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An explicit dimension formula for Siegel cusp forms with respect to the non-split symplectic groups

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Abstract. We give an explicit dimension formula for the spaces of vector valued Siegel cusp forms of degree two with respect to a certain kind of arithmetic subgroups of the non-split $\boldsymbol{Q}$-forms of $Sp(2, \boldsymbol{R})$.

1. Introduction.

The purpose of this paper is to give an explicit dimension formula for the spaces of vector valued Siegel cusp forms of degree two with respect to a certain kind of discrete subgroups of the non-split $\boldsymbol{Q}$-forms of $Sp(2, \boldsymbol{R})$.

In general, the dimensions of the spaces of Siegel modular forms can be calculated, in principle, by Selberg trace formula or Riemann-Roch theorem if the weights are sufficiently large, but there are many difficulties in actual calculations. Our concern is explicit dimension formulae, *i.e.*, elementary functions of weights which give numerical values of dimensions. Such formulae are very useful for determining explicit ring structures of Siegel modular forms. In addition, they can be used also for studying a possible correspondence between Siegel modular forms for different discrete subgroups by means of comparisons of dimension formulae.

Our main result is Theorem 3.1, which is shortly explained below. Let B be an indefinite division quaternion algebra over $\boldsymbol{Q}$ with discriminant D. Let $\mathfrak{O}$ be the maximal order of B, which is unique up to conjugation. If we take a positive divisor D_1 of D and put $D_2 := D/D_1$, then there is the unique maximal two-sided ideal $\mathfrak{A}$ of $\mathfrak{O}$ such that $\mathfrak{A} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p = \mathfrak{O}_p$ if $p \mid D_1$ or $p \nmid D$, and $\mathfrak{A} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p = \pi\mathfrak{O}_p$ if $p \mid D_2$, where π is a prime element of $\mathfrak{O}_p$. We consider the unitary group of the quaternion hermitian space of rank two and denote by $\Gamma(D_1, D_2)$ the stabilizer of the maximal lattice $(\mathfrak{A}, \mathfrak{O})$, namely we define

$$\Gamma(D_1, D_2) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2; B) \,\middle|\, g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{array}{l} a, d \in \mathfrak{O}, \\ b \in \mathfrak{A}^{-1}, c \in \mathfrak{A} \end{array} \right\}.$$

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(see Section 2.3.) This group can be regarded as a discrete subgroup of $Sp(2;\boldsymbol{R})$. Our main theorem (Theorem 3.1) is an explicit formula for dimension of $S_{k,j}(\Gamma(D_1,D_2))$, the space of vector valued Siegel cusp forms of weight $\det^k \otimes \mathrm{Sym}_j$ with respect to $\Gamma(D_1,D_2)$ for $k \geq 5$. For example, we have

$$\begin{aligned}\dim_{\boldsymbol{C}} S_{k,0}(\Gamma(1,6)) = {} & \frac{4k^3-18k^2+696k-1737+(-1)^k\cdot 225}{1440} \\ & + \frac{[0,-1,1;3]_k}{9} + \frac{[1,0,0,-1;4]_k}{4} + \frac{4[1,0,0,-1,0;5]_k}{5},\end{aligned}$$

where $[a_0,\dots,a_{m-1};m]_k$ is the function of k which takes the value a_i if $k \equiv i \bmod m$.

Explicit dimension formula for the spaces of Siegel cusp forms of degree two has been studied by many mathematicians. Among them, Arakawa [**Ara75**], [**Ara81**], Hashimoto [**Has84**] and Wakatsuki [**Wak**, Theorem 6.1] treated the non-split $\boldsymbol{Q}$-forms. Hashimoto [**Has84**] obtained an explicit dimension formula for scalar valued Siegel cusp forms for $\Gamma(D,1)$ by using Selberg trace formula, and Wakatsuki [**Wak**, Theorem 6.1] generalized it to the vector valued Siegel cusp forms for $\Gamma(D,1)$. Our main result of this paper (Theorem 3.1) is a generalization of [**Has84**] and [**Wak**, Theorem 6.1] to any $\Gamma(D_1,D_2)$. We obtain the result by essentially the same method as [**Has84**], but our situation is much more complicated.

We make some remarks on some known facts which are used to obtain our result. We divide $\Gamma=\Gamma(D_1,D_2)$ into disjoint union of four subsets $\Gamma^{(e)}$, $\Gamma^{(u)}$, $\Gamma^{(qu)}$ and $\Gamma^{(h)}$ as follows:

- (i) $\Gamma^{(e)}$ consists of torsion elements of Γ.
- (ii) $\Gamma^{(u)}$ consists of non-semi-simple elements of Γ whose semi-simple factors are 1_4 or -1_4.
- (iii) $\Gamma^{(qu)}$ consists of non-semi-simple elements of Γ whose semi-simple factors belong to $\Gamma^{(e)}$ other than $\pm 1_4$.
- (iv) $\Gamma^{(h)}$ consists of the other elements of Γ than the above three types.

We denote the contributions to dimension formula of each subset above by $I(\Gamma^{(e)})_{k,j}$, $I(\Gamma^{(u)})_{k,j}$, $I(\Gamma^{(qu)})_{k,j}$ and $I(\Gamma^{(h)})_{k,j}$. It is known that $I(\Gamma^{(h)})_{k,j}=0$ and

$$\dim_{\boldsymbol{C}} S_{k,j}(\Gamma) = I(\Gamma^{(e)})_{k,j} + I(\Gamma^{(u)})_{k,j} + I(\Gamma^{(qu)})_{k,j}.$$

As for the contribution $I(\Gamma^{(e)})_{k,j}$, we can evaluate it by means of the method developed by Hashimoto and Ibukiyama. It is shown by Hashimoto and Ibukiyama

[**HI80**], [**Has83**] that the formula for $I(\Gamma^{(e)})_{k,j}$ can be expressed adelically and can be reduced to local computation (cf. Theorem 4.1). We do not need to calculate the local data "$c_p(g, R_p, \Lambda_p)$" since they have been obtained in [**HI80**] and [**HI83**]. (Although B is definite in the case of [**HI80**] and [**HI83**], there is no difference if being localized.) So, our main task is to combine the local data and determine G-conjugacy classes which appear in the first sum in Theorem 4.1. It is still a complicated work, and the details will be explained in Section 4. (The way of combining local data is different from that of [**HI80**] and [**HI83**] since B is indefinite in our case.) On the other hand, as for the contributions $I(\Gamma^{(u)})_{k,j}$ and $I(\Gamma^{(qu)})_{k,j}$, we can not reduce them to local calculations. Wakatsuki [**Wak**] gave an arithmetic formula for the contributions of them, but one still have to carry out detailed calculation to obtain an explicit formula. More precisely, we need to determine a complete system of representatives of Γ-conjugacy classes of "families" (cf. Proposition 5.7, 5.8, 5.9) and calculate some data for them. Arakawa has calculated the contribution $I(\Gamma^{(u)})_{k,j}$ in his master thesis [**Ara75**], but we prove it again in Section 5 by means of Wakatsuki's formula (e-2) since [**Ara75**] was not published with enough generality. Hashimoto calculated the contribution $I(\Gamma^{(qu)})_{k,j}$ in the case where $D_1 = D$ and $D_2 = 1$ in [**Has84**]. We can calculate $I(\Gamma^{(qu)})_{k,j}$ also in the general case by almost the same way.

We organize this paper as follows. In Section 2, we will review Siegel cusp forms in Subsection 2.1, and give in Subsection 2.2 and 2.3 the precise definition of the discrete subgroup $\Gamma(D_1, D_2)$. In Section 3, we will state our main theorem (Theorem 3.1) which will be proved in Section 4 and 5. In Section 4, we evaluate the contribution $I(\Gamma^{(e)})_{k,j}$. First, we quote the formula of Hashimoto and Ibukiyama (Theorem 4.1), and then we evaluate $H_1, \ldots, H_{12}$ of Theorem 3.1 in Subsection 4.1–4.12. In Section 5, we evaluate the contribution $I(\Gamma^{(u)})_{k,j}$ and $I(\Gamma^{(qu)})_{k,j}$. We evaluate I_1, I_2 and I_3 of Theorem 3.1 in Subsection 5.1, 5.2 and 5.3 respectively. In Section 6, we give some numerical examples for our main theorem.

Finally, we want to refer to a possible application of our result. In the case where B is definite, Ibukiyama has been studying a generalization of Eichler-Jacquet-Langlands correspondence to the case of $Sp(2)$ (cf. [**Ibu85**], [**HI85**], [**Ibu07a**]). He obtained some relations of dimension formulae and conjectured a correspondence of discrete subgroups. Moreover, he calculated some numerical examples of coincidence of Euler factors of L-functions in [**Ibu84**]. On the other hand, in our case B is indefinite. The author constructed explicit generators of the graded ring of scalar valued Siegel modular forms for $\Gamma(1, 6)$ as an application of our dimension formula, which will appear in a forthcoming paper and can be used for calculating Hecke operators concretely. Our dimension formula and numerical calculations of Hecke eigenvalues will be eventually used for comparisons similar to the above.

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2. Preliminaries.

2.1. Siegel cusp forms.

Let $Sp(2;\boldsymbol{R})$ be the real symplectic group of degree two, i.e.

$$Sp(2;\boldsymbol{R}) = \left\{ g \in GL(4,\boldsymbol{R}) \,\middle|\, g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} {}^t g = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}.$$

Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two, i.e.

$$\mathfrak{H}_2 = \{ Z \in M(2;\boldsymbol{C}) \mid {}^t Z = Z,\ \mathrm{Im}(Z) \text{ is positive definite} \}.$$

The group $Sp(2;\boldsymbol{R})$ acts on $\mathfrak{H}_2$ by

$$\gamma\langle Z \rangle := (AZ+B)(CZ+D)^{-1}$$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(2;\boldsymbol{R})$ and $Z \in \mathfrak{H}_2$.

Let Γ be a discrete subgroup of $Sp(2;\boldsymbol{R})$ such that $\mathrm{vol}(\Gamma\backslash\mathfrak{H}_2) < \infty$. Let $\rho_{k,j} : GL(2;\boldsymbol{C}) \to GL(j+1;\boldsymbol{C})$ be the irreducible rational representation of the signature $(j+k,k)$ for $k,j \in \boldsymbol{Z}_{\geq 0}$, i.e. $\rho_{k,j} = \det^k \otimes Sym_j$, where Sym_j is the symmetric j-tensor representation of $GL(2;\boldsymbol{C})$. We denote by $S_{k,j}(\Gamma)$ the space of Siegel cusp forms of weight $\rho_{k,j}$ with respect to Γ, i.e. the space which consists of holomorphic function $f : \mathfrak{H}_2 \to \boldsymbol{C}^{j+1}$ satisfying the following two conditions:

(i) $f(\gamma\langle Z \rangle) = \rho_{k,j}(CZ+D) f(Z), \quad \text{for } \forall\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma,\ \forall Z \in \mathfrak{H}_2,$

(ii) $\left|\rho_{k,j}(\mathrm{Im}(Z)^{1/2}) f(Z)\right|_{\boldsymbol{C}^{j+1}}$ is bounded on $\mathfrak{H}_2$,

where we define $|u|_{\boldsymbol{C}^{j+1}} = ({}^t u \overline{u})^{1/2}$ for $u \in \boldsymbol{C}^{j+1}$. It is known that $S_{k,j}(\Gamma)$ is a finite dimensional $\boldsymbol{C}$-vector space.

2.2. The non-split $\boldsymbol{Q}$-forms of $Sp(2;\boldsymbol{R})$.

Let B be an indefinite quaternion algebra over $\boldsymbol{Q}$. We fix an isomorphism $B \otimes_{\boldsymbol{Q}} \boldsymbol{R} \simeq M(2;\boldsymbol{R})$ and we identify B with a subalgebra of $M(2;\boldsymbol{R})$. Let D be a product of all prime numbers p for which $B \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$ is a division algebra. We call

D the discriminant of B. Let W be a left free B-module of rank 2. Let f be a map on $W \times W$ to B defined by

$$f(x,y) = x_1\overline{y_2} + x_2\overline{y_1}, \quad x = (x_1, x_2), y = (y_1, y_2) \in W,$$

where $^-$ is the canonical involution of B. Any non-degenerate quaternion hermitian form on W is equivalent to f. (cf. [**Shi63**]). Let $U(2;B)$ be the unitary group with respect to this hermitian space (W,f), that is,

$$\begin{aligned} U(2;B) &= \{g \in GL(2;B) \mid f(xg, yg) = f(x,y) \text{ for } \forall x, y \in W\} \\ &= \left\{ g \in GL(2;B) \,\middle|\, g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \end{aligned}$$

where ${}^t\overline{g} = \left(\begin{smallmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{smallmatrix}\right)$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. It is known that $U(2;B) \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ is isomorphic to $Sp(2;\boldsymbol{R})$ by

$$\phi : U(2;B) \otimes_{\boldsymbol{Q}} \boldsymbol{R} \xrightarrow{\sim} Sp(2;\boldsymbol{R})$$

$$\phi(g) = \begin{pmatrix} a_1 & a_2 & b_2 & -b_1 \\ a_3 & a_4 & b_4 & -b_3 \\ c_3 & c_4 & d_4 & -d_3 \\ -c_1 & -c_2 & -d_2 & d_1 \end{pmatrix}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2;B) \otimes_{\boldsymbol{Q}} \boldsymbol{R}$$

where $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \end{smallmatrix}\right) \in B \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. It is known that each $\boldsymbol{Q}$-form of $Sp(2;\boldsymbol{R})$ can be obtained as $U(2;B)$ for some indefinite quaternion algebra B (cf. [**PR94**]). If $B = M(2;\boldsymbol{Q})$, then $U(2;B)$ is isomorphic to $Sp(2;\boldsymbol{Q})$ by ϕ. In this paper, we treat the case where B is a division algebra.

2.3.

Let $\mathfrak{O}$ be the maximal order of B, which is unique up to inner automorphisms. We fix a quaternion hermitian space (W,f). Let L be a left $\mathfrak{O}$-lattice in W, that is, L is a finitely generated $\boldsymbol{Z}$-module satisfying $L \otimes_{\boldsymbol{Z}} \boldsymbol{Q} = W$ and $aL \subset L$ for any $a \in \mathfrak{O}$. We put

$$U(2;B)_L := \{g \in U(2;B) \mid Lg = L\}.$$

Then it is a discrete subgroup of $Sp(2,\boldsymbol{R})$ such that $\mathrm{vol}(U(2;B)_L \backslash \mathfrak{H}_2) < \infty$ by identifying it with its image by ϕ in $Sp(2;\boldsymbol{R})$.

The two-sided $\mathfrak{O}$-ideal generated by the elements $f(x,y)$ for $x, y \in L$ is called the norm of L. We call L a maximal lattice if L is maximal among the left $\mathfrak{O}$-

lattices having the same norm. For any maximal lattice L and any prime number p, it is known by [**Shi63**] that

$$L \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p = \begin{cases} (\mathfrak{O}_p, \mathfrak{O}_p) g_p & \text{if } p \nmid D \\ (\mathfrak{O}_p, \mathfrak{O}_p) g_p \text{ or } (\pi \mathfrak{O}_p, \mathfrak{O}_p) g_p & \text{if } p \mid D \end{cases}$$

for some $g_p \in U(2;B) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$, where $\mathfrak{O}_p := \mathfrak{O} \otimes \boldsymbol{Z}_p$ and π is a prime element of $\mathfrak{O}_p$. Hence there are exactly 2^s genera of maximal lattices in W if D is a product of s prime numbers. We put $D = D_1 D_2$, where $D_1, D_2 \in \boldsymbol{N}$ such that $L \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p = (\mathfrak{O}_p, \mathfrak{O}_p) g_p$ if $p \mid D_1$, and $L \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p = (\pi \mathfrak{O}_p, \mathfrak{O}_p) g_p$ if $p \mid D_2$, for some $g_p \in U(2;B) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$. It is known that if two maximal lattices L_1 and L_2 correspond to the same pair (D_1, D_2), then L_1 and L_2 belong to the same class (i.e. $L_1 = L_2 g$ for some $g \in U(2;B)$) since B is indefinite, and therefore $U(2;B)_{L_1} = U(2;B)_{L_2}$. For simplicity, we put

$$\Gamma(D_1, D_2) := U(2;B)_L$$

for the maximal lattice L corresponding to the pair (D_1, D_2).

3. Main result.

Our main result is Theorem 3.1 below. It is an explicit dimension formula of the spaces of Siegel cusp forms of weight $\rho_{k,j}$ with respect to $\Gamma(D_1, D_2)$ defined above. This formula is a generalization of [**Has84**] and [**Wak**, Theorem 6.1]. We prove Theorem 3.1 in Sections 4 and 5. We suppose that j is even. If j is odd, we have $S_{k,j}(\Gamma(D_1, D_2)) = \{0\}$ for any k since $\Gamma(D_1, D_2)$ contains -1_4. For natural number m and n, we denote by $[a_0, \ldots, a_{m-1}; m]_n$ the function on n which takes the value a_i if $n \equiv i \bmod m$. We define the set $T(m;n) := \{p \mid T \; ; \; p \equiv m \bmod n\}$ for $T = D$, D_1 or D_2.

Theorem 3.1. *If $k \geq 5$ and j is an even non-negative integer, then we have*

$$\dim_{\boldsymbol{C}} S_{k,j}(\Gamma(D_1, D_2)) = \sum_{i=1}^{12} H_i + \sum_{i=1}^{3} I_i,$$

where H_i and I_i are given as follows:

$$H_1 = \frac{(j+1)(k-2)(j+k-1)(j+2k-3)}{2^7 \cdot 3^3 \cdot 5} \cdot \prod_{p \mid D_1} (p-1)(p^2+1) \cdot \prod_{p \mid D_2} (p^2-1)$$

$$H_2 = \frac{(-1)^k(j+k-1)(k-2)}{2^7 \cdot 3^2} \cdot \prod_{p|D}(p-1)^2 \times \begin{cases} 7 & \text{if } 2 \nmid D_1, D_2 = 1 \\ 13 & \text{if } 2 \mid D_1, D_2 = 1 \\ 3 & \text{if } D_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$H_3 = \frac{[(-1)^{j/2}(k-2), -(j+k-1), (-1)^{j/2+1}(k-2), j+k-1; 4]_k}{2^5 \cdot 3}$$
$$\times \prod_{p|D_1}(p-1)\left(1-\left(\frac{-1}{p}\right)\right) \times \begin{cases} 1 & \text{if } D_2 = 1 \\ 3 & \text{if } D_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$H_4 = \frac{[j+k-1, -(j+k-1), 0; 3]_k + [k-2, 0, -(k-2); 3]_{j+k}}{2^3 \cdot 3^3}$$
$$\times \prod_{p|D_1}(p-1)\left(1-\left(\frac{-3}{p}\right)\right) \times \begin{cases} 1 & \text{if } D_2 = 1 \\ 8 & \text{if } D_2 = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$H_5 = 2^{-3} \cdot 3^{-2} \cdot \big([-(j+k-1), -(j+k-1), 0, j+k-1, j+k-1, 0; 6]_k$$
$$+ [k-2, 0, -(k-2), -(k-2), 0, k-2; 6]_{j+k}\big)$$
$$\times \prod_{p|D_1}(p-1)\left(1-\left(\frac{-3}{p}\right)\right) \times \begin{cases} 1 & \text{if } D_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_6 = \sum_{n|2D} A \prod_{p|n}(p-1) \prod_{\substack{p \nmid n \\ p \mid D_1 \\ p \neq 2}} \left(1-\left(\frac{-1}{p}\right)\right) \prod_{\substack{p \nmid n \\ p \mid D_2 \\ p \neq 2}} \left(\frac{p+1}{2}\left(1-\left(\frac{-1}{p}\right)\right)\right) \cdot B$$

For each n, A and B are defined as follows;

$$A = \begin{cases} 2^{-7}3^{-1}(-1)^{k+j/2}(j+1) \\ \qquad \text{if } n \text{ has odd numbers of prime divisors} \\ 2^{-7}3^{-1}(-1)^{j/2}(j+2k-3) \\ \qquad \text{if } n \text{ has even numbers of prime divisors} \end{cases}.$$

If D_2 has a prime divisor p such that $p \mid n$ and $(-1/p) = -1$, then $B = 0$, otherwise,

$$B = \begin{cases} 5 & \text{if } 2 \mid D_1 \text{ and } 2 \mid n \\ 11 & \text{if } 2 \mid D_1 \text{ and } 2 \nmid n \\ 7 & \text{if } 2 \mid D_2 \text{ and } 2 \mid n \\ 9 & \text{if } 2 \mid D_2 \text{ and } 2 \nmid n \\ 3 & \text{if } 2 \nmid D \text{ and } 2 \mid n \\ 5 & \text{if } 2 \nmid D \text{ and } 2 \nmid n \end{cases}$$

$$H_7 = \sum_{n|3D} A \prod_{p|n}(p-1) \prod_{\substack{p \nmid n \\ p \mid D_1 \\ p \neq 3}} \left(1 - \left(\frac{-3}{p}\right)\right) \prod_{\substack{p \nmid n \\ p \mid D_2 \\ p \neq 3}} \left(\frac{p+1}{2}\left(1 - \left(\frac{-3}{p}\right)\right)\right) \cdot B$$

For each n, A and B are defined as follows;

$$A = \begin{cases} 2^{-3}3^{-3}(j+1)[0,1,-1;3]_{j+2k} \\ \qquad\qquad\qquad \text{if } n \text{ has odd numbers of prime divisors} \\ 2^{-3}3^{-3}(j+2k-3)[1,-1,0;3]_j \\ \qquad\qquad\qquad \text{if } n \text{ has even numbers of prime divisors} \end{cases}.$$

If D_2 has a prime divisor p such that $p \mid n$ and $(-3/p) = -1$, then $B = 0$, otherwise,

$$B = \begin{cases} 1 & \text{if } 3 \mid D_1 \text{ and } 3 \mid n \\ 16 & \text{if } 3 \mid D_1 \text{ and } 3 \nmid n \\ 4 & \text{if } 3 \mid D_2 \text{ and } 3 \mid n \\ 10 & \text{if } 3 \mid D_2 \text{ and } 3 \nmid n \\ 1 & \text{if } 3 \nmid D \text{ and } 3 \mid n \\ 4 & \text{if } 3 \nmid D \text{ and } 3 \nmid n \end{cases}$$

$$H_8 = \frac{C_1}{2^2 \cdot 3} \cdot \prod_{p|D} \left(1 - \left(\frac{-1}{p}\right)\right)\left(1 - \left(\frac{-3}{p}\right)\right) \times \begin{cases} 1 & \text{if } D_2 = 1 \\ 0 & \text{otherwise} \end{cases},$$

where we put

$$C_1 = \begin{cases} [1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]_k & \text{if } j \equiv 0 \bmod 12 \\ [-1,1,0,1,1,0,1,-1,0,-1,-1,0;12]_k & \text{if } j \equiv 2 \bmod 12 \\ [1,-1,0,0,-1,1,-1,1,0,0,1,-1;12]_k & \text{if } j \equiv 4 \bmod 12 \\ [-1,0,0,-1,1,-1,1,0,0,1,-1,1;12]_k & \text{if } j \equiv 6 \bmod 12 \\ [1,1,0,1,-1,0,-1,-1,0,-1,1,0;12]_k & \text{if } j \equiv 8 \bmod 12 \\ [-1,-1,0,0,1,1,1,1,0,0,-1,-1;12]_k & \text{if } j \equiv 10 \bmod 12 \end{cases}$$

$$H_9 = \frac{C_2}{2 \cdot 3^2} \times \prod_{p|D_1, p\neq 2} \left(1 - \left(\frac{-3}{p}\right)\right)^2 \times \begin{cases} 2 & \text{if } 2 \nmid D_1 \text{ and } D_2 = 1 \\ 5 & \text{if } 2 \mid D_1 \text{ and } D_2 = 1 \\ 3 & \text{if } 2 \nmid D_1 \text{ and } D_2 = 2 \\ 0 & \text{otherwise} \end{cases},$$

where we put

$$C_2 = \begin{cases} [1,0,0,-1,0,0;6]_k & \text{if } j \equiv 0 \bmod 6 \\ [-1,1,0,1,-1,0;6]_k & \text{if } j \equiv 2 \bmod 6 \\ [0,-1,0,0,1,0;6]_k & \text{if } j \equiv 4 \bmod 6 \end{cases}.$$

$$H_{10} = \frac{C_3}{2\cdot 5} \times \prod_{p\mid D} 2 \times \prod_{p\in D(4;5)} 2 \times \begin{cases} 0 & \text{if } \bigcup_{i=1}^{3} D_1(i;5) \cup \bigcup_{i\in\{1,-1\}} D_2(i;5) \neq \emptyset \\ 1 & \text{if } \bigcup_{i=1}^{3} D_1(i;5) \cup \bigcup_{i\in\{1,-1\}} D_2(i;5) = \emptyset \\ & \qquad \text{and } 5 \mid D \\ 2 & \text{if } \bigcup_{i=1}^{3} D_1(i;5) \cup \bigcup_{i\in\{1,-1\}} D_2(i;5) = \emptyset \\ & \qquad \text{and } 5 \nmid D \end{cases},$$

where we put

$$C_3 = \begin{cases} [1,0,0,-1,0;5]_k & \text{if } j \equiv 0 \bmod 10 \\ [-1,1,0,0,0;5]_k & \text{if } j \equiv 2 \bmod 10 \\ 0 & \text{if } j \equiv 4 \bmod 10 \\ [0,0,0,1,-1;5]_k & \text{if } j \equiv 6 \bmod 10 \\ [0,-1,0,0,1;5]_k & \text{if } j \equiv 8 \bmod 10 \end{cases}.$$

$$H_{11} = \frac{C_4}{2^3} \times \prod_{p\mid D, p\neq 2} 2 \times \prod_{p\in D_1(7;8)} 2 \times \begin{cases} 0 & \text{if } D(1;8) \sqcup D_2(7;8) \neq \emptyset \\ 1 & \text{otherwise} \end{cases},$$

where we put

$$C_4 = \begin{cases} [1,0,0,-1;4]_k & \text{if } j \equiv 0 \bmod 8 \\ [-1,1,0,0;4]_k & \text{if } j \equiv 2 \bmod 8 \\ [-1,0,0,1;4]_k & \text{if } j \equiv 4 \bmod 8 \\ [1,-1,0,0;4]_k & \text{if } j \equiv 6 \bmod 8 \end{cases}.$$

$$\begin{aligned} H_{12} &= 2^{-2}3^{-1}(-1)^{j/2+k}[1,-1,0;3]_j \times \prod_{p\mid D} 2 \times \prod_{p\in D_1(11;12)} 2 \times A \\ &\quad + 2^{-2}3^{-1}(-1)^{j/2}[0,-1,1;3]_{j+2k} \times \prod_{p\mid D} 2 \times \prod_{p\in D_1(11;12)} 2 \times B, \end{aligned}$$

where A and B are defined as follows.

(i) If $D(1;12) \sqcup D_2(11;12) \neq \emptyset$, then $A = B = 0$.

(ii) If $D(1;12) \sqcup D_2(11;12) = \emptyset$, then A (resp. B) are given by the following table, depending on the conditions of D, D_1 and D_2.

	case (I)	case (II)	case (III)
$2 \nmid D, 3 \nmid D$	0	1/2	1
$2 \nmid D, 3 \mid D_1$	1/2	3/4	1
$2 \nmid D, 3 \mid D_2$	0	1/4	1/2
$2 \mid D_1, 3 \nmid D$	1	3/4	1/2
$2 \mid D_1, 3 \mid D_1$	5/4	9/8	1
$2 \mid D_1, 3 \mid D_2$	1/2	3/8	1/4
$2 \mid D_2, 3 \nmid D$	1/2	1/4	0
$2 \mid D_2, 3 \mid D_1$	1/2	3/8	1/4
$2 \mid D_2, 3 \mid D_2$	1/4	1/8	0

where case (I), (II) and (III) are given as follows:

$$\begin{cases} (\ \mathrm{I}\ )\ \ D_1(11;12) = \emptyset \ \text{ and } \sharp D(5;12) = \text{even (resp. odd)} \\ (\ \mathrm{II}\ )\ \ D_1(11;12) \neq \emptyset \\ (\mathrm{III})\ \ D_1(11;12) = \emptyset \ \text{ and } \sharp D(5;12) = \text{odd (resp. even)} \end{cases}.$$

$$I_1 = \frac{j+1}{2^3 \cdot 3} \prod_{p \mid D} (p-1)$$

$$I_2 = -\frac{(-1)^{j/2}}{2^3} \prod_{p \mid D} \left(1 - \left(\frac{-1}{p}\right)\right)$$

$$I_3 = -\frac{[1,-1,0;3]_j}{2 \cdot 3} \prod_{p \mid D} \left(1 - \left(\frac{-3}{p}\right)\right).$$

4. The contribution of semi-simple conjugacy classes.

In this section, we evaluate $I(\Gamma^{(e)})_{k,j}$, i.e. the contributions of torsion elements (cf. Section 1). The principal polynomials of torsion elements of $G = U(2;B)$ are as follows, and each H_i in Theorem 3.1 means the contribution of Γ-conjugacy classes whose principal polynomials are of the form $f_i(\pm x)$.

$$f_1(x) = (x-1)^4, \qquad f_1(-x) = (x+1)^4,$$

$$f_2(x) = (x-1)^2(x+1)^2,$$

$$f_3(x) = (x-1)^2(x^2+1), \qquad f_3(-x) = (x+1)^2(x^2+1),$$

$$f_4(x) = (x-1)^2(x^2+x+1), \qquad f_4(-x) = (x+1)^2(x^2-x+1),$$

$$f_5(x) = (x-1)^2(x^2-x+1), \qquad f_5(-x) = (x+1)^2(x^2+x+1),$$
$$f_6(x) = (x^2+1)^2,$$
$$f_7(x) = (x^2+x+1)^2, \qquad f_7(-x) = (x^2-x+1)^2,$$
$$f_8(x) = (x^2+1)(x^2+x+1), \qquad f_8(-x) = (x^2+1)(x^2-x+1),$$
$$f_9(x) = (x^2+x+1)(x^2-x+1),$$
$$f_{10}(x) = x^4+x^3+x^2+x+1, \qquad f_{10}(-x) = x^4-x^3+x^2-x+1,$$
$$f_{11}(x) = x^4+1,$$
$$f_{12}(x) = x^4-x^2+1.$$

We will evaluate each H_i in Subsection 4.1–4.12. The method was developed by Hashimoto and Ibukiyama [**Has80**], [**HI80**], [**HI82**], [**HI83**], [**Has83**], [**Has84**]. We quote the formula for $I(\Gamma^{(e)})_{k,j}$.

Theorem 4.1.

$$I(\Gamma^{(e)})_{k,j} = c_{k,j} \sum_{\{g\}_G} J_0'(g) \sum_{L_G(\Lambda)} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p),$$

where notations are as follows:

(i) The first sum is extended over the G-conjugacy classes $\{g\}_G$ of torsion elements of G which satisfies $\{g\}_G \cap \Gamma \neq \emptyset$.

(ii) The second sum is extended over the G-genus $L_G(\Lambda)$ of $\boldsymbol{Z}$-orders in $Z(g)$ for each $g \in G$. Here $Z(g)$ is the commutor algebra of g in $M(2;B)$. The G-genus $L_G(\Lambda)$ represented by a $\boldsymbol{Z}$-order Λ of $Z(g)$ consists of all $\boldsymbol{Z}$-orders in $Z(g)$ which are $(Z(g)^\times \cap G) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$-conjugate to $\Lambda \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$ for all p.

(iii) The constant $c_{k,j}$ for k, j are defined by

$$c_{k,j} := 2^{-6}\pi^{-3}(k-2)(j+k-1)(j+2k-3).$$

(iv) We define $J_0'(g)$ for each $\{g\}_G$ as follows. We put

$$H_g^{k,j}(Z) := \operatorname{tr}\left[\rho_{k,j}(CZ+D)^{-1}\rho_{k,j}\left(\frac{g\langle Z\rangle - \overline{Z}}{2i}\right)^{-1}\rho_{k,j}(Y)\right],$$

$$J_0(g) := \int_{C_0(g;Sp(2;\boldsymbol{R}))\backslash\mathfrak{H}_2} H_g^{k,j}(\hat{Z})d\hat{Z},$$

where $dZ = \det(Y)^{-3}dXdY$, $dX = dx_1dx_{12}dx_2$, $dY = dy_1dy_{12}dy_2$ for $Z = X + iY \in \mathfrak{H}_2$, $X = \left(\begin{smallmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{smallmatrix}\right)$, and $d\hat{Z}$ is an invariant measure on $C_0(g; Sp(2;\boldsymbol{R}))\backslash\mathfrak{H}_2$ induced from dZ and a Haar measure on $C_0(g; Sp(2;\boldsymbol{R}))$. The definition of $C_0(g; Sp(2;\boldsymbol{R}))$ is given for each g in each Subsection 4.1–4.12. We define $J_0'(g) := J_0(g)$ if $-1_4 \notin C_0(g; Sp(2;\boldsymbol{R}))$, and $J_0'(g) := 2^{-1}\cdot J_0(g)$ if $-1_4 \in C_0(g; Sp(2;\boldsymbol{R}))$.

(v) We define $M_G(\Lambda)$ for each $\{g\}_G$ and $L_G(\Lambda)$ as follows. We decompose the group $(Z(g)^\times \cap G)_{\boldsymbol{A}}$ into the disjoint union

$$(Z(g)^\times \cap G)_{\boldsymbol{A}} = \sqcup_{i=1}^h \big(Z(g)^\times \cap G\big) y_i \big(\Lambda_{\boldsymbol{A}}^\times \cap G_{\boldsymbol{A}}\big),$$

where $\Lambda_{\boldsymbol{A}} = \Lambda \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_{\boldsymbol{A}}$. We put $\Lambda_i = y_i\Lambda y_i^{-1} = \cap_p((y_i)_p\Lambda_p(y_i)_p^{-1} \cap Z(g))$ and define

$$M_G(\Lambda) := \sum_{i=1}^h \mathrm{vol}\big((\Lambda_i^\times \cap G)\backslash C_0(g; Sp(2;\boldsymbol{R}))\big).$$

(vi) We define $c_p(g, R_p, \Lambda_p)$ for each $\{g\}_G$, $L_G(\Lambda)$ and p as

$$c_p(g, R_p, \Lambda_p) = \sharp\big((Z(g)^\times \cap G)_p \backslash M_p(g, R_p, \Lambda_p)/(R_p^\times \cap G_p)\big),$$

where

$$R_p := M(2;\mathfrak{O}_p) \text{ if } p \nmid D_2 \text{ and } R_p := \begin{pmatrix} \mathfrak{O}_p & \pi^{-1}\mathfrak{O}_p \\ \pi\mathfrak{O}_p & \mathfrak{O}_p \end{pmatrix} \text{ if } p \mid D_2,$$

$$M_p(g, R_p, \Lambda_p) := \left\{ x \in G_p \,\middle|\, \begin{array}{l} x^{-1}gx \in R_p, \text{ and } Z(g)_p \cap xR_px^{-1} \\ \text{is } (Z(g)^\times \cap G)_p\text{-conjugate to } \Lambda_p \end{array} \right\}.$$

If $M_p(g, R_p, \Lambda_p) = \emptyset$, then we put $c_p(g, R_p, \Lambda_p) = 0$.

Remark 4.2. We give some remarks about Theorem 4.1.

(1) We do not need to calculate the local data $c_p(g, R_p, \Lambda_p)$ since they have been obtained in [**HI80**] and [**HI83**]. We have only to combine the data depending on the cases.
(2) We need to determine G-conjugacy classes which appear in the first sum in Theorem 4.1. It is known that $\{g\}_G \cap \Gamma \neq \emptyset$ if and only if $\{g\}_{G_p} \cap R_p \neq \emptyset$ for all p. (cf. Theorem 1-3 in [**Has83**]). We can obtain the result by using [**HI80**, Section 2], [**Has84**] and the results of c_p mentioned above.

(3) The integral $J_0'(g)$ depends only on $Sp(2;\boldsymbol{R})$-conjugacy classes. Langlands [**Lan63**] gave a formula for $J_0(g)$. We can evaluate $J_0'(g)$ by applying explicit formulae in [**Has83**] and [**Wak**].

We will evaluate each H_i in Subsection 4.1–4.12. We denote by $G[f_i]$ the set of torsion elements of G whose principal polynomials are $f_i(x)$. For $i = 1, 3, 4, 5, 7, 8, 10$, we have only to evaluate the contribution of $G[f_i]$ and double it to obtain H_i because the contribution of g is equal to that of $-g$. We use the notation:

$$\alpha(\theta_1,\theta_2) = \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}.$$

4.1. The contribution $\mathbf{H_1}$.

In this subsection, we consider the contribution of $\pm 1_4$. If $\gamma = \pm 1_4$, we have $C(\gamma; Sp(2;\boldsymbol{R})) = C_0(\gamma; Sp(2;\boldsymbol{R})) = Sp(2;\boldsymbol{R})$, $C(\gamma;\Gamma) = C_0(\gamma;\Gamma) = \Gamma$ and $H_\gamma^{k,j}(Z) = 1$, so $J_0'(\gamma) = (1/2)\int_{Sp(2;\boldsymbol{R})\backslash\mathfrak{H}_2} d\hat{Z}$. Also, we have

$$c_p(\gamma, R_p, \Lambda_p) = \begin{cases} 1 & \text{if } \Lambda_p \sim R_p \\ 0 & \text{otherwise.} \end{cases}$$

Hence from Theorem 4.1 we have

$$H_1 = 2^{-6}\pi^{-3}(k-2)(j+k-1)(j+2k-3)\cdot \mathrm{vol}(Sp(2;\boldsymbol{R})\backslash\mathfrak{H}_2)\cdot \mathrm{vol}(\Gamma\backslash Sp(2;\boldsymbol{R})).$$

We have only to multiple the value H_1 in the case of $D_2 = 1$ in [**Wak**] by $\prod_{p|D_2}$ $(p+1)/(p^2+1)$ because we have the indexes as follows:

$$\left[G_p \cap \begin{pmatrix} \mathfrak{O}_p & \mathfrak{O}_p \\ \mathfrak{O}_p & \mathfrak{O}_p \end{pmatrix}^\times : G_p \cap \begin{pmatrix} \mathfrak{O}_p & \mathfrak{O}_p \\ \pi\mathfrak{O}_p & \mathfrak{O}_p \end{pmatrix}^\times \right] = p+1,$$

$$\left[G_p \cap \begin{pmatrix} \mathfrak{O}_p & \pi^{-1}\mathfrak{O}_p \\ \pi\mathfrak{O}_p & \mathfrak{O}_p \end{pmatrix}^\times : G_p \cap \begin{pmatrix} \mathfrak{O}_p & \mathfrak{O}_p \\ \pi\mathfrak{O}_p & \mathfrak{O}_p \end{pmatrix}^\times \right] = p^2+1.$$

Hence we obtain H_1 as in Theorem 3.1.

4.2. The contribution $\mathbf{H_2}$.

In this subsection, we evaluate the contribution of $G[f_2]$, where $f_2(x) = (x-1)^2(x+1)^2$. The set $G[f_2]$ consists of only one G-conjugacy class represented by an element g. We have $Z(g) \simeq B \oplus B$. We fix g and this isomorphism until the end of this subsection. We put

$$L := \{(x,y) \in \mathfrak{O} \oplus \mathfrak{O} \mid x - y \in \pi\mathfrak{O}_2\},$$

where π is a prime element of $\mathfrak{O}_2$. We have the following proposition.

Proposition 4.3.

(1) *The class $\{g\}_G$ appears in the first sum of Theorem 4.1, i.e. $\{g\}_G \cap \Gamma \neq \emptyset$, if and only if $D_2 = 1$ or 2.*

(2) *We assume $D_2 = 2$. If Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as L, then we have $\prod_p c_p(g, R_p, \Lambda_p) = 1$. If Λ does not belong to the same G-genus as L, then $\prod_p c_p(g, R_p, \Lambda_p) = 0$.*

Proof. We can prove (1) and the latter part of (2) easily by [**HI83**, Proposition 2.4]. If $D_2 = 2$ and Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as L, then it follows from [**HI80**, Proposition 13] and [**HI83**, Proposition 2.4] that $c_p(g, R_p, \Lambda_p) = 1$ for any p. $\square$

From Proposition 4.3, we have $H_2 = 0$ if $D_2 \neq 1, 2$. In the case where $D_2 = 1$, H_2 has been evaluated in [**Has84**] and [**Wak**]. Hereafter, we assume $D_2 = 2$. From Proposition 4.3, we have

$$H_2 = c_{k,j} \cdot J_0'(g) \cdot M_G(L).$$

We see that g is $Sp(2; \boldsymbol{R})$-conjugate to $\alpha(\pi, 0)$ and

$$C_0(\alpha(\pi,0); Sp(2;\boldsymbol{R})) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix} \middle| \begin{array}{l} ad - bc = 1 \\ a'd' - b'c' = 1 \end{array} \right\},$$

$$J_0'(g) = \frac{1}{2} J_0(\alpha(\pi,0)) = {c_{k,j}}^{-1} 2^{-7} \pi^{-4} (-1)^k (j+k-1)(k-2)$$

if j is even. (cf. (b-5) in [**Wak**]). If we put $L_0 = \mathfrak{O} \oplus \mathfrak{O}$, then we have

$$\begin{aligned} M_G(L) &= \mathrm{vol}\big((L^\times \cap G)\backslash C(g; Sp(2;\boldsymbol{R}))\big) \\ &= \big[L_0^\times \cap G : L^\times \cap G\big] \cdot \mathrm{vol}\big((L_0^\times \cap G)\backslash C(g; Sp(2;\boldsymbol{R}))\big) \\ &= 3^{-1}\pi^4 \prod_{p \mid D} (p-1)^2. \end{aligned}$$

(cf. (3.6), (3.7) of [**Has84**]). Hence we can obtain H_2 as in Theorem 3.1.

4.3. The contribution $\mathbf{H_3}$.

In this subsection, we evaluate the contribution of $G[f_3]$, where $f_3(x) = (x-1)^2(x^2+1)$. We have only to double it to obtain H_3. Note that $G[f_3] \neq \emptyset$ if and only if $(-1/p) \neq 1$ for any prime divisor p of D. Hereafter, we assume that $G[f_3] \neq \emptyset$. The set $G[f_3]$ consists of two G-conjugacy classes represented by g and g^{-1} for an element g. We have $Z(g) \simeq B \oplus F$ with $F = \boldsymbol{Q}(\sqrt{-1})$. We fix g and this isomorphism until the end of this subsection. We put

$$L := \{(x,y) \in \mathfrak{O} \oplus \mathscr{O} \mid x - y \in \pi\mathscr{O}_2\},$$

where $\mathscr{O}$ is the ring of integers of F and π is a prime element of $\mathscr{O}_2$. Then we have the following proposition.

Proposition 4.4.

(1) *The classes $\{g\}_G$ and $\{g^{-1}\}_G$ appear in the first sum of Theorem* 4.1 *if and only if $D_2 = 1$ or* 2.

(2) *We assume $D_2 = 2$. If Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as L, then we have*

$$\prod_p c_p(g, R_p, \Lambda_p) = \prod_p c_p(g^{-1}, R_p, \Lambda_p) = 2^{\sharp D_1(3;4)}.$$

If Λ does not belong to the same G-genus as L, then $\prod_p c_p(g, R_p, \Lambda_p) = 0$.

Proof. We can prove (1) and the latter part of (2) easily by [**HI83**, Proposition 2.4]. If $D_2 = 2$ and Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as L, then it follows from [**HI80**, Proposition 14] and [**HI83**, Proposition 2.4] that

$$c_p(g, R_p, \Lambda_p) = c_p(g^{-1}, R_p, \Lambda_p) = \begin{cases} 2 & \text{if } p \mid D_1 \text{ and } \left(\dfrac{-1}{p}\right) = -1 \\ 1 & \text{otherwise.} \end{cases} \qquad \square$$

From Proposition 4.4, we have $H_3 = 0$ if $D_2 \neq 1, 2$. In the case where $D_2 = 1$, H_3 has been evaluated in [**Has84**] and [**Wak**]. Hereafter, we assume $D_2 = 2$. From Proposition 4.4, we have

$$
\begin{aligned}
H_3 &= 2 \cdot c_{k,j} \cdot \sum_{\gamma \in \{g, g^{-1}\}} J_0'(\gamma) \cdot M_\gamma(L) \prod_p c_p(\gamma, R_p, L_p) \\
&= 2 \cdot c_{k,j} \cdot \big(J_0'(g) + J_0'(g^{-1})\big) \cdot M_g(L) \cdot 2^{\sharp D_1(3;4)}.
\end{aligned}
$$

We see that g and g^{-1} are $Sp(2;\boldsymbol{R})$-conjugate to $\alpha(\pi/2, 0)$ and $\alpha(-\pi/2, 0)$ respectively and

$$
\begin{aligned}
C_0\Big(\alpha\Big(\frac{\pi}{2}, 0\Big), Sp(2;\boldsymbol{R})\Big) &= C_0\Big(\alpha\Big(-\frac{\pi}{2}, 0\Big), Sp(2;\boldsymbol{R})\Big) \\
&= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\},
\end{aligned}
$$

$$
\begin{aligned}
J_0'(g) + J_0'(g^{-1}) = {c_{k,j}}^{-1} \cdot 2^4 \cdot \pi^2 \cdot \big[&(-1)^{j/2}(k-2), -(j+k-1), \\
&(-1)^{j/2+1}(k-2), j+k-1; 4\big]_k
\end{aligned}
$$

(cf. (b-4) in [**Wak**]). If we put $L_0 = \mathfrak{O} \oplus \mathscr{O}$, then we have

$$
\begin{aligned}
M_g(L) &= \mathrm{vol}\big((L^\times \cap G)\backslash C(g; Sp(2;\boldsymbol{R}))\big) \\
&= \big[L_0^\times \cap G : L^\times \cap G\big] \cdot \mathrm{vol}\big((L_0^\times \cap G)\backslash C(g; Sp(2;\boldsymbol{R}))\big) \\
&= 2^{-2}\pi^2 \prod_{p \mid D}(p-1).
\end{aligned}
$$

(cf. (3.10), (3.11) of [**Has84**]). Hence we can obtain H_3 as in Theorem 3.1.

4.4. The contribution $\mathbf{H_4}$.

In this subsection, we evaluate the contribution of $G[f_4]$, where $f_4(x) = (x-1)^2(x^2+x+1)$. We have only to double it to obtain H_4. Note that $G[f_4] \neq \emptyset$ if and only if $(-3/p) \neq 1$ for any prime divisor p of D. Hereafter, we assume that D does not have such a prime divisor. The set $G[f_4]$ consists of two G-conjugacy classes represented by g and g^{-1} for an element g. We have $Z(g) \simeq B \oplus F$ with $F = \boldsymbol{Q}(\sqrt{-3})$. We fix g and this isomorphism until the end of this subsection. We put

$$
L := \{(x, y) \in \mathfrak{O} \oplus \mathscr{O} \mid x - y \in \pi\mathscr{O}_3\},
$$

where $\mathscr{O}$ is the ring of integers of F and π is a prime element of $\mathscr{O}_3$. Then we have the following proposition.

PROPOSITION 4.5.

(1) *The classes* $\{g\}_G$ *and* $\{g^{-1}\}_G$ *appear in the first sum of Theorem* 4.1 *if and only if* $D_2 = 1$ *or* 3.
(2) *We assume* $D_2 = 3$. *If* Λ *is a* $\boldsymbol{Z}$*-order of* $Z(g)$ *belonging to the same* G*-genus as* L*, then we have*

$$\prod_p c_p(g, R_p, \Lambda_p) = \prod_p c_p(g^{-1}, R_p, \Lambda_p) = 2^{\sharp D_1(2;3)}.$$

If Λ *does not belong to the same* G*-genus as* L*, then* $\prod_p c_p(g, R_p, \Lambda_p) = 0$.

PROOF. We can prove (1) and the latter part of (2) easily by [**HI83**, Proposition 2.4]. If $D_2 = 3$ and Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as L, then it follows from [**HI80**, Proposition 14] and [**HI83**, Proposition 2.4] that

$$c_p(g, R_p, \Lambda_p) = c_p(g^{-1}, R_p, \Lambda_p) = \begin{cases} 2 & \text{if } p \mid D_1 \text{ and } \left(\dfrac{-3}{p}\right) = -1 \\ 1 & \text{otherwise.} \end{cases} \qquad \square$$

From Proposition 4.5, we have

$$\begin{aligned} H_4 &= 2 \cdot c_{k,j} \cdot \sum_{\gamma \in \{g, g^{-1}\}} J_0'(\gamma) \cdot M_\gamma(L) \prod_p c_p(\gamma, R_p, L_p) \\ &= 2 \cdot c_{k,j} \cdot \left(J_0'(g) + J_0'(g^{-1})\right) \cdot M_g(L) \cdot 2^{\sharp D_1(2;3)}. \end{aligned}$$

We see that g and g^{-1} are $Sp(2;\boldsymbol{R})$-conjugate to $\alpha(2\pi/3, 0)$ and $\alpha(-2\pi/3, 0)$ respectively and

$$\begin{aligned} C_0\left(\alpha\left(\frac{2\pi}{3}, 0\right), Sp(2;\boldsymbol{R})\right) &= C_0\left(\alpha\left(-\frac{2\pi}{3}, 0\right), Sp(2;\boldsymbol{R})\right) \\ &= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \, ad - bc = 1 \right\}, \end{aligned}$$

$$\begin{aligned} J_0'(g) + J_0'(g^{-1}) = {} & {c_{k,j}}^{-1} \cdot 2^{-3} \cdot 3^{-1} \cdot \pi^{-2} \\ & \times \big\{[j+k-1, -(j+k-1), 0; 3]_k \\ & \qquad + [k-2, 0, -(k-2); 3]_{j+k}\big\} \end{aligned}$$

(cf. (b-4) in [**Wak**]). If we put $L_0 = \mathfrak{O} \oplus \mathscr{O}$, then we have

$$\begin{aligned} M_g(L) &= \mathrm{vol}\big((L^\times \cap G)\backslash C(g; Sp(2; \boldsymbol{R}))\big) \\ &= \big[L_0^\times \cap G : L^\times \cap G\big] \cdot \mathrm{vol}\big((L_0^\times \cap G)\backslash C(g; Sp(2; \boldsymbol{R}))\big) \\ &= 2^2 \cdot 3^{-2} \cdot \pi^2 \prod_{p \mid D} (p-1). \end{aligned}$$

(cf. (3.10), (3.11) of [**Has84**]). Hence we can obtain H_4 as in Theorem 3.1.

4.5. The contribution $\mathbf{H_5}$.

In this subsection, we consider the contribution of $G[f_5]$, where $f_5(x) = (x-1)^2(x^2 - x + 1)$. We have only to double it to obtain H_5. The set $G[f_5]$ consists of two G-conjugacy classes represented by g and g^{-1} for an element g. We see from [**HI80**, Proposition 14] and [**HI83**, Proposition 2.4] that the classes $\{g\}_G$ and $\{g^{-1}\}_G$ do not appear in the first sum of Theorem 4.1 if $D_2 \neq 1$. In the case where $D_2 = 1$, H_5 has been evaluated in [**Has84**] and [**Wak**].

4.6. The contribution $\mathbf{H_6}$.

In this subsection, we consider the contribution of $G[f_6]$, where $f_6(x) = (x^2+1)^2$. Note that $G[f_6] = \emptyset$ if and only if D_2 has a prime divisor p with $(-1/p) = 1$. Hereafter, we assume that D does not have such a prime divisor. Then there are infinitely many G-conjugacy classes in $G[f_6]$. As in [**Has84**, Theorem 3.2 (i),(ii)], we have a correspondence between the set of G-conjugacy classes $\{g\}_G$'s in $G[f_6]$ and the set of isomorphism classes of quaternion algebras $Z_0(g)$'s over $\boldsymbol{Q}$ which are contained in $B \otimes_{\boldsymbol{Q}} F$, with $F = \boldsymbol{Q}(\sqrt{-1})$. This correspondence is two-to-one or one-to-one according as $Z_0(g)$ is definite or indefinite. We denote by $D(Z_0(g))$ the discriminant of $Z_0(g)$. If $Z_0(g)$ is definite, two G-conjugacy classes $\{g\}_G$ and $\{g^{-1}\}_G$ correspond to $Z_0(g)$. In this case, g is $Sp(2; \boldsymbol{R})$-conjugate to $\alpha(\pi/2, \pi/2)$ and

$$J_0'(g) + J_0'(g^{-1}) = {c_{k,j}}^{-1} \cdot 2^{-2} \cdot (j+1) \cdot (-1)^{k+j/2},$$

if j is even (cf. (b-2) in [**Wak**]),

$$M_G(\Lambda) = \frac{1}{48} \prod_{p \mid D(Z_0(g))} (p-1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)}$$

for a $\boldsymbol{Z}$-order Λ of $Z(g)$ (cf. [**HI80**, Proposition 12]), where $\mathfrak{O}_0$ is a maximal order of $Z_0(g)$ and

$$d_p(\Lambda) = \left[\mathfrak{O}_{0p}^{\times} : (\Lambda \cap Z_0(g))_p^{\times}\right], \qquad e_p(\Lambda) = \left[\Lambda_p^{\times} \cap G_p : \mathfrak{O}_{0p}^{\times} \cdot \mathscr{O}_{Fp}^{\times}\right].$$

On the other hand, if $Z_0(g)$ is indefinite, then only one G-conjugacy class $\{g\}_G$ corresponds to $Z_0(g)$. In this case, g is $Sp(2;\boldsymbol{R})$-conjugate to $\alpha(\pi/2, -\pi/2)$ and

$$J_0'(g) = c_{k,j}{}^{-1} \cdot 2^{-5} \cdot \pi^{-2} \cdot (j + 2k - 3) \cdot (-1)^{j/2},$$

if j is even (cf. (b-3) in [**Wak**]),

$$M_G(\Lambda) = \frac{\pi^2}{6} \prod_{p \mid D(Z_0(g))} (p-1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)}$$

for a $\boldsymbol{Z}$-order Λ of $Z(g)$ (cf. [**Has84**, (3.16)]).

PROPOSITION 4.6.

(1) *The class* $\{g\}_G$ *appears in the first sum of Theorem* 4.1 *if and only if* $D(Z_0(g)) \mid 2D$.

(2) (i) *the case where* $2 \mid D_1$,

- *If a* G*-conjugacy class* $\{g\}_G$ *satisfies* $2 \mid D(Z_0(g))$, *then two* G*-genus of* $\boldsymbol{Z}$*-orders of* $Z(g)$ *appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 2}} (p+1) \cdot \frac{3}{2} \text{ (resp. 1)}$$

- *If a* G*-conjugacy class* $\{g\}_G$ *satisfies* $2 \nmid D(Z_0(g))$, *then three* G*-genus of* $\boldsymbol{Z}$*-orders of* $Z(g)$ *appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 2}} (p+1) \cdot \frac{3}{2} \text{ (resp. 1 and 3)}$$

(ii) *the case where* $2 \mid D_2$,

- *If a* G*-conjugacy class* $\{g\}_G$ *satisfies* $2 \mid D(Z_0(g))$, *then two* G*-genus of* $\boldsymbol{Z}$*-orders of* $Z(g)$ *appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 2}} (p+1) \cdot \frac{1}{2} \text{ (resp. 3)}$$

- *If a G-conjugacy class $\{g\}_G$ satisfies $2 \nmid D(Z_0(g))$, then two G-genus of $\boldsymbol{Z}$-orders of $Z(g)$ appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 2}} (p+1) \cdot 3 \text{ (resp. } 3/2)$$

(iii) *the case where $2 \nmid D$,*

- *If a G-conjugacy class $\{g\}_G$ satisfies $2 \mid D(Z_0(g))$, then only one G-genus of $\boldsymbol{Z}$-orders of $Z(g)$ appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 2}} (p+1) \cdot \frac{3}{2}$$

- *If a G-conjugacy class $\{g\}_G$ satisfies $2 \nmid D(Z_0(g))$, then two G-genus of $\boldsymbol{Z}$-orders of $Z(g)$ appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 2}} (p+1) \cdot \frac{3}{2} \text{ (resp. } 1)$$

(3) *For any case and any G-genus, we have* $\displaystyle\prod_p c_p(g, R_p, \lambda_p) = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_1,\ p \neq 2}} 2.$

Proof. We see from [**HI80**, Proposition 15 and 16] and [**HI83**, Proposition 2.5 and 2.6] that

$$\begin{aligned}
&p \mid D_1 &&\Longrightarrow \{g\}_{G_p} \cap R_p \neq \emptyset,\\
&p \nmid D_1 \text{ and } p \nmid Z_0(g) &&\Longrightarrow \{g\}_{G_p} \cap R_p \neq \emptyset,\\
&p \nmid D_1,\ p \mid Z_0(g) \text{ and } p = 2 &&\Longrightarrow \{g\}_{G_p} \cap R_p \neq \emptyset,\\
&p \nmid D_1,\ p \mid Z_0(g) \text{ and } p \neq 2 &&\Longrightarrow \{g\}_{G_p} \cap R_p = \emptyset.
\end{aligned}$$

Hence we obtain (1) (cf. [**Has80**, Theorem 1-3]). Also we can obtain (2), (3) from the above four propositions. □

4.7. The contribution $\mathbf{H_7}$.

In this subsection, we consider the contribution of $G[f_7]$, where $f_7(x) = (x^2 + x + 1)^2$. We have only to double it to obtain H_7. Note that $G[f_7] = \emptyset$ if and only if D_2 has a prime divisor p with $(-3/p) = 1$. Hereafter, we assume that D does not have such a prime divisor. We can use the same method as in the case of H_6. we have a correspondence between the set of G-conjugacy classes $\{g\}_G$'s in $G[f_7]$ and the set of isomorphism classes of quaternion algebras $Z_0(g)$'s over $\boldsymbol{Q}$ which are contained in $B \otimes_{\boldsymbol{Q}} F$, with $F = \boldsymbol{Q}(\sqrt{-3})$. If $Z_0(g)$ is definite, then

$$J_0'(g) + J_0'(g^{-1}) = {c_{k,j}}^{-1} 3^{-1}(j+1)[0,1,-1;3]_{j+2k},$$

if j is even (cf. (b-2) in [**Wak**]), and

$$M_G(\Lambda) = \frac{1}{72} \prod_{p \mid D(Z_0(g))} (p-1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)}$$

for a $\boldsymbol{Z}$-order Λ of $Z(g)$. If $Z_0(g)$ is indefinite, then

$$J_0'(g) = {c_{k,j}}^{-1} 2^3 \cdot 3\pi^2 (j+2k-3)[1,-1,0;3]_j,$$

if j is even (cf. (b-3) in [**Wak**]), and

$$M_G(\Lambda) = \frac{\pi^2}{3^2} \prod_{p \mid D(Z_0(g))} (p-1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)}.$$

PROPOSITION 4.7.

(1) *The class $\{g\}_G$ appears in the first sum of Theorem* 4.1 *if and only if $D(Z_0(g)) \mid 3D$.*

(2) (i) *the case where $3 \mid D_1$,*

- *If a G-conjugacy class $\{g\}_G$ satisfies $3 \mid D(Z_0(g))$, then only one G-genus of $\boldsymbol{Z}$-orders of $Z(g)$ appears in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 3}} (p+1) \cdot \frac{1}{2}$$

- *If a G-conjugacy class $\{g\}_G$ satisfies $3 \nmid D(Z_0(g))$, then two G-genus of $\boldsymbol{Z}$-orders of $Z(g)$ appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 3}} (p+1) \cdot 2 \text{ (resp. 6)}$$

(ii) *the case where* $3 \mid D_2$,

- *If a G-conjugacy class* $\{g\}_G$ *satisfies* $3 \mid D(Z_0(g))$, *then only one G-genus of* $\boldsymbol{Z}$*-orders of* $Z(g)$ *appears in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 3}} (p+1) \cdot 1$$

- *If a G-conjugacy class* $\{g\}_G$ *satisfies* $3 \nmid D(Z_0(g))$, *then two G-genus of* $\boldsymbol{Z}$*-orders of* $Z(g)$ *appear in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 3}} (p+1) \cdot 1 \text{ (resp. 4)}$$

(iii) *the case where* $3 \nmid D$,

- *If a G-conjugacy class* $\{g\}_G$ *satisfies* $3 \mid D(Z_0(g))$, *then only one G-genus of* $\boldsymbol{Z}$*-orders of* $Z(g)$ *appears in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 3}} (p+1) \cdot \frac{1}{2}$$

- *If a G-conjugacy class* $\{g\}_G$ *satisfies* $3 \nmid D(Z_0(g))$, *then only one G-genus of* $\boldsymbol{Z}$*-orders of* $Z(g)$ *appears in the second sum of Theorem* 4.1, *and*

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_2,\ p \neq 3}} (p+1) \cdot 1$$

(3) *For any case and any G-genus, we have* $\displaystyle\prod_p c_p(g, R_p, \lambda_p) = \prod_{\substack{p \nmid D(Z_0(g)) \\ p \mid D_1,\ p \neq 3}} 2.$

Proof. We can prove this by the same way as Proposition 4.6. □

4.8. The contribution $\mathbf{H_8}$.

In this subsection, we evaluate the contribution of $G[f_8]$, where $f_8(x) = (x^2+1)(x^2+x+1)$. We have only to double it to obtain H_8. We see from [**HI83**, Proposition 2.7] that no G-conjugacy classes corresponding to $f_8(\pm x)$ appear in Theorem 4.1 if $D_2 \neq 1$. In the cases where $D_2 = 1$, H_8 has been evaluated in [**Has84**] and [**Wak**].

4.9. The contribution $\mathbf{H_9}$.

In this subsection, we evaluate the contribution of $G[f_9]$, where $f_9(x) = (x^2 + x + 1)(x^2 - x + 1)$. Note that $G[f_9] \neq \emptyset$ if and only if $(-3/p) \neq 1$ for any prime divisor p of D. Hereafter, we assume that $G[f_9] \neq \emptyset$. We put $F := \boldsymbol{Q}(\sqrt{-3})$, then $Z(g) \simeq F \oplus F$ for any g. We put

$$L := \{(x, y) \in \mathscr{O} \oplus \mathscr{O} \mid x - y \in 2\mathscr{O}\},$$

where $\mathscr{O}$ is the ring of integers of F, then we have the following proposition.

Proposition 4.8.

(1) *If $D_2 \neq 1, 2$, then no G-conjugacy classes in $G[f_9]$ appear in the first sum of Theorem* 4.1*.*

(2) *If $D_2 = 2$, then the followings hold.*

 (i) *The number of G-conjugacy classes in $G[f_9]$ which appear in the first sum of Theorem* 4.1 *is* $4 \cdot 2^{\sharp D_1(2;3)}$.

 (ii) *Let $\{g\}_G$ be any one of them. If Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as L, then*

$$\prod_p c_p(g, R_p, \Lambda_p) = 2 \cdot 2^{\sharp D_1(2;3)}.$$

 If Λ does not belong to the same G-genus as L, then $\prod_p c_p(g, R_p, \Lambda_p) = 0$.

Proof. We can obtain (1) and the latter part of (2 ii) from [**HI83**, Proposition 2.7]. For an element g of $G[f_9]$, g is G_p-conjugate to

$$\begin{cases} \gamma_p & \text{if } \left(\dfrac{-3}{p}\right) = 1 \\ \gamma_p \text{ or } \delta_p & \text{if } \left(\dfrac{-3}{p}\right) \neq 1 \end{cases}$$

for some elements γ_p and $\delta_p \in G_p$. It follows from [**HI80**, Proposition 18] and [**HI83**, Proposition 2.7] that if $D_2 = 2$ and Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as L, then c_p is as in the following table for each prime number p satisfying the first column:

p	$c_p(\gamma_p, R_p, \Lambda_p)$	$c_p(\delta_p, R_p, \Lambda_p)$
$p \mid D_1$ and $\left(\frac{-3}{p}\right) = -1$	2	2
$p \nmid D$ and $\left(\frac{-3}{p}\right) = 1$	1	$\times$
$p \nmid D$ and $\left(\frac{-3}{p}\right) = -1$	1	0
$p = 2$	2	0
$p = 3$	1	1

Also, g is $Sp(2;\boldsymbol{R})$-conjugate to $g_1 := \alpha(\pi/3, 2\pi/3)$, ${g_1}^{-1} = \alpha(-\pi/3, -2\pi/3)$, $g_2 := \alpha(\pi/3, -2\pi/3)$, and ${g_2}^{-1} = \alpha(-\pi/3, 2\pi/3)$. Since g^2 belongs to $G[f_7]$, g is $Sp(2;\boldsymbol{R})$-conjugate to g_1 or g_1^{-1} if $Z_0(g^2)$ is indefinite, and g is $Sp(2;\boldsymbol{R})$-conjugate to g_2 or g_2^{-1} if $Z_0(g^2)$ is definite. We take all combinations of G_p-conjugations, and also we take $Sp(2;\boldsymbol{R})$-conjugation out of "g_1 or g_1^{-1}" or "g_2 or g_2^{-1}", according as $Z_0(g^2)$ is indefinite or definite. Then G-conjugacy class is determined uniquely for them by Hasse principle ([**Has80**, Theorem 1-2]). □

We see from Proposition 4.8 that $H_9 = 0$ if $D_2 \neq 1, 2$. In the case where $D_2 = 1$, H_2 has been evaluated in [**Has84**] and [**Wak**]. Hereafter, we assume $D_2 = 2$. We obtain from Proposition 4.8 that

$$H_9 = c_{k,j} \cdot \sum_{\{g\}_G} J_0'(g) \cdot M_G(L) \cdot \prod_p c_p(g, R_p, L_p),$$

$$\sum_{\{g\}_G} J_0'(g) = \left(J_0'(g_1) + J_0'(g_1^{-1}) + J_0'(g_2) + J_0'(g_2^{-1})\right) \cdot 2^{\sharp D_1(2;3)}.$$

We have $C_0(g; Sp(2;\boldsymbol{R})) = \{1_4\}$ for any g, and

$$\begin{aligned} & J_0'(g_1) + J_0'({g_1}^{-1}) + J_0'(g_2) + J_0'({g_2}^{-1}) \\ & = c_{k,j}^{-1} \cdot \begin{cases} [1,0,0,-1,0,0;6]_k & \text{if } j \equiv 0 \bmod 6 \\ [-1,1,0,1,-1,0;6]_k & \text{if } j \equiv 2 \bmod 6 \\ [0,-1,0,0,1,0;6]_k & \text{if } j \equiv 4 \bmod 6 \end{cases}. \end{aligned}$$

(cf. (b-1) in [**Wak**]). We have

$$M_G(L) = \frac{1}{12}, \quad \text{(cf. (3.21) in [\textbf{Has84}])}$$

$$\prod_p c_p(g, R_p, L_p) = 2 \cdot 2^{\sharp D_1(2;3)}$$

for any g. Hence we can obtain H_9 as in Theorem 3.1.

4.10. The contribution $\mathbf{H_{10}}$.

In this subsection, we evaluate the contribution of $G[f_{10}]$, where $f_{10}(x) = x^4 + x^3 + x^2 + x + 1$. We have only to double it to obtain H_{10}. Note that if $D(1;5) \neq \emptyset$, then $G[f_{10}] = \emptyset$. Hereafter, we assume $D(1;5) = \emptyset$. We have $Z(g) = \boldsymbol{Q}(g) \simeq \boldsymbol{Q}(\zeta_5)$ for any g. We have the following proposition.

PROPOSITION 4.9.

(1) *If $D_1(2;5) \sqcup D_1(3;5) \sqcup D_2(4;5) \neq \emptyset$, then no G-conjugacy classes in $G[f_{10}]$ appear in the first sum of Theorem 4.1.*

(2) *If $D_1(2;5) \sqcup D_1(3;5) \sqcup D_2(4;5) = \emptyset$, then the followings hold.*

(i) *The number of G-conjugacy classes in $G[f_{10}]$ which appear in the first sum of Theorem 4.1 is $4 \cdot 2^{\sharp D_1(4;5)}$.*

(ii) *Let $\{g\}_G$ be any one of them. If Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as $\mathscr{O}$, where $\mathscr{O}$ is the ring of integers of $Z(g)$, then*

$$\prod_p c_p(g, R_p, \Lambda_p) = 2^{\sharp D_1(4;5)} \cdot 2^{\sharp D_2(2;5)} \cdot 2^{\sharp D_2(3;5)}.$$

If Λ does not belong to the same G-genus as $\mathscr{O}$, then $\prod_p c_p(g, R_p, \Lambda_p) = 0$.

PROOF. We can obtain (1) and the latter part of (2 ii) from [**HI80**, Proposition 19] and [**HI83**, Proposition 2.8]. For any element g of $G[f_{10}]$, g is G_p-conjugate to

$$\begin{cases} \gamma_p & \text{if } p \notin D_1(4;5) \\ \gamma_p \text{ or } \delta_p & \text{if } p \in D_1(4;5) \end{cases}$$

for some elements γ_p and $\delta_p \in G_p$. It follows from [**HI80**, Proposition 19] and [**HI83**, Proposition 2.8] that if $D_1(2;5) \sqcup D_1(3;5) \sqcup D_2(4;5) = \emptyset$ and Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as $\mathscr{O}$, then c_p is as in the following table for each prime number p satisfying the first column:

p	$c_p(\gamma_p, R_p, \Lambda_p)$	$c_p(\delta_p, R_p, \Lambda_p)$
$p \mid D_1$ and $p \equiv 2, 3 \bmod 5$	0	$\times$
$p \mid D_1$ and $p \equiv 4 \bmod 5$	2	2
$p \mid D_1$ and $p = 5$	1	$\times$
$p \mid D_2$ and $p \equiv 2, 3 \bmod 5$	2	$\times$
$p \mid D_2$ and $p \equiv 4 \bmod 5$	0	0
$p \mid D_2$ and $p = 5$	1	$\times$
$p \nmid D$ and $p \equiv 1 \bmod 5$	1	$\times$
$p \nmid D$ and $p \equiv 2, 3 \bmod 5$	1	$\times$
$p \nmid D$ and $p \equiv 4 \bmod 5$	1	0
$p \nmid D$ and $p = 5$	1	$\times$

Also, g is $Sp(2; \boldsymbol{R})$-conjugate to $g_1 := \alpha(2\pi/5, 4\pi/5)$, ${g_1}^{-1} = \alpha(-2\pi/5, -4\pi/5)$, $g_2 := \alpha(2\pi/5, -4\pi/5)$, and ${g_2}^{-1} = \alpha(-2\pi/5, 4\pi/5)$. We can take all combinations of $Sp(2; \boldsymbol{R})$-conjugation and G_p-conjugations. $\square$

We see from Proposition 4.9 that $H_{10} = 0$ if $D_1(2; 5) \sqcup D_1(3; 5) \sqcup D_2(4; 5) \neq \emptyset$. In the other cases, we obtain from Proposition 4.9 that

$$H_{10} = c_{k,j} \cdot \sum_{\{g\}_G} J_0'(g) \cdot M_G(\mathscr{O}) \cdot \prod_p c_p(g, R_p, \mathscr{O}_p),$$

$$\sum_{\{g\}_G} J_0'(g) = \left(J_0'(g_1) + J_0'(g_1^{-1}) + J_0'(g_2) + J_0'(g_2^{-1})\right) \cdot 2^{\sharp D_1(4;5)}.$$

We have $C_0(g; Sp(2; \boldsymbol{R})) = \{1_4\}$ for any g, and

$$J_0'(g_1) + J_0'({g_1}^{-1}) + J_0'(g_2) + J_0'({g_2}^{-1}) = \begin{cases} [1, 0, 0, -1, 0; 5]_k & \text{if } j \equiv 0 \bmod 10 \\ [-1, 1, 0, 0, 0; 5]_k & \text{if } j \equiv 2 \bmod 10 \\ 0 & \text{if } j \equiv 4 \bmod 10 \\ [0, 0, 0, 1, -1; 5]_k & \text{if } j \equiv 6 \bmod 10 \\ [0, -1, 0, 0, 1; 5]_k & \text{if } j \equiv 8 \bmod 10 \end{cases}.$$

(cf. (b-1) in [**Wak**]). We have

$$M_G(\mathscr{O}) = \frac{1}{10}, \quad \text{(cf. (3.23) in [\textbf{Has84}])}$$

$$\prod_p c_p(g, R_p, \mathscr{O}_p) = 2^{\sharp D_1(4;5)} \cdot 2^{\sharp D_2(2;5)} \cdot 2^{\sharp D_2(3;5)}$$

for any g. Hence we can obtain H_{10} as in Theorem 3.1.

4.11. The contribution $\mathbf{H_{11}}$.

In this subsection, we evaluate the contribution of $G[f_{11}]$, where $f_{11}(x) = x^4+1$. Note that if $D(1;8) \neq \emptyset$, then $G[f_{11}] = \emptyset$. Hereafter, we assume $D(1;8) = \emptyset$. We have $Z(g) = \boldsymbol{Q}(g) \simeq \boldsymbol{Q}(\zeta_8)$ for any g. We have the following proposition.

PROPOSITION 4.10.

(1) *If $D_2(7;8) \neq \emptyset$, then no G-conjugacy classes of $G[f_{11}]$ appear in the first sum of Theorem* 4.1.
(2) *If $D_2(7;8) = \emptyset$, then the followings hold.*
 (i) *The number of G-conjugacy classes in $G[f_{11}]$ which appear in the first sum of Theorem* 4.1 *is* $4 \cdot 2^{\sharp D_1(7;8)}$.
 (ii) *Let $\{g\}_G$ be any one of them. If Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as $\mathscr{O}$, where $\mathscr{O}$ is the ring of integers of $Z(g)$, then*

$$\prod_p c_p(g, R_p, \Lambda_p) = \prod_{\substack{p \mid D \\ p \neq 2}} 2.$$

If Λ does not belong to the same G-genus as $\mathscr{O}$, then $\prod_p c_p(g, R_p, \Lambda_p) = 0$.

PROOF. We can prove (1) and the latter part of (2 ii) easily by [**HI83**, Proposition 2.9]. For any element g of $G[f_{11}]$, g is G_p-conjugate to

$$\begin{cases} \gamma_p & \text{if } p \equiv 1, 3 \text{ or } 5 \bmod 8 \\ \gamma_p \text{ or } \delta_p & \text{if } p = 2 \text{ or } p \equiv 7 \bmod 8 \end{cases}$$

for some elements γ_p and $\delta_p \in G_p$. It follows from [**HI80**, Proposition 20] and [**HI83**, Proposition 2.9] that if Λ is a $\boldsymbol{Z}$-order of $Z(g)$ belonging to the same G-genus as $\mathscr{O}$, then c_p is as in the following table for each prime number p satisfying the first column:

(i) If $p \mid D_1$, then we have

	$c_p(\gamma_p, R_p, \mathscr{O}_p)$	$c_p(\delta_p, R_p, \mathscr{O}_p)$
$p \equiv 3 \bmod 8$	2	$\times$
$p \equiv 5 \bmod 8$	2	$\times$
$p \equiv 7 \bmod 8$	2	2
$p = 2$	1	1

(ii) If $p \mid D_2$, then we have

	$c_p(\gamma_p, R_p, \mathscr{O}_p)$	$c_p(\delta_p, R_p, \mathscr{O}_p)$
$p \equiv 3 \bmod 8$	2	$\times$
$p \equiv 5 \bmod 8$	2	$\times$
$p \equiv 7 \bmod 8$	0	0
$p = 2$	1	1

(iii) If $p \nmid D$, then we have

	$c_p(\gamma_p, R_p, \mathscr{O}_p)$	$c_p(\delta_p, R_p, \mathscr{O}_p)$
$p \equiv 1 \bmod 8$	1	$\times$
$p \equiv 3 \bmod 8$	1	$\times$
$p \equiv 5 \bmod 8$	1	$\times$
$p \equiv 7 \bmod 8$	1	0
$p = 2$	1	1

Also, g is $Sp(2;\boldsymbol{R})$-conjugate to $g_1 := \alpha(\pi/4, 3\pi/4)$, ${g_1}^{-1} = \alpha(-\pi/4, -3\pi/4)$, $g_2 := \alpha(\pi/4, -3\pi/4)$, or ${g_2}^{-1} = \alpha(-\pi/4, 3\pi/4)$. Since g^2 belongs to $G[f_6]$, g is $Sp(2;\boldsymbol{R})$-conjugate to g_1 or g_1^{-1} if $Z_0(g^2)$ is indefinite, and g is $Sp(2;\boldsymbol{R})$-conjugate to g_2 or g_2^{-1} if $Z_0(g^2)$ is definite. We take all combinations of G_p-conjugacy classes for all p, and also take $Sp(2;\boldsymbol{R})$-conjugation out of "g_1 or g_1^{-1}" or "g_2 or g_2^{-1}", according as $Z_0(g^2)$ is indefinite or definite. $\square$

We see from Proposition 4.10 that $H_{11} = 0$ if $D_2(7;8) \neq \emptyset$. Hereafter, we assume $D_2(7;8) = \emptyset$. We obtain from Proposition 4.10 that

$$H_{11} = c_{k,j} \cdot \sum_{\{g\}_G} J_0'(g) \cdot M_G(\mathscr{O}) \cdot \prod_p c_p(g, R_p, \mathscr{O}_p),$$

$$\sum_{\{g\}_G} J_0'(g) = \left(J_0'(g_1) + J_0'(g_1^{-1}) + J_0'(g_2) + J_0'(g_2^{-1})\right) \cdot 2^{\sharp D_1(7;8)}.$$

We have $C_0(g; Sp(2;\boldsymbol{R})) = \{1_4\}$ for any g, and

$$J_0'(g_1) + J_0'({g_1}^{-1}) + J_0'(g_2) + J_0'({g_2}^{-1}) = \begin{cases} [1,0,0,-1;4]_k & \text{if } j \equiv 0 \bmod 8 \\ [-1,1,0,0;4]_k & \text{if } j \equiv 2 \bmod 8 \\ [-1,0,0,1;4]_k & \text{if } j \equiv 4 \bmod 8 \\ [1,-1,0,0;4]_k & \text{if } j \equiv 6 \bmod 8 \end{cases}.$$

(cf. (b-1) in [**Wak**]). We have

$$M_G(\mathscr{O}) = \frac{1}{8}, \quad \text{(cf. (3.25) in [\textbf{Has84}])}$$
$$\prod_p c_p(g, R_p, \mathscr{O}_p) = \prod_{\substack{p|D \\ p\neq 2}} 2$$

for any g. Hence we can obtain H_{11} as in Theorem 3.1.

4.12. The contribution $\mathbf{H_{12}}$.

In this subsection, we evaluate the contribution of $G[f_{12}]$, where $f_{12}(x) = x^4 - x^2 + 1$. Note that $G[f_{12}] = \emptyset$ if and only if $D(1;12) \neq \emptyset$. Hereafter, we assume $D(1;12) = \emptyset$. We see from [**HI80**, Proposition 21] and [**HI83**, Proposition 2.10] that if $D_2(11;12) \neq \emptyset$, then no G-conjugacy classes of $G[f_{12}]$ appear in the first sum of Theorem 4.1. Hereafter, we assume that $D_2(11;12) = \emptyset$.

The set $G[f_{12}]$ consists of four $Sp(2;\boldsymbol{R})$-conjugacy classes represented by $h := \alpha(\pi/6, 5\pi/6)$, $h^{-1} = \alpha(-\pi/6, -5\pi/6)$, $h' := \alpha(\pi/6, -5\pi/6)$, $h'^{-1} = \alpha(-\pi/6, 5\pi/6)$. We have $C_0(g; Sp(2;\boldsymbol{R})) = \{1_4\}$ for any $g \in G[f_{12}]$ and

$$J_0'(h) + J_0'(h^{-1}) = (-1)^{j/2+k} \cdot [1,-1,0;3]_j,$$
$$J_0'(h') + J_0'(h'^{-1}) = (-1)^{j/2} \cdot [0,-1,1;3]_{j+2k}.$$

(cf. (b-1) in [**Wak**]). Only one G-genus represented by $\mathscr{O}$, where $\mathscr{O}$ is the ring of integers of $Z(g) \simeq \boldsymbol{Q}(\zeta_{12})$, appears in the second sum of Theorem 4.1 and $M_G(\mathscr{O}) = 1/12$ (cf. (3.27) in [**Has84**]). If g is an element of $G[f_{12}]$, then g^2 belongs to $G[f_7]$. We can obtain the following proposition from [**HI80**, Proposition 21] and [**HI83**, Proposition 2.10].

Proposition 4.11.

(I) *the case where* $D_1(11;12)=\emptyset$ *and* $\sharp D(5;12)$ *is even* (*resp. odd*)

(i) *the case where* $2 \nmid D$ *and* $3 \nmid D_1$:
Two G*-conjugacy classes* $\{g\}_G$ *and* $\{g^{-1}\}_G$ *appear in the first sum of Theorem* 4.1. *They are* $Sp(2;\boldsymbol{R})$*-conjugate to* h' *and* h'^{-1} (*resp.* h *and* h^{-1}) *respectively, and*

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

(ii) *the case where* $2 \mid D_2$ *and* $3 \nmid D_1$:
Two G*-conjugacy classes* $\{g\}_G$ *and* $\{g^{-1}\}_G$ *appear in the first sum of Theorem* 4.1. *They are* $Sp(2;\boldsymbol{R})$*-conjugate to* h *and* h^{-1} (*resp.* h' *and* h'^{-1}) *respectively, and*

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

(iii) *the case where* $2 \nmid D_1$ *and* $3 \mid D_1$:
Four G*-conjugacy classes appear in the first sum of Theorem* 4.1. *They are* $Sp(2;\boldsymbol{R})$*-conjugate to* h,h^{-1},h',h'^{-1} *respectively.*
If g *is* $Sp(2;\boldsymbol{R})$*-conjugate to* h (*resp.* h'), *then*

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

If g *is* $Sp(2;\boldsymbol{R})$*-conjugate to* h' (*resp.* h), *then*

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=2\cdot\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

(iv) *the case where* $2 \nmid D_1$ *and* $3 \mid D_1$:
Four G*-conjugacy classes appear in the first sum of Theorem* 4.1. *They are* $Sp(2;\boldsymbol{R})$*-conjugate to* h,h^{-1},h',h'^{-1} *respectively.*
If g *is* $Sp(2;\boldsymbol{R})$*-conjugate to* h (*resp.* h'), *then*

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=2\cdot\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

If g is $Sp(2;\boldsymbol{R})$-conjugate to h' (resp. h), then

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

(v) *the case where $2\mid D_1$ and $3\nmid D_1$:*
Four G-conjugacy classes appear in the first sum of Theorem 4.1. They are $Sp(2;\boldsymbol{R})$-conjugate to h,h^{-1},h',h'^{-1} respectively.
If g is $Sp(2;\boldsymbol{R})$-conjugate to h (resp. h'), then

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=2\cdot\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

If g is $Sp(2;\boldsymbol{R})$-conjugate to h' (resp. h), then

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.$$

(vi) *the case where $2\mid D_1$ and $3\mid D_1$:*
Eight G-conjugacy classes appear in the first sum of Theorem 4.1. Each two of them are $Sp(2;\boldsymbol{R})$-conjugate to h,h^{-1},h',h'^{-1} respectively.
If g is $Sp(2;\boldsymbol{R})$-conjugate to h (resp. h'), then

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=\begin{cases}4\cdot\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2\\ \prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.\end{cases}$$

If g is $Sp(2;\boldsymbol{R})$-conjugate to h' (resp. h), then

$$\prod_p c_p(g,R_p,\mathscr{O}_p)=\prod_p c_p(g^{-1},R_p,\mathscr{O}_p)=\begin{cases}2\cdot\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2\\ 2\cdot\prod_{p\in D(5;12)}2\cdot\prod_{p\in D(7;12)}2.\end{cases}$$

(II) *the case where $D_1(11;12)\neq\emptyset$*
(i) *the case where $2\nmid D_1$ and $3\nmid D_1$:*
The number of G-conjugacy classes which appear in the first sum of Theorem 4.1 is $2^{\sharp D_1(11;12)+1}$. They are $Sp(2;\boldsymbol{R})$-conjugate to h,h^{-1},h', h'^{-1}. All of them satisfy

$$\prod_{p} c_p(g, R_p, \mathscr{O}_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2.$$

(ii) *the case where "$2 \mid D_1$ and $3 \nmid D_1$" or "$2 \nmid D_1$ and $3 \mid D_1$":*
The number of G-conjugacy classes which appear in the first sum of Theorem 4.1 is $2^{\sharp D_1(11;12)+2}$. They are $Sp(2; \boldsymbol{R})$-conjugate to h, h^{-1}, h', h'^{-1}. In each case, $2^{\sharp D_1(11;12)-1}$ G-conjugacy classes satisfy

$$\prod_{p} c_p(g, R_p, \mathscr{O}_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2 \cdot 2,$$

$2^{\sharp D_1(11;12)-1}$ G-conjugacy classes satisfy

$$\prod_{p} c_p(g, R_p, \mathscr{O}_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2.$$

(iii) *the case where $2 \mid D_1$ and $3 \mid D_1$:*
The number of G-conjugacy classes which appear in the first sum of Theorem 4.1 is $2^{\sharp D_1(11;12)+3}$. They are $Sp(2; \boldsymbol{R})$-conjugate to h, h^{-1}, h', h'^{-1}. In each case, $2^{\sharp D_1(11;12)-1}$ G-conjugacy classes satisfy

$$\prod_{p} c_p(g, R_p, \mathscr{O}_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2 \cdot 2^2,$$

$2^{\sharp D_1(11;12)}$ G-conjugacy classes satisfy

$$\prod_{p} c_p(g, R_p, \mathscr{O}_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2 \cdot 2,$$

$2^{\sharp D_1(11;12)-1}$ G-conjugacy classes satisfy

$$\prod_{p} c_p(g, R_p, \mathscr{O}_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2.$$

5. The contribution of non-semi-simple conjugacy classes.

In this section, we evaluate $I(\Gamma^{(u)})_{k,j}$ and $I(\Gamma^{(qu)})_{k,j}$, i.e. the contributions of non-semi-simple conjugacy classes (cf. Section 1). We prove I_1, I_2 and I_3 of Theorem 3.1. Since the class number of $\mathfrak{O}$ is one, any maximal two-sided ideal $\mathfrak{A}$ can be written as $\mathfrak{A} = \mathfrak{O}\pi = \pi\mathfrak{O}$ for some $\pi \in \mathfrak{O}$. By taking conjugation by

$\left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{smallmatrix}\right) \in G$, we may regard $\Gamma = G \cap \left(\begin{smallmatrix} \mathfrak{O} & \mathfrak{A} \\ \mathfrak{A}^{-1} & \mathfrak{O} \end{smallmatrix}\right)$.

We put

$$P = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in B^{\times}, b \in B^0 \right\}.$$

Then P is the unique parabolic subgroup of G up to G-conjugation since we consider the case where $B \neq M_2(\boldsymbol{Q})$ in this paper. We can prove that $\Gamma \backslash \mathfrak{H}_2$ has only one 0-dimensional cusp, up to equivalence, in the same way as [**Ara81**, Proposition 2]. Arakawa proved Lemma 5.1 below in his master thesis [**Ara75**, Proposition 7].

Lemma 5.1. *We have $G = P \cdot \Gamma$.*

Proof. We take any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. There are some γ, $\delta \in \mathfrak{O}$ such that $c^{-1}d = \pi^{-1}\gamma^{-1}\delta$. We can assume that there are some $u, v \in \mathfrak{O}$ such that $\gamma u + \delta v = 1$. If we put

$$\tau := \begin{pmatrix} \overline{v} - \overline{\pi^{-1}u} v\gamma\pi & \overline{\pi^{-1}u}(1 - v\delta) \\ \gamma\pi & \delta \end{pmatrix},$$

then we have $\tau \in \Gamma$ and $\sigma\tau^{-1} \in P$. □

By using Lemma 5.1, we can prove Proposition 5.2 below in the same way that Hashimoto proved it when $D_2 = 1$ in [**Has84**, Lemma 1.2].

Proposition 5.2. *If γ is an element of $\Gamma^{(u)} \sqcup \Gamma^{(qu)}$, then γ is Γ-conjugate to an element of the form*:

$$\gamma(a, b) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where $a \in \mathfrak{O}^{\times}$ is a root of unity and $b \in \mathfrak{A}^0 - \{0\}$.

If $\gamma \in \Gamma^{(u)}$, then $a = \pm 1$ and the principal polynomial of γ is $f_1(x) = (x-1)^4$ or $f_1(-x)$. We put $I_1 = I(\Gamma^{(u)})_{k,j}$. If $\gamma \in \Gamma^{(qu)}$, then a is a primitive 4-th, 3-rd, or 6-th root of unity and the principal polynomial of γ is $f_6(x) = (x^2+1)^2$, $f_7(x) = (x^2+x+1)^2$ or $f_7(-x)$ respectively. We denote by I_2 (resp. I_3) the contribution of elements of $\Gamma^{(qu)}$ whose principal polynomial is $f_6(x)$ (resp. $f_7(\pm x)$). We evaluate I_1 in Subsection 5.1, and I_2 and I_3 in Subsection 5.2 and 5.3. We use the notation

$$\gamma(\theta,t) := \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We summarize some lemmas which are used in Subsection 5.2. These lemmas were proved in the case of $D_2 = 1$ by Hashimoto [**Has84**]. The following lemmas are easy generalizations of them and can be proved in the almost same method, so we omit the proof. Let a be a primitive 3-rd or 4-th or 6-th root of unity. We put $F := \boldsymbol{Q}(a)$ and denote by $\mathscr{O}_F$ the ring of integers of F and by d the discriminant of F. F is isomorphic to $\boldsymbol{Q}(\sqrt{-1})$ or $\boldsymbol{Q}(\sqrt{-3})$.

Lemma 5.3. *Let γ be an element of Γ of the form $\gamma(a,b)$ in Proposition* 5.2. *Then we have*

(1) *If β is an element of $B^\times$ such that $\beta a = \overline{a}\beta$, then we have $B = F \oplus F\beta$.*
(2) *If we express $b \in \mathfrak{A}^0$ as $b = x\sqrt{d} + y\beta$ $(x \in \boldsymbol{Q}, y \in F)$, then the Jordan decomposition $\gamma(a,b) = \gamma_s \cdot \gamma_u$ is given by*

$$\gamma_s = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & y\beta \\ 0 & 1 \end{pmatrix}, \quad \gamma_u = \begin{pmatrix} 1 & x\sqrt{d} \\ 0 & 1 \end{pmatrix}.$$

Lemma 5.4. *If we put, for a fixed a as above,*

$$C(a) := \{x^{-1}ax \mid x \in B^\times\},$$

then we have

$$\sharp\big((C(a) \cap \mathfrak{O})/\sim_{\mathfrak{O}^\times}\big) = \prod_{p|D} \left(1 - \left(\frac{F}{p}\right)\right).$$

Lemma 5.5. *Let $\gamma_i = \gamma(a_i, b_i)$ $(i = 1,2)$ be two elements of Γ^{qu} of the form of Proposition* 5.2. *If γ_1 and γ_2 are Γ-conjugate, then a_1 and a_2 are $\mathfrak{O}^\times$-conjugate.*

Lemma 5.6. *Let $\gamma_i = \gamma(a, b_i)$ $(i = 1,2)$ be two elements of $\Gamma^{(qu)}$ of the form of Proposition* 5.2. *We put*

$$L_{\mathfrak{A}}(a) := \{a^{-1}za - z \mid z \in \mathfrak{A}^0\}.$$

Then $\gamma(a,b_1)$ and $\gamma(a,b_2)$ are Γ-conjugate if and only if $b_1 - b_2 \in L_{\mathfrak{A}}(a)$.

5.1. The contribution $\mathbf{I_1}$.

In this subsection, we evaluate the contribution I_1. We define the following four subsets of Γ:

$$F_1 := \left\{ \begin{pmatrix} 1_2 & S \\ 0_2 & 1_2 \end{pmatrix} \,\middle|\, S \in SM_2(\boldsymbol{R}), \det S \neq 0, S : \text{definite} \right\} \cap \Gamma,$$

$$F_2 := \left\{ \begin{pmatrix} 1_2 & S \\ 0_2 & 1_2 \end{pmatrix} \,\middle|\, S \in SM_2(\boldsymbol{R}), \det S \neq 0, S : \text{indefinite}, -\det S \notin (\boldsymbol{Q}^\times)^2 \right\} \cap \Gamma,$$

$$F_3 := \left\{ \begin{pmatrix} 1_2 & S \\ 0_2 & 1_2 \end{pmatrix} \,\middle|\, S \in SM_2(\boldsymbol{R}), \det S \neq 0, S : \text{indefinite}, -\det S \in (\boldsymbol{Q}^\times)^2 \right\} \cap \Gamma,$$

$$F_4 := \left\{ \begin{pmatrix} 1_2 & S \\ 0_2 & 1_2 \end{pmatrix} \,\middle|\, S \in SM_2(\boldsymbol{R}), \det S = 0 \right\} \cap \Gamma,$$

where we denote by $SM_2(\boldsymbol{R})$ the set of all symmetric matrices of degree 2 over $\boldsymbol{R}$. We denote by $C^{(u)}$ the set of all Γ-conjugacy classes of $\Gamma^{(u)}$. We can prove the following proposition by Proposition 5.2.

Proposition 5.7. *We can decompose* $C^{(u)}$ *as*

$$C^{(u)} = \bigsqcup_{i=1}^{4} \Bigg(\bigsqcup_{\gamma \in F_i/\sim_\Gamma} \{\{\gamma\}_\Gamma\} \Bigg),$$

where $F_i/\sim_\Gamma$ *denotes a complete system of representatives of* Γ*-conjugacy classes of* F_i *and* $\{\gamma\}_\Gamma$ *denotes the* Γ*-conjugacy class represented by* γ.

Proof. Take an arbitrary $\{\gamma'\}_\Gamma \in C_u$. By Proposition 5.2, we have some $x \in \Gamma$ such that $x^{-1}\gamma' x = \pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, $b \in \mathfrak{A}^0 - \{0\}$. Identifying $x^{-1}\gamma' x$ and its image by ϕ in $Sp_2(\boldsymbol{R})$, it is contained in some F_i, so we have

$$\{\gamma'\}_\Gamma = \{x^{-1}\gamma' x\}_\Gamma \in \bigsqcup_{\gamma \in F_i/\sim_\Gamma} \{\{\gamma\}_\Gamma\}. \qquad \square$$

However, especially in the case of our Γ, we have $F_3 = F_4 = \emptyset$ and

$$I_1 = c_{k,j} \cdot \sum_{i=1}^{2} \mathrm{vol}\big(C_0(\gamma_i;\Gamma)\backslash C_0(\gamma_i; Sp(2;\boldsymbol{R}))\big) \cdot \lim_{s\to+0} \sum_{\gamma' \in F_i/\sim} \frac{J_0(\gamma';s)}{[C(\gamma';\Gamma) : \pm C_0(\gamma';\Gamma)]},$$

where γ_i is an any element of F_i (cf. [**Wak**, Theorem 3.1]). By using the formula of [**Wak**, (e-2), (e-3)], we have

$$\lim_{s\to+0}\sum_{\gamma'\in F_i/\sim}\frac{J_0(\gamma';s)}{[C(\gamma';\Gamma):\pm C_0(\gamma';\Gamma)]}$$
$$=\begin{cases} c_{k,j}^{-1}\cdot\dfrac{j+1}{2^2\pi}\cdot\dfrac{1}{[\tilde{\Gamma}:\tilde{\Gamma}_+]}\cdot\dfrac{\mathrm{vol}(\tilde{\Gamma}_+\backslash\mathfrak{H}_1)}{\mathrm{vol}(L)} & i=1\\ 0 & i=2\end{cases}$$

Here, we define the notations as follows.

We define a lattice L in $SM_2(\boldsymbol{R})$ by

$$\left\{\begin{pmatrix}1_2 & X\\ 0_2 & 1_2\end{pmatrix}\,\middle|\, X\in L\right\}=\left\{\begin{pmatrix}1_2 & S\\ 0_2 & 1_2\end{pmatrix}\,\middle|\, S\in SM_2(\boldsymbol{R})\right\}\cap\Gamma.$$

We put

$$C_0(\gamma_1;Sp(2;\boldsymbol{R}))=\left\{\begin{pmatrix}1_2 & S\\ 0_2 & 1_2\end{pmatrix}\,\middle|\, S\in SM_2(\boldsymbol{R})\right\},$$
$$C_0(\gamma_1;\Gamma)=C_0(\gamma_1;Sp(2;\boldsymbol{R}))\cap\Gamma=\left\{\begin{pmatrix}1_2 & X\\ 0_2 & 1_2\end{pmatrix}\,\middle|\, X\in L\right\}$$

and

$$\mathrm{vol}(L):=\mathrm{vol}(C_0(\gamma_1;\Gamma)\backslash C_0(\gamma_1;G(\boldsymbol{R})))=\int_{L\backslash SM_2(\boldsymbol{R})}dx_{11}dx_{12}dx_{22}$$

for $\left(\begin{smallmatrix}x_{11} & x_{12}\\ x_{12} & x_{22}\end{smallmatrix}\right)\in SM_2(\boldsymbol{R})$. We put

$$\tilde{\Gamma}=\left\{\begin{pmatrix}x & 0\\ 0 & \overline{x}^{-1}\end{pmatrix}\,\middle|\, x\in B^\times\right\}\cap\Gamma,\quad \tilde{\Gamma}_+=\left\{\begin{pmatrix}x & 0\\ 0 & \overline{x}^{-1}\end{pmatrix}\in\tilde{\Gamma}\,\middle|\, x\overline{x}>0\right\}.$$

We can identify $\tilde{\Gamma}_+$ as the subgroup of $GL_+(2;\boldsymbol{R})=\{g\in GL(2;\boldsymbol{R})\mid\det(g)>0\}$ and we define

$$\mathrm{vol}(\tilde{\Gamma}_+\backslash\mathfrak{H}_1)=\int_{\tilde{\Gamma}_+\backslash\mathfrak{H}_1}y^{-2}dxdy$$

for $x+iy \in \mathfrak{H}_1$, where $\mathfrak{H}_1$ is the upper half plane $\{z \in \boldsymbol{C} | \mathrm{Im}(z) > 0\}$.

It follows that we have

$$I_1 = \frac{j+1}{2^3\pi} \cdot \frac{\mathrm{vol}(\tilde{\Gamma}_+ \backslash \mathfrak{H}_1)}{[\tilde{\Gamma} : \tilde{\Gamma}_+]}.$$

Noting that $\tilde{\Gamma}$ and $\tilde{\Gamma}_+$ are independent on a choice of pairs $(D_1,\ D_2)$ for a fixed D, we see that the value I_1 is also independent on it. Hence we have

$$I_1 = 2^{-3}3^{-1}(j+1)\prod_{p|D}(p-1),$$

which is the same value as in [**Wak**, Theorem 6.1].

5.2. The contribution $\mathbf{I_2}$.

In this section, we evaluate the contribution I_2. Let γ be an element of $\Gamma^{(qu)}$ whose principal polynomial is $f_6(x) = (x^2+1)^2$. Then γ is $Sp(2;\boldsymbol{R})$-conjugate to an element of the form

$$\gamma\left(\frac{\pi}{2}, s\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(cf. Proposition 5.2) and corresponds to (f-3) of [**Wak**].

We denote by C_6 the set of all Γ-conjugacy classes of $\Gamma^{(qu)}$ whose principal polynomial is $f_6(x)$. Then we have the following proposition:

Proposition 5.8. *We can decompose C_6 into disjoint union of $4N$ subsets as*

$$C_6 = \bigsqcup_{i=1}^{N}\bigsqcup_{j=1}^{4}\left(\bigsqcup_{\gamma \in F_{i,j}} \{\{\gamma\}_\Gamma\}\right),$$

where $N := \prod_{p|D}(1-(-1/p))$ and $F_{i,j}$ is defined as follows.

Let $a_1, \dots, a_N$ be a complete system of $\mathfrak{O}^\times$-conjugacy classes of elements of $\mathfrak{O}$ of order 4. (cf. Lemma 5.4). There exist some $x_i \in \boldsymbol{Q}_{>0}$ and $\beta_i \in \mathfrak{O}^0$ depending on each a_i such that $F_{i,j}$'s are given as one of the following four cases. Here we put

$$\delta(a_i, \gamma_1, \gamma_2) := \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 1 & \gamma_1\beta_i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \gamma_2 x_i a_i \\ 0 & 1 \end{pmatrix},$$

where the symbol " $\cdot$ " means the Jordan decomposition.

Case 1:

$$F_{i,1} = \{\delta(a_i, 0, l) \mid l \in \boldsymbol{Z} - \{0\}\}, \qquad F_{i,2} = \{\delta(a_i, 1, l) \mid l \in \boldsymbol{Z} - \{0\}\}$$
$$F_{i,3} = \{\delta(a_i, a_i, l) \mid l \in \boldsymbol{Z} - \{0\}\}, \qquad F_{i,4} = \{\delta(a_i, 1 + a_i, l) \mid l \in \boldsymbol{Z} - \{0\}\}$$

All elements of $F_{i,j}$ are conjugate to $\gamma(\pi/2, l)$ in $Sp(2, \boldsymbol{R})$.

Case 2:

$$F_{i,1} = \{\delta(a_i, 0, 2l) \mid l \in \boldsymbol{Z} - \{0\}\}, \qquad F_{i,2} = \{\delta(a_i, 1, 2l) \mid l \in \boldsymbol{Z} - \{0\}\}$$
$$F_{i,3} = \left\{\delta\left(a_i, \frac{1}{2}a_i, 2l + 1\right) \middle| \, l \in \boldsymbol{Z}\right\}, \quad F_{i,4} = \left\{\delta\left(a_i, 1 + \frac{1}{2}a_i, 2l + 1\right) \middle| \, l \in \boldsymbol{Z}\right\}$$

All elements of $F_{i,1}$ and $F_{i,2}$ are conjugate to $\gamma(\pi/2, l)$ in $Sp(2, \boldsymbol{R})$. All elements of $F_{i,3}$ and $F_{i,4}$ are conjugate to $\gamma(\pi/2, l + (1/2))$ in $Sp(2, \boldsymbol{R})$.

Case 3:

$$F_{i,1} = \{\delta(a_i, 0, 2l) \mid l \in \boldsymbol{Z} - \{0\}\}, \qquad F_{i,2} = \{\delta(a_i, a_i, 2l) \mid l \in \boldsymbol{Z} - \{0\}\}$$
$$F_{i,3} = \left\{\delta\left(a_i, \frac{1}{2}, 2l + 1\right) \middle| \, l \in \boldsymbol{Z}\right\}, \qquad F_{i,4} = \left\{\delta\left(a_i, \frac{1}{2} + a_i, 2l + 1\right) \middle| \, l \in \boldsymbol{Z}\right\}$$

All elements of $F_{i,1}$ and $F_{i,2}$ are conjugate to $\gamma(\pi/2, l)$ in $Sp(2, \boldsymbol{R})$. All elements of $F_{i,3}$ and $F_{i,4}$ are conjugate to $\gamma(\pi/2, l + (1/2))$ in $Sp(2, \boldsymbol{R})$.

Case 4:

$$F_{i,1} = \{\delta(a_i, 0, 2l) \mid l \in \boldsymbol{Z} - \{0\}\}, \qquad F_{i,2} = \{\delta(a_i, 1, 2l) \mid l \in \boldsymbol{Z} - \{0\}\}$$
$$F_{i,3} = \left\{\delta\left(a_i, \frac{1}{2} + \frac{1}{2}a_i, 2l + 1\right) \middle| \, l \in \boldsymbol{Z}\right\}, \quad F_{i,4} = \left\{\delta\left(a_i, \frac{1}{2} + \frac{3}{2}a_i, 2l + 1\right) \middle| \, l \in \boldsymbol{Z}\right\}$$

All elements of $F_{i,1}$ and $F_{i,2}$ are conjugate to $\gamma(\pi/2, l)$ in $Sp(2, \boldsymbol{R})$. All elements of $F_{i,3}$ and $F_{i,4}$ are conjugate to $\gamma(\pi/2, l + (1/2))$ in $Sp(2, \boldsymbol{R})$.

Proof. We take an arbitrary $\{\gamma\}_\Gamma \in C_6$. By Proposition 5.2, we have

$$\gamma \sim_\Gamma \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for some $a \in \mathfrak{O}$ of order 4 and $b \in \mathfrak{A}^0 - \{0\}$. By taking Γ-conjugation, we may have $a = a_i$ for some $i \in \{1, \ldots, N\}$. Hence we have

$$C_6 = \bigsqcup_{i=1}^{N} X_i, \quad X_i = \left\{ \left\{ \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}_\Gamma \middle| \, b \in \mathfrak{A}^0 - \{0\} \right\}.$$

For each X_i, we simply put $a = a_i$. By Lemma 5.6, each X_i can be decomposed as

$$X_i = \bigsqcup_{b \in \mathfrak{A}^0 / L_{\mathfrak{A}}(a)} \left\{ \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}_\Gamma \right\}.$$

We can describe the structure of $\mathfrak{A}^0/L_{\mathfrak{A}}(a)$ by the same way as Hashimoto [**Has84**] as follows. From Proposition 2.5 of [**Has84**], we have

$$\mathfrak{O}^0 = \begin{cases} \boldsymbol{Z} \cdot \dfrac{a+\beta}{2} + \mathscr{O}_F \beta & \text{if } 2 \nmid D, \\ \boldsymbol{Z} \cdot a + \mathscr{O}_F \beta & \text{if } 2 \nmid D \end{cases}$$

for some β. So we have $\mathfrak{O}^0 \cap F^\perp = \mathscr{O}_F \beta$ and $\mathscr{A}^0 \cap F^\perp$ is a $\mathscr{O}_F$-submodule of $\mathfrak{O}^0 \cap F^\perp$. Since $\mathscr{O}_F$ is P.I.D. and $\mathfrak{O}^0 \cap F^\perp$ is a free $\mathscr{O}_F$-module of rank 1, $\mathfrak{A}^0 \cap F^\perp$ is also a free $\mathscr{O}_F$-module of rank 1. So we can write $\mathfrak{A}^0 \cap F^\perp = \mathscr{O}_F \beta'$ with some β'. Since $\mathfrak{A}^0/(\mathfrak{A}^0 \cap F^\perp)$ is a torsion-free $\boldsymbol{Z}$-module, $\mathfrak{A}^0 \cap F^\perp$ is a direct summand of $\mathfrak{A}^0$, that is, there exists some sub$\boldsymbol{Z}$-module M of $\mathfrak{A}^0$ and we can write $\mathfrak{A}^0 = M \oplus (\mathfrak{A}^0 \cap F^\perp)$. The $\boldsymbol{Z}$-module M is free of rank 1. A basis of M can be expressed as the form: $xa + y\beta'$ ($x \in \boldsymbol{Q} - \{0\}$, $y \in F$) because we have $B^0 = \boldsymbol{Q}a + F\beta'$ with β' mentioned above. So we can take $\rho_1 := xa + y\beta'$, $\rho_2 := \beta'$ and $\rho_3 := a\beta'$ as a basis of $\mathfrak{A}^0$. From the relation $-2y\beta' = a^{-1}\rho_1 a - \rho_1 \in L_{\mathfrak{A}}(a) \subset \mathfrak{A}^0 \cap F^\perp = \mathscr{O}_F \beta'$, we have $2y \in \mathscr{O}_F = \boldsymbol{Z} + \boldsymbol{Z}a$. We divide the situation into two cases according as $y \in \mathscr{O}_F$ of $\notin \mathscr{O}_F$.

(i) The case of $y \in \mathscr{O}_F$. We can write $\rho_1 = xa + y_1\beta' + y_2 a\beta'$ with some $y_1, y_2 \in \boldsymbol{Z}$. So by replacing ρ_1, $\rho_1 = xa$, $\rho_2 = \beta'$, $\rho_3 = a\beta'$ forms a basis of $\mathfrak{A}^0$, that is

$$\mathfrak{A}^0 = \boldsymbol{Z}xa \oplus \boldsymbol{Z}\beta' \oplus \boldsymbol{Z}a\beta'.$$

(ii) The case of $y \notin \mathscr{O}_F$. We have $y = y_1 + y_2 a$, $2y_1, 2y_2 \in \boldsymbol{Z}$ and $\mathfrak{A}^0 = \boldsymbol{Z} \cdot (xa + y_1\beta' + y_1 a\beta') \oplus \boldsymbol{Z}\beta' \oplus \boldsymbol{Z}a\beta'$. So $\mathfrak{A}^0$ is one of the following three cases:

$$\text{Case (ii a)} \qquad \mathfrak{A}^0 = \boldsymbol{Z} \cdot \left(xa + \frac{1}{2}a\beta'\right) \oplus \boldsymbol{Z}\beta' \oplus \boldsymbol{Z}a\beta'$$

$$\text{Case (ii b)} \qquad \mathfrak{A}^0 = \boldsymbol{Z} \cdot \left(xa + \frac{1}{2}\beta'\right) \oplus \boldsymbol{Z}\beta' \oplus \boldsymbol{Z}a\beta'$$

$$\text{Case (ii c)} \qquad \mathfrak{A}^0 = \boldsymbol{Z} \cdot \left(xa + \frac{1}{2}\beta' + \frac{1}{2}a\beta'\right) \oplus \boldsymbol{Z}\beta' \oplus \boldsymbol{Z}a\beta'.$$

In each case, the structure of $L_{\mathfrak{A}}(a)$ and $\mathfrak{A}^0/L_{\mathfrak{A}}(a)$ are given as follows:

Case (i) :

$$L_{\mathfrak{A}}(a) = \{2m\beta' + 2na\beta' \mid m, n \in \boldsymbol{Z}\},$$

$$\begin{aligned}\mathfrak{A}^0/L_{\mathfrak{A}}(a) = {} & \{lxa \mid l \in \boldsymbol{Z}\} \sqcup \{lxa + \beta' \mid l \in \boldsymbol{Z}\} \sqcup \{lxa + a\beta' \mid l \in \boldsymbol{Z}\} \\ & \sqcup \{lxa + \beta' + a\beta' \mid l \in \boldsymbol{Z}\}.\end{aligned}$$

Case (ii a) :

$$L_{\mathfrak{A}}(a) = \{2m\beta' + na\beta' \mid m, n \in \boldsymbol{Z}\},$$

$$\begin{aligned}\mathfrak{A}^0/L_{\mathfrak{A}}(a) = {} & \{lxa \mid l \in 2\boldsymbol{Z}\} \sqcup \{lxa + \beta' \mid l \in 2\boldsymbol{Z}\} \\ & \sqcup \left\{lxa + \frac{1}{2}a\beta' \,\middle|\, l \in 2\boldsymbol{Z} + 1\right\} \sqcup \left\{lxa + \beta' + \frac{1}{2}a\beta' \,\middle|\, l \in 2\boldsymbol{Z} + 1\right\}.\end{aligned}$$

Case (ii b) :

$$L_{\mathfrak{A}}(a) = \{m\beta' + 2na\beta' \mid m, n \in \boldsymbol{Z}\},$$

$$\begin{aligned}\mathfrak{A}^0/L_{\mathfrak{A}}(a) = {} & \{lxa \mid l \in 2\boldsymbol{Z}\} \sqcup \{lxa + a\beta' \mid l \in 2\boldsymbol{Z}\} \\ & \sqcup \left\{lxa + \frac{1}{2}\beta' \,\middle|\, l \in 2\boldsymbol{Z} + 1\right\} \sqcup \left\{lxa + \frac{1}{2}\beta' + a\beta' \,\middle|\, l \in 2\boldsymbol{Z} + 1\right\}.\end{aligned}$$

Case (ii c) :

$$L_{\mathfrak{A}}(a) = \{m\beta' + na\beta' \mid m, n \in \boldsymbol{Z}\},$$

$$\begin{aligned}\mathfrak{A}^0/L_{\mathfrak{A}}(a) = \{lxa \mid l \in 2\boldsymbol{Z}\} &\sqcup \{lxa + \beta' \mid l \in 2\boldsymbol{Z}\}\\ &\sqcup \left\{lxa + \frac{1}{2}\beta' + \frac{1}{2}a\beta' \middle| \, l \in 2\boldsymbol{Z} + 1\right\}\\ &\sqcup \left\{lxa + \frac{1}{2}\beta' + \frac{3}{2}a\beta' \middle| \, l \in 2\boldsymbol{Z} + 1\right\}.\end{aligned}$$

Thus we have completed the proof of Proposition 5.8. □

The sets $F_{i,l}$'s are called *families* in [**Has83**], [**Has84**], [**Wak**], etc. For each $F_{i,l}$, there exist $g_{i,l} \in Sp(2; \boldsymbol{R})$ and $\lambda \in \boldsymbol{R}$ with $0 \le \lambda_{i,l} < 1$, such that

$$F_{i,l} = g_{i,l} \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & n+\lambda_{i,l} & 0 \\ 0 & 1 & 0 & n+\lambda_{i,l} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{c} n \in \boldsymbol{Z} \\ n+\lambda_{i,l} \neq 0 \end{array} \right\} g_{i,l}^{-1}.$$

We define

$$C(F_{i,l}; Sp(2;\boldsymbol{R})) := g_{i,l} \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, \theta, t \in \boldsymbol{R} \right\} g_{i,l}^{-1},$$

$$C_0(F_{i,l}; Sp(2;\boldsymbol{R})) := g_{i,l} \left\{ \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, t \in \boldsymbol{R} \right\} g_{i,l}^{-1},$$

$$C(F_{i,l}; \Gamma) := C(F_{i,l}; Sp(2;\boldsymbol{R})) \cup \Gamma,$$

$$C_0(F_{i,l}; \Gamma) := C_0(F_{i,l}; Sp(2;\boldsymbol{R})) \cup \Gamma.$$

Then, from (f-3) in [**Wak**], we have

$$\begin{aligned} I_2 = \sum_{i=1}^{N} \sum_{l=1}^{4} \frac{1}{2} \cdot \frac{\mathrm{vol}(C_0(F_{i,l};\Gamma) \backslash C_0(F_{i,l}; Sp(2,\boldsymbol{R})))}{[C(F_{i,l};\Gamma) : \pm C_0(F_{i,l};\Gamma)]} \\ \cdot (-2^{-3}(-1)^{j/2}) \cdot \big(1 - \sqrt{-1}\cot^* \pi\lambda_{i,l}\big), \end{aligned}$$

where we put

$$\cot^* \pi\lambda := \begin{cases} 0 & \text{if } \lambda = 0, \\ \cot \pi\lambda & \text{if } 0 \leq \lambda < 1. \end{cases}$$

We can verify that

$$C(F_{i,l};\Gamma) = g_{i,l} \left\{ \pm \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, t \in \boldsymbol{Z} \right\} g_{i,l}^{-1},$$

$$C_0(F_{i,l};\Gamma) = g_{i,l} \left\{ \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, t \in \boldsymbol{Z} \right\} g_{i,l}^{-1}$$

and

$$\mathrm{vol}\big(C_0(F_{i,l};\Gamma)\backslash C_0(F_{i,l};Sp(2;\boldsymbol{R}))\big) = 1,$$
$$[C(F_{i,l};\Gamma) : \pm C_0(F_{i,l};\Gamma)] = 2.$$

Hence we have $I_2 = -4N \cdot 2^{-5}(-1)^{j/2}$.

5.3. The contribution $\mathbf{I_3}$.

In this section, we evaluate the contribution I_3. We consider the contribution of elements whose principal polynomials are $f_7(x) = (x^2 + x + 1)^2$ and double it to obtain I_3. Let γ be an element of $\Gamma^{(qu)}$ whose principal polynomial is $f_7(x)$. Then γ is $Sp(2;\boldsymbol{R})$-conjugate to an element of the form

$$\gamma\left(\frac{2\pi}{3}, s\right) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & \sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(cf. Proposition 5.2) and corresponds to (f-3) of [**Wak**].

We denote by C_7 the set of all Γ-conjugacy classes of $\Gamma^{(qu)}$ whose principal polynomial is $f_7(x)$. By the same way as Proposition 5.8, we can prove the following proposition:

Proposition 5.9. *We can decompose C_7 into disjoint union of $3N$ subsets as*

$$C_7 = \coprod_{i=1}^{N} \coprod_{l=1}^{3} \Bigg(\coprod_{\gamma \in F_{i,l}} \{\{\gamma\}_\Gamma\} \Bigg),$$

where $N := \prod_{p|D}(1-(-3/p))$ *and* $F_{i,j}$ *is defined as follows.*

Let $a_1, \dots a_N$ be a complete system of $\mathfrak{O}^\times$-conjugacy classes of elements of $\mathfrak{O}$ of order 3 (cf. Lemma 5.4). There exist some $x_i \in \boldsymbol{Q}_{>0}$ and $\beta_i \in \mathfrak{O}^0$ depending on each a_i such that $F_{i,j}$'s are given as one of the following two cases. Here we put

$$\delta(a_i, \gamma_1, \gamma_2) := \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 1 & \gamma_1 \beta_i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \gamma_2 x_i \sqrt{-3} \\ 0 & 1 \end{pmatrix},$$

where the symbol " $\cdot$ " means the Jordan decomposition.

Case 1:

$$F_{i,1} = \big\{\delta(a_i, 0, n) \mid n \in \boldsymbol{Z} - \{0\}\big\}, \qquad F_{i,2} = \big\{\delta(a_i, 1, n) \mid n \in \boldsymbol{Z} - \{0\}\big\}$$
$$F_{i,3} = \big\{\delta(a_i, 2, n) \mid n \in \boldsymbol{Z} - \{0\}\big\},$$

All elements of $F_{i,l}$'s are $Sp(2;\boldsymbol{R})$-conjugate to $\gamma(2\pi/3, n)$.

Case 2:

$$F_{i,1} = \big\{\delta(a_i, 0, 3n) \mid n \in \boldsymbol{Z} - \{0\}\big\}, \qquad F_{i,2} = \left\{\delta\left(a_i, \frac{1+2a}{3}, 3n+1\right) \middle| \, n \in \boldsymbol{Z}\right\}$$
$$F_{i,3} = \left\{\delta\left(a_i, \frac{2+a}{3}, 3n+2\right) \middle| \, l \in \boldsymbol{Z}\right\},$$

All elements of each $F_{i,l}$ are $Sp(2;\boldsymbol{R})$-conjugate to $\gamma(2\pi/3, n+(l-1)/3)$.

For each $F_{i,l}$, we define $g_{i,l}$, $\lambda_{i,l}$, $C(F_{i,l}; Sp(2;\boldsymbol{R}))$, $C_0(F_{i,l}; Sp(2;\boldsymbol{R}))$, $C(F_{i,l};\Gamma)$ and $C_0(F_{i,l};\Gamma)$ in the same way as in Subsection 5.2. Then, from (f-3) in [**Wak**], we have

$$I_3 = \sum_{i=1}^{N} \sum_{l=1}^{3} \cdot \frac{\mathrm{vol}(C_0(F_{i,l};\Gamma) \backslash C_0(F_{i,l}; Sp(2,\boldsymbol{R})))}{[C(F_{i,l};\Gamma) : \pm C_0(F_{i,l};\Gamma)]}$$
$$\cdot (-2^{-1}3^{-1}[1,-1,0;3]_j) \cdot \big(1 - \sqrt{-1}\cot^* \pi\lambda_{i,l}\big).$$

We can verify that

$$C(F_{i,l};\Gamma)=g_{i,l}\left\{\pm\gamma(\theta,t)\,\middle|\,\theta=0,\frac{\pi}{3},\frac{2\pi}{3},t\in\boldsymbol{Z}\right\}g_{i,l}^{-1},$$
$$C_0(F_{i,l};\Gamma)=g_{i,l}\{\gamma(0,t)\mid t\in\boldsymbol{Z}\}g_{i,l}^{-1}$$

and

$$\mathrm{vol}\big(C_0(F_{i,l};\Gamma)\backslash C_0(F_{i,l};Sp(2;\boldsymbol{R}))\big)=1,\quad [C(F_{i,l};\Gamma):\pm C_0(F_{i,l};\Gamma)]=3.$$

Hence we have $I_3=-3N\cdot 2^{-1}3^{-2}[1,-1,0;3]_j$.

6. Numerical examples.

In this section, we give some numerical examples of $\dim_{\boldsymbol{C}} S_{k,j}(\Gamma(D_1,D_2))$ for various D_1,D_2. The tables for $D=D_1=6,10,15$ appeared in [**Wak**]. Our theorem can not be applied for $k\leq 4$. In the following tables, we formally substitute $k\leq 4$ in the formula of Theorem 3.1. Hashimoto conjectured that the dimension of $S_{4,0}(\Gamma(D,1))$ (resp. $S_{3,0}(\Gamma(D,1))$) can be obtained by substituting $k=4$ in Theorem 3.1 (resp. by substituting $k=3$ and adding $+1$). (Conjecture 4.3, 4.4 in [**Has84**]. cf. [**Ibu07b**] in the split case).

(**I**) $\mathbf{D=2\cdot 3}$

(i) $D_1=2\cdot 3$, $D_2=1$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	-1	0	-1	2	0	4	2	8	5	15	10	25	15	34	26
2	-1	2	0	1	2	2	5	7	15	17	33	34	53	58	91	96
4	0	-1	0	2	4	6	14	19	35	42	67	77	114	126	179	200
6	-2	-1	1	5	9	17	30	40	65	82	118	145	195	224	299	341
8	-3	-2	2	7	19	27	49	67	106	131	188	223	298	346	448	514

(ii) $D_1=3$, $D_2=2$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-1	-1	0	0	2	1	3	4	7	5	9	11	17	14	21	24
2	0	1	0	1	0	1	3	6	7	10	18	23	29	36	52	61
4	0	-1	0	1	2	2	7	12	19	23	36	48	65	75	100	122
6	0	0	1	5	6	11	19	29	39	51	72	93	116	140	180	214
8	-1	-2	2	5	12	16	30	44	64	79	110	139	179	211	265	315

(iii) $D_1 = 2,\ D_2 = 3$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-1	-1	0	0	1	1	3	2	4	6	6	7	12	11	14	19
2	0	1	0	0	0	1	1	3	4	7	10	14	18	25	31	39
4	1	0	0	2	1	3	7	8	13	20	24	34	45	53	69	86
6	0	-1	1	3	2	8	12	16	25	36	43	60	77	92	115	143
8	0	0	2	3	9	13	21	30	43	56	75	94	119	146	178	212

(iv) $D_1 = 1,\ D_2 = 2 \cdot 3$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	-1	0	-1	1	2	2	2	3	4	6	6	8	8	11	13
2	-1	2	0	0	0	0	1	2	2	4	5	9	10	15	18	22
4	1	0	0	1	1	1	4	5	7	11	15	19	26	32	40	50
6	0	0	1	3	1	6	7	11	17	21	27	38	46	58	70	86
8	0	0	2	1	8	8	12	19	27	34	47	56	72	89	109	127

(II) $\mathbf{D = 2 \cdot 5}$

(i) $D_1 = 2 \cdot 5,\ D_2 = 1$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	-1	0	-1	4	2	13	5	26	19	56	41	98	70	149	123
2	-2	3	0	3	9	12	28	39	82	99	170	185	285	316	470	513
4	0	-3	0	8	23	33	76	99	180	227	346	408	587	675	926	1051
6	-8	-7	3	18	46	83	150	203	330	423	607	742	1004	1173	1534	1771
8	-22	-12	3	31	88	141	246	347	532	684	955	1157	1522	1805	2302	2669

(ii) $D_1 = 5,\ D_2 = 2$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	-1	0	-1	2	3	7	7	15	16	30	32	53	55	84	88
2	-2	3	0	1	4	8	16	28	45	61	93	118	164	203	269	316
4	2	-1	0	5	13	21	45	64	102	140	201	253	344	418	539	643
6	-3	-4	3	11	25	53	88	128	196	259	355	456	592	721	909	1079
8	-12	-5	3	17	53	88	146	218	315	415	564	706	905	1105	1367	1616

(iii) $D_1 = 2,\ D_2 = 5$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-1	-1	0	0	2	2	4	5	8	10	14	17	23	28	35	42
2	-1	2	0	0	2	4	5	12	16	24	35	47	60	81	100	124
4	2	0	0	2	4	7	16	24	36	53	73	96	127	160	200	247
6	-1	-1	3	7	10	25	35	53	78	106	137	184	229	285	352	426
8	-3	-1	3	6	23	35	57	86	122	161	218	275	347	430	524	626

(iv) $D_1 = 1,\ D_2 = 2 \cdot 5$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-1	-1	0	0	2	3	4	5	7	9	12	14	18	21	26	31
2	-1	2	0	0	1	2	3	7	9	14	20	28	35	48	59	73
4	2	0	0	1	2	3	9	13	20	30	42	55	74	93	117	145
6	0	0	3	6	7	17	23	34	50	66	85	114	141	175	215	260
8	-1	0	3	4	16	22	35	53	75	98	133	166	210	260	317	377

(III) $\mathbf{D = 3 \cdot 5}$

(i) $D_1 = 3 \cdot 5,\ D_2 = 1$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	-1	1	0	9	8	34	29	86	85	183	178	331	318	536	531
2	-1	3	0	7	30	52	117	170	311	405	640	775	1120	1324	1821	2100
4	-3	-6	1	28	84	149	298	431	703	934	1357	1694	2316	2789	3644	4283
6	-29	-24	3	63	174	323	574	834	1281	1702	2373	2985	3936	4757	6044	7136
8	-79	-54	6	119	330	575	979	1416	2091	2756	3752	4681	6044	7305	9117	10746

(ii) $D_1 = 5,\ D_2 = 3$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-1	-1	1	1	3	6	15	17	30	50	63	86	126	150	194	254
2	0	2	0	4	9	24	44	75	115	172	239	327	429	555	699	869
4	3	-3	1	14	29	63	118	176	271	388	520	698	908	1134	1426	1751
6	-8	-10	3	32	64	137	229	344	503	705	927	1219	1559	1935	2384	2909
8	-24	-23	6	50	131	237	390	579	827	1121	1481	1899	2397	2960	3613	4343

(iii) $D_1 = 3,\ D_2 = 5$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-1	-1	1	1	5	6	11	15	24	32	45	58	78	98	124	152
2	0	2	0	2	5	14	24	43	65	98	137	187	245	319	401	499
4	1	-1	1	6	17	35	64	102	153	218	300	398	516	654	816	1001
6	-4	-2	3	20	42	83	133	206	295	409	543	711	901	1127	1384	1681
8	-12	-11	6	30	79	139	228	337	481	649	859	1099	1387	1712	2089	2509

(iv) $D_1 = 1,\ D_2 = 3 \cdot 5$

$j\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	-1	1	0	3	4	6	7	12	15	21	26	35	42	54	65
2	-1	3	0	1	2	6	9	18	25	39	54	75	96	128	159	198
4	3	0	1	4	8	13	28	41	61	88	121	158	208	261	326	401
6	-1	0	3	11	16	37	54	84	121	166	217	289	362	453	556	676
8	-3	-2	6	11	38	57	93	138	197	260	350	441	558	689	841	1004

References

[AI05] H. Aoki and T. Ibukiyama, Simple graded rings of Siegel modular forms, differential operators and Borcherds products, Internat. J. Math., **16** (2005), 249–279.

[Ara75] T. Arakawa, Automorphic forms on quaternion unitary group of degree 2 (in Japanese), Master thesis, University of Tokyo, 1975.

[Ara81] T. Arakawa, The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary group, J. Math. Soc. Japan, **33** (1981), 125–145.

[Has80] K. Hashimoto, On Brandt matrices associated with the positive definite quaternion Hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **27** (1980), 227–245.

[Has83] K. Hashimoto, The dimension of the spaces of cusp forms on Siegel upper half plane of degree two I, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1983), 403–488.

[Has84] K. Hashimoto, The dimension of the spaces of cusp forms on Siegel upper half plane of degree two II, The $\boldsymbol{Q}$-rank one case, Math. Ann., **266** (1984), 539–559.

[HI80] K. Hashimoto and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **27** (1980), 549–601.

[HI82] K. Hashimoto and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms (II), J. Fac. Sci. Univ. Tokyo Sect. IA Math., **28** (1982), 695–699.

[HI83] K. Hashimoto and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms (III), J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1983), 393–401.

[HI85] K. Hashimoto and T. Ibukiyama, On relations of dimensions of automorphic forms of $Sp(2,\boldsymbol{R})$ and its compact twist $Sp(2)$ (II), Automorphic forms and number theory, Adv. Stud. Pure Math., **7** (1985), 31–102.

[Hir99] Y. Hirai, On Eisenstein series on quaternion unitary groups of degree 2, J. Math. Soc. Japan, **51** (1999), 93–128.

[Ibu84] T. Ibukiyama, On symplectic Euler factors of genus two, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1984), 587–614.

[Ibu85] T. Ibukiyama, On relations of dimensions of automorphic forms of $Sp(2,\boldsymbol{R})$ and its compact twist $Sp(2)$ (I), Automorphic forms and number theory, Adv. Stud. Pure Math., **7** (1985), 7–30.

[Ibu99] T. Ibukiyama, On differential operators on automorphic forms and invariant pluriharmonic polynomials, Comment. Math. Univ. St. Pauli, **48** (1999), 103–118.

[Ibu07a] T. Ibukiyama, Paramodular forms and compact twist, In: Automorphic Forms on $GSp(4)$, Proceedings of the 9th Autumn Workshop on Number Theory, (ed. M. Furusawa), 2007, pp. 37–48.

[Ibu07b] T. Ibukiyama, Dimension formulas of Siegel modular forms of weight 3 and supersingular abelian surfaces, Proceedings of the 4-th Spring Conference, 2007, pp. 39–60.

[Lan63] R. P. Langlands, The dimension of spaces of automorphic forms, Amer. J. Math., **85** (1963), 99–125.

[Mor74] Y. Morita, An explicit formula for the dimension of spaces of Siegel modular forms of degree two, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **21** (1974), 167–248.

[PR94] V. Platonov and A. Rapinchuk, Algebraic groups and number theory, Academic Press, 1994.

[Shi63] G. Shimura, Arithmetic of alternating forms and quaternion hermitian forms, J. Math. Soc. Japan, **15** (1963), 33–65.

[Wak] S. Wakatsuki, Dimension formulas for spaces of vector valued Siegel cusp forms of degree two, to appear in J. Number Theory.

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