

## Descendent theory for stable pairs on toric 3-folds

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**Abstract.** We prove the rationality of the descendent partition function for stable pairs on nonsingular toric 3-folds. The method uses a geometric reduction of the 2- and 3-leg descendent vertices to the 1-leg case. As a consequence, we prove the rationality of the relative stable pairs partition functions for all log Calabi-Yau geometries of the form  $(X, K3)$  where  $X$  is a nonsingular toric 3-fold.

### 0. Introduction.

#### 0.1. Descendents.

Let  $X$  be a nonsingular 3-fold, and let  $\beta \in H_2(X, \mathbb{Z})$  be a nonzero class. We will study here the moduli space of stable pairs

$$[\mathcal{O}_X \xrightarrow{s} F] \in P_n(X, \beta)$$

where  $F$  is a pure sheaf supported on a Cohen-Macaulay subcurve of  $X$ ,  $s$  is a morphism with 0-dimensional cokernel, and

$$\chi(F) = n, \quad [F] = \beta.$$

The space  $P_n(X, \beta)$  carries a virtual fundamental class obtained from the deformation theory of complexes in the derived category [19].

Since  $P_n(X, \beta)$  is a fine moduli space, there exists a universal sheaf

$$\mathbb{F} \rightarrow X \times P_n(X, \beta),$$

see Section 2.3 of [19]. For a stable pair  $[\mathcal{O}_X \rightarrow F] \in P_n(X, \beta)$ , the restriction of

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$\mathbb{F}$  to the fiber

$$X \times [\mathcal{O}_X \rightarrow F] \subset X \times P_n(X, \beta)$$

is canonically isomorphic to  $F$ . Let

$$\begin{aligned} \pi_X &: X \times P_n(X, \beta) \rightarrow X, \\ \pi_P &: X \times P_n(X, \beta) \rightarrow P_n(X, \beta) \end{aligned}$$

be the projections onto the first and second factors. Since  $X$  is nonsingular and  $\mathbb{F}$  is  $\pi_P$ -flat,  $\mathbb{F}$  has a finite resolution by locally free sheaves. Hence, the Chern character of the universal sheaf  $\mathbb{F}$  on  $X \times P_n(X, \beta)$  is well-defined. By definition, the operation

$$\pi_{P*}(\pi_X^*(\gamma) \cdot \text{ch}_{2+i}(\mathbb{F}) \cap (\pi_P^*(\cdot))) : H_*(P_n(X, \beta)) \rightarrow H_*(P_n(X, \beta))$$

is the action of the descendent  $\tau_i(\gamma)$ , where  $\gamma \in H^*(X, \mathbb{Z})$ .

For nonzero  $\beta \in H_2(X, \mathbb{Z})$  and arbitrary  $\gamma_i \in H^*(X, \mathbb{Z})$ , define the stable pairs invariant with descendent insertions by

$$\begin{aligned} \left\langle \prod_{j=1}^k \tau_{i_j}(\gamma_j) \right\rangle_{n, \beta}^X &= \int_{[P_n(X, \beta)]^{vir}} \prod_{j=1}^k \tau_{i_j}(\gamma_j) \\ &= \int_{P_n(X, \beta)} \prod_{j=1}^k \tau_{i_j}(\gamma_j) ([P_n(X, \beta)]^{vir}). \end{aligned}$$

The partition function is

$$\mathbf{Z}_\beta^X \left( \prod_{j=1}^k \tau_{i_j}(\gamma_j) \right) = \sum_n \left\langle \prod_{j=1}^k \tau_{i_j}(\gamma_j) \right\rangle_{n, \beta}^X q^n.$$

Since  $P_n(X, \beta)$  is empty for sufficiently negative  $n$ ,  $\mathbf{Z}_\beta^X \left( \prod_{j=1}^k \tau_{i_j}(\gamma_j) \right)$  is a Laurent series in  $q$ . The following conjecture was made in [20].

CONJECTURE 1. *The partition function  $\mathbf{Z}_\beta^X \left( \prod_{j=1}^k \tau_{i_j}(\gamma_j) \right)$  is the Laurent expansion of a rational function in  $q$ .*

If only primary field insertions  $\tau_0(\gamma)$  appear, Conjecture 1 is known for toric

$X$  by [9], [13] and for Calabi-Yau  $X$  by [1], [22] together with [5]. In case  $X$  is a local curve, Conjecture 1 has been proven for descendent insertions  $\tau_{>0}(\gamma)$  in [16].

Let  $T$  be a 3-dimensional algebraic torus acting on a nonsingular toric 3-fold  $X$ .<sup>1</sup> Let  $s_1, s_2, s_3 \in H_T^*(\bullet)$  be the first Chern classes of the standard representations of the three factors of  $T$ . The  $T$ -equivariant stable pairs invariants of  $X$  take values in  $\mathbb{Q}(s_1, s_2, s_3)$ . Let

$$Z_\beta^X \left( \prod_{j=1}^k \tau_{i_j}(\gamma_j) \right)^T \in \mathbb{Q}(s_1, s_2, s_3)((q))$$

be the  $T$ -equivariant partition function with  $\gamma_j \in H_T^*(X, \mathbb{Q})$ . The main result of the present paper is the proof of a stronger  $T$ -equivariant version of Conjecture 1 in the toric case.

**THEOREM 1.** *Let  $X$  be a nonsingular toric 3-fold. The partition function  $Z_\beta^X \left( \prod_{j=1}^k \tau_{i_j}(\gamma_j) \right)^T$  is the Laurent expansion in  $q$  of a rational function in the field  $\mathbb{Q}(q, s_1, s_2, s_3)$ .*

**0.2. Capped descendent vertex.**

Capped vertices were introduced in [9] to study the Gromov-Witten and Donaldson-Thomas theories of toric 3-folds. By the same construction, we define here capped stable pairs vertices with descendent insertions. The 1-leg case was already treated in [16].

Let  $T$  be a 3-dimensional algebraic torus, and let  $s_1, s_2, s_3 \in H_T^*(\bullet)$  be first Chern classes of the standard representations of the three factors of  $T$ . Let  $T$  act diagonally on  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ,

$$(\xi_1, \xi_2, \xi_3) \cdot ([x_1, y_1], [x_2, y_2], [x_3, y_3]) = ([x_1, \xi_1 y_1], [x_2, \xi_2 y_2], [x_3, \xi_3 y_3]).$$

Let  $0, \infty \in \mathbb{P}^1$  be the points  $[1, 0]$  and  $[0, 1]$  respectively. The tangent weights<sup>2</sup> of  $T$  at the point

$$p = (0, 0, 0) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

are  $s_1, s_2$ , and  $s_3$ .

Let  $U \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  be the  $T$ -invariant 3-fold obtained by removing the

<sup>1</sup> $X$  need not be compact. In the open case, the stable pairs invariants are defined by  $T$ -equivariant residues so long as the  $T$ -fixed locus  $X^T \subset X$  is compact.

<sup>2</sup>Our sign conventions here follow [9] and disagree with [16]. Since we will not require explicit vertex calculations here, the sign conventions will not play a significant role.

three  $T$ -invariant lines

$$L_1, L_2, L_3 \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

passing through the point  $(\infty, \infty, \infty)$ ,

$$U = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \setminus \bigcup_{i=1}^3 L_i.$$

Let  $D_i \subset U$  be the divisor with  $i^{\text{th}}$  coordinate  $\infty$ . For  $i \neq j$ , the divisors  $D_i$  and  $D_j$  are disjoint in  $U$ .

The capped descendent vertex is the stable pairs partition function of  $U$  with integrand

$$\tau_{\alpha_1}(\mathbf{p}) \cdots \tau_{\alpha_\ell}(\mathbf{p})$$

and free relative conditions imposed at the divisors  $D_i$ . While the relative geometry  $U/\bigcup_i D_i$  is not compact, the moduli spaces  $P_n(U/\bigcup_i D_i, \beta)$  have compact  $T$ -fixed loci. The stable pairs invariants of  $U/\bigcup_i D_i$  are well-defined by  $T$ -equivariant residues. In the localization formula for the reduced theories of  $U/\bigcup_i D_i$ , nonzero degrees can occur *only* on the edges meeting the origin  $\mathbf{p} \in U$ .

We denote the capped stable pair descendent vertex by

$$\begin{aligned} C(\alpha|\lambda, \mu, \nu) &= \mathbf{Z} \left( U / \bigcup_i D_i, \prod_{i=1}^{\ell} \tau_{\alpha_i}(\mathbf{p}) \mid \lambda, \mu, \nu \right)^{\mathbf{T}} \\ &= \sum_n \left\langle \prod_{i=1}^{\ell} \tau_{\alpha_i}(\mathbf{p}) \right\rangle_{n, \lambda, \mu, \nu}^{\mathbf{T}} q^n \end{aligned} \tag{1}$$

where the partition  $\alpha$  specifies the descendent integrand and the partitions  $\lambda, \mu, \nu$  denote relative conditions imposed at  $D_1, D_2, D_3$  in the Nakajima basis. The curve class  $\beta$  in (1) is determined by the relative conditions:  $\beta$  is the sum of the three axes passing through  $\mathbf{p} \in U$  with coefficients  $|\lambda|$ ,  $|\mu|$ , and  $|\nu|$  respectively. The superscript  $\mathbf{T}$  after the bracket denotes  $T$ -equivariant integration on  $P_n(U/\bigcup_i D_i, \beta)$ .

Since the parts of the partition  $\alpha$  are positive, our capped descendent vertices have no  $\tau_0(\mathbf{p})$  insertions. For a stable pair  $(F, s)$  on  $X$ , a direct calculation shows

$$\text{ch}_2(F) \cap [X] = \beta \in H_2(X, \mathbb{Z}).$$

Hence,  $\tau_0(\mathbf{p})$  acts simply as the scalar

$$|\lambda|s_2s_3 + s_1|\mu|s_3 + s_1s_2|\nu|.$$

Restricting  $\alpha$  to be a partition is therefore no loss.

If  $\alpha = \emptyset$ , there are no descendents and our capped descendent vertex reduces to the capped vertex of [9]. The basic  $\Sigma_3$ -action permuting the axes of  $U$  implies an  $\Sigma_3$ -symmetry of the capped descendent vertex. The 2-leg and 1-leg vertices are the restrictions

$$C(\alpha|\lambda, \mu, \emptyset), \quad C(\alpha|\lambda, \emptyset, \emptyset)$$

respectively. For stable pairs, we always require  $|\lambda| + |\mu| + |\nu| > 0$ . However, we follow the conventions

$$C(\emptyset | \emptyset, \emptyset, \emptyset) = 1, \quad C(\alpha \neq \emptyset | \emptyset, \emptyset, \emptyset) = 0,$$

for convenience in formulas.

We will prove Theorem 1 by a refined rationality result for the capped descendent vertex.

**THEOREM 2.** *For all partitions  $\alpha, \lambda, \mu, \nu$ , the vertex  $C(\alpha|\lambda, \mu, \nu)$  is the Laurent expansion in  $q$  of a rational function in the field  $\mathbb{Q}(q, s_1, s_2, s_3)$ .*

The proof of Theorem 2 uses two geometric constraints to reduce the capped descendent vertex  $C(\alpha|\lambda, \mu, \nu)$  to the 1-leg case studied in [16]. The first involves the  $\mathcal{A}_n$ -surfaces as in [9]. The second, for large partitions  $\alpha$ , involves Hirzebruch surfaces and the relative/descendent correspondence in the 1-leg case. The final outcome is an effective computation of  $C(\alpha|\lambda, \mu, \nu)$  in terms of the capped 1-leg descendent vertex.

While we are interested here in the theory of stable pairs, the geometric constraints used in the proof of Theorem 2 are equally valid for Gromov-Witten theory and Donaldson-Thomas theory. In the latter theories, the constraints determine the capped 2- and 3-leg descendent vertices in terms of capped 1-leg descendent vertices.<sup>3</sup> However, rationality does *not* hold for the capped descendent vertices in the Gromov-Witten or Donaldson-Thomas cases.

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<sup>3</sup>The Donaldson-Thomas constraints also involve the 0-leg descendent vertex. The 0-leg descendent vertex concerns degree 0 contributions which may be removed in Gromov-Witten theory by requiring stable maps to have no connected components contracted to a point. The 0-leg descendent vertex is absent in the theory of stable pairs by definition.

The Gromov-Witten, Donaldson-Thomas, and stable pairs descendent theories are all conjectured to be equivalent [11], [19], [20]. The geometric constraints studied here show the differences between the three descendent theories in the toric case should be viewed as occurring in the 1-leg descendent vertex. A descendent correspondence, rooted in 1-leg geometry, is proven for the  $T$ -equivariant Gromov-Witten and stable pairs theories of all nonsingular toric 3-folds in [18]. By the rationality result of Theorem 1, the MNOP [10] variable change

$$-q = e^{iu}$$

is well-defined for the stable pairs partition function. Rationality plays a crucial role in the study of the correspondence in [18].

**0.3. Log Calabi-Yau geometries.**

Let  $X$  be a nonsingular projective 3-fold and let  $S \subset X$  be a nonsingular anti-canonical divisor isomorphic to a  $K3$  surface. The pair  $(X, S)$  is log Calabi-Yau,

$$K_X(S) = \mathcal{O}_X.$$

The most basic example is  $(\mathbf{P}^3, Q)$  where  $Q \subset \mathbf{P}^3$  is a nonsingular quartic  $K3$  surface.

There is a natural notion of pure counting in log Calabi-Yau geometries  $(X, S)$ . Let  $\beta \in H_2(X, \mathbb{Z})$  be a curve class, and let

$$d = \int_{\beta} [S] \in \mathbb{Z}$$

be the intersection number. Let  $\text{Hilb}(S, d)$  denote the Hilbert scheme of  $d$  points on  $S$ , and let

$$\mathcal{L}_{\beta} \in H^{2d}(\text{Hilb}(S, d), \mathbb{Q})$$

be a middle dimensional class. The analogue of the partition function of Calabi-Yau invariants in the log Calabi-Yau situation is

$$\mathbf{Z}_{\beta}^{X/S}(1 \mid \mathcal{L}_{\beta}) = \sum_n \langle 1_{/n, \mathcal{L}_{\beta}} \rangle^{X/S} q^n. \tag{2}$$

The partition function (2) counts stable pairs of curve class  $\beta$  with the relative condition determined by the class  $\mathcal{L}_{\beta}$ . As a consequence of Theorem 1, we obtain

the following result.

**THEOREM 3.** *Let  $X$  be a nonsingular projective toric 3-fold with an anti-canonical K3 section  $S$ . The partition function  $Z_{\beta}^{X/S}(1 \mid \mathcal{L}_{\beta})$  is the Laurent expansion of a rational function in  $q$ .*

The Hilbert scheme  $\text{Hilb}(S, d)$  is well-known to carry a canonical holomorphic symplectic form. If  $\mathcal{L}_{\beta}$  is obtained from a Lagrangian cycle, a very natural approach to Theorem 3, following the successful arguments for rationality in Calabi-Yau cases, is to study Behrend functions and wall-crossing for sheaf counting invariants associated to relative geometries. D. Maulik and R. P. Thomas have recently initiated a study of Behrend functions for log Calabi-Yau geometries.

In fact, our proof of Theorem 3 yields the rationality of all partition functions for the relative geometry  $(X, S)$  with arbitrary descendent insertions and non-Lagrangian boundary conditions.

#### 0.4. Further directions.

The techniques of [12] allow the use of degeneration to exchange relative conditions and descendent invariants. Theorem 3 is an easy application of the ideas of [12] to the theory of stable pairs. New results including rationality of the stable pairs partition functions in many non-toric settings will be established in a sequel.

The arguments of [12] are for Gromov-Witten theory where target dimension reduction plays a basic role. For stable pairs, several aspects have to be redone without dimension reduction.

#### 0.5. Plan of the paper.

We start with a review of capped localization in Section 1. The full capped localization formula for descendents is given in Section 1.3. The capped 2-leg descendent vertex is studied in Section 2. The first half of the reduction to 1-leg uses results about  $\mathcal{A}_n$ -surface geometries in Section 2.2. The second half of the reduction uses constraints obtained from Hirzebruch surfaces in Section 2.3. The equivalence between relative conditions and stationary descendents for local curves (used before in [15]) is reviewed in Section 2.4. The proofs of Theorems 1 and 2 are completed in Section 3 with the analysis of the capped 3-leg descendent vertex. The proof of Theorem 3 for log Calabi-Yau geometries is presented in Section 4.

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## 1. Capped localization.

### 1.1. Toric geometry.

Let  $X$  be a nonsingular toric 3-fold. Virtual localization with respect to the action of the full 3-dimensional torus  $\mathbf{T}$  reduces all stable pairs invariants of  $X$  to local contributions of the vertices and edges of the associated toric polytope. However, the standard constituent pieces of the localization formula yield transcendental functions. We will use the regrouped localization procedure introduced in [9] with capped vertex and edge contributions. The capped vertex and edge terms are equivalent building blocks for global toric calculations, but are much better behaved.

Let  $\Delta$  denote the polytope associated to  $X$ . The vertices of  $\Delta$  are in bijection with  $\mathbf{T}$ -fixed points  $X^{\mathbf{T}}$ . The edges  $e$  correspond to  $\mathbf{T}$ -invariant curves

$$C_e \subset X.$$

The three edges incident to any vertex carry canonical  $\mathbf{T}$ -weights — the tangent weights of the torus action.

We will consider both compact and noncompact toric varieties  $X$ . In the latter case, edges may be compact or noncompact. Every compact edge is incident to two vertices.

### 1.2. Capping.

Capped localization expresses the  $\mathbf{T}$ -equivariant stable pairs descendents of  $X$  as a sum of capped descendent vertex and capped edge data.

A half-edge  $h = (e, v)$  is a compact edge  $e$  together with the choice of an incident vertex  $v$ . A partition assignment

$$h \mapsto \lambda(h)$$

to half-edges is *balanced* if the equality

$$|\lambda(e, v)| = |\lambda(e, v')|$$

always holds for the two halves of  $e$ . For a balanced assignment, let

$$|e| = |\lambda(e, v)| = |\lambda(e, v')|$$

denote the *edge degree*.

The outermost sum in the capped localization formula runs over all balanced assignments of partitions  $\lambda(h)$  to the half-edges  $h$  of  $\Delta$  satisfying

$$\beta = \sum_e |e| \cdot [C_e] \in H_2(X, \mathbb{Z}). \tag{3}$$

Such a partition assignment will be called a *capped marking* of  $\Delta$ . The weight of each capped marking in the localization sum for the stable pairs descendent partition function equals the product of three factors:

- (i) capped descendent vertex contributions,
- (ii) capped edge contributions,
- (iii) gluing terms.

Each vertex determines up to three half-edges specifying the partitions for the capped vertex. Each compact edge determines two half-edges specifying the partitions of the capped edge. The capped edge contributions (ii) and gluing terms (iii) here are *exactly* the same as for the capped localization formula in [9]. Precise formulas are written in Section 1.3.

The capped localization formula is easily derived from the standard localization formula (with roots in [4], [10]). Indeed, the capped objects are obtained from the uncapped objects by rubber integral<sup>4</sup> factors. The rubber integrals cancel in pairs in capped localization to yield standard localization.

**1.3. Formulas.**

The  $T$ -equivariant cohomology of  $X$  is generated (after localization) by the classes of the  $T$ -fixed points  $X^T \subset X$ . Let  $\alpha$  be a partition with parts  $\alpha_1, \dots, \alpha_\ell$ , and let

$$\sigma : \{1, \dots, \ell\} \rightarrow X^T.$$

Let  $\mathbf{p}_{\sigma(i)} \in H_T^6(X, \mathbb{Q})$  denote the class of the  $T$ -fixed point  $\sigma(i)$ . We consider the capped localization formula for the  $T$ -equivariant stable pairs descendent partition function

$$\mathbf{Z}_\beta \left( X, \prod_{i=1}^\ell \tau_{\alpha_i}(\mathbf{p}_{\sigma(i)}) \right) = \sum_n \left\langle \prod_{i=1}^\ell \tau_{\alpha_i}(\mathbf{p}_{\sigma(i)}) \right\rangle_{n, \beta}^X q^n. \tag{4}$$

Let  $\mathcal{V}$  be the set of vertices of  $\Delta$  which we identify with  $X^T$ . For  $v \in \mathcal{V}$ , let  $\alpha^v$  be the collection of parts  $\alpha_i$  of  $\alpha$  satisfying  $\sigma(i) = v$ . The partition  $\alpha^v$  has size bounded by  $|\alpha|$ .

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<sup>4</sup>Rubber integrals  $\langle \lambda | 1/(1 - \psi_\infty) | \mu \rangle^\sim$  arise in the localization formulas for relative geometries. See [16] for a discussion.

For  $v \in \mathcal{V}$ , let  $h_1^v, h_2^v, h_3^v$  be the associated half-edges<sup>5</sup> with tangent weights  $s_1^v, s_2^v, s_3^v$  respectively. Let  $\Gamma_\beta$  be the set of capped markings satisfying the degree condition (3). Each  $\Gamma \in \Gamma_\beta$  associates a partition  $\lambda(h)$  to every half-edge  $h$ . Let

$$|h| = |\lambda(h)|$$

denote the half-edge degree.

For each  $v \in \mathcal{V}$ , the assignments  $\sigma$  and  $\Gamma$  determines an evaluation of the capped vertex,

$$C(v, \sigma, \Gamma) = C(\alpha^v |\lambda(h_1^v), \lambda(h_2^v), \lambda(h_3^v)|_{s_1=s_1^v, s_2=s_2^v, s_3=s_3^v}.$$

Let  $h_1^e$  and  $h_2^e$  be the half-edges associated to the edge  $e$ . The assignment  $\Gamma$  also determines an evaluation of the capped edge,

$$E(e, \Gamma) = E(\lambda(h_1^e), \lambda(h_2^e)).$$

The edge factors and weights are identical to the corresponding Donaldson-Thomas edge terms in [9]. A gluing factor is specified by  $\Gamma$  at each half-edge  $h_i^v \in \mathcal{H}$  by

$$G(h_i^v, \Gamma) = (-1)^{|h_i^v| - \ell(\lambda(h_i^v))} \mathfrak{z}(\lambda(h_i^v)) \left( \frac{\prod_{j=1}^3 s_j^v}{s_i^v} \right)^{\ell(\lambda(h_i^v))} q^{-|h_i^v|}$$

where  $\mathfrak{z}(\lambda)$  is the order of the centralizer in the symmetric group of an element with cycle type  $\lambda$ .

The capped localization formula for stable pairs can be written exactly in the form presented in Section 1.2,

$$Z_\beta \left( X, \prod_{i=1}^\ell \tau_{\alpha_i}(\mathbf{p}_{\sigma(i)}) \right) = \sum_{\Gamma \in \Gamma_\beta} \prod_{v \in \mathcal{V}} \prod_{e \in \mathcal{E}} \prod_{h \in \mathcal{H}} C(v, \sigma, \Gamma) E(e, \Gamma) G(h, \Gamma)$$

where the product is over the sets of vertices  $\mathcal{V}$ , edges  $\mathcal{E}$ , and half-edges  $\mathcal{H}$  of the polytope  $\Delta$ .

The most basic example of capped localization occurs for the 3-fold total space of

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \rightarrow P^1. \tag{5}$$

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<sup>5</sup>For simplicity, we assume  $X$  is projective so each vertex is incident to 3 compact edges.

The standard localization formula has vertices over  $0, \infty \in \mathbf{P}^1$  and a single edge. To write the answer in terms of capped localization, we consider a  $T$ -equivariant degeneration of (5) to a chain

$$(0, 0) \cup (a, b) \cup (0, 0)$$

of total spaces of bundles over  $\mathbf{P}^1$  denoted here by splitting degrees. The first  $(0, 0)$ -geometry is relative over  $\infty \in \mathbf{P}^1$ , the central  $(a, b)$ -geometry is relative on both sides, and the last  $(0, 0)$ -geometry is relative over  $0 \in \mathbf{P}^1$ . The degeneration formula exactly expresses the stable pair theory of (5) as capped localization with 2 capped vertices and a single capped edge in the middle.

**2. Capped descendent 2-leg vertex.**

**2.1. Induction strategy.**

We will prove Theorem 2 for capped 2-leg descendent vertices by induction. Using the  $\Sigma_3$ -symmetry of the vertex, we may assume 2-leg vertices are of the form

$$C(\alpha|\lambda, \mu, \emptyset), \quad |\lambda| \geq |\mu| \geq 0.$$

We know  $C(\alpha|\lambda, \mu, \emptyset)$  is the Laurent expansion in  $q$  of a rational function if

$$\alpha = \emptyset \quad \text{or} \quad \mu = \emptyset. \tag{6}$$

In the former case, there are no descendents and rationality is a central result of [9]. In the latter case, rationality for the capped 1-leg descendent vertex is a central result of [16].

Define a partial ordering on capped 2-leg descendent vertices by the following rules. We say

$$C(\alpha|\lambda, \mu, \emptyset) \triangleright C(\alpha'|\lambda', \mu', \emptyset),$$

if we have

- $|\alpha| > |\alpha'|$ ,
- or  $|\alpha| = |\alpha'|$  and  $|\mu| > |\mu'|$ ,
- or  $|\alpha| = |\alpha'|$ ,  $|\mu| = |\mu'|$  and  $|\lambda| > |\lambda'|$ .

The relationship  $\triangleright$  is just the lexicographic ordering on the triples  $(|\alpha|, |\mu|, |\lambda|)$ .

To prove Theorem 2 in the 2-leg case for  $C(\alpha|\lambda, \mu, \emptyset)$ , we assume rationality holds for all vertices  $C(\alpha'|\lambda', \mu', \emptyset)$  occurring earlier in the partial ordering  $\triangleright$ . The

ground cases of the induction are (6) so we may assume  $|\alpha|, |\mu| > 0$ .

To prove rationality for  $C(\alpha|\lambda, \mu, \emptyset)$ , we will use geometric constraints. The approach depends upon whether  $|\alpha| < |\lambda|$  or  $|\alpha| \geq |\lambda|$ . In the former case, we will use  $\mathcal{A}_1$ -surface geometry. In the latter case, we will use Hirzebruch surfaces.

**2.2. Case  $|\alpha| < |\lambda|$ .**

**2.2.1.  $\mathcal{A}_1$  geometry.**

Let  $\zeta$  be a primitive  $(n + 1)^{th}$  root of unity, for  $n \geq 0$ . Let the generator of the cyclic group  $\mathbb{Z}_{n+1}$  act on  $\mathbb{C}^2$  by

$$(z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1} z_2).$$

Let  $\mathcal{A}_n$  be the minimal resolution of the quotient

$$\mathcal{A}_n \rightarrow \mathbb{C}^2 / \mathbb{Z}_{n+1}.$$

The diagonal  $(\mathbb{C}^*)^2$ -action on  $\mathbb{C}^2$  commutes with the action of  $\mathbb{Z}_{n+1}$ . As a result, the surfaces  $\mathcal{A}_n$  are toric.

The surface  $\mathcal{A}_1$  is isomorphic to the total space of

$$\mathcal{O}(-2) \rightarrow \mathbb{P}^1$$

and admits a toric compactification

$$\mathcal{A}_1 \subset \mathbb{P}(\mathcal{O} + \mathcal{O}(-2)) = \mathcal{F}_2$$

by the Hirzebruch surface.

Let  $C \subset \mathcal{A}_1$  be the 0-section of  $\mathcal{O}(-2)$ , and let  $\star, \bullet \in C$  be the  $(\mathbb{C}^*)^2$ -fixed points. Let

$$\bar{\star}, \bar{\bullet} \in \mathcal{F}_2 \setminus \mathcal{A}_1$$

be the  $(\mathbb{C}^*)^2$ -fixed points lying above  $\star, \bullet$  respectively. We fix our  $(\mathbb{C}^*)^2$ -action by specifying tangent weights at the four  $(\mathbb{C}^*)^2$ -points:

$$\begin{aligned} T_\star(\mathcal{F}_2) &: s_1 - s_2, & 2s_2 \\ T_\bullet(\mathcal{F}_2) &: s_2 - s_1, & 2s_1 \\ T_{\bar{\star}}(\mathcal{F}_2) &: s_1 - s_2, & -2s_2 \\ T_{\bar{\bullet}}(\mathcal{F}_2) &: s_2 - s_1, & -2s_1. \end{aligned} \tag{7}$$

None of the tangent weights are divisible by  $s_1 + s_2$ .

Consider the nonsingular projective toric variety  $\mathcal{F}_2 \times \mathbf{P}^1$ . The 3-torus

$$T = (\mathbb{C}^*)^3$$

acts on  $\mathcal{F}_2$  as above via the first two factors and acts on  $\mathbf{P}^1$  via the third factor with tangent weights  $s_3$  and  $-s_3$  at the points  $0, \infty \in \mathbf{P}^1$  respectively. The two  $T$ -invariant divisors of  $\mathcal{F}_2 \times \mathbf{P}^1$

$$\mathcal{D}_0 = \mathcal{F}_2 \times \{0\}, \quad \mathcal{D}_\infty = \mathcal{F}_2 \times \{\infty\}$$

will play a basic role. The 3-fold  $\mathcal{F}_2 \times \mathbf{P}^1$  has eight  $T$ -fixed points which we denote by

$$\star_0, \bar{\star}_0, \bullet_0, \bar{\bullet}_0, \star_\infty, \bar{\star}_\infty, \bullet_\infty, \bar{\bullet}_\infty \in \mathcal{F}_2 \times \mathbf{P}^1$$

where the subscript indicates the coordinate in  $\mathbf{P}^1$ .

Let  $L_0 \subset \mathcal{F}_2 \times \mathbf{P}^1$  be the  $T$ -invariant line connecting  $\star_0$  and  $\bar{\star}_0$ . Similarly, let  $L_\infty \subset \mathcal{F}_2 \times \mathbf{P}^1$  be the  $T$ -invariant line connecting  $\star_\infty$  and  $\bar{\star}_\infty$ . The lines  $L_0$  and  $L_\infty$  are  $\mathbf{P}^1$ -fibers of the Hirzebruch surfaces  $\mathcal{D}_0$  and  $\mathcal{D}_\infty$ . We have

$$H_2(\mathcal{F}_2 \times \mathbf{P}^1, \mathbb{Z}) = \mathbb{Z}[C] \oplus \mathbb{Z}[L_0] \oplus \mathbb{Z}[P]$$

where  $P$  is the fiber of the projection to  $\mathcal{F}_2$ .

### 2.2.2. Integration.

We will find relations which express  $C(\alpha|\lambda, \mu, \emptyset)$  in terms of inductively treated vertices. Let  $\mu'$  be any partition. The relations will be obtained from vanishing stable pairs invariants of the relative geometry  $\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty$  in curve class

$$\beta = |\mu| \cdot [C] + (|\lambda| + |\mu'|) \cdot [P].$$

The virtual dimension of the associated moduli space is

$$\dim^{vir} P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \beta) = 2|\lambda| + 2|\mu'|.$$

Relative conditions in  $\text{Hilb}(\mathcal{D}_\infty, |\lambda| + |\mu'|)$  are best expressed in terms of the Nakajima basis given by a  $T$ -equivariant cohomology weighted partition of  $|\lambda| + |\mu'|$ . We impose the relative condition determined by the partition

$$\lambda \cup \mu' = \lambda_1 + \cdots + \lambda_{\ell(\lambda')} + \mu'_1 + \cdots + \mu'_{\ell(\mu')}$$

weighted by  $[L_\infty] \in H_{\mathbf{T}}^2(\mathcal{D}_\infty, \mathbb{Q})$  for the parts of  $\lambda$  and  $[\bullet_\infty] \in H_{\mathbf{T}}^4(\mathcal{D}_\infty, \mathbb{Q})$  for the parts of  $\mu'$ . We denote the relative condition by  $\mathbf{r}(\lambda, \mu')$ . After imposing  $\mathbf{r}(\lambda, \mu')$ , the virtual dimension drops to

$$\dim^{vir} P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \beta)_{\mathbf{r}(\lambda, \mu')} = |\lambda| + |\mu'| - \ell(\mu') \geq |\lambda|.$$

We now specify descendent insertion. Since  $|\alpha| > 0$ , there is a positive part  $\alpha_1$ . We consider the descendent insertion

$$\tau_{\alpha_1}([L_0]) \cdot \prod_{i=2}^{\ell} \tau_{\alpha_i}([D_0]).$$

The descendent insertion imposes  $|\alpha| + 1$  conditions. Therefore, the integral

$$\int_{[P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \beta)_{\mathbf{r}(\lambda, \mu')}]^{vir}} \tau_{\alpha_1}([L_0]) \cdot \prod_{i=2}^{\ell} \tau_{\alpha_i}([D_0]), \tag{8}$$

viewed as  $\mathbf{T}$ -equivariant push-forward to a point, has dimension at least

$$|\lambda| - |\alpha| - 1 \geq 0.$$

PROPOSITION 1. *The  $\mathbf{T}$ -equivariant integral (8) vanishes for all  $n$ .*

PROOF. If the integral has dimension greater than 0, then  $\mathbf{T}$ -equivariant push-forward with values in  $\mathbb{Q}[s_1, s_2, s_3]$  vanishes since the moduli space

$$P_n(\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty, \beta)_{\mathbf{r}(\lambda, \mu')}$$

is compact.

If the integral has dimension 0, then  $\mathbf{T}$ -equivariant push-forward is a constant in  $\mathbb{Q} \subset \mathbb{Q}[s_1, s_2, s_3]$ . In particular, the integral can be computed by  $\mathbf{T}$ -equivariant localization followed by the specialization

$$s_1 + s_2 = 0.$$

The  $\mathbf{T}$ -equivariant localization formula for  $\mathcal{F}_2 \times \mathbf{P}^1/\mathcal{D}_\infty$  will be discussed carefully below. In fact, very little is needed for the vanishing here.

Since the  $[C]$  coefficient of  $\beta$  is positive, a  $T$ -fixed stable pair in the moduli space must contain a component in the open set  $\mathcal{A}_1 \times \mathbf{P}^1$  relative to the divisor over  $\infty$ . The  $s_1 + s_2 = 0$  specialization of localization on  $\mathcal{A}_1 \times \mathbf{P}^1$  relative to  $\infty$  is well-known to vanish since  $\mathcal{A}_1$  is holomorphic symplectic (and the  $[C]$  coefficient is positive). In addition, there may be components of  $T$ -fixed stable pairs with support over  $\mathcal{F}_2 \setminus \mathcal{A}_1$ . The latter give rise to descendent 1-leg vertex contributions which, because of the tangent weight analysis (7), have no poles at  $s_1 + s_2 = 0$ . Hence, the substitution  $s_1 + s_2 = 0$  after localization is well-defined and kills all contributions.  $\square$

**2.2.3. Relation.**

We define the  $T$ -equivariant series

$$Z_\beta(\alpha, \lambda, \mu') = \sum_n q^n \int_{[P_n(\mathcal{F}_2 \times \mathbf{P}^1 / \mathcal{D}_\infty, \beta)_{r(\lambda, \mu')}]^{vir}} \tau_{\alpha_1}([L_0]) \cdot \prod_{i=2}^\ell \tau_{\alpha_i}([D_0])$$

obtained from the integrals (8). By Proposition 1, the series  $Z(\alpha, \lambda, \mu')_\beta$  vanishes identically. We will calculate the left side of

$$Z_\beta(\alpha, \lambda, \mu') = 0 \tag{9}$$

by capped localization to obtain a relation constraining capped descendent vertices.

The stable pairs theory of the relative geometry  $\mathcal{F}_2 \times \mathbf{P}^1 / \mathcal{D}_\infty$  admits a capped localization formula. Over  $0 \in \mathbf{P}^1$ , capped descendent vertices occur as in the capped localization formula of Section 1.3. Over  $\infty \in \mathbf{P}^1$ , capped rubber terms for  $T$ -equivariant localization in the relative geometry arise. Capped rubber is discussed in Section 3.4 of [9]. Since all our descendent insertions lie over  $0 \in \mathbf{P}^1$ , our capped rubber has the same definition as the capped rubber of [9].

Two types of capped rubber contributions arise over  $\infty \in \mathbf{P}^1$  in the  $T$ -equivariant localization formula for  $Z_\beta(\alpha, \lambda, \mu')$ ,

- (i) capped 1-leg rubber corresponding to  $T$ -fixed stable pairs with components contracted to  $\mathcal{F}_2 \setminus \mathcal{A}_1$ ,
- (ii) capped rubber contributions of  $\mathcal{A}_1 \times \mathbf{P}^1$  relative to the divisor over  $\infty$ .

The capped contributions (i) are just 1-leg with no descendents, so are rational. The capped contributions (ii) are proven to be rational in Lemma 6 of [9] relying on the results of [7], [8]. See Section 5 of [13] for the stable pairs results.

We now analyze the capped localization of  $Z_\beta(\alpha, \lambda, \mu')$  over  $0 \in \mathbf{P}^1$ . A term in the capped localization formula is said to be *principal* if not all the capped

descendent vertices which arise are known inductively to be rational.

First consider the descendent insertions. The descendents

$$\tau_{\alpha_1}([L_0]) \cdot \prod_{i=2}^{\ell} \tau_{\alpha_i}([D_0])$$

are free to distribute<sup>6</sup> to the  $T$ -fixed points over  $0 \in \mathbf{P}^1$ . By the choice of  $\beta$ , capped 2-leg descendent vertices can only occur at  $\star_0$  and  $\bullet_0$ . Descendents which distribute to  $\bar{\star}_0$  and  $\bar{\bullet}_0$  will lie on capped 1-leg descendent vertices. The first descendent  $\tau_{\alpha_1}(L_0)$  has to lie on  $\star_0$  or  $\bar{\star}_0$ . We conclude all capped descendent vertices are known inductively to be rational except possibly when all the descendents lie on  $\star_0$ .

Next consider the edge degree  $d$  of  $C$  over  $0 \in \mathbf{P}^1$  in the capped localization formula. If  $d < |\mu|$ , then the capped descendent vertex at  $\star_0$  is known inductively to be rational since the minimal relative partition is lower. Hence, we restrict ourselves to the principal terms where  $d = |\mu|$ .

Since all of  $|\mu| \cdot [C]$  occurs over  $0 \in \mathbf{P}^1$ , the rubber over  $\infty \in \mathbf{P}^1$  is all 1-leg. By the relative conditions imposed by  $\mu'$  with weights  $[\bullet_\infty]$ , the principal terms all have a capped 2-leg vertex with no descendents at  $\bullet_0$  with outgoing partitions of size  $|\mu|$  and  $|\mu'|$ .

Finally, consider the relative conditions  $\lambda$  weighted by  $[L_\infty]$ . The weight allows the parts to distribute to  $\bar{\star}_\infty$ . Such a distribution would result in an inductively treated capped descendent vertex at  $\star_0$  with lower maximal relative partition.

In the principal terms of the capped localization of (9), precisely the following set of capped 2-leg descendent vertices occur at  $\star_0$ :

$$\{C(\alpha|\lambda, \hat{\mu}, \emptyset) \mid |\hat{\mu}| = |\mu|\}. \tag{10}$$

The principal terms arise as displayed in Figure 1. In addition to the vertex  $C(\alpha|\lambda, \hat{\mu}, \emptyset)$  at  $\star_0$ , there is a capped edge with partitions

$$|\hat{\mu}| = |\hat{\mu}'|$$

along the curve  $C$  over  $0 \in \mathbf{P}^1$ . Finally, there is capped 2-leg vertex with no descendents at  $\bullet_0$  with outgoing partitions  $\hat{\mu}'$  and  $\mu'$ .

The system of equations as the partition  $\mu'$  varies has unknowns (10) param-

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<sup>6</sup>The classes  $[L_0], [D_0] \in H_T^*(\mathcal{F}_2 \times \mathbf{P}^1, \mathbb{Q})$  are expressed as  $\mathbb{Q}(s_1, s_2, s_3)$ -linear combinations of the  $T$ -fixed points incident to  $L_0$  and  $D_0$  respectively.

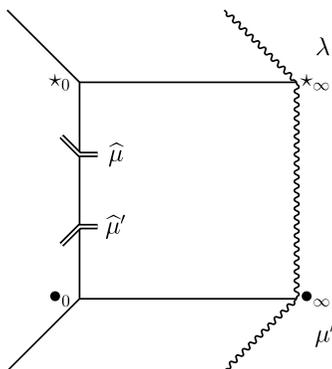


Figure 1. Principal terms.

eterized by partitions of  $|\mu|$ . However, the number of equations is infinite. The induction step is established if the set of equations as  $\mu'$  varies has maximal rank<sup>7</sup> with respect to the unknowns (10).

The maximal rank statement is proven with two observations. First, the capped edge matrix along  $C$  has maximal rank [9]. Second, the matrix of capped 2-leg vertices

$$C(\emptyset | \hat{\mu}', \mu', \emptyset)$$

has maximal rank  $\mathcal{P}(|\mu|)$  even when  $\mu'$  varies only among partitions of size at most  $|\mu| - 1$  by Lemma 9 of [9]. □

**2.3. Case  $|\alpha| \geq |\lambda|$ .**

**2.3.1. Hirzebruch surfaces.**

We require a different geometric construction for the inductive relation in case  $|\alpha| \geq |\lambda|$ .

Let  $k > 0$  be an integer. Let  $\mathcal{F}_k$  be the Hirzebruch surfaces given by the projective bundle

$$P(\mathcal{O} \oplus \mathcal{O}(k)) \rightarrow P^1.$$

The split presentation distinguishes two sections

$$C^+, C^- \subset \mathcal{F}_k$$

---

<sup>7</sup>Maximal rank here is equal to the number  $\mathcal{P}(|\mu|)$  of partitions of size  $|\mu|$ .

of self-intersections  $k$  and  $-k$  respectively. Let  $(\mathbb{C}^*)^2$  act on  $\mathcal{F}_k$  with fixed points

$$\star^+, \bullet^+, \star^-, \bullet^- \in \mathcal{F}_k$$

where the first two lie on  $C^+$  and the last two lie on  $C^-$ . The 2-torus invariant curves of  $\mathcal{F}_k$  are then

$$C^+, C^-, L^*, L^\bullet \subset \mathcal{F}_k \tag{11}$$

where  $L^*$  is a fiber of the projective bundle connecting  $\star^+$  and  $\star^-$  and similarly for  $L^\bullet$ .

Consider the nonsingular projective toric variety  $\mathcal{F}_k \times \mathbf{P}^1$ . The 3-torus

$$\mathbf{T} = (\mathbb{C}^*)^3$$

acts on  $\mathcal{F}_k$  as above via the first two factors and acts on  $\mathbf{P}^1$  via the third factor with tangent weights  $s_3$  and  $-s_3$ . The 3-fold  $\mathcal{F}_k \times \mathbf{P}^1$  has eight  $\mathbf{T}$ -fixed points which we denote by

$$\star_0^+, \bullet_0^+, \star_0^-, \bullet_0^-, \star_\infty^+, \bullet_\infty^+, \star_\infty^-, \bullet_\infty^- \in \mathcal{F}_k \times \mathbf{P}^1$$

where the subscript indicates the coordinate in  $\mathbf{P}^1$ .

There are twelve  $\mathbf{T}$ -invariant curves of  $\mathcal{F}_k \times \mathbf{P}^1$ . There are four of type (11) lying over  $0 \in \mathbf{P}^1$ , four lying over  $\infty \in \mathbf{P}^1$  and four fibers

$$P_{\star^+}, P_{\bullet^+}, P_{\star^-}, P_{\bullet^-}$$

of the projection of  $\mathcal{F}_k \times \mathbf{P}^1$  to  $\mathcal{F}_k$ . We have

$$H_2(\mathcal{F}_k \times \mathbf{P}^1, \mathbb{Z}) = \mathbb{Z}[C_0^+] \oplus \mathbb{Z}[L_0^*] \oplus \mathbb{Z}[P_{\star^+}].$$

**2.3.2. Integration.**

We will find relations which express  $C(\alpha|\lambda, \mu, \emptyset)$  in terms of inductively treated vertices. Let  $k$  be a positive integer satisfying

$$k > 3|\alpha| + 3|\lambda| + 3|\mu|.$$

The relations will be obtained from vanishing stable pairs invariants of the nonsingular projective toric 3-fold  $\mathcal{F}_k \times \mathbf{P}^1$  in curve class

$$\beta = |\mu| \cdot [C_0^+] + |\lambda| \cdot [L_0^*].$$

The virtual dimension of the associated moduli space is

$$\dim^{vir} P_n(\mathcal{F}_k \times \mathbf{P}^1, \beta) > 3|\alpha| + 3|\lambda| + 3|\mu|$$

since we easily compute

$$c_1(\mathcal{F}_k \times \mathbf{P}^1) \cdot [C_0^+] = k + 2, \quad c_1(\mathcal{F}_k \times \mathbf{P}^1) \cdot [L_0^*] = 2.$$

Our relations will be parameterized by two partitions  $\lambda'$  and  $\mu'$  which satisfy

$$|\lambda'| \leq |\lambda| - 1, \quad |\mu'| \leq |\mu| - 1.$$

We consider the descendent insertion

$$\prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i}([\star_0^+]) \cdot \prod_{i=1}^{\ell(\lambda')} \tau_{\lambda'_i}([\star_0^-]) \cdot \prod_{i=1}^{\ell(\mu')} \tau_{\mu'_i}([\bullet_0^+]). \tag{12}$$

Since  $\tau_r(\mathbf{p})$  imposes  $r + 2$  conditions, the descendent insertion (12) imposes at most

$$3|\alpha| + 3|\lambda'| + 3|\mu'|$$

conditions.

PROPOSITION 2. *The  $\mathbf{T}$ -equivariant integral*

$$\int_{[P_n(\mathcal{F}_k \times \mathbf{P}^1, \beta)]^{vir}} \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i}([\star_0^+]) \cdot \prod_{i=1}^{\ell(\lambda')} \tau_{\lambda'_i}([\star_0^-]) \cdot \prod_{i=1}^{\ell(\mu')} \tau_{\mu'_i}([\bullet_0^+])$$

vanishes for all  $n$ .

PROOF. The integral, viewed as  $\mathbf{T}$ -equivariant push-forward to a point, has dimension at least

$$3|\alpha| + 3|\lambda| + 3|\mu| - (3|\alpha| + 3|\lambda'| + 3|\mu'|) > 0,$$

so always vanishes. □

**2.3.3. Relation.**

We define the  $T$ -equivariant series

$$\mathbf{Z}_\beta(\alpha, \lambda', \mu') = \sum_n q^n \int_{[P_n(\mathcal{F}_k \times \mathbf{P}^1, \beta)]^{vir}} \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i}([\star_0^+]) \cdot \prod_{i=1}^{\ell(\lambda')} \tau_{\lambda'_i}([\star_0^-]) \cdot \prod_{i=1}^{\ell(\mu')} \tau_{\mu'_i}([\bullet_0^+]).$$

By Proposition 2, the series  $\mathbf{Z}_\beta(\alpha, \lambda', \mu')$  vanishes identically. We will calculate the left side of

$$\mathbf{Z}_\beta(\alpha, \lambda', \mu') = 0 \tag{13}$$

by capped localization to obtain a relation constraining capped descendent vertices.

As before, a term in the capped localization formula is principal if not all the capped descendent vertices which arise are known inductively to be rational. Because the curve class  $\beta$  lies in the fiber of the projection of  $\mathcal{F}_k \times \mathbf{P}^1$  to  $\mathbf{P}^1$ , no capped descendent vertex in the localization formula will have more than 2 legs. Capped descendent vertices at  $T$ -fixed points other than  $\star_0^+$  will have descendent partition of size less than  $|\alpha|$  and thus are known inductively to be rational.

Consider the capped 2-leg descendent vertex at  $\star_0^+$ . Let  $d_C$  be the associated edge degree of  $C_0^+$  in the capped localization formula. By the geometry of  $\mathcal{F}_k$ , the class

$$\beta - d_C[C_0^+] \in H_2(\mathcal{F}_k, \mathbb{Z})$$

is not effective if  $d_C > |\mu|$ . If  $d_C < |\mu|$ , then the vertex at  $\star_0$  is known inductively to be rational since the minimal relative partition is lower. Hence, we restrict ourselves to the principal terms where  $d_C = |\mu|$ .

Finally, consider the edge degree  $d_L$  of  $L_0^*$ . If  $d_L < |\lambda|$ , then the vertex at  $\star_0$  is known inductively to be rational since the maximal relative partition is lower. Certainly  $d_L > |\lambda|$  is not permitted. We restrict ourselves to the principal terms where  $d_L = |\lambda|$ .

In the principal terms of the capped localization of (13), precisely the following set of capped 2-leg descendent vertices occur at  $\star_0^+$ :

$$\{\mathbf{C}(\alpha|\widehat{\lambda}, \widehat{\mu}, \emptyset) \mid |\widehat{\lambda}| = |\lambda|, |\widehat{\mu}| = |\mu|\}. \tag{14}$$

The principal terms arise as displayed in Figure 2. In addition to the vertex  $\mathbf{C}(\alpha|\widehat{\lambda}, \widehat{\mu}, \emptyset)$  at  $\star_0^+$ , there are capped edges along  $C_0^+$  and  $L_0^*$  with partitions

$$|\widehat{\lambda}| = |\widehat{\lambda}'|, \quad |\widehat{\mu}| = |\widehat{\mu}'|$$

respectively. Finally, there are capped 1-leg descendent vertices

$$C(\lambda'|\widehat{\lambda}', \emptyset, \emptyset), \quad C(\mu'|\widehat{\mu}', \emptyset, \emptyset)$$

at  $\star_0^-$  and  $\bullet_0^+$ .

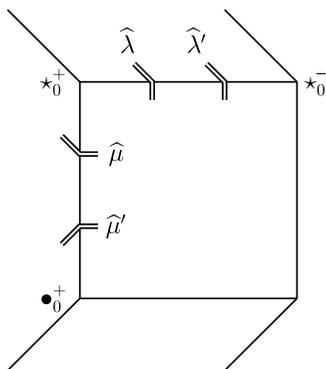


Figure 2. Principal terms.

The system of equations as the partitions  $\lambda'$  and  $\mu'$  vary has unknowns (14) parameterized by partitions of  $|\lambda|$  and  $|\mu|$ . The induction step is established if the set of equations as  $\lambda'$  and  $\mu'$  vary has maximal rank with respect to the unknowns (14).

The maximal rank statement is proven by the following observations. The capped edge matrices along  $C_0^+$  and  $L_0^*$  have maximal rank [9]. The matrix of capped 1-leg vertices

$$C(\lambda'|\widehat{\lambda}', \emptyset, \emptyset)$$

has maximal rank  $\mathcal{P}(|\lambda|)$  when  $\lambda'$  varies among partitions of size at most  $|\lambda| - 1$  by the relative/descendent correspondence discussed in Section 2.4 below. The matrix of capped 1-leg vertices

$$C(\mu'|\widehat{\mu}', \emptyset, \emptyset)$$

has maximal rank  $\mathcal{P}(|\mu|)$  when  $\mu'$  varies among partitions of size at most  $|\mu| - 1$  for the same reason. □

#### 2.4. Relative/descendent correspondence.

The remaining step in the analysis of the relations in Section 2.3.3. is to show the following maximal rank statement.

PROPOSITION 3. *Let  $d > 0$  be an integer. The matrix with coefficients*

$$C(\alpha|\lambda, \emptyset, \emptyset)$$

*as  $\alpha$  varies among partitions of size at most  $d - 1$  and  $\lambda$  varies among partitions of size  $d$  has maximal rank  $\mathcal{P}(d)$ .*

PROOF. Consider the Hilbert scheme of points  $\text{Hilb}(\mathbb{C}^2, d)$  of the plane. Let  $\mathbb{F}$  be the universal quotient sheaf on

$$\text{Hilb}(\mathbb{C}^2, d) \times \mathbb{C}^2,$$

and define the descendent<sup>8</sup>

$$\tau_c = \pi_*(\text{ch}_{2+c}(\mathbb{F})) \in H^{2c}(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q}) \quad (15)$$

where  $\pi$  is the projection

$$\pi : \text{Hilb}(\mathbb{C}^2, d) \times \mathbb{C}^2 \rightarrow \text{Hilb}(\mathbb{C}^2, d).$$

Define the tautological sheaf by

$$\pi_*(\mathbb{F}) = \mathcal{E}_d \rightarrow \text{Hilb}(\mathbb{C}^2, d).$$

The tautological sheaf is a rank  $d$  vector bundle with Chern classes

$$c_1, \dots, c_{d-1} \in H^*(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q}). \quad (16)$$

Since  $\mathcal{E}_d$  has a nonvanishing section,  $c_d$  vanishes.

By a basic result of Ellingsrud and Strömme [2],  $H^*(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q})$  is generated as an algebra by the classes (16). Hence, by Grothendieck-Riemann-Roch,  $H^*(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q})$  is also generated by  $\tau_1, \dots, \tau_{d-1}$ . Since

$$H^{>2(d-1)}(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q}) = 0,$$

---

<sup>8</sup>The Chern character of  $\mathbb{F}$  is properly supported over  $\text{Hilb}(\mathbb{C}^2, d)$ .

graded homogeneous polynomials of degree at most  $d - 1$  in the  $\tau_c$  span  $H^*(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q})$  additively.

We examine the leading  $q^d$  coefficients of the matrix coefficients,

$$\text{Coeff}_{q^d}(C(\alpha|\lambda, \emptyset, \emptyset)) = (s_1 s_2)^\ell \langle \tau_{\alpha_1} \cdots \tau_{\alpha_\ell} \mid \lambda \rangle_{\text{Hilb}(\mathbb{C}^2, d)} \tag{17}$$

where the bracket on the right denotes the  $(\mathbb{C}^*)^2$ -equivariant intersection pairing on the Hilbert scheme. On the left side of the bracket are the classes  $\tau_c$  defined above, and on the right side are the Nakajima basis elements. Since the  $(\mathbb{C}^*)^2$ -equivariant intersection pairing is non-degenerate, we need only show graded homogeneous polynomials of degree at most  $d - 1$  in the classes  $\tau_c$  span

$$H_T^*(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q}) \otimes \mathbb{Q}(s_1, s_2).$$

The result follows from the non-equivariant spanning statement since

$$\begin{aligned} \dim_{\mathbb{Q}} H^*(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q}) \\ = \dim_{\mathbb{Q}(s_1, s_2)} H_T^*(\text{Hilb}(\mathbb{C}^2, d), \mathbb{Q}) \otimes \mathbb{Q}(s_1, s_2) = \mathcal{P}(d). \end{aligned}$$

Another approach to Proposition 3 is explained in Proposition 9 of [16]. To each partition  $\gamma$  of  $d$ , we associate a partition  $\alpha(\gamma)$  of at most  $d - 1$  by removing 1 from each part of  $\gamma$ . The leading  $q^d$  term of the square matrix

$$C(\alpha(\gamma)|\lambda, \emptyset, \emptyset)$$

is easily seen to be triangular when ordered by length of partition. The diagonal elements can be shown to not vanish by the single calculation

$$s_1 s_2 \langle \tau_{c-1} \mid (c) \rangle_{\text{Hilb}(\mathbb{C}^2, d)} = \frac{1}{c!} \tag{18}$$

which appears in [17]. □

### 3. Capped descendent 3-leg vertex.

#### 3.1. Induction strategy.

We will prove Theorem 2 for capped 3-leg descendent vertices by induction by a method parallel to the 2-leg case. We may assume 3-leg vertices are of the form

$$C(\alpha|\lambda, \mu, \nu), \quad |\lambda| \geq |\mu| \geq |\nu|.$$

We know  $C(\alpha|\lambda, \mu, \nu)$  is the Laurent expansion in  $q$  of a rational function if

$$\alpha = \emptyset \quad \text{or} \quad \nu = \emptyset. \tag{19}$$

The latter case uses the established rationality for the capped 2-leg descendent vertex.

Define a partial ordering on capped 3-leg descendent via the lexicographic ordering on the vectors  $(|\alpha|, |\nu|, |\mu|, |\lambda|)$ . To prove Theorem 2 in the 3-leg case for  $C(\alpha|\lambda, \mu, \nu)$ , we assume rationality holds for all vertices  $C(\alpha'|\lambda', \mu', \nu')$  occurring earlier in the partial ordering. The ground cases of the induction are (19) so we may assume  $|\alpha|, |\nu| > 0$ . Again, our strategy depends upon whether  $|\alpha| < |\lambda|$  or  $|\alpha| \geq |\lambda|$ .

**3.2. Case  $|\alpha| < |\lambda|$ .**

**3.2.1.  $\mathcal{A}_2$  geometry.**

Let  $\mathcal{A}_2 \subset \mathcal{F}$  be any nonsingular projective toric compactification. We will only be interested in the two  $(-2)$ -curves of  $\mathcal{A}_2$ ,

$$C, \widehat{C} \subset \mathcal{A}_2.$$

No other curves of  $\mathcal{F}$  will play a role in the construction.

Let  $\widehat{\bullet}, \star, \bullet \in \mathcal{A}_2$  be the  $(\mathbb{C}^*)^2$ -fixed points. The curve  $\widehat{C}$  connects  $\widehat{\bullet}$  to  $\star$  and  $C$  connects  $\star$  to  $\bullet$ . The other  $(\mathbb{C}^*)^2$ -fixed points in  $\mathcal{F} \setminus \mathcal{A}_2$  will not play an important role.

Consider the nonsingular projective toric variety  $\mathcal{F} \times \mathbf{P}^1$ . The 3-torus

$$\mathbf{T} = (\mathbb{C}^*)^3$$

acts on  $\mathcal{F}$  via the first two factors and acts on  $\mathbf{P}^1$  via the third factor with tangent weights  $s_3$  and  $-s_3$  at the points  $0, \infty \in \mathbf{P}^1$  respectively. Let

$$\mathcal{D}_0 = \mathcal{F} \times \{0\}, \quad \mathcal{D}_\infty = \mathcal{F} \times \{\infty\}$$

be  $\mathbf{T}$ -invariant divisors of  $\mathcal{F}_2 \times \mathbf{P}^1$ . The 3-fold  $\mathcal{F} \times \mathbf{P}^1$  has six important  $\mathbf{T}$ -fixed points which we denote by

$$\widehat{\bullet}_0, \star_0, \bullet_0, \widehat{\bullet}_\infty, \star_\infty, \bullet_\infty \in \mathcal{F} \times \mathbf{P}^1$$

where the subscript indicates the coordinate in  $\mathbf{P}^1$ .

Let  $L_\infty \subset \mathcal{F} \times \mathbf{P}^1$  be the  $T$ -invariant line connecting  $\star_\infty$  to  $(\mathcal{F} \setminus \mathcal{A}_2)_\infty$ . We have

$$H_2(\mathcal{F} \times \mathbf{P}^1, \mathbb{Z}) \supset \mathbb{Z}[C] \oplus \mathbb{Z}[\widehat{C}] \oplus \mathbb{Z}[P]$$

where  $P$  is the fiber of the projection to  $\mathcal{F}$ .

**3.2.2. Integration.**

We will find relations which express  $C(\alpha|\lambda, \mu, \nu)$  in terms of inductively treated vertices. Let  $\mu'$  and  $\nu'$  be any partitions. The relations will be obtained from vanishing stable pairs invariants of  $\mathcal{F} \times \mathbf{P}^1/\mathcal{D}_\infty$  in curve class

$$\beta = |\mu| \cdot [C] + |\nu| \cdot [\widehat{C}] + (|\lambda| + |\mu'| + |\nu'|) \cdot [P].$$

The virtual dimension of the associated moduli space is

$$\dim^{vir} P_n(\mathcal{F} \times \mathbf{P}^1/\mathcal{D}_\infty, \beta) = 2|\lambda| + 2|\mu'| + 2|\nu'|.$$

We impose relative conditions along  $\mathcal{D}_\infty$  in the Nakajima basis by the partition  $\lambda \cup \mu' \cup \nu'$  weighted by  $[L_\infty] \in H_T^2(\mathcal{D}_\infty, \mathbb{Q})$  for the parts of  $\lambda$  and

$$[\bullet_\infty], [\widehat{\bullet}_\infty] \in H_T^4(\mathcal{D}_\infty, \mathbb{Q})$$

for the parts of  $\mu'$  and  $\nu'$  respectively. We denote the relative condition by  $\mathbf{r}(\lambda, \mu', \nu')$ . After imposing  $\mathbf{r}(\lambda, \mu', \nu')$ , the virtual dimension drops to

$$\dim^{vir} P_n(\mathcal{F} \times \mathbf{P}^1/\mathcal{D}_\infty, \beta)_{\mathbf{r}(\lambda, \mu', \nu')} = |\lambda| + |\mu'| - \ell(\mu') + |\nu'| - \ell(\nu') \geq |\lambda|.$$

We now specify the descendent insertion<sup>9</sup> by

$$\prod_{i=1}^{\ell} \tau_{\alpha_i}([D_0]).$$

The descendent insertion imposes  $|\alpha|$  conditions. Therefore, the integral

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<sup>9</sup>Unlike the 2-leg case, the descendent  $\tau_{\alpha_1}$  is not treated separately.

$$\int_{[P_n(\mathcal{F} \times \mathbf{P}^1 / \mathcal{D}_{\infty, \beta})_{r(\lambda, \mu', \nu')}]^{vir}} \prod_{i=1}^{\ell} \tau_{\alpha_i}([D_0]), \tag{20}$$

viewed as  $\mathbf{T}$ -equivariant push-forward to a point, has dimension at least

$$|\lambda| - |\alpha| > 0.$$

Hence, we conclude the following vanishing.

PROPOSITION 4. *The  $\mathbf{T}$ -equivariant integral (20) vanishes for all  $n$ .*

**3.2.3. Relation.**

We define the  $\mathbf{T}$ -equivariant series

$$\mathbf{Z}_{\beta}(\alpha, \lambda, \mu', \nu') = \sum_n q^n \int_{[P_n(\mathcal{F} \times \mathbf{P}^1 / \mathcal{D}_{\infty, \beta})_{r(\lambda, \mu', \nu')}]^{vir}} \prod_{i=1}^{\ell} \tau_{\alpha_i}([D_0])$$

obtained from the integrals (20). By Proposition 4, the series  $\mathbf{Z}_{\beta}(\alpha, \lambda, \mu', \nu')$  vanishes identically. We will calculate the left side of

$$\mathbf{Z}_{\beta}(\alpha, \lambda, \mu', \nu') = 0 \tag{21}$$

by capped localization to obtain a relation constraining capped descendent vertices.

Two types of capped rubber contributions arise over  $\infty \in \mathbf{P}^1$  in the  $\mathbf{T}$ -equivariant localization formula for  $\mathbf{Z}_{\beta}(\alpha, \lambda, \mu', \nu')$ ,

- (i) capped 1-leg rubber corresponding to  $\mathbf{T}$ -fixed stable pairs with components contracted to  $\mathcal{F} \setminus \mathcal{A}_2$ ,
- (ii) capped rubber contributions of  $\mathcal{A}_2 \times \mathbf{P}^1$  relative to the divisor over  $\infty$ .

The capped contributions (i) are just 1-leg with no descendents, so are rational. The capped contributions (ii) are proven to be rational in Lemma 6 of [9] relying on the results of [7], [8]. See Section 5 of [13] for the stable pairs results.

We now analyze the capped localization of  $\mathbf{Z}_{\beta}(\alpha, \lambda, \mu', \nu')$  over  $0 \in \mathbf{P}^1$ . As before, a term in the capped localization formula is principal if not all the capped descendent vertices which arise are known inductively to be rational.

First consider the descendent insertions. The descendents

$$\prod_{i=1}^{\ell} \tau_{\alpha_i}([D_0])$$

are free to distribute to the  $T$ -fixed points over  $0 \in \mathbf{P}^1$ . By the choice of  $\beta$ , a capped 3-leg descendent vertex can only occur at  $\star_0$ . Descendents which distribute away from to  $\star_0$  will lie on capped 1-leg or 2-leg descendent vertices. We conclude all capped descendent vertices are known inductively to be rational except possibly when all the descendents lie on  $\star_0$ .

Next consider the edge degree  $\widehat{d}$  of  $\widehat{C}$  over  $0 \in \mathbf{P}^1$  in the capped localization formula. If  $\widehat{d} < |\nu|$ , then the capped descendent vertex at  $\star_0$  is known inductively to be rational since the minimal relative partition is lower. Hence, we restrict ourselves to the principal terms where  $\widehat{d} = |\nu|$ .

Similarly, consider the edge degree  $d$  of  $C$  over  $0 \in \mathbf{P}^1$  in the capped localization formula. If  $d < |\mu|$ , then the capped descendent vertex at  $\star_0$  is known inductively to be rational since the middle relative partition is lower. Hence, we further restrict ourselves to the principal terms where  $d = |\mu|$ .

Since all of  $|\mu| \cdot [C] + |\nu| \cdot [\widehat{C}]$  occurs over  $0 \in \mathbf{P}^1$ , the rubber over  $\infty \in \mathbf{P}^1$  is all 1-leg. By the relative conditions imposed by  $\nu'$  with weights  $[\widehat{\bullet}_\infty]$ , the principal terms all have a capped 2-leg vertex with no descendents at  $\widehat{\bullet}_0$  with outgoing partitions of size  $|\nu|$  and  $|\nu'|$ . Similarly, the principal terms all have a capped 2-leg vertex with no descendents at  $\bullet_0$  with outgoing partitions of size  $|\mu|$  and  $|\mu'|$ .

Finally, consider the relative conditions  $\lambda$  weighted by  $[L_\infty]$ . The weight allows the parts to distribute to  $\mathcal{F} \setminus \mathcal{A}_2$ . Such a distribution would result in an inductively treated capped descendent vertex at  $\star_0$  with lower maximal relative partition.

In the principal terms of the capped localization of (21), precisely the following set of capped 3-leg descendent vertices occur at  $\star_0$ :

$$\{C(\alpha|\lambda, \widehat{\mu}, \widehat{\nu}) \mid |\widehat{\mu}| = |\mu|, |\widehat{\nu}| = |\nu|\}. \tag{22}$$

The induction step is established as before if the set of equations as  $\mu'$  and  $\nu'$  varies has maximal rank<sup>10</sup> with respect to the unknowns (22). Again, the maximal rank statement follows from Lemma 8 of [9]. □

**3.3. Case  $|\alpha| \geq |\lambda|$ .**

**3.3.1. Integration.**

We will find relations which express  $C(\alpha|\lambda, \mu, \nu)$  in terms of inductively treated vertices. Let  $k$  be a positive integer satisfying

$$k > 3|\alpha| + 3|\lambda| + 3|\mu| + 3|\nu|.$$

The relations will be obtained from vanishing stable pairs invariants of the non-

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<sup>10</sup>Maximal rank here is equal to the number  $\mathcal{P}(|\mu|) \cdot \mathcal{P}(|\nu|)$ .

singular projective toric 3-fold  $\mathcal{F}_k \times \mathbf{P}^1$  in curve class

$$\beta = |\nu| \cdot [C_0^+] + |\mu| \cdot [L_0^*] + |\lambda| \cdot [P_{\star+}]$$

following the conventions of Section 2.3.1. The virtual dimension of the associated moduli space is

$$\dim^{vir} P_n(\mathcal{F}_k \times \mathbf{P}^1, \beta) > 3|\alpha| + 3|\lambda| + 3|\mu| + 3|\nu|$$

since we easily compute

$$c_1(\mathcal{F}_k \times \mathbf{P}^1) \cdot [C_0^+] = k + 2, \quad c_1(\mathcal{F}_k \times \mathbf{P}^1) \cdot [L_0^*] = 2, \quad c_1(\mathcal{F}_k \times \mathbf{P}^1) \cdot [P_{\star+}] = 2.$$

Our relations will be parameterized by three partitions  $\lambda', \mu', \nu'$  which satisfy

$$|\lambda'| \leq |\lambda| - 1, \quad |\mu'| \leq |\mu| - 1, \quad |\nu'| \leq |\nu| - 1.$$

Consider the descendent insertion

$$\prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i}([\star_0^+]) \cdot \prod_{i=1}^{\ell(\lambda')} \tau_{\lambda'_i}([\star_\infty^+]) \cdot \prod_{i=1}^{\ell(\mu')} \tau_{\mu'_i}([\star_0^-]) \cdot \prod_{i=1}^{\ell(\nu')} \tau_{\nu'_i}([\bullet_0^+]). \tag{23}$$

Since  $\tau_r(\mathbf{p})$  imposes  $r + 2$  conditions, the descendent insertion (23) imposes at most

$$3|\alpha| + 3|\lambda'| + 3|\mu'| + 3|\nu'|$$

conditions.

PROPOSITION 5. *The  $\mathbf{T}$ -equivariant integral*

$$\int_{[P_n(\mathcal{F}_k \times \mathbf{P}^1, \beta)]^{vir}} \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i}([\star_0^+]) \cdot \prod_{i=1}^{\ell(\lambda')} \tau_{\lambda'_i}([\star_\infty^+]) \cdot \prod_{i=1}^{\ell(\mu')} \tau_{\mu'_i}([\star_0^-]) \cdot \prod_{i=1}^{\ell(\nu')} \tau_{\nu'_i}([\bullet_0^+])$$

vanishes for all  $n$ .

PROOF. The integral, viewed as  $\mathbf{T}$ -equivariant push-forward to a point, has dimension at least

$$3|\alpha| + 3|\lambda| + 3|\mu| + 3|\nu| - (3|\alpha| + 3|\lambda'| + 3|\mu'| + 3|\nu'|) > 0,$$

so always vanishes. □

**3.3.2. Relation.**

We define the  $T$ -equivariant series

$$\begin{aligned} \mathbf{Z}_\beta(\alpha, \lambda', \mu', \nu') = \sum_n q^n \int_{[P_n(\mathcal{F}_k \times \mathbf{P}^1, \beta)]^{vir}} & \prod_{i=1}^{\ell(\alpha)} \tau_{\alpha_i}([\star_0^+]) \cdot \prod_{i=1}^{\ell(\lambda')} \tau_{\lambda'_i}([\star_\infty^+]) \\ & \cdot \prod_{i=1}^{\ell(\mu')} \tau_{\mu'_i}([\star_0^-]) \cdot \prod_{i=1}^{\ell(\nu')} \tau_{\nu'_i}([\bullet_0^+]). \end{aligned}$$

By Proposition 5, the series  $\mathbf{Z}_\beta(\alpha, \lambda', \mu', \nu')$  vanishes identically. We will calculate the left side of

$$\mathbf{Z}_\beta(\alpha, \lambda', \mu', \nu') = 0 \tag{24}$$

by capped localization to obtain a relation constraining capped descendent vertices.

Capped descendent vertices at  $T$ -fixed points other than  $\star_0^+$  will have descendent partition of size less than  $|\alpha|$  and thus are known inductively to be rational. Consider the capped 3-leg descendent vertex at  $\star_0^+$ . Let  $d_C$  be the associated edge degree of  $C_0^+$  in the capped localization formula. By the geometry of  $\mathcal{F}_k$ , the class

$$\beta - d_C[C_0^+] \in H_2(\mathcal{F}_k, \mathbb{Z})$$

is not effective if  $d_C > |\nu|$ . If  $d_C < |\nu|$ , then the vertex at  $\star_0$  is known inductively to be rational since the minimal relative partition is lower. Hence, we restrict ourselves to the principal terms where  $d_C = |\nu|$ .

Consider the edge degree  $d_L$  of  $L_0^*$ . If  $d_L < |\mu|$ , then the vertex at  $\star_0$  is known inductively to be rational since the middle relative partition is lower. As before,  $d_L > |\mu|$  is not permitted. We restrict ourselves to the principal terms where  $d_L = |\mu|$ .

Finally, consider the edge degree  $d_P$  of  $P_{\star^+}$ . If  $d_P < |\lambda|$ , then the vertex at  $\star_0$  is known inductively to be rational since the maximal relative partition is lower. Again,  $d_P > |\lambda|$  is not permitted. We restrict ourselves to the principal terms where  $d_P = |\lambda|$ .

In the principal terms of the capped localization of (24), precisely the following

set of capped 3-leg descendent vertices occur at  $\star_0^+$ :

$$\{C(\alpha|\widehat{\lambda}, \widehat{\mu}, \widehat{\nu}) \mid |\widehat{\lambda}| = |\lambda|, |\widehat{\mu}| = |\mu|, |\widehat{\nu}| = |\nu|\}.$$

In the principal terms, the vertex  $C(\alpha|\widehat{\lambda}, \widehat{\mu}, \widehat{\nu})$  at  $\star_0^+$ , is accompanied by capped edges along  $C_0^+$ ,  $L_0^*$ , and  $P_{\star+}$ . Finally, there are capped 1-leg descendent vertices

$$C(\lambda'|\widehat{\lambda}', \emptyset, \emptyset), \quad C(\mu'|\widehat{\mu}', \emptyset, \emptyset), \quad C(\nu'|\widehat{\nu}', \emptyset, \emptyset),$$

at  $\star_\infty^+$ ,  $\star_0^-$ , and  $\bullet_0^+$  respectively. The required maximal rank condition follows from Proposition 3. □

**4. Log Calabi-Yau geometry.**

**4.1. Relative/descendent correspondence.**

Let  $X$  be a nonsingular projective 3-fold, and let  $S \subset X$  be a nonsingular anti-canonical  $K3$  surface. Let  $N$  be the normal bundle of  $S$  in  $X$ . Let

$$S_0, S_\infty \subset P(\mathcal{O}_S \oplus N) \rightarrow S$$

be the sections determined by the summand  $\mathcal{O}_S$  and  $N$  respectively. Let

$$\iota_0 : S \hookrightarrow P(\mathcal{O}_S \oplus N)$$

be the section onto  $S_0$ .

Let  $\mathcal{B}$  be a fixed self-dual basis of the cohomology of  $S$ . Recall a Nakajima basis element in the Hilbert scheme  $\text{Hilb}(S, n)$  is a cohomology weighted partition  $\mu$  of  $n$ ,

$$((\mu_1, \gamma_1), \dots, (\mu_\ell, \gamma_\ell)), \quad n = \sum_{i=1}^{\ell} \mu_i, \quad \gamma_i \in \mathcal{B}.$$

Such a weighted partition determines a descendent insertion

$$\tau[\mu] = \prod_{i=1}^{\ell} \tau_{\mu_i-1}(\iota_{0*}(\gamma_i)).$$

By  $K3$  vanishing arguments explained in Section 4.3, the stable pairs invariants of the relative 3-fold geometry  $P(\mathcal{O}_S \oplus N)/S_\infty$  are nontrivial only for curves

classes in the fibers of

$$P(\mathcal{O}_S \oplus N)/S_\infty \rightarrow S. \tag{25}$$

Define the partition function for the relative geometry by

$$Z_d^{P(\mathcal{O}_S \oplus N)/S_\infty}(\mu, \nu) = \sum_n \langle \tau[\mu] \mid \nu \rangle_{n,d}^{P(\mathcal{O}_S \oplus N)/S_\infty} q^n. \tag{26}$$

Here,  $\mu$  and  $\nu$  are partitions of  $d$  weighted by  $\mathcal{B}$ . The curve class is  $d$  times a fiber of (25). By further vanishing, only the leading  $q^d$  terms of (26) are possibly nonzero.

PROPOSITION 6. *Let  $d > 0$  be an integer. The square matrix indexed by  $\mathcal{B}$ -weighted partitions of  $d$  with coefficients*

$$Z_d^{P(\mathcal{O}_S \oplus N)/S_\infty}(\mu, \nu)$$

*has maximal rank.*

PROOF. We need only consider the leading  $q^d$  coefficients to prove the maximal rank statement. By standard arguments, dimension constraints imply the matrix is upper triangular with respect to a suitable length ordering. The diagonal elements are non-zero by the calculation (18).

More precisely, the lowest Euler characteristic moduli space of stable pairs

$$P_d(P(\mathcal{O}_S \oplus N)/S_\infty, dF)$$

in  $d$  times the fiber class is canonically isomorphic to  $\text{Hilb}(S, d)$ . The  $q^d$  coefficients of  $Z_d^{P(\mathcal{O}_S \oplus N)/S_\infty}(\mu, \nu)$  are simply the pairings<sup>11</sup> of the descendents  $\tau[\mu]$  with the Nakajima basis elements  $\nu$ ,

$$\int_{\text{Hilb}(S,d)} \tau[\mu] \cdot \nu. \tag{27}$$

The codimension of the class  $\tau[\mu]$  on  $\text{Hilb}(S, d)$  is

$$d - \ell(\mu) + \sum_i \text{codim}_{\mathbb{C}}(\gamma_i).$$

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<sup>11</sup>See Lemma of [16] for an equivariant study the same pairing for  $\text{Hilb}(\mathbb{C}, d)$ .

Similarly the dimension of  $\nu$  is

$$d - \ell(\nu) + \sum_j \dim_{\mathbb{C}}(\delta_j)$$

where the cohomology weights of  $\nu$  are  $\delta_j$ . Certainly the pairing (27) vanishes unless

$$\sum_i \operatorname{codim}_{\mathbb{C}}(\gamma_i) = \sum_j \dim_{\mathbb{C}}(\delta_j) + \ell(\mu) - \ell(\nu). \quad (28)$$

We first establish the vanishing

$$\int_{\operatorname{Hilb}(S, 2d)} \tau[\mu] \cdot \nu = 0 \quad (29)$$

if  $\ell(\mu) > \ell(\nu)$ . The proof is by considering cycles in the symmetric product

$$\epsilon : \operatorname{Hilb}(S, d) \rightarrow \operatorname{Sym}^d(S).$$

The cycle  $\nu$  has image<sup>12</sup> under  $\epsilon$  of dimension  $\sum_i \dim_{\mathbb{C}}(\delta_j)$ . On the other hand,  $\tau[\mu]$  is supported on  $\operatorname{Hilb}(S, d)$  over a cycle of codimension at least  $\sum_i \operatorname{codim}_{\mathbb{C}}(\gamma_i)$  in  $\operatorname{Sym}^d(S)$  determined just by the incidence conditions with the cycles  $\gamma_i$ . If  $\ell(\mu) > \ell(\nu)$ , the latter codimension in  $\operatorname{Sym}^d(S)$  exceeds the dimension of  $\epsilon(\nu)$  in  $\operatorname{Sym}^d(S)$  by (28). The corresponding cycles in  $\operatorname{Sym}^d(S)$  can then be chosen with empty incidence — proving the vanishing (29).

If  $\ell(\mu) = \ell(\nu)$ , the corresponding cycles  $\operatorname{Sym}^d(S)$  will intersect in finitely many points. In fact, the intersection can easily be shown to vanish unless there is a bijection  $\sigma$  satisfying  $\gamma_{\sigma(j)} = \delta_j^{\vee}$ . Further dimension constraints then require  $\mu_{\sigma(j)} = \nu_j$  for nonvanishing. Finally, the diagonal pairings are determined by (18).  $\square$

We note Proposition 6 holds for any nonsingular projective surface  $S$ . The proof does not use any special properties of  $K3$  surfaces.

#### 4.2. Rationality.

Let  $X$  be a nonsingular projective toric 3-fold with an anti-canonical  $K3$  divisor

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<sup>12</sup>We are considering the actual set theoretic image for general choices of the classes  $\delta_j$ .

$$\iota : S \hookrightarrow X.$$

Let  $\beta \in H_2(X, \mathbb{Z})$  be a curve class, and let

$$d = \int_{\beta} [S].$$

Let  $\mu$  be a  $\mathcal{B}$ -weighted partition of  $d$ . We associate a descendent insertion<sup>13</sup> to  $\mu$  as before,

$$\tau[\mu] = \prod_{i=1}^{\ell} \tau_{\mu_i - 1}(\iota_*(\gamma_i)).$$

Consider the descendent partition function of  $X$ ,

$$\mathbf{Z}_{\beta}^X(\mathbf{I} \cdot \tau[\mu]) = \sum_n \langle \mathbf{I} \cdot \tau[\mu] \rangle_{n, \beta}^X q^n \tag{30}$$

where  $\mathbf{I}$  is any descendent insertion for  $X$ .

We may degenerate  $X$  to the normal cone of  $S$ . Then, the degeneration formula expresses  $\mathbf{Z}_{\beta}^X(\mathbf{I} \cdot \tau[\mu])$  in terms of the relative geometries

$$(X/S) \quad \text{and} \quad (\mathbf{P}(\mathcal{O}_S \oplus N), S_{\infty}).$$

The partition function  $\mathbf{Z}_{\beta}^X(\mathbf{I} \cdot \tau[\mu])$  is rational by Theorem 1. The relative theory of  $(\mathbf{P}(\mathcal{O}_S \oplus N), S_{\infty})$  yields rational functions in  $q$  by the vanishings discussed above. The invertibility of Proposition 6 applied to the degeneration formula inductively implies the following result (of which Theorem 3 is a special case).

**THEOREM 4.** *Let  $X$  be a nonsingular projective toric 3-fold with an anti-canonical K3 section  $S$ . The partition function  $\mathbf{Z}_{\beta}^{X/S}(\mathbf{I} \mid \mu)$  is the Laurent expansion of a rational function in  $q$ .*

**4.3. K3 vanishing.**

The vanishing<sup>14</sup> of the stable pairs invariants of the relative 3-fold geometry  $\mathbf{P}(\mathcal{O}_S \oplus N)/S_{\infty}$  in all cases except for the minimal Euler characteristic in the

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<sup>13</sup>We consider here descendents of the cohomology  $H^*(X, \mathbb{Q})$ , not the  $\mathbf{T}$ -equivariant cohomology.

<sup>14</sup>The results here were communicated to the authors by R. Thomas.

fibers classes of

$$\epsilon : \mathbf{P}(\mathcal{O}_S \oplus N)/S_\infty \rightarrow S \tag{31}$$

can be seen by constructing a trivial 1-dimension quotient of the obstruction theory, see [6], [13].

We consider first the absolute geometry  $\mathbf{P} = \mathbf{P}(\mathcal{O}_S \oplus N)$ . To start, fix a trivialization of the canonical bundle of the  $K3$  surface  $S$ ,

$$\omega_S = \mathcal{O}_S. \tag{32}$$

Let  $T_\epsilon$  be the tangent bundle to the fibers of  $\epsilon$ , and let

$$\mathcal{I}^\bullet = [\mathcal{O}_X \xrightarrow{s} F]$$

be a stable pair on  $\mathbf{P}$ . A canonical map

$$T_\epsilon \rightarrow \mathcal{E}xt^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \tag{33}$$

is obtained by infinitesimal translation along the vector field. On the right,  $\mathcal{E}xt^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0$  is traceless sheaf Ext. After tensoring (33) with  $\omega_\epsilon$ , the dual of  $T_\epsilon$ , we find

$$\mathcal{O}_\mathbf{P} \rightarrow \mathcal{E}xt^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0 \otimes \omega_\epsilon = \mathcal{E}xt^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet \otimes \omega_\mathbf{P})_0$$

since  $\omega_\epsilon$  is canonically isomorphic to  $\omega_\mathbf{P}$  using the fixed trivialization (32). Since all the lower  $\mathcal{E}xt$  sheaves vanish [19], we get a canonical map

$$\mathcal{O}_\mathbf{P} \rightarrow \mathcal{R}Hom(\mathcal{I}^\bullet, \mathcal{I}^\bullet)_0[-1] \quad \text{via} \quad \mathcal{E}xt^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet \otimes \omega_\mathbf{P})_0 \rightarrow \mathcal{R}Hom(\mathcal{I}^\bullet, \mathcal{I}^\bullet \otimes \omega_\mathbf{P})_0[-1].$$

After taking hypercohomology and applying Serre duality, we obtain

$$\mathbb{C} \rightarrow \text{Ext}^1(\mathcal{I}^\bullet, \mathcal{I}^\bullet \otimes \omega_\mathbf{P}), \quad \text{Ext}^2(\mathcal{I}^\bullet, \mathcal{I}^\bullet) \rightarrow \mathbb{C}. \tag{34}$$

If the stable pair  $\mathcal{I}^\bullet$  is not of minimal Euler characteristic in a fiber class, the map on the left in (34) is not trivial (as vertical translation then induces a nontrivial deformation of  $\mathcal{I}^\bullet$ ). Hence, the map on the right in (34) provides the desired trivial quotient of the obstruction theory.

For the relative geometry  $\mathbf{P}(\mathcal{O}_S \oplus N)/S_\infty$ , we consider the deformation theory

relative to the Artin stack of destabilizations of the target. Then, the construction of the trivial quotient is exactly as above. A destabilization  $\mathcal{D}$  of  $\mathbf{P}(\mathcal{O}_S \oplus N)/S_\infty$  maps canonically to  $S$ ,

$$\epsilon : \mathcal{D} \rightarrow S.$$

We work with the relative dualizing sheaf  $\omega_\epsilon$  and define  $T_\epsilon = \omega_\epsilon^*$ . The rest is the same.

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