, \ldots, \alpha_n)$ is well-defined. We identify $(f_1, \ldots, f_{n-1}) \in \boldsymbol{Z}^{n-1}$ with $(f_1, \ldots, f_{n-1}, 0) \in \boldsymbol{Z}^n$. We note that if the conductor of π is positive, then we have $s_f(\alpha) = s_{(f_1,\ldots,f_{n-1},0)}(\alpha_1, \ldots, \alpha_{n-1}, 0) = s_{(f_1,\ldots,f_{n-1})}(\alpha_1, \ldots, \alpha_{n-1})$, for $f \in \boldsymbol{Z}^{n-1}$ such that $f_1 \geq \cdots \geq f_{n-1} \geq 0$.

We denote by δ_B the modulus character of B. We have $\delta_B(\varpi^f) = q^{-\sum_{j=1}^{n-1}(n+1-2j)f_j}$, for $f \in \boldsymbol{Z}^{n-1}$.

Theorem 4.1. *Let π be an irreducible generic representation of G and W its newform in $\mathcal{W}(\pi,\psi)$. For $f \in \boldsymbol{Z}^{n-1}$, we have*

$$W(\varpi^f) = \begin{cases} \delta_B^{1/2}(\varpi^f) s_f(\alpha) W(1), & \text{if } f_1 \geq \cdots \geq f_{n-1} \geq 0; \\ 0, & \text{otherwise}. \end{cases}$$

Proof. If $c(\pi) = 0$, then the theorem follows from [**12**]. So we may assume that π has positive conductor. For $f \in \boldsymbol{Z}^{n-1}$, we set

$$\widetilde{w}(f) = q^{\sum_{j=1}^{n-1}(n-1-j)f_j} W(\varpi^f).$$

By Proposition 2.2 and Lemma 3.2, the function $\widetilde{w}$ on $\boldsymbol{Z}^{n-1}$ satisfies the following system of difference equations:

$$\begin{cases} q^{i(i-1)/2-i}\lambda_i \widetilde{w}(f) = \sum_{\varepsilon\in I_i} \widetilde{w}(f+\varepsilon), & \text{if } f_1 \geq \cdots \geq f_{n-1} \geq 0; \\ \widetilde{w}(f) = 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

As in [**12**, p. 182], the solution of the above difference equations is unique and given by

$$\widetilde{w}(f) = \begin{cases} s_f(\mu)W(1), & \text{if } f_1 \geq \cdots \geq f_{n-1} \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu_1, \ldots, \mu_{n-1}$ are complex numbers whose i-th elementary symmetric polynomial equals to $q^{i(i-1)/2-i}\lambda_i$, for $1 \leq i \leq n-1$, and $\mu_n = 0$.

By [**9**, Theorem 3.5], we have

$$L(s,\pi) = \left(\sum_{i=0}^{n-1} (-1)^i \lambda_i q^{i(i-1)/2 - i((n-1)/2+s)} \right)^{-1}.$$

Hence we may assume $\mu_i = q^{(n-1)/2-1}\alpha_i$, for $1 \leq i \leq n$. Thus, if $f \in \boldsymbol{Z}^{n-1}$ satisfies $f_1 \geq \cdots \geq f_{n-1} \geq 0$, then we obtain

$$\begin{aligned} W(\varpi^f) &= q^{-\sum_{j=1}^{n-1}(n-1-j)f_j} s_f(\mu)W(1) \\ &= q^{-\sum_{j=1}^{n-1}(n-1-j)f_j + ((n-1)/2-1)\sum_{j=1}^{n-1} f_j} s_f(\alpha)W(1) \\ &= q^{-\sum_{j=1}^{n-1}((n+1)/2-j)f_j} s_f(\alpha)W(1) \\ &= \delta_B^{1/2}(\varpi^f) s_f(\alpha)W(1). \end{aligned}$$

This completes the proof. □

Remark 4.3. Set $D_1 = \{\varpi^f \mid f \in \boldsymbol{Z}^{n-1}\}$. Since the center Z of G acts on $\mathcal{W}(\pi,\psi)$ by the central character of π, Theorem 4.1 gives an explicit formula for newforms in $\mathcal{W}(\pi,\psi)$ on $BK_{c(\pi)} = UZD_1K_{c(\pi)}$.

Corollary 4.4. *Let π be an irreducible generic representation of G. Then we have $W(1) \neq 0$ for all non-zero newforms W in $\mathcal{W}(\pi,\psi)$.*

Proof. By Theorem 4.1, $W(1) = 0$ implies $W(\varpi^f) = 0$ for all $f \in \boldsymbol{Z}^{n-1}$. This contradicts the remark before Proposition 2.2. □

5. An application to zeta integral.

In this section, we give an integral representation of L-factors by using our formula for Whittaker functions associated to newforms. Let (π, V) be an irreducible generic representation of G. For $W \in \mathcal{W}(\pi, \psi)$, we set

$$Z(s, W) = \int_{F^\times} W(t(a))|a|^{s-(n-1)/2} d^\times a, \quad s \in \boldsymbol{C},$$

where $t(a) = \mathrm{diag}(a, 1, \dots, 1)$, for $a \in F^\times$. Here we normalize Haar measure $d^\times a$ on $F^\times$ so that $\int_{\mathfrak{o}^\times} d^\times a = 1$. The integral $Z(s, W)$ absolutely converges to a rational function in q^{-s} when the real part of s is sufficiently large. By [**8**, Theorem 2.7 (ii)], the set $\{Z(s, W) \mid W \in \mathcal{W}(\pi, \psi)\}$ coincides with the fractional ideal of $\boldsymbol{C}[q^{-s}, q^s]$ generated by $L(s, \pi)$. We shall show that $Z(s, W)$ attains $L(s, \pi)$ when W is a newform.

Theorem 5.1. *Let π be an irreducible generic representation of G and W the newform in $\mathcal{W}(\pi, \psi)$ such that $W(1) = 1$. Then we have*

$$Z(s, W) = L(s, \pi).$$

Proof. By Proposition 2.2 and Theorem 4.1, we obtain

$$\begin{aligned} Z(s, W) &= \sum_{k=0}^{\infty} W(t(\varpi^k))|\varpi^k|^{s-(n-1)/2} \\ &= \sum_{k=0}^{\infty} \delta_B^{1/2}(t(\varpi^k)) s_{(k,0,\dots,0)}(\alpha)|\varpi^k|^{s-(n-1)/2} \\ &= \sum_{k=0}^{\infty} s_{(k,0,\dots,0)}(\alpha) q^{-ks}. \end{aligned}$$

It follows from [**10**, Chapter I (3.4)] that $s_{(k,0,\dots,0)}(\alpha)$ is the complete homogeneous symmetric polynomial of degree k, that is,

$$s_{(k,0,\dots,0)}(\alpha) = \sum_{k_1 + \cdots + k_n = k} \alpha_1^{k_1} \cdots \alpha_n^{k_n}.$$

Hence we get

$$\begin{aligned} Z(s, W) &= \sum_{k=0}^{\infty} \left(\sum_{k_1 + \cdots + k_n = k} \alpha_1^{k_1} \cdots \alpha_n^{k_n} \right) q^{-ks} = \prod_{i=1}^{n} \left(\sum_{k_i=0}^{\infty} \alpha_i^{k_i} q^{-k_i s} \right) \\ &= \prod_{i=1}^{n} (1 - \alpha_i q^{-s})^{-1} = L(s, \pi), \end{aligned}$$

as required. $\square$

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