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On Siegel-Eisenstein series attached to certain cohomological representations

By Takuya Miyazaki

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Abstract. We introduce a Siegel-Eisenstein series of degree 2 which generates a cohomological representation of Saito-Kurokawa type at the real place. We study its Fourier expansion in detail, which is based on an investigation of the confluent hypergeometric functions with spherical harmonic polynomials. We will also consider certain Mellin transforms of the Eisenstein series, which are twisted by cuspidal Maass wave forms, and show their holomorphic continuations to the whole plane.

Introduction.

In this paper we will study a real analytic Siegel Eisenstein series on $G_{\boldsymbol{A}} = Sp(2, \boldsymbol{A})$, $\boldsymbol{A} = \boldsymbol{A}_{\boldsymbol{Q}}$, which generates a nontempered cohomological representation $\pi(d)$ of $G_\infty = Sp(2, \boldsymbol{R})$ of Saito-Kurokawa type. The representation $\pi(d)$ has the minimal K_∞-type of dimension $2d+1$, where d is an even positive integer.

This paper may be viewed as a continuation of our previous work [**HM**]. The distinction from [**HM**] is that we are concerned here with the unitary representation $\pi(d)$ whose infinitesimal character is regular, whereas our motivation is kept the same as before. Namely, our purpose is to find out some specific properties of a real analytic automorphic form of Saito-Kurokawa type, through an investigation of the Fourier expansion, or the twisted Mellin transforms, of the Eisenstein series which is suitably attached to the $\pi(d)$. The similar problem is discussed in [**HM**] for a certain unitary representation, which can be understood as the limit $\pi(0)$.

Let P_∞ denote the maximal Siegel parabolic subgroup in G_∞. According to a result in [**L**] by Lee, it is known that the cohomological representation $\pi(d)$ can be embedded into a degenerate principal series representation $I_\infty(s_d) = I_{P_\infty}^{G_\infty}(|\det|^{s_d})$ with a suitably chosen parameter $s_d \in \boldsymbol{C}$. Then our idea is as following. Let $\tau(d)$ denote the minimal K_∞-type of $\pi(d)$. For any $s \in \boldsymbol{C}$, we consider the equivalent K_∞-type $\tau(d)$ of $I_\infty(s)$, and take a function $\Lambda_\infty(g_\infty, s)(\varphi)$ which belongs to it.

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Here we note that the K_∞-type $\tau(d)$ of $I_\infty(s)$ can be realized explicitly by using spherical harmonic polynomials φ of homogeneous degree d. We consider an unramified Eisenstein series

$$E(g,s)(\varphi)=\sum_{\gamma\in P\backslash G}\Lambda(\gamma g,s)(\varphi),\quad g\in G_{\boldsymbol{A}},$$

which is convergent if $\mathrm{Re}(s)$ is sufficiently large. Then we define the Eisenstein series $E(g,s_d)(\varphi)$, which will be our main ingredient, by means of the analytic continuation of the $E(g,s)(\varphi)$ with respect to the parameter s. It has the Fourier expansion

$$E(n(x)g,s_d)(\varphi)=\sum_{B={}^tB\in L}\boldsymbol{e}(\mathrm{tr}(Bx))E_B(g,s_d)(\varphi),\tag{0.1}$$

where L stands for the set of all semi-integral symmetric 2×2 matrices. See Sections 2 and 3 of this paper for the precise definitions of all these above.

The followings are our main results.

Theorem 1. *Suppose that $B\in L$ is a non-degenerate matrix. Then one has the vanishing $E_B(g,s_d)(\varphi)=0$, unless the matrix B is indefinite, that is, it has one positive and one negative eigenvalues.*

It is to be noted that the choice $\Lambda_\infty(g_\infty,s_d)(\varphi)\in\pi(d)$ is crucial for this assertion to hold, whereas our method of the proof does not appeal to the representation theoretic language. Indeed, the proof will be based on the direct exploration of the confluent hypergeometric function twisted by φ. The function, $\xi(h,\alpha,\beta;\varphi)$, is defined by the integral

$$\int_{S_\infty}e^{-2\pi i\mathrm{tr}(hx)}\det(\varepsilon(x))^{-\alpha}\det(\overline{\varepsilon(x)})^{-\beta}\varphi(\varepsilon(x)^{-1}\overline{\varepsilon(x)})dx.$$

Here S_∞ denotes the set of all real symmetric matrix of size 2, $h\in S_\infty$, $\varepsilon(x)=e_2-ix$, and $(\alpha,\beta)\in\boldsymbol{C}^2$. It is convergent if $\mathrm{Re}(\alpha+\beta)>2$. Following the arguments by Shimura in [**S1**], [**S2**], we will prove its analytic continuation in (α,β) to the whole $\boldsymbol{C}^2$. This will be our Theorem 5.2. In addition, we will give a criterion of its vanishing at $(\alpha,\beta)=(1/2,d+(1/2))$ in terms of the signature of h (see Theorem 5.4), which yields the theorem above. These results give the heart of this paper.

In Section 6, we will investigate the confluent hypergeometric functions when the matrices h have the low rank. For instance, the case $h=0_2$ will be treated in Proposition 6.2. Those observations will be applied in the next section to describe the Fourier coefficients $E_B(g, s_d)(\varphi)$ for the degenerate matrices B.

Let

$$E^{(2)}(g; \varphi) = \sum_{B\colon \text{regular}} E_B(g, s_d)(\varphi),$$

be the regular part of the Fourier expansion (0.1). In Section 8, we will define its integral transform by

$$\mathscr{M}(\phi, s; \varphi_l) = \int_{SL_2(\boldsymbol{Z})\backslash\mathfrak{H}_1} \left(\int_0^\infty t^s E^{(2)}(g_{\tau,t}; \varphi_l) d^\times t \right) \phi(\tau) d\mu(\tau), \tag{0.2}$$

which is referred to as the Mellin transform twisted by ϕ. Here ϕ is a cuspidal Maass wave form for $SL_2(\boldsymbol{Z})$ of weight $-2l$ with $-d \le l \le d$, and φ_l is a homogeneous harmonic polynomial of degree d of $SO(2)$-weight $2l$. This turns out to be zero when l is odd (Lemma 8.1). For even l, we will obtain the following.

Theorem 2. *Let ϕ be a cuspidal Maass wave form for $SL_2(\boldsymbol{Z})$ of weight $-2l$ with an even integer l, $-d \le l \le d$. Then the twisted Mellin transform $\mathscr{M}(\phi, s; \varphi_l)$ has an entire continuation in s. Moreover, it satisfies the functional equation*

$$\mathscr{M}(\phi, -s; \varphi_l) = \mathscr{M}(\phi, s; \varphi_l).$$

This will be our Theorem 8.3.

Let $-4\nu(\nu + 1)$ denote the Casimir eigenvalue of ϕ. Then we will be able to describe $\mathscr{M}(\phi, s; \varphi_l)$ in the following way.

Theorem 3. *The twisted Mellin transform $\mathscr{M}(\phi, s; \varphi_l)$ becomes the product*

$$\Gamma(s; l, \nu)\zeta(2s + 1)D_l(s, \phi)$$

up to a constant multiple, where $\Gamma(s; l, \nu)$ is the function which will be defined in (9.8), *and*

$$D_l(s, \phi) = \sum_{N=1}^{\infty} \frac{H(d, N)b(N)}{N^{s+(d/2)-(1/4)}}$$

is the Rankin product Dirichlet series attached to Cohen's Eisenstein series for $\Gamma_0(4)$ of weight $d + (1/2)$ and the cuspidal lifting θ_ϕ of ϕ of weight $l + (1/2)$.

This will be given in Theorem 9.2. See (9.1), or (9.9), (9.5) and (9.6), for the definitions of $H(d,N)$ and $b(N)$ in the above statement. We also give an analytic property of $\Gamma(s;l,\nu)$ in Corollary 9.4.

The following problems are still remaining: (i) to formulate and to study an integral transform, which should be different from (0.2), when l is odd; (ii) to study the Mellin transforms twisted by non-cuspidal Maass forms ϕ. This should involve a discussion of a regularization process. We hope to come back to these problems in future.

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NOTATIONS. For a field F let $M_n(F)$ be the set of $n \times n$ matrices with coefficients in F. For every $x = (x_{ij}) \in M_n(F)$ let ${}^t x$, $\sigma(x)$, or $\delta(x)$ stand for the transpose, the trace, or the determinant, of x, respectively. Also we put $\delta_1(x) = x_{11}$. For any $x \in M_n(\boldsymbol{C})$, $\overline{x}$ denotes its complex conjugation, and $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are defined as usual. We use the symbol $GL_n(F)$, or $SL_n(F)$, in the usual way. Let e_n, or 0_n, denote the unit, or the zero, matrix of size n.

For each matrix $x = {}^t x \in M_n(F)$ and $a \in GL_n(F)$ we use $x[a] = {}^t a x a$. The symbol $x > y$ means that $x - y$ is positive definite for real symmetric matrices $x = {}^t x$ and $y = {}^t y \in M_n(\boldsymbol{R})$.

Let $\{\infty\}$, or $\boldsymbol{f}$, denote the sets of the archimedean prime, or the non-archimedean primes, of $\boldsymbol{Q}$, respectively. For each $v \in \{\infty\} \cup \boldsymbol{f}$ we set $\boldsymbol{Q}_v$ to be the v-adic completion of $\boldsymbol{Q}$. Let $\boldsymbol{A}$ denote the ring of adeles of $\boldsymbol{Q}$.

Put $i = \sqrt{-1} \in \boldsymbol{C}$. Let $\mathfrak{H}_1 = \{\tau \in \boldsymbol{C} \mid \mathrm{Im}(\tau) > 0\}$ be the upper half plane. Let $\mathfrak{H}$ be the Siegel upper half space of degree 2, that is,

$$\mathfrak{H} = \big\{ z \in M_2(\boldsymbol{C}) \mid z = {}^t z,\ 2\mathrm{Im}(z) = -i(z - \overline{z}) > 0_2 \big\}.$$

Also let $\mathfrak{H}' = -i\mathfrak{H}$ be the right half space of degree 2, which is defined by

$$\mathfrak{H}' = \big\{ z \in M_2(\boldsymbol{C}) \mid z = {}^t z,\ 2\mathrm{Re}(z) = z + \overline{z} > 0_2 \big\}.$$

1. Preliminaries.

Let $G = Sp(2,\boldsymbol{Q})$ be the symplectic group of degree 2, which is defined by

$$G = \big\{ g \in GL_4(\boldsymbol{Q}) \mid {}^t g J g = J \big\} \quad \text{with} \quad J = \begin{pmatrix} 0 & e_2 \\ -e_2 & 0 \end{pmatrix}.$$

It gives an algebraic group over $\boldsymbol{Q}$. For each $v \in \{\infty\} \cup \boldsymbol{f}$, we put $G_v = G(\boldsymbol{Q}_v)$. Let us write $G_{\boldsymbol{A}}$ for the adelization of G and put $G_{\boldsymbol{f}} = G_{\boldsymbol{A}} \cap \prod_{v \in \boldsymbol{f}} G_v$. If $x \in G_{\boldsymbol{A}}$, then we define $x_\infty \in G_\infty$ and $x_{\boldsymbol{f}} \in G_{\boldsymbol{f}}$ to write $x = x_\infty x_{\boldsymbol{f}}$. We use the symbol G to denote the group of $\boldsymbol{Q}$-points embedded in $G_{\boldsymbol{A}}$. We will keep these conventions for any algebraic group in the following.

The real group G_∞ acts on $\mathfrak{H}$ in the standard way. For each $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_\infty$ and $z \in \mathfrak{H}$, we put

$$\mu_g(z) = cz + d \quad \text{and} \quad j_g(z) = \delta(\mu_g(z)). \tag{1.1}$$

A maximal compact subgroup K_v of G_v at each prime v is defined by

$$K_v = \begin{cases} \{r \in G_\infty \mid r\boldsymbol{i} = \boldsymbol{i}\} & (v = \infty), \\ G_v \cap GL_4(\boldsymbol{Z}_v) & (v \in \boldsymbol{f}), \end{cases}$$

where $\boldsymbol{i} = ie_2 \in \mathfrak{H}$. Then K_∞ is isomorphic to the unitary group $U(2)$ by the map sending $r \in K_\infty$ to $\mu_r(\boldsymbol{i})$. Take the product group $K = \prod_v K_v$ over all primes, and define a discrete subgroup Γ of $G_{\boldsymbol{A}}$ by $\Gamma = G \cap G_\infty K$, which is identified with the Siegel modular group $Sp(2, \boldsymbol{Z})$.

Let S be the $\boldsymbol{Q}$-vector space defined by

$$S = \{u = {}^t u \in M_2(\boldsymbol{Q})\}.$$

Consider the standard action of $G_1 = GL_2(\boldsymbol{Q})$ on S, which sends u to $u[a] = {}^t aua$, $a \in G_1$. The completion S_v at every $v \in \{\infty\} \cup \boldsymbol{f}$ and the adelization $S_{\boldsymbol{A}}$ are defined as before.

Let us take subgroups N and M of G by $N = \{n(u) \mid u \in S\}$ and $M = \{m(a) \mid a \in G_1\}$, where

$$n(u) = \begin{pmatrix} e_2 & u \\ 0 & e_2 \end{pmatrix} \quad \text{and} \quad m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}.$$

Then $P = NM$ gives the Siegel maximal parabolic subgroup of G. We have the Iwasawa decomposition $G_{\boldsymbol{A}} = P_{\boldsymbol{A}} K$ and $G = P\Gamma$. Define a function δ_v of $p \in P_v$, $p = n(u)m(a)$, at each place v by

$$\delta_v(p) = |\delta(a)|_v, \tag{1.2}$$

where $|\cdot|_v$ stands for the normalized absolute value on $\boldsymbol{Q}_v$. For any $s \in \boldsymbol{C}$, let us consider the degenerate principal series representation $I_v(s) = I_{P_v}^{G_v}(s)$ of G_v, whose space consists of all smooth functions $f \in C^\infty(G_v)$ satisfying the condition

$$f(pg) = \delta_v(p)^s f(g) \quad \text{for} \quad p \in P_v \quad \text{and} \quad g \in G_v$$

on which G_v acts by right translations.

Consider the principal series $I_\infty(s)$ at the real place. We recall that it has the multiplicity free K_∞-type decomposition. All K_∞-types of $I_\infty(s)$ are parameterized by the highest weights (d_1, d_2) with *even* integers $d_1 \geq d_2$, each of which is assigned to an irreducible $U(2)$-module being equivalent to $\det^{d_2} \otimes Symm^{d_1-d_2}(\boldsymbol{C}^2)$. Here $\boldsymbol{C}^2$ stands for the standard representation of $U(2)$.

The G_∞-module structure of $I_\infty(s)$ was studied in detail by Lee, [**L**], which we recall in a form suitable for our use. Let us put

$$s_d = d + 1 \tag{1.3}$$

for an even integer $d > 0$. Then

Proposition 1.1 ([**L**, Theorems 4.1 and 5.2]). *The principal series $I_\infty(s_d)$ is reducible. It contains an irreducible subrepresentation $\pi(d)$, which has the minimal K_∞-type of the highest weight $(d, -d)$. In fact, all the K-types of $\pi(d)$ are given by K_∞-modules of the highest weights $(d+2i, -d-2j)$ for all integers $i, j \geq 0$. Moreover, $\pi(d)$ is unitarizable.*

It is well known that every cohomological representation is characterized by the unitarizability, the K_∞-type decomposition, and its infinitesimal character, among all the irreducible $(\mathfrak{sp}(2), K_\infty)$-modules, see [**VZ**, Proposition 6.1], for example. Now, besides the above proposition, we know the infinitesimal character of $\pi(d)$ which is, of course, equal to the one of $I_\infty(s_d)$. Using these data, one can identify the $\pi(d)$ with a nontempered cohomological representation $A_{\mathfrak{q}}(\lambda_d)$. Here $\mathfrak{q}$ is a θ-stable maximal parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{sp}(2)$ of Siegel type, and λ_d is a suitable linear form on the Levi subalgebra $\mathfrak{l} \simeq \mathfrak{u}(1,1) \otimes \boldsymbol{C}$ of $\mathfrak{q}$. This $A_{\mathfrak{q}}(\lambda_d)$ belongs to a local A-packet of Saito-Kurokawa type.

2. Eisenstein series with spherical harmonics polynomials.

We set $T = \{u = {}^t u \in M_2(\boldsymbol{C})\}$. Let $\mathscr{P}_d$ denote the space of homogeneous polynomials $p(u)$ on T of degree d. The unitary group $U(2)$ acts on $\mathscr{P}_d$ by sending $p(u)$ to $p({}^t rur)$, $r \in U(2)$, whose restriction to the subgroup $SU(2)$ factors through the quotient $SU(2)/\{\pm e_2\} \simeq SO(3)$.

We shall put coordinates $\{u_1, u_2, u_3\}$ on T by

$$u = \begin{pmatrix} u_3 + iu_1 & iu_2 \\ iu_2 & u_3 - iu_1 \end{pmatrix}.$$

Recall that a polynomial $\varphi \in \boldsymbol{C}[u_1, u_2, u_3]$ is said to be harmonic, if it satisfies the equation

$$\Delta\varphi = \left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2}\right)\varphi = 0.$$

Let $\mathscr{H}_d \subset \mathscr{P}_d$ denote the subspace of homogeneous harmonic polynomials of degree d. Then the following facts are well known.

LEMMA 2.1.

(i) *The space $\mathscr{H}_d$ is stable under the action of $U(2)$. It realizes an irreducible representation of $U(2)$ of the dimension $2d+1$, which is isomorphic to $Symm^{2d}(\boldsymbol{C}^2)$.*

(ii) *Consider a vector space $\delta^j \otimes \mathscr{H}_{d-2j}$ for every integer $0 \le j \le [d/2]$, which is linearly spanned by the polynomials $\delta(u)^j \varphi(u)$, $\varphi \in \mathscr{H}_{d-2j}$. Then it gives an irreducible $U(2)$-submodule of $\mathscr{P}_d$ of dimension $2d - 4j + 1$. We have an irreducible decomposition*

$$\mathscr{P}_d \simeq \bigoplus_{j=0}^{[d/2]} \delta^j \otimes \mathscr{H}_{d-2j}, \tag{2.1}$$

which is multiplicity free.

Let τ_d denote the irreducible $U(2)$-action on $\mathscr{H}_d$: $[\tau_{2d}(r)\varphi](u) = \varphi({}^t rur)$ for $r \in U(2)$ and $\varphi(u) \in \mathscr{H}_d$. Also let $L(\mathscr{H}_d)$ denote the space of linear forms on $\mathscr{H}_d$ with the dual $U(2)$-action τ_{2d}^* given by $[\tau_{2d}^*(r)\lambda](\varphi) = \lambda(\tau_{2d}({}^t r)\varphi)$ for $r \in U(2)$, $\lambda \in L(\mathscr{H}_d)$, and $\varphi \in \mathscr{H}_d$.

Let us provide a basis of $\mathscr{H}_d$ in the following. For every integer k, $0 \le k \le d$, we define a function $\widehat{C}_k$ on $\{u \in T \mid \delta(u) > 0\}$ by

$$\widehat{C}_k(u) = \delta(u)^{(d-k)/2} C_{d-k}^{k+(1/2)}\left(\delta(u)^{-1/2}\sigma\left(\frac{u}{2}\right)\right). \tag{2.2}$$

Here

$$C_n^p(t) = \frac{\Gamma(n+2p)}{n!\Gamma(2p)} {}_2F_1\left(-n,\ n+2p,\ p+\frac{1}{2};\ \frac{1-t}{2} \right)$$

is the Gegenbauer polynomial; see Chapter IX of [**V**]. The formula (3) in Subsection 1 of [**V**, Chapter IX-3]:

$$C_n^p(t) = \frac{2^n\Gamma(n+p)}{n!\Gamma(p)}\Bigg(t^n - \frac{n(n-1)}{2^2(n+p-1)}t^{n-2} \\ + \frac{n(n-1)(n-2)(n-3)}{2^4\cdot 1\cdot 2(n+p-1)(n+p-2)}t^{n-4} + \cdots\Bigg) \tag{2.3}$$

gives us the following fact.

Lemma 2.2. *We find that $\widehat{C}_k(u) \in \boldsymbol{C}[\sigma(u/2), \delta(u)]$. Hence, it is a polynomial on T, which also belongs to $\mathscr{P}_{d-k}$.*

Let us put

$$\delta_1^c(u) = \delta_1(cuc) \quad \text{with} \quad c = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Then we have $\delta_1^c(u)^k \in \mathscr{H}_k$ for every integer $k \geq 0$. Now, for each k, $0 \leq k \leq d$, we define the polynomials $\varphi_k(u)$ and $\varphi_{-k}(u)$ by the products

$$\varphi_k(u) = \delta_1^c(u)^k\widehat{C}_k(u) = (iu_1 - u_2)^k\widehat{C}_k(u), \quad \text{and}$$
$$\varphi_{-k}(u) = (iu_1 + u_2)^k\widehat{C}_k(u) = \varphi_k(\iota u\iota), \quad \iota = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Lemma 2.3. *The polynomials $\varphi_k(u)$ and $\varphi_{-k}(u)$, $0 \leq k \leq d$, are all harmonic and homogeneous of degree d. They form a basis of $\mathscr{H}_d$.*

For the proof, see Chap. IX of [**V**], for example.

We give some remarks. Firstly, we find that

$$\varphi_k(e_2) = \varphi_{-k}(e_2) = 0 \quad \text{for every} \quad k \neq 0. \tag{2.4}$$

The action of $SO(2) \subset U(2)$ on the polynomial φ_k, or φ_{-k}, is described by a character multiplication, indeed, one finds that

$$[\tau_{2d}(r_\theta)\varphi_k](u) = e^{i2k\theta}\varphi_\kappa(u), \quad \text{or} \quad [\tau_{2d}(r_\theta)\varphi_{-k}](u) = e^{-i2k\theta}\varphi_\kappa(u),$$

for every $r_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in SO(2)$.

Let $d > 0$ be an even integer as before. For every $s \in \boldsymbol{C}$ and $\varphi \in \mathscr{H}_d$, we define a function $\Lambda_\infty(g,s)(\varphi)$ on G_∞ by

$$\Lambda_\infty(g,s)(\varphi) = |j_g(\boldsymbol{i})|_\infty^{-s+d} \cdot \overline{j_g(\boldsymbol{i})}^{-d} \varphi\big(\mu_g(\boldsymbol{i})^{-1}\overline{\mu_g(\boldsymbol{i})}\big), \quad g \in G_\infty,$$

which is belonging to $I_\infty(s)$. It will be convenient to regard $\Lambda_\infty(g,s)(\cdot)$ as a linear form on $\mathscr{H}_d$, namely as $\Lambda_\infty(g,s)(\cdot) \in L(\mathscr{H}_d)$. Then each right translation $\Lambda_\infty(gr,s)(\varphi)$ by $r \in K_\infty$ is written as

$$\Lambda_\infty(gr,s)(\varphi) = \delta\big(\overline{\mu_r(\boldsymbol{i})}\big)^{-d} \cdot \big[\tau_{2d}^*\big({}^t\overline{\mu_r(\boldsymbol{i})}\big)\Lambda_\infty(g,s)\big](\varphi),$$

which is identified with the self dual module $\det^{-d} \otimes Symm^{2d}(\boldsymbol{C}^2)$.

According to Proposition 1.1, we have the following observation.

Lemma 2.4. *Keep the notations above. Then, for every $\varphi \in \mathscr{H}_d$, the function $\Lambda_\infty(g,s_d)(\varphi)$ belongs to the subrepresentation $\pi(d)$ of $I_\infty(s_d)$. More precisely, it belongs to the minimal K_∞-type of $\pi(d)$.*

At each finite prime $v \in \boldsymbol{f}$ we define a function $\Lambda_v(g,s) \in I_v(s)$ on G_v by

$$\Lambda_v(g,s) = \delta_v(p)^s$$

for $g = pw$ with $p \in P_v$ and $w \in K_v$. This is a right K_v-invariant function. Then we define the function $\Lambda(g,s)(\varphi)$ on $G_{\boldsymbol{A}}$ by the product

$$\Lambda(g,s)(\varphi) = \Lambda_\infty(g_\infty,s)(\varphi) \prod_{v \in \boldsymbol{f}} \Lambda_v(g_v,s), \quad g = (g_v) \in G_{\boldsymbol{A}}.$$

It is easy to check that $\Lambda(\gamma g,s)(\varphi) = \Lambda(g,s)(\varphi)$ for any $\gamma \in P$. Thus we can define an Eisenstein series on $G_{\boldsymbol{A}}$ by

$$E(g,s)(\varphi) = \sum_{\gamma \in P\backslash G} \Lambda(\gamma g,s)(\varphi), \quad g \in G_{\boldsymbol{A}} \text{ and } s \in \boldsymbol{C} \tag{2.5}$$

for every $\varphi \in \mathscr{H}_d$. This is convergent locally uniformly and absolutely if $\mathrm{Re}(s)$ is sufficiently large. We will study the Fourier expansion of (2.5) in the following section in order to discuss the behavior of

$$E(g,s_d)(\varphi), \quad \varphi \in \mathscr{H}_d$$

by means of the analytic continuation. By Lemma 2.4, we know that $E(g, s_d)(\varphi)$, $\varphi \in \mathscr{H}_d$, belongs to the cohomological representation $\pi(d)$ at the archimedean place.

3. The Fourier expansion.

Fix an additive character $\boldsymbol{e}(\cdot)$ on $\boldsymbol{A}$, which is trivial on $\boldsymbol{Q}$, in the standard way: let us put $\boldsymbol{e}_\infty(t) = e^{2\pi i t}$ for $t \in \boldsymbol{R}$ and $\boldsymbol{e}_p(t) = e^{-2\pi i t_0}$ for $t \in \boldsymbol{Q}_p$ with $t_0 \in \boldsymbol{Z}[1/p]$ so that $t_0 \equiv t$ modulo $\boldsymbol{Z}_p$, then we define $\boldsymbol{e} = \boldsymbol{e}_\infty \prod_{p \in \boldsymbol{f}} \boldsymbol{e}_p$. We identify S_v with its $\boldsymbol{Q}_v$-dual at every place v by using the bilinear form $(u_1, u_2) \mapsto \sigma(u_1 u_2)$. Let $L \subset S$ be the dual lattice of $S \cap S_\infty \prod_{p \in \boldsymbol{f}} S(\boldsymbol{Z}_p)$; which is explicitly given by

$$L = \left\{ B = \begin{pmatrix} m & \dfrac{r}{2} \\ \dfrac{r}{2} & n \end{pmatrix} \middle| \, m, n, r \in \boldsymbol{Z} \right\}.$$

For each $B \in L$ we put $D_B = -\delta(2B)$, and set $e_B = \gcd(m, n, r)$ if $B \neq 0_2$.

Consider a subgroup KG_∞ of $G_{\boldsymbol{A}}$. Then, $E(n(x)g, s)(\varphi)$, $g \in KG_\infty$, has the Fourier series expansion with respect to $x \in S_{\boldsymbol{A}}$

$$E(n(x)g, s)(\varphi) = \sum_{B \in L} \boldsymbol{e}(\sigma(Bx)) E_B(g, s)(\varphi), \tag{3.1}$$

where every coefficient $E_B(g, s)(\varphi)$ is given by the integral

$$E_B(g, s)(\varphi) = \int_{S_{\boldsymbol{A}}/S} \boldsymbol{e}(-\sigma(Bu)) E(n(u)g, s)(\varphi) du. \tag{3.2}$$

Here we take a measure du on $S_{\boldsymbol{A}}$ such that $\int_{S_{\boldsymbol{A}}/S} du = 1$. Since the right K_∞-action has been specified, it suffices to take φ as the base φ_k or the φ_{-k} for all $0 \leq k \leq d$ in order to study the behavior of (3.2).

Recall a disjoint decomposition of G

$$G = \coprod_{j=0,1,2} P w_j P \quad \text{with} \quad w_j = \begin{pmatrix} e_{2-j} & & & \\ & 0_j & & e_j \\ & & e_{2-j} & \\ & -e_j & & 0_j \end{pmatrix}$$

which we apply into the definition (2.5). Then one finds that the sum over $P \backslash P w_2 P$ contributes to the coefficient (3.2) by

$$W_B^{(2)}(g,s)(\varphi) = \int_{S_{\boldsymbol{A}}} \boldsymbol{e}(-\sigma(Bu))\Lambda(w_2 n(u)g,s)(\varphi)du. \tag{3.3}$$

Let $P_1 = \left\{\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \in G_1\right\}$ be the Borel subgroup of $G_1 = GL_2(\boldsymbol{Q})$, and let $\Gamma_1 = SL_2(\boldsymbol{Z})$. If the rank of $B \in L$ is equal to 1, then there exists a unique coset $\langle\gamma_B\rangle \in \Gamma_1 \cap P_1\backslash\Gamma_1$ so that one has

$$B\left[\gamma_B^{-1}\right] = {}^t\gamma_B^{-1}B\gamma_B^{-1} = \mathrm{diag}(0, e_B), \quad \text{or} \quad \mathrm{diag}(0, -e_B)$$

for any representative γ_B of it. We set $\gamma_{0_2} = e_2$ if $B = 0_2$. Also we define a subgroup S_1 of S by

$$S_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \middle| \, x \in \boldsymbol{Q} \right\}.$$

Now assume that B is degenerate, that is, $D_B = 0$. Then, the partial sum over $P\backslash Pw_1P$ contributes to the integral (3.2) by

$$W_B^{(1)}(g,s)(\varphi) = \int_{S_{1,\boldsymbol{A}}} \boldsymbol{e}\big(-\sigma(B[\gamma_B^{-1}]u)\big)\Lambda(w_1 n(u)m(\gamma_B)g,s)(\varphi)du. \tag{3.4}$$

One can verify directly the following facts.

Lemma 3.1.

(i) *Suppose $B \in L$ is regular, that is, B has the rank 2. Then one has*

$$E_B(g,s) = W_B^{(2)}(g,s)(\varphi) \quad \textit{for} \quad g \in KG_\infty.$$

(ii) *Suppose $B \in L$ has the rank 1. Then one has*

$$E_B(g,s)(\varphi) = W_B^{(2)}(g,s)(\varphi) + W_B^{(1)}(g,s)(\varphi) \quad \textit{for} \quad g \in KG_\infty.$$

For the constant term with $B = 0_2$ we have

Lemma 3.2. *The constant term $E_{0_2}(g,s)(\varphi)$, $g \in KG_\infty$, is equal to the sum*

$$W_{0_2}^{(2)}(g,s)(\varphi) + \sum_{\gamma \in P_1\backslash G_1} W_{0_2}^{(1)}(m(\gamma)g,s)(\varphi) + \Lambda(g,s)(\varphi).$$

Each of the $W_B^{(j)}(g,s)(\varphi)$, $j=1,2$, in (3.3), (3.4) is a product of the corresponding local integrals:

$$W_B^{(j)}(g,s)(\varphi) = c_j \cdot W_{B,\infty}^{(j)}(g_\infty,s)(\varphi) \cdot \prod_{p\in \boldsymbol{f}} W_{B,p}^{(j)}(e_2,s), \quad g\in KG_\infty \tag{3.5}$$

with a constant c_j, where the non-archimedean local integral at $p\in\boldsymbol{f}$ is defined by

$$W_{B,p}^{(2)}(e_2,s) = \int_{S_p} \boldsymbol{e}_p(-\sigma(Bu))\Lambda_p(w_2 n(u),s)du$$

for every $B\in L$, or

$$W_{B,v}^{(1)}(e_2,s) = \int_{S_{1,p}} \boldsymbol{e}_p\big(-\sigma(B[\gamma_B^{-1}]u)\big)\Lambda_p(w_1 n(u),s)du$$

for every $B\in L$ of lower rank. Here we take a measure du on S_p, or $S_{1,p}$, so that the volume $\int_{S(\boldsymbol{Z}_p)} du$, or $\int_{S_1(\boldsymbol{Z}_p)} du$, is equal to 1.

Kaufhold, in [**K**], gave explicit formulas of these non-archimedean local integrals. We recall it now.

We need some notations. For each regular matrix $B\in L$, we put $\mathfrak{d}_B$ to be the fundamental discriminant of the quadratic extension $\boldsymbol{Q}(\sqrt{D_B})/\boldsymbol{Q}$, and thus we have $D_B=\mathfrak{d}_B\mathfrak{f}_B^2$ with a positive integer $\mathfrak{f}_B$. For every $p\in\boldsymbol{f}$ we set $\alpha_{1,p}=\mathrm{ord}_p e_B$ and $\alpha_p=\mathrm{ord}_p\mathfrak{f}_B$. Let $\chi_B(\cdot)=(\frac{\mathfrak{d}_B}{\cdot})$ denote the Kronecker symbol. Then we define

$$F_p(B;s) = \sum_{j=0}^{\alpha_{1,p}} p^{j(2-s)}\Bigg(\sum_{m=0}^{\alpha_p-j} p^{m(3-2s)} - \chi_B(p)p^{1-s}\sum_{l=0}^{\alpha_p-j-1} p^{l(3-2s)}\Bigg)$$

for $s\in\boldsymbol{C}$. This gives the constant function 1 for almost all primes. The important functional equation

$$F_p(B;3-s) = p^{\alpha_p(2s-3)}F_p(B;s)$$

was proved in [**K**], and thus, in particular, we have

$$F_B(3-s) = \mathfrak{f}_B^{2s-3}F_B(s)$$

by putting $F_B(s) = \prod_{p \in \boldsymbol{f}} F_p(B; s)$. Also we define

$$F_p(b; s) = \sum_{j=0}^{\mathrm{ord}_p b} p^{j(2-s)}$$

for every integer $b \neq 0$. Set the Euler factors $L_p(s, \chi_B) = (1 - \chi_B(p)p^{-s})^{-1}$ and $\zeta_p(s) = (1 - p^{-s})^{-1}$ in the usual way.

PROPOSITION 3.3 (Kaufhold). *Keep the notations above. Then each local integral $W^{(j)}_{B,p}(e_4, s)$, $j = 1, 2$, has the following expression.*

(i) *Suppose $D_B \neq 0$. Then we have*

$$W^{(2)}_{B,p}(e_4, s) = \frac{L_p(s-1, \chi_B)F_p(B; s)}{\zeta_p(s)\zeta_p(2s-2)}.$$

(ii) *Suppose $D_B = 0$. Then we have*

$$W^{(2)}_{B,p}(e_4, s) = \begin{cases} \dfrac{\zeta_p(2s-3)F_p(e_B; s)}{\zeta_p(s)\zeta_p(2s-2)} & (B \neq 0_2), \\[2ex] \dfrac{\zeta_p(s-2)\zeta_p(2s-3)}{\zeta_p(s)\zeta_p(2s-2)} & (B = 0_2). \end{cases}$$

(iii) *Suppose $D_B = 0$. Then $W^{(1)}_{B,p}(e_4, s)$ is equal to*

$$\frac{F_p(e_B; s+1)}{\zeta_p(s)} \quad (B \neq 0_2), \quad \text{or} \quad \frac{\zeta_p(s-1)}{\zeta_p(s)} \quad (B = 0_2),$$

respectively.

See the paper [**K**] for the proofs.

4. Some auxiliary formulas.

We shall give some auxiliary lemmas, which will be used in the analysis of the archimedean local integrals $W^{(j)}_{B,\infty}(g, s)(\varphi)$, $j = 1, 2$.

For each $x \in S_\infty$ we set

$$\varepsilon(x) = e_2 - ix \in \mathfrak{H}', \quad \rho(x) = \varepsilon(x)^{-1}\overline{\varepsilon(x)} \in U(2),$$

and

$$\nu(x) = \frac{\sigma\left(\dfrac{\varepsilon(x)^{-1}}{2}\right)}{\delta(\varepsilon(x)^{-1})^{1/2}} = \frac{\sigma\left(\dfrac{\varepsilon(x)}{2}\right)}{\delta(\varepsilon(x))^{1/2}}.$$

The following identities (4.1), (4.2), and (4.3) can be checked easily: for every $z \in \mathfrak{H}'$ one has

$$\delta_1^c(z^{-1}\overline{z}) = 2\delta_1^c(z^{-1}\mathrm{Re}(z)). \tag{4.1}$$

Also one has

$$\begin{aligned} \delta(z)\sigma\left(\frac{z^{-1}\overline{z}}{2}\right) &= \delta(\mathrm{Re}(z)) + \delta(\mathrm{Im}(z)) \\ &= 2\delta(\mathrm{Re}(z)) - \frac{1}{2}(\delta(z) + \delta(\overline{z})) \\ &= 2\delta(\mathrm{Im}(z)) + \frac{1}{2}(\delta(z) + \delta(\overline{z})), \end{aligned} \tag{4.2}$$

and

$$\frac{\sigma\left(\rho\left(\dfrac{x}{2}\right)\right)}{\delta(\rho(x))^{1/2}} = \nu(x)\overline{\nu(x)} - \left[(1-\nu(x)^2)(1-\overline{\nu(x)}^2)\right]^{1/2}. \tag{4.3}$$

Lemma 4.1.

(i) *For each $\varphi_k \in \mathscr{H}_d$, $0 \le k \le d$, the function $\delta(\varepsilon(x))^d \varphi_k(\rho(x))$ can be written into a finite sum*

$$\delta_1^c(\varepsilon(x)^{-1})^k \sum_{l_1,l_2} c_{l_1,l_2} \delta(\varepsilon(x))^{k+l_1} \delta(\overline{\varepsilon(x)})^{l_2}$$

with suitable $c_{l_1,l_2} \in \boldsymbol{C}$ for non-negative integers l_1 and l_2.

(ii) *For each $\varphi_k \in \mathscr{H}_d$, $0 \le k \le d$, the function $\delta(\rho(x))^{-d/2}\varphi_k(\rho(x))$ can be written into a finite sum*

$$|\delta(\varepsilon(x))|^d \delta(\overline{\varepsilon(x)})^{-k} \sum_{l,r} \varphi_{k+l}(\varepsilon(x)^{-1}) \psi_{l,r}(\overline{\varepsilon(x)}^{-1})$$

with suitable choices of polynomials $\psi_{l,r} \in \delta^r \otimes \mathscr{H}_{d-k-2r}$, *where* l *and* r *are integers satisfying* $0 \leq l \leq d-k$ *and* $0 \leq r \leq [(d-k)/2]$.

PROOF. By Lemma 2.2 $\varphi_k(\rho(x))$ is written as a product of $\delta_1^c(\rho(x))^k$ and some polynomial in $\delta(\rho(x))$, $\sigma(\rho(x)/2)$. Then the equations (4.1) and (4.2), which are applied to $z = \varepsilon(x)$, concludes the first assertion.

To prove (ii) we recall the addition theorem for the Gegenbauer polynomial:

$$\begin{aligned} &C_n^p(\cos(\theta+\psi)) \\ &\quad = \frac{1}{[\Gamma(p)]^2} \sum_{l=0}^{n} \frac{(-1)^l 2^{2l}[\Gamma(p+l)]^2(n-l)!\Gamma(2p+l-1)(2l+2p-1)}{\Gamma(n+l+2p)l!} \\ &\qquad\qquad \times C_{n-l}^{p+l}(\cos\psi)\sin^l\psi \cdot C_{n-l}^{p+l}(\cos\theta)\sin^l\theta, \end{aligned}$$

which is the formula (3) in Subsection 2 of Chapter IX-4 of [**V**]. Using this formula and (4.1) and (4.3), we write $\delta(\rho(x))^{-d/2}\varphi_k(\rho(x))$ into a finite linear sum of

$$|\delta(\varepsilon(x))|^d \delta(\overline{\varepsilon(x)})^{-k}\varphi_{k+l}(\varepsilon(x)^{-1}) \cdot \delta_1^{\overline{c}}(\overline{\varepsilon(x)}^{-1})^l \widehat{C}_{k+l}(\overline{\varepsilon(x)}^{-1})$$

for $0 \leq l \leq d-k$. Notice that the polynomial $\delta_1^{\overline{c}}(u)^l \widehat{C}_{k+l}(u) \in \mathscr{P}_{d-k}$ can be expressed as a sum of polynomials in $\delta^r \otimes \mathscr{H}_{d-k-2r}$ for $0 \leq r \leq [(d-k)/2]$ according to the decomposition of $\mathscr{P}_{d-k}$ given in Lemma 2.1. We apply this expression to each factor $\delta_1^{\overline{c}}(\overline{\varepsilon(x)}^{-1})^l \widehat{C}_{k+l}(\overline{\varepsilon(x)}^{-1})$ occurring above, which concludes the second assertion of the lemma. $\square$

Next we recall some integral transforms of spherical harmonic polynomials. To each $U(2)$-module $\delta^j \otimes \mathscr{H}_{d-2j}$ of the highest weight $(2d-2j, 2j)$, $0 \leq j \leq d/2$, we assign a product of the classical Gamma-functions defined by

$$\Gamma_{d-j,j}(s) = \sqrt{\pi}\,\Gamma(s+d-j)\Gamma\left(s+j-\frac{1}{2}\right). \tag{4.4}$$

Let $S_\infty^+ \subset S_\infty$ denote the subset of all positive definite matrices. Then we recall the following formulas given in [**S2**].

LEMMA 4.2.

(i) *For every* $z \in \mathfrak{H}'$, $s \in \boldsymbol{C}$, *and* $f \in \delta^j \otimes \mathscr{H}_{d-2j}$, *one has*

$$\int_{S_\infty^+} e^{-\sigma(zv)}\delta(v)^{s-(3/2)} f(v)dv = \Gamma_{d-j,j}(s)\delta(z)^{-s} f(z^{-1})$$

when $\mathrm{Re}(s) > 1/2$.

(ii) *For every* $v \in S_\infty^+$, $x \in S_\infty$, $s \in \boldsymbol{C}$, *and* $f \in \delta^j \otimes \mathscr{H}_{d-2j}$, *one has*

$$2\Gamma_{d-j,j}(s) \int_{S_\infty} e^{2\pi i \sigma(ux)} \delta(v + 2\pi i u)^{-s} f((v + 2\pi i u)^{-1}) du$$
$$= \begin{cases} e^{-\sigma(vx)} \delta(x)^{s-(3/2)} f(x) & (x \in S_\infty^+), \\ 0 & (x \notin S_\infty^+), \end{cases}$$

when $\mathrm{Re}(s) > 2$.

For the proofs, see [**S1**, Section 1], and [**S2**, Proposition 3.1].

5. Confluent hypergeometric functions.

We recall the integral

$$\eta(z; \alpha, \beta) = \int_0^\infty e^{-zt} (t+1)^{\alpha-1} t^{\beta-1} dt$$

and set

$$\omega(z; \alpha, \beta) = \Gamma(\beta)^{-1} z^\beta \eta(z; \alpha, \beta)$$

for $\alpha, \beta \in \boldsymbol{C}$ satisfying $\mathrm{Re}(\beta) > 0$ and $z \in \mathfrak{H}_1' = \{y + ix \mid y > 0,\ x \in \boldsymbol{R}\}$. The following facts are well known; (i) $\omega(z; \alpha, \beta)$ can be continued as a holomorphic function to the whole $\mathfrak{H}_1' \times \boldsymbol{C}^2$; (ii) for every compact subset U of $\boldsymbol{C}^2$, there exist positive constants A and B depending only on U so that one has the uniform bound

$$|\omega(g; \alpha, \beta)| \leq A(1 + g^{-B}) \tag{5.1}$$

for all $g > 0$ and $(\alpha, \beta) \in U$. For the proofs, see [**S1**, p. 282].

We define the archimedean local integrals $W_{B,\infty}^{(j)}(g, s)(\varphi)$, $\varphi \in \mathscr{H}_d$, $j = 1, 2$, by

$$W_{B,\infty}^{(2)}(g, s)(\varphi) = \int_{S_\infty} e^{-2\pi i \sigma(Bx)} \Lambda_\infty(w_2 n(x) g, s)(\varphi) dx, \tag{5.2}$$

for all $B \in S_\infty$, and

$$W^{(1)}_{B,\infty}(g,s)(\varphi) = \int_{S_{1,\infty}} e^{-2\pi i\sigma(B[\gamma_B^{-1}]x)} \Lambda_\infty(w_1 n(x) m(\gamma_B) g, s)(\varphi) dx \tag{5.3}$$

for every $B \in L$ of lower rank. Here dx denotes the Euclidean measure on $S_\infty = \boldsymbol{R}^3$, or on $S_{1,\infty} = \boldsymbol{R}$, respectively.

In this section we study the behavior of the integrals (5.2) at $s = s_d = d+1$ with a positive even integer d, while those for (5.3) will be discussed in the next section.

We remark that

$$W^{(2)}_{B,\infty}(m(a)g,s)(\varphi) = |\delta(a)|^{3-s}_\infty W^{(2)}_{B[a],\infty}(g,s)(\varphi) \tag{5.4}$$

for all $m(a) \in M_\infty$ and $g \in G_\infty$.

Now let $g = m(a) \in M_\infty$. Then we can write

$$\mu_{w_2 n(x) g}(\boldsymbol{i}) = -ia \cdot \varepsilon(a^{-1} x^t a^{-1})$$

using the matrix $\varepsilon(x)$ defined in Section 4, and thus get an expression

$$W^{(2)}_{B,\infty}(m(a),s)(\varphi) = |\delta(a)|^{3-s}_\infty \xi\biggl(B[a], \frac{s-d}{2}, \frac{s+d}{2}; \varphi\biggr). \tag{5.5}$$

Here $\xi(h,\alpha,\beta;\varphi)$ is the confluent hypergeometric function, which is defined by

$$\xi(h,\alpha,\beta;\varphi) = \int_{S_\infty} e^{-2\pi i\sigma(hx)} \delta(\varepsilon(x))^{-\alpha} \delta(\overline{\varepsilon(x)})^{-\beta} \varphi(\rho(x)) dx$$

for $h \in S_\infty$, $(\alpha,\beta) \in \boldsymbol{C}^2$, and $\varphi \in \mathscr{H}_d$. Since $\rho(x) \in U(2)$ and φ is a polynomial, this is convergent if $\mathrm{Re}(\alpha+\beta) > 2$, see [**S1**, Lemma 1.1].

Recall the basis $\{\varphi_k,\ \varphi_{-k} \mid 0 \le k \le d\}$ of $\mathscr{H}_d$. Then we note the following.

Lemma 5.1. *For each $\varphi_k \in \mathscr{H}_d$, $0 \le k \le d$, and $B \in L$, one has*

$$W^{(2)}_{-B,\infty}(m(a),s)(\varphi_k) = (-1)^k W^{(2)}_{B,\infty}(m(a),s)(\varphi_k).$$

The similar equalities hold also for φ_{-k}.

Proof. Consider the change of the variables from x to $-x$ for the above integral $\xi(B[a],(s-d)/2,(s+d)/2;\varphi)$. Since

$$\delta(\overline{\rho(x)})^{-(d/2)}\varphi_k(\overline{\rho(x)}) = (-1)^k \delta(\rho(x))^{-(d/2)}\varphi_k(\rho(x)),$$

which can be verified directly from the definition, we obtain the assertion. $\square$

By Lemma 4.1 (i) we remark that $\xi(h,\alpha,\beta;\varphi_k)$ is a finite linear sum of

$$\int_{S_\infty} e^{-2\pi i\sigma(hx)}\delta(\varepsilon(x))^{-\alpha-d+k+l_1}\delta(\overline{\varepsilon(x)})^{-\beta+l_2}\delta_1^c(\varepsilon(x)^{-1})^k dx.$$

Now we follow the arguments in [**K**] and [**S1**] to get the following result.

Theorem 5.2. *For every $h \in S_\infty$ and $\varphi \in \mathscr{H}_d$, the function $\xi(h,\alpha,\beta;\varphi)$ has a meromorphic continuation in (α,β) to the whole $\boldsymbol{C}^2$. Moreover, if h is non-degenerate, this continuation is holomorphic.*

Proof. By the above remark it suffices to prove the continuation of

$$\int_{S_\infty} e^{-2\pi i\sigma(hx)}\delta(\varepsilon(x))^{-\alpha}\delta(\overline{\varepsilon(x)})^{-\beta}\delta_1^c(\varepsilon(x)^{-1})^k dx.$$

Applying Lemma 4.2 (i) to $f(u) = \delta_1^c(u)^k \in \mathscr{H}_k$, one obtains that this equals

$$\frac{(2\pi)^3}{\Gamma_{k,0}(\alpha)}\int_{S_\infty^+} e^{-\sigma(v)}\delta(v)^{\alpha-(3/2)}\delta_1^c(v)^k \int_{S_\infty} e^{2\pi i\sigma(v-2\pi h)x}\delta(e_2+2\pi i x)^{-\beta}dxdv.$$

Then, by Lemma 4.2 (ii), it further equals

$$\frac{4\pi^{2(\alpha+\beta)+k}}{\Gamma_{k,0}(\alpha)\Gamma_{0,0}(\beta)}\int_{V(0,2h)} e^{-2\pi\sigma(v-h)}\delta(v)^{\alpha-(3/2)}\delta(v-2h)^{\beta-(3/2)}\delta_1^c(v)^k dv, \quad (5.6)$$

where we set $V(h_1,h_2) = \{v \in S_\infty \mid v > h_1,\ v > h_2\}$ for $h_1, h_2 \in S_\infty$.

We shall analyze the integral (5.6) case by case according to the signature of h.

(1) Firstly, let us assume $h > 0_2$ with eigenvalues $q' \geq q > 0$. Set $h_0 = \mathrm{diag}(q',q) > 0_2$. Then there exists a matrix $g \in GL_2(\boldsymbol{R})$ so that $h[g] = e_2$ and ${}^t gg = h_0^{-1}$ hold. By a change of variables, the integral (5.6) is written in the form

$$4(2\pi)^k\delta(2\pi h)^{\alpha+\beta-(3/2)}e^{-2\pi\sigma(h)}\Gamma_{k,0}(\alpha)^{-1}\Gamma_{0,0}(\beta)^{-1}$$
$$\times\int_{v>0_2} e^{-4\pi\sigma(h_0 v)}\delta(v+e_2)^{\alpha-(3/2)}\delta(v)^{\beta-(3/2)}\delta_1^c\big((v+e_2)[g^{-1}]\big)^k dv.$$

Now we take the expansion

$$\delta_1^c\big((v+e_2)[g^{-1}]\big)^k = \sum_{j=0}^{k} f_j(x,y)z^j, \quad v = \begin{pmatrix} x & z \\ z & y \end{pmatrix} > 0_2$$

with suitable polynomials $f_j(x,y) \in \boldsymbol{C}[x,y]$ and use it in the analysis of the factor

$$\Gamma_{0,0}(\beta)^{-1} \int_{v>0_2} e^{-4\pi\sigma(h_0 v)} \delta(v+e_2)^{\alpha-(3/2)} \delta(v)^{\beta-(3/2)} \delta_1^c\big((v+e_2)[g^{-1}]\big)^k dv, \quad (5.7)$$

which occurs in the above expression of (5.6).

Recall that the domain $\big\{v = \big(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\big) > 0_2\big\}$ maps bijectively onto $(v_1, v_2, w) \in \boldsymbol{R}_{>0} \times \boldsymbol{R}_{>0} \times \boldsymbol{R}$ by putting

$$x = v_1, \quad y = v_2(1+w^2) + (v_1+1)w^2, \quad z = w\sqrt{v_1(v_1+1)},$$

[**K**, p. 465], or [**S1**, p. 283]. We also recall that

$$\delta(v+e_2) = (v_1+1)(v_2+1)(w^2+1), \quad \delta(v) = v_1 v_2 (w^2+1),$$
$$dxdydz = \sqrt{v_1(v_1+1)}(w^2+1)dv_1 dv_2 dw.$$

This change of variables takes (5.7) to the finite sum of functions

$$C_{j_1,j_2,j_3,j_4} \Gamma_{0,0}(\beta)^{-1} \int_{-\infty}^{\infty} e^{-4\pi q w^2} (1+w^2)^{\alpha+\beta+j_3-2} w^{2j_4}$$
$$\times\, \eta(4\pi g_1; \alpha+j_1, \beta+j_2)\eta\bigg(4\pi g_2; \alpha-\frac{1}{2}, \beta-\frac{1}{2}+j_3\bigg)dw$$

for non-negative integers $j_2 \le j_4 \le j_1 \le k$ and $j_3 \le k$, where we put

$$g_1 = q' + qw^2, \quad g_2 = q(1+w^2),$$

and the constants C_{j_1,j_2,j_3,j_4} do not depend on α and β. According to (5.1), we can choose constants A, $B > 0$ for every compact subset U of $\boldsymbol{C}^2$ to get the uniform bound

$$\left|\Gamma_{0,0}(\beta)^{-1} g_1^{\beta+j_2}\eta(4\pi g_1;\alpha+j_1,\beta+j_2) g_2^{\beta-(1/2)+j_3}\eta\left(4\pi g_2;\alpha-\frac{1}{2},\beta-\frac{1}{2}+j_3\right)\right|$$
$$\leq\ A(1+q^{-B})$$

for all $(\alpha,\beta)\in U$. Then each function above is bounded by

$$A(1+q^{-B})\left|q^{-\beta+(1/2)-j_3}\right|\int_{-\infty}^{\infty} e^{-4\pi q w^2}\left|(q'+qw^2)^{-\beta-j_2}(1+w^2)^{\alpha-(3/2)+j_4}\right|dw,$$

uniformly on U, where the integral part is further estimated by

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} e^{-4\pi q w^2}\left|(1+w^2)^{\alpha+j_4-(3/2)}\right|dw, & \text{if } \mathrm{Re}(\beta)\geq -j_2, \\ \displaystyle\int_{-\infty}^{\infty} e^{-4\pi q w^2}\left|(1+w^2)^{\alpha-\beta-j_2+j_4-(3/2)}\right|dw, & \text{if } \mathrm{Re}(\beta)< -j_2. \end{cases}$$

The last one is convergent for general α,β in the respective case. Thus the proof in the case $h>0_2$ is completed by varying U.

If $h<0_2$, we use Lemma 5.1 to replace it with $-h>0_2$, thus the proof is reduced to the above case.

(2) Let h have the signature $(1,1)$, and write its eigenvalues as $q', -q$ with $q, q'>0$. We set $h_0=\mathrm{diag}(q',q)>0_2$. Take a matrix $g\in GL_2(\boldsymbol{R})$ so that $h[g]=\mathrm{diag}(1,-1)$ and ${}^t gg=h_0^{-1}$ hold. In this case (5.6) becomes

$$\begin{aligned} &4(2\pi)^k|\delta(2\pi h)|^{\alpha+\beta-(3/2)}e^{-2\pi\sigma(h_0)}\Gamma_{k,0}(\alpha)^{-1}\Gamma_{0,0}(\beta)^{-1} \\ &\quad\times\int_{V(\varepsilon_1,\varepsilon_2)} e^{-4\pi\sigma(h_0 v)}\delta(v-\varepsilon_1)^{\alpha-(3/2)}\delta(v-\varepsilon_2)^{\beta-(3/2)}\delta_1^c\big((v-\varepsilon_1)[g^{-1}]\big)^k dv, \end{aligned} \tag{5.8}$$

where we put $\varepsilon_1=\mathrm{diag}(-1,0)$ and $\varepsilon_2=\mathrm{diag}(0,-1)$.

The domain $V(\varepsilon_1,\varepsilon_2)=\left\{\left(\begin{smallmatrix} x+1 & z \\ z & y\end{smallmatrix}\right)>0_2 \text{ and } \left(\begin{smallmatrix} x & z \\ z & y+1\end{smallmatrix}\right)>0_2\right\}$ maps bijectively onto $(v_1,v_2,w)\in\boldsymbol{R}_{>0}\times\boldsymbol{R}_{>0}\times\boldsymbol{R}$ by

$$\begin{gathered} x=v_1(1+w^2)+w^2,\quad y=v_2(1+w^2)+w^2, \\ z=w\sqrt{(1+v_1)(1+v_2)(1+w^2)}, \end{gathered} \tag{5.9}$$

[**K**, p. 466], for which we have

$$\delta(v-\varepsilon_1) = (1+v_1)v_2(1+w^2), \quad \delta(v-\varepsilon_2) = v_1(1+v_2)(1+w^2),$$
$$dxdydz = \sqrt{(1+v_1)(1+v_2)(1+w^2)}(1+w^2)dv_1dv_2dw.$$

We put

$$g_1 = q'(1+w^2) \text{ and } g_2 = q(1+w^2).$$

Then, (5.8) becomes a finite sum of functions

$$\Gamma_{k,0}(\alpha)^{-1}\Gamma_{0,0}(\beta)^{-1}\int_{-\infty}^{\infty} e^{-4\pi(q+q')w^2}w^{2j_5}(1+w^2)^{\alpha+\beta+j_4-(3/2)}$$
$$\times \eta\biggl(4\pi g_1; \alpha+j_1, \beta-\frac{1}{2}+j_2\biggr)\eta\biggl(4\pi g_2; \beta+j_1, \alpha-\frac{1}{2}+j_3\biggr)dw$$

up to holomorphic factors, for non-negative integers $j_1, j_2, j_3, j_4, j_5 \le k$. Again by (5.1), we get uniform bounds for

$$\left|\Gamma_{0,0}(\beta)^{-1}g_1^{\beta-(1/2)+j_2}\eta\biggl(4\pi g_1; \alpha+j_1, \beta-\frac{1}{2}+j_2\biggr)\right| \text{ and}$$
$$\left|\Gamma_{k,0}(\alpha)^{-1}g_2^{\alpha-(1/2)+j_3}\eta\biggl(4\pi g_2; \beta+j_1, \alpha-\frac{1}{2}+j_3\biggr)\right|$$

on every compact subset U in $\boldsymbol{C}^2$. Thus the proof is completed by the similar arguments as given in the case (1).

(3) Lastly, we treat the case that h is degenerate. We take $g \in SO(2)$ so that $h[g] = h_0 = \mathrm{diag}(q, 0)$ with $q \in \boldsymbol{R}$ holds. Then (5.6) is equal to

$$4\pi^{2(\alpha+\beta)+k}\Gamma_{k,0}(\alpha)^{-1}\Gamma_{0,0}(\beta)^{-1}$$
$$\times \int_{V(h_0,-h_0)} e^{-2\pi\sigma(v)}\delta(v+h_0)^{\alpha-(3/2)}\delta(v-h_0)^{\beta-(3/2)}\delta_1^c\bigl((v+h_0)[g^{-1}]\bigr)dv.$$

The domain $\left\{\left(\begin{smallmatrix} x\pm q & z \\ z & y\end{smallmatrix}\right) > 0_2\right\}$ maps bijectively onto $v_1 \pm q > 0$, $v_2 > 0$, $-\infty < w < \infty$ by

$$x = v_1 + w^2v_2^{-1}, \quad y = v_2, \ z = w \tag{5.10}$$

for which we have

$$\delta(v+h_0)=(v_1+q)v_2, \quad \delta(v-h_0)=(v_1-q)v_2.$$

Consequently, the above integral becomes the finite sum of integrals

$$\frac{4\pi^{2(\alpha+\beta)+k}}{\Gamma_{k,0}(\alpha)\Gamma_{0,0}(\beta)}\left(\int_{v_1>|q|} e^{-2\pi v_1}(v_1+q)^{\alpha-(3/2)+j_1}(v_1-q)^{\beta-(3/2)}dv_1\right)$$
$$\times\left(\int_0^\infty e^{-2\pi v_2}v_2^{\alpha+\beta-3+j_2-j_3}\int_{-\infty}^\infty e^{-2\pi w^2 v_2^{-1}}w^{2(j_3+j_4)}dwdv_2\right)$$

for non-negative integers j_1, j_2, j_3, j_4. Each integral can be computed as

$$\frac{\Gamma\left(\alpha+\beta-\dfrac{3}{2}+j_2+j_4\right)}{\Gamma_{k,0}(\alpha)\Gamma_{0,0}(\beta)}\int_{v_1>|q|} e^{-2\pi v_1}(v_1+q)^{\alpha-(3/2)+j_1}(v_1-q)^{\beta-(3/2)}dv_1 \quad (5.11)$$

up to a multiple of a simple holomorphic function of $(\alpha,\beta)\in \boldsymbol{C}^2$. These all have meromorphic continuations in (α,β) to the whole $\boldsymbol{C}^2$, and thus the proof is completed. □

Corollary 5.3. *The integral $W_{B,\infty}^{(2)}(g,s)(\varphi)$, $g\in G_\infty$, has a meromorphic continuation in s to the whole $\boldsymbol{C}$ for all $B\in L$ and $\varphi\in\mathscr{H}_d$. In particular, the continuation is holomorphic if $D_B\neq 0$.*

Proof. This is obtained by Theorem 5.2 and (5.5). □

We give the criterion of vanishing at $s=s_d$.

Theorem 5.4. *Suppose that $B\in L$ satisfies one of the following conditions*; (i) *B is positive, or negative, definite*; *or* (ii) *B is degenerate. Then we have*

$$W_{B,\infty}^{(2)}(g,s_d)(\varphi)=0$$

for all $g\in G_\infty$ and $\varphi\in\mathscr{H}_d$.

Proof. It suffices to prove that $\xi(B[a],(s_d-d)/2,(s_d+d)/2;\varphi_k)=0$ for all $0\le k\le d$ under the conditions on B. Let us put $h=B[a]$. According to Lemma 4.1 (ii), $\xi(h,(s-d)/2,(s+d)/2;\varphi_k)$ can be written as a sum of integrals

$$\int_{S_\infty} e^{-2\pi i\sigma(hx)}|\delta(\varepsilon(x))|^{-s+d}\delta(\overline{\varepsilon(x)})^{-k}\varphi_{k+l}(\varepsilon(x)^{-1})\psi_{l,r}(\overline{\varepsilon(x)}^{-1})dx$$

with $0 \le l \le d-k$ and $0 \le r \le [(d-k)/2]$. By Lemma 4.2 (i) and (ii) this is equal to

$$4\pi^{2s+k}\Gamma_{d,0}\left(\frac{s-d}{2}\right)^{-1}\Gamma_{d-k-r,r}\left(\frac{s-d}{2}+k\right)^{-1} \times \int_{V(0,2h)} e^{-2\pi\sigma(v-h)}\delta(v)^{(s-d-3)/2}\delta(v-2h)^{(s-d-3)/2+k}\varphi_{k+l}(v)\psi_{l,r}(v-2h)dv. \tag{5.12}$$

Firstly we assume that $k > 0$. In this case, the key point in the proof will be the finiteness of the integral occurring in (5.12) at $s = s_d$ for all l, r, which will be checked case by case:

(1) We assume $k > 0$ and $h > 0_2$. We consider the integral

$$\int_{v>0_2} e^{-2\pi\sigma(v+h)}\delta(v+2h)^{(s-d-3)/2}\delta(v)^{(s-d-3)/2+k}\varphi_{k+l}(v+2h)\psi_{l,r}(v)dv$$

within (5.12). Since $k > 0$ this is obviously finite at $s = s_d$ for each l, r. On the other hand, the factor $\Gamma_{d,0}((s-d)/2)^{-1}$ in (5.12) has a simple zero at $s = s_d$. Consequently, the function (5.12) specializes to 0 at $s = s_d$ for all l and r, and thus we get the conclusion in this case.

(2) Assume $k > 0$ and that h is degenerate which is conjugate by $SO(2)$-action to $h_0 = \mathrm{diag}(q, 0)$ with $q > 0$. To show the finiteness at $s = s_d$ of the integrals

$$\int_{V(h_0,-h_0)} e^{-2\pi\sigma(v)}\delta(v+h_0)^{(s-d-3)/2}\delta(v-h_0)^{(s-d-3)/2+k}\varphi(v+h_0)\psi(v-h_0)dv$$

for polynomials $\varphi \in \mathscr{H}_d$ and $\psi \in \delta^r \otimes \mathscr{H}_{d-k-2r}$, we repeat the same argument as in the proof of the case (3) of Theorem 5.2. Then it shows that each of the integrals above equals a sum of integrals of the type (5.11) with parameters $\alpha = \beta - k = (s-d)/2$. Since $k > 0$, they are all finite at $s = s_d$, which completes the conclusion.

(3) Let us assume that $k > 0$ and $h < 0_2$, or that h is degenerate with one negative eigenvalue $-q < 0$. Then we can apply Lemma 5.1 to reduce each to the case (1), or (2), above respectively. Thus the proof is done.

(4) Next we assume $B = 0_2$ and $k > 0$. If, moreover, k is odd, then Lemma 5.1 implies that $W^{(2)}_{0_2,\infty}(m(a), s)(\varphi_k) = 0$ for s in the range of absolute convergence of the integral expression (5.2). Consequently, it vanishes on the whole $\boldsymbol{C}$, because

of the analytic continuation which was proved in Corollary 5.3.

Hence, we assume that $k > 1$ is even. If we put $B = 0_2$, then the factor of integral in (5.12) becomes

$$\int_{S_\infty^+} e^{-2\pi\sigma(v)}\delta(v)^{s-d-3+k}\varphi_{k+l}(v)\psi_{l,r}(v)dv.$$

We apply Lemma 2.1 to express $\varphi_{k+l}\psi_{l,r} \in \mathscr{P}_{2d-k}$ as a sum of polynomials $\psi_{l,r,r'}$ in $\delta^{r'} \otimes \mathscr{H}_{2d-k-2r'}$ for $0 \le r' \le d-(k/2)$. Then Lemma 4.2 (i) gives that the above integral is equal to the sum of

$$c_{l,r,r'}(s)\Gamma_{2d-k-r',r'}\left(s-d+k-\frac{3}{2}\right),$$

where $c_{l,r,r'}(s)$ are elementary entire functions in s. Since $k > 1$, these terms are all finite at $s = s_d$. Thus the proof is completed in this case.

Now we consider the remaining case that $k = 0$. Here we can write the function $\delta(\rho(x))^{-(d/2)}\varphi_0(\rho(x))$ into the finite sum

$$|\delta(\varepsilon(x))|^d \sum_{l=0}^{d} c_l \cdot \varphi_l(\varepsilon(x)^{-1})\varphi_{-l}(\overline{\varepsilon(x)}^{-1}) \quad \text{with} \quad c_l \in \boldsymbol{C}.$$

Correspondingly, $\xi(h,(s-d)/2,(s+d)/2;\varphi_0)$ becomes a finite linear sum of the functions

$$\frac{4\pi^{2s}}{\Gamma_{d,0}\left(\dfrac{s-d}{2}\right)^2}\int_{V(h,-h)} e^{-2\pi\sigma(v)}\delta(v+h)^{(s-d-3)/2} \times \delta(v-h)^{(s-d-3)/2}\varphi_l(v+h)\varphi_{-l}(v-h)dv.$$

As concerns these functions, one notes that the factor $\Gamma_{d,0}((s-d)/2)^2$ in the denominator has a double pole at $s = s_d$, whereas each integral part provides a simple pole at $s = s_d$ under the assumptions on B. Thus the proof is completed. $\square$

Let us consider the archimedean local integral attached to the zonal polynomial $\varphi_0 \in \mathscr{H}_d$. It turns out the analysis of its specialization at $s = s_d$ can be carried out explicitly, which provides the following result.

Proposition 5.5. *Suppose $B \in L$ is regular and indefinite; i.e. $D_B > 0$.*

We put $y = a^t a$ for $a \in GL_2(\boldsymbol{R})$. Then we have

$$W^{(2)}_{B,\infty}(m(a), s_d)(\varphi_0) = \frac{4(2\pi)^d}{(2d)!} D_B^{d-(1/2)} \delta(y)^{(d+1)/2} K_0\big(2\pi\sqrt{[\sigma(By)]^2 + D_B\delta(y)}\big),$$

where $K_\nu(z)$ denotes the K-Bessel function.

Proof. We extend $\varphi_0(u) = \widehat{C}_0(u)$ to the function $\varphi_0(vu) = \varphi_0(uv) = \varphi_0(v^{1/2}uv^{1/2})$ for all $v \in S^+_\infty$ and $u \in S_\infty$. Then one recalls the integral formula

$$\int_{S^+_\infty} e^{-\sigma(\varepsilon(x)v)} \delta(v)^{s-(3/2)} \varphi_0(v\overline{\varepsilon(x)}) dv = \Gamma_{d,0}(s) \delta(\varepsilon(x))^{-s} \varphi_0(\rho(x)),$$

see [**Mu**, Theorem 7.2.7]. Using this formula and the identity

$$\varphi_0(v\overline{\varepsilon(x)}) = \delta(v)^d \delta(\overline{\varepsilon(x)})^d \varphi_0\big(v^{-1}\overline{\varepsilon(x)}^{-1}\big)$$

we write $\xi(h, (s-d)/2, (s+d)/2; \varphi_0)$, $h = B[a]$, as

$$\begin{aligned} &\frac{(2\pi)^3}{\Gamma_{d,0}\left(\dfrac{s-d}{2}\right)} \int_{S^+_\infty} e^{-\sigma(v)} \delta(v)^{(s+d-3)/2} \\ &\quad \times \int_{S_\infty} e^{2\pi i \sigma(v-2\pi h)x} \delta(\overline{\varepsilon(2\pi x)})^{(d-s)/2} \varphi_0\big(v^{-1}\overline{\varepsilon(2\pi x)}^{-1}\big) dx dv. \end{aligned}$$

Now Fourier's inversion formula is applied to show that $\xi(h, (s-d)/2, (s+d)/2; \varphi_0)$ equals

$$\begin{aligned} 4\pi^{2s}\Gamma_{d,0}\left(\frac{s-d}{2}\right)^{-2} \int_{V(h,-h)} e^{-2\pi\sigma(v)} \delta(v+h)^{(s-d-3)/2} & \\ \times\, \delta(v-h)^{(s-d-3)/2} \Phi_h(v) dv, & \quad (5.13) \end{aligned}$$

where we set

$$\Phi_h(v) = \delta(v+h)^d \varphi_0((v+h)^{-1}(v-h)).$$

According to Lemma 2.2 and a variant of the identity (4.2), we can write $\Phi_h(v)$ as a polynomial in $\delta(v+h)$ and $\delta(v-h)$, whose constant term is given by

$$\Gamma\left(d+\frac{1}{2}\right)(d!\sqrt{\pi})^{-1}[-\delta(2h)]^d$$

(see the expansion formula (2.3)). Consequently, the value of (5.13) at $s = s_d$, that is $\xi(h, 1/2, d+(1/2); \varphi_0)$, coincides with the value of

$$\frac{4\pi^{2s-(3/2)}\Gamma\left(d+\frac{1}{2}\right)}{\Gamma\left(\frac{s+d}{2}\right)^2 d!} \times \frac{[-\delta(2h)]^d}{\Gamma\left(\frac{s-d-1}{2}\right)^2} \times \int_{V(h,-h)} e^{-2\pi\sigma(v)}\delta(v+h)^{(s-d-3)/2}\delta(v-h)^{(s-d-3)/2}dv$$

at the same point.

Let q' and $-q$ (q, $q' > 0$) denote the eigenvalues of the indefinite matrix h, and set $h_0 = \operatorname{diag}(q', q) > 0_2$ as before. Take a $g \in GL_2(\boldsymbol{R})$ so that $h[g] = \operatorname{diag}(1, -1)$ and ${}^t gg = h_0^{-1}$ hold. Then, by the change of variables (5.9), we can compute the latter integral as

$$\begin{aligned}
&[-\delta(2h)]^d\Gamma\left(\frac{s-d-1}{2}\right)^{-2}\int_{V(h,-h)} e^{-2\pi\sigma(v)}\delta(v+h)^{(s-d-3)/2}\delta(v-h)^{(s-d-3)/2}dv \\
&\quad = [-\delta(2h)]^{s-(3/2)}e^{-2\pi\sigma(h_0)}(4\pi g_1)^{(d+1-s)/2}(4\pi g_2)^{(d+1-s)/2} \\
&\qquad \times \int_{-\infty}^{\infty} e^{-4\pi\sigma(h_0)w^2}(1+w^2)^{s-d-(3/2)} \\
&\qquad\qquad \times \omega\left(4\pi g_1; \frac{s-d}{2}, \frac{s-d-1}{2}\right)\omega\left(4\pi g_2; \frac{s-d}{2}, \frac{s-d-1}{2}\right)dw
\end{aligned}$$

with the previously defined notations. Since $\omega(z; \alpha, 0) = 1$ (see [**S1**, (3.15)]), this concludes that $\xi(h, 1/2, d+(1/2); \varphi_0)$ is equal to

$$\frac{4(2\pi)^{2d}[-\delta(2h)]^{d-(1/2)}}{(2d)!}e^{-2\pi\sigma(h_0)}\eta\left(4\pi\sigma(h_0); \frac{1}{2}, \frac{1}{2}\right).$$

Finally, the relation

$$e^{-2\pi(q+q')}\eta\left(4\pi(q+q'); \frac{1}{2}, \frac{1}{2}\right) = K_0(2\pi(q+q'))$$

is verified by direct computation, which completes the proof of the proposition. □

6. Degenerate confluent hypergeometric functions.

We study the archimedean local integral $W^{(1)}_{B,\infty}(g,s)(\varphi)$, (5.3), for a matrix $B \in S_\infty$ of lower rank.

LEMMA 6.1. *For every base polynomial $\varphi_k \in \mathscr{H}_d$, $0 \le k \le d$, and a degenerate matrix $B \in S_\infty$, we have*

$$W^{(1)}_{-B,\infty}(m(a),s)(\varphi_k) = (-1)^k W^{(1)}_{B,\infty}(m(a),s)(\varphi_k).$$

The similar equalities hold also for φ_{-k}.

PROOF. By definition we find that

$$\varphi_k\left(\begin{pmatrix} x-i & 0 \\ 0 & x+i \end{pmatrix}\right) = (-1)^k \varphi_k\left(\begin{pmatrix} x+i & 0 \\ 0 & x-i \end{pmatrix}\right),$$

which implies $\Lambda_\infty(w_1 n(-u),s)(\varphi_k) = (-1)^k \Lambda_\infty(w_1 n(u),s)(\varphi_k)$ for $u = \left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right)$. This remark together with the change of variables from u to $-u$ in (5.3) concludes the assertion. □

Let us attach the matrix $a_\tau \in SL_2(\boldsymbol{R})$ to each $\tau = u+iv \in \mathfrak{H}_1$ by

$$a_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} v^{1/2} & \\ & v^{-1/2} \end{pmatrix} = \begin{pmatrix} v^{1/2} & v^{-1/2}u \\ 0 & v^{-1/2} \end{pmatrix}.$$

Also one sets $a_{\tau,t} = t^{1/4}a_\tau \in GL_2(\boldsymbol{R})$ for $t > 0$ and $\tau \in \mathfrak{H}_1$.

Now let us consider the case $B = 0_2$. Then we have

$$W^{(1)}_{0_2,\infty}(m(a_{\tau,t}r_\theta),s)(\varphi_{\pm k}) = e^{\pm 2ik\theta}t^{1/2}v^{s-1}W^{(1)}_{0_2,\infty}(e_4,s)(\varphi_{\pm k}) \tag{6.1}$$

for every $r_\theta \in SO(2)$. Here $W^{(1)}_{0_2,\infty}(e_4,s)(\varphi_k) = W^{(1)}_{0_2,\infty}(e_4,s)(\varphi_{-k})$ are equal to

$$i^k \int_{-\infty}^{\infty}(x^2+1)^{-(s+k)/2}C^{k+(1/2)}_{d-k}\left(\frac{x}{\sqrt{x^2+1}}\right)dx. \tag{6.2}$$

These give rise to the following evaluations at $s = s_d$.

Proposition 6.2. *For every $\varphi_k \in \mathscr{H}_d$, $0 \le k \le d$, one has*

$$W^{(1)}_{0_2,\boldsymbol{R}}(m(a_{\tau,t}), s_d)(\varphi_k) = \begin{cases} i^d \cdot 2\left(\dfrac{(4\pi)^d d!}{(2d)!}\right)^2 \dfrac{\Gamma(2d)}{(2\pi i)^{2d}} t^{1/2} v^d & (k = d), \\ 0 & (k \neq d). \end{cases}$$

Proof. Recall the identity $C^{k+(1/2)}_{d-k}(-z) = (-1)^k C^{k+(1/2)}_{d-k}(z)$ for Gegenbauer polynomial. Then the integral (6.2) obviously vanishes, if k is odd.

Suppose $k < d$ even. Then (6.2) at $s = s_d$ becomes

$$\int_{-1}^{1} (1-y^2)^{(d-k)/2-1}(1-y^2)^k C^{k+(1/2)}_{d-k}(y)dy.$$

Now we recall the formula (7) in p. 485 of [**V**];

$$(1-y^2)^k C^{k+(1/2)}_{d-k}(y) = \frac{1}{2^{d-k}} \frac{k!(d+k)!}{d!(d-k)!} \frac{d^{d-k}}{dy^{d-k}}(1-y^2)^d.$$

Then integration by parts concludes the vanishing of the above integral for each $k \le d-2$. If $k = d$, then the formula is obtained by a direct computation. □

7. The lower rank parts of the Fourier expansion.

In this section we discuss the part of the Fourier expansion (3.1) of the Eisenstein series with the matrices $B \in L$ of lower rank.

We begin with the constant term $E_{0_2}(g, s_d)(\varphi)$ for $B = 0_2$. According to Lemma 3.2 and Theorem 5.4, one has

$$E_{0_2}(g, s_d)(\varphi) = \Lambda(g)(\varphi) + \sum_{\gamma \in P_1 \backslash G_1} \Lambda_M(m(\gamma)g)(\varphi), \qquad g \in KG_\infty, \tag{7.1}$$

by putting $\Lambda(g)(\varphi) = \Lambda(g, s_d)(\varphi)$, and $\Lambda_M(g)(\varphi) = \Lambda_M(g, s_d)(\varphi)$ with

$$\Lambda_M(g, s)(\varphi) = W^{(1)}_{0_2}(g, s)(\varphi).$$

Let us also put

$$g_{\tau,t} = m(a_{\tau,t}) \prod_{v \in \boldsymbol{f}} e_4 \in KG_\infty$$

for $\tau \in \mathfrak{H}_1$ and $t > 0$.

By the remark (2.4), $\Lambda(g)(\varphi)$ in (7.1) can be evaluated as following.

LEMMA 7.1. *Keep the above notations. Then one has*

$$\Lambda(g_{\tau,t})(\varphi_{\pm k}) = \begin{cases} 0 & (k \neq 0), \\ C_d^{1/2}(1)t^{(d+1)/2} = t^{(d+1)/2} & (k = 0). \end{cases}$$

Next we are concerned with the latter infinite sum in (7.1): we put

$$E_M(g)(\varphi) = \sum_{\gamma \in P_1 \backslash G_1} \Lambda_M(m(\gamma)g)(\varphi). \tag{7.2}$$

It is easily verified that the function $\Lambda_M(g)(\varphi)$ restricted to $g \in M_{\boldsymbol{A}}$ defines a function belonging to a certain principal series representation of $GL_2(\boldsymbol{A}) \simeq M_{\boldsymbol{A}}$, cf. (6.1). Therefore, the function $E_M(g)(\varphi)$ is indeed an Eisenstein series on $GL_2(\boldsymbol{A})$. Let us consider its Fourier expansion

$$E_M(n_1(u)g)(\varphi) = \Lambda_M(g)(\varphi) + \sum_{N=-\infty}^{\infty} \boldsymbol{e}(Nu)W_N(g;\varphi),$$

where we put $n_1(u) = m\left(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\right) \in M_{\boldsymbol{A}}$ and $g \in (K \cap M_{\boldsymbol{A}})M_\infty$. The N-th Fourier coefficient $W_N(g;\varphi)$ is given by

$$W_N(g;\varphi) = \int_{\boldsymbol{A}} \boldsymbol{e}(-Nx)\Lambda_M(w_M n_1(x)g)(\varphi)dx \tag{7.3}$$

with $w_M = m\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) \in M$, which is a product of the local integrals:

$$W_N(g;\varphi) = c_1 \cdot W_{N,\infty}(g_\infty;\varphi) \prod_{p \in \boldsymbol{f}} W_{N,p}(e_4) \quad \text{for} \quad g \in (K \cap M_{\boldsymbol{A}})M_\infty$$

with some constant c_1 in the usual way. For the non-archimedean local integral we get the following formula.

LEMMA 7.2. *For every finite prime $v = p \in \boldsymbol{f}$ one has*

$$W_{N,p}(e_4) = \frac{\zeta_p(d)}{\zeta_p(d+1)\zeta_p(2d)} \times \begin{cases} \zeta_p(2d-1) & (N = 0), \\ F_p(N; 2d+1) & (N \neq 0). \end{cases}$$

Proof. The result is obtained directly by computing the integral

$$W_{N,p}(e_4) = \int_{\boldsymbol{Q}_p} \boldsymbol{e}_p(-Nx)\biggl(\int_{S_{1,p}} \varLambda_p(w_1 n(u) w_M n_1(x), s_d)du\biggr)dx.$$

We omit the details here. □

As for the archimedean local integrals

$$W_{N,\infty}(g;\varphi) = \int_{-\infty}^{\infty} e^{-2\pi i N x}\biggl(\int_{S_{1,\infty}} \varLambda_\infty(w_1 n(u) w_M n_1(x) g, s_d)(\varphi)du\biggr)dx$$

for $g \in M_\infty$, we obtain the following result.

Lemma 7.3. *For each base polynomial φ_k, or φ_{-k}, in $\mathscr{H}_d$, $0 \le k \le d$, we have the following formula.*

(i) $W_{N,\infty}(m(a_{\tau,t});\varphi_k) = 0$ *if $k \neq d$ and $N \in \boldsymbol{Z}$, or if $k = d$ and $N \le 0$.*
(ii) $W_{N,\infty}(m(a_{\tau,t});\varphi_{-k}) = 0$ *if $k \neq d$ and $N \in \boldsymbol{Z}$, or if $k = d$ and $N \ge 0$.*
(iii) *For every $N > 0$ we have that $W_{N,\infty}(m(a_{iv,t});\varphi_d) = W_{-N,\infty}(m(a_{iv,t});\varphi_{-d})$, which is nonvanishing. The explicit value is given by*

$$i^d \cdot 2\biggl(\frac{(4\pi)^d d!}{(2d)!}\biggr)^2 t^{1/2} v^d N^{2d-1} e^{-2\pi N v}.$$

Proof. It suffices to prove the assertion for every $\varphi = \varphi_k$, $0 \le k \le d$. The integral $W_{N,\infty}(m(a_{iv,t});\varphi_k)$ was defined by the value at $s = s_d$ of

$$t^{1/2} v^{s-1}\biggl(\int_{-\infty}^{\infty} \frac{e^{-2\pi i N x}(x-iv)^k}{(x^2+v^2)^{s-1}(x+iv)^k}dx\biggr) W^{(1)}_{0_2,\infty}(e_4, s)(\varphi_k). \tag{7.4}$$

Here we note that the middle integral equals

$$\frac{2\pi(-1)^k}{\Gamma(s+k-1)\Gamma(s-k-1)} \int_{\substack{t>0\\ t>2\pi N}} e^{-2v(t-\pi N)} t^{s+k-2}(t-2\pi N)^{s-k-2}dt. \tag{7.5}$$

This is finite at $s = s_d$ for any $0 \le k \le d$. Thus the vanishings for all $k \neq d$ and all N are deduced from Proposition 6.2, which says that $W^{(1)}_{0_2,\infty}(e_4, s_d)(\varphi_k) = 0$ for $0 \le k < d$ in (7.4). If $k = d$ and $N \le 0$ we have the vanishing of (7.5) at $s = s_d$. These prove the assertion (i).

On the other hand, a direct computation of the integral, which is combined with the formula in Proposition 6.2, provides the assertion (iii) for $k = d$ and $N > 0$. We omit the details here. □

By Proposition 3.3 and Lemmas 7.1, 7.2, and 7.3, we obtain the following description of the constant term.

PROPOSITION 7.4 (The constant term). *Keep the previous notations. Then we have the following formulas.*

(i) $E_{0_2}(g_{\tau,t}, s_d)(\varphi_k) = E_{0_2}(g_{\tau,t}, s_d)(\varphi_{-k}) = 0$ *for all* $0 < k < d$;
(ii) $E_{0_2}(g_{\tau,t}, s_d)(\varphi_0) = t^{(d+1)/2}$; *and*
(iii) $E_{0_2}(g_{\tau,t}, s_d)(\varphi_d) = E_M(g_{\tau,t})(\varphi_d)$, *and this is equal to*

$$\frac{(4i)^d \kappa_d \cdot \zeta(1-d) t^{1/2} v^d}{\zeta(1+d)\zeta(1-2d)} \left(\frac{\zeta(1-2d)}{2} + \sum_{N=1}^{\infty} \sigma_{2d-1}(N) q^N \right),$$

where we put $q = e^{2\pi i \tau}$, $\kappa_d = (2\pi i)^d d!/(2d)!$, *and* $\sigma_\nu(N) = \sum_{l|N} l^\nu$.
(iv) $E_{0_2}(g_{\tau,t}, s_d)(\varphi_{-d})$ *is the complex conjugation of the formula in* (iii) *above.*

Next we shall give some remarks on the rank 1 part of the Fourier expansion (3.1) at $s = s_d$: we set

$$E^{(1)}(n(u)g;\varphi) = \sum_{B \text{ of rank } 1} \boldsymbol{e}(\sigma(Bu)) E_B(g, s_d)(\varphi)$$

for $u \in S_\infty$ and $g \in KG_\infty$. By Lemma 3.1 and Proposition 5.4, this is equal to

$$\sum_{\gamma \in \Gamma_1 \cap P_1 \backslash \Gamma_1} \sum_{N \neq 0} \boldsymbol{e}(\sigma(C_N[\gamma]u)) W^{(1)}_{C_N[\gamma]}(g, s_d)(\varphi) = \sum_{\gamma \in \Gamma_1 \cap P_1 \backslash \Gamma_1} G(m(\gamma)n(u)g)(\varphi),$$

where we set $\Gamma_1 = SL_2(\boldsymbol{Z})$, $C_N = \mathrm{diag}(0, N)$, and

$$G(n(u)g)(\varphi) = \sum_{N \neq 0} \boldsymbol{e}(\sigma(C_N u)) W^{(1)}_{C_N}(g, s_d)(\varphi).$$

Thus we have

$$E^{(1)}(g;\varphi) = \sum_{\gamma \in \Gamma_1 \cap P_1 \backslash \Gamma_1} G(m(\gamma)g)(\varphi). \tag{7.6}$$

Take an embedding $m' : SL_2(\boldsymbol{Q}) \to G$ defined by

$$m'(h) = \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix} \quad \text{for} \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Q}).$$

Let $M' \simeq SL_2(\boldsymbol{Q})$ denote its image. Then we find that

$$G(n(u)g_{\tau,t})(\varphi) = t^{(d+1)/4} v^{(d+1)/2} G\left(m'\begin{pmatrix} \dfrac{t^{1/4}}{v^{1/2}} & \dfrac{v^{1/2}x}{t^{1/4}} \\ 0 & \dfrac{v^{1/2}}{t^{1/4}} \end{pmatrix} \prod_{v \in \boldsymbol{f}} e_4 \right)(\varphi)$$

for every $u = \left(\begin{smallmatrix} * & * \\ * & x \end{smallmatrix}\right) \in S_\infty$, where the latter factor gives a function on $(K \cap M'_{\boldsymbol{A}})M'_\infty$, which can be related to an Eisenstein series on $M'_{\boldsymbol{A}}$ as below.

We define a function $E_{M'}(n(u)g)(\varphi)$, $g \in KG_\infty$, for each $u \in S_\infty$, $\varphi \in \mathscr{H}_d$ by

$$E_{M'}(n(u)g)(\varphi) = G(n(u)g)(\varphi) + \Lambda(g)(\varphi) + \Lambda_M(g)(\varphi). \tag{7.7}$$

This is equal to

$$\Lambda(g)(\varphi) + \sum_{N=-\infty}^{\infty} e^{2\pi i \sigma(C_N u)} W_{C_N}^{(1)}(g, s_d)(\varphi) \qquad (C_0 = 0_2),$$

which implies the equality

$$E_{M'}(g)(\varphi) = \sum_{\gamma \in \Gamma_1 \cap P_1 \backslash \Gamma_1} \Lambda(m'(\gamma)g)(\varphi)$$

for every $g \in (K \cap M'_{\boldsymbol{A}})M'_\infty$. This is also right $K \cap M'_{\boldsymbol{A}}$-finite, and thus it gives an Eisenstein series on $(K \cap M'_{\boldsymbol{A}})M'_\infty$.

8. Twisted Mellin transforms.

Let us define the regular part of the Fourier expansion (3.1) at $s = s_d$ by

$$E^{(2)}(n(u)g; \varphi) = \sum_{B \text{ of rank } 2} \boldsymbol{e}(\sigma(Bu)) E_B(g, s_d)(\varphi)$$

for $u \in S_{\boldsymbol{A}}$, $g \in KG_\infty$, and $\varphi \in \mathscr{H}_d$. Using the previous notations, we can write

$$E(n(u)g,s_d)(\varphi)=E_{0_2}(g,s_d)(\varphi)+E^{(1)}(n(u)g;\varphi)+E^{(2)}(n(u)g;\varphi)$$
$$=\Lambda(g)(\varphi)+E_M(g)(\varphi)+E^{(1)}(n(u)g;\varphi)+E^{(2)}(n(u)g;\varphi)$$

(see (7.1) and (7.2)).

Let ϕ be a Maass wave form on $SL_2(\boldsymbol{R})/\{\pm e_2\}$ for the group $\Gamma_1=SL_2(\boldsymbol{Z})$. We regard it as an unramified automorphic form on $M^1_{\boldsymbol{A}}$ with $M^1=\{m(a)\in M \mid a\in SL_2(\boldsymbol{Q})\}$ in the standard way, which is also denoted by ϕ. We will assume that ϕ is cuspidal in the following arguments.

Put $g_\tau=g_{\tau,1}\in M^1_{\boldsymbol{A}}$ for $\tau\in\mathfrak{H}_1$. We take measures $d\mu(\tau)=dudv/v^2$ on $\mathfrak{H}_1$ and $d^\times t=dt/t$ on $\boldsymbol{R}^\times_+$. Also we take a measure dr on $K^1_M=\prod_{v\in\boldsymbol{f}\cup\{\infty\}}(K_v\cap M^1_v)$ so that $\int_{K^1_M}dr=1$.

For every $s\in\boldsymbol{C}$ with $\mathrm{Re}(s)$ sufficiently large and $\varphi\in\mathscr{H}_d$, we define an integral transform $\mathscr{M}(\phi,s;\varphi)$ of $E^{(2)}(g;\varphi)$ by

$$\mathscr{M}(\phi,s;\varphi)=\int_{\Gamma_1\backslash\mathfrak{H}_1}\int_{K^1_M}\left(\int_0^\infty t^sE^{(2)}(g_{\tau,t}r;\varphi)d^\times t\right)\phi(g_\tau r)drd\mu(\tau). \tag{8.1}$$

We call it the Mellin transform of $E(g,s_d)(\varphi)$ twisted by the cuspidal Maass wave form ϕ.

Lemma 8.1.

(i) *If k, $0\le k\le d$, is odd, then $E^{(2)}(g_{\tau,t};\varphi_k)$ and $E^{(2)}(g_{\tau,t};\varphi_{-k})$ are both equal to 0. In particular, if k is odd, then $\mathscr{M}(\phi,s;\varphi_k)=\mathscr{M}(\phi,s;\varphi_{-k})=0$ for any cuspidal ϕ.*

(ii) *If k is even, then $E^{(2)}(g_{\tau,t};\varphi_k)=E^{(2)}(w_2g_{\tau,t^{-1}};\varphi_k)$.*

Proof. The first assertion is verified by using Lemma 3.1, Proposition 3.3, Lemma 5.1 and Theorem 5.4, where we remark that

$$\prod_{v\in\boldsymbol{f}}W_{B,v}(e_4,s_d)=(2\kappa_d)^{-1}D_B^{(1/2)-d}\frac{L(1-d,\chi_B)F_B(2-d)}{\zeta(1+d)\zeta(1-2d)}$$

depends only on D_B and e_B (see also (9.2)).

In order to prove (ii) we take Iwasawa decompositions

$$w_Mm(a_{\tau,t})=m(a_{-\tau^{-1},t})m(r_1)\quad\text{and}\quad w_2m(a_{\tau,t^{-1}})w_2^{-1}=m(a_{-\tau^{-1},t})m(r_2)$$

in M_∞, where r_1 and $r_2\in SO(2)$ are explicitly given by

$$r_1 = |\tau|^{-1} \begin{pmatrix} -u & v \\ -v & -u \end{pmatrix}, \quad r_2 = |\tau|^{-1} \begin{pmatrix} v & u \\ -u & v \end{pmatrix}.$$

Then we find that

$$E^{(2)}(g_{\tau,t}; \varphi_k) = E^{(2)}(w_M g_{\tau,t}; \varphi_k) = \overline{\tau}^k \tau^{-k} E^{(2)}(g_{-\tau^{-1},t}; \varphi_k)$$

and

$$E^{(2)}(w_2 g_{\tau,t^{-1}}; \varphi_k) = E^{(2)}\big(w_2 g_{\tau,t^{-1}} w_2^{-1}; \varphi_k\big) = (-1)^k \overline{\tau}^k \tau^{-k} E^{(2)}(g_{-\tau^{-1},t}; \varphi_k).$$

Comparing these two, we obtain the assertion (ii). □

In order to prove the analytic continuation of $\mathscr{M}(\phi, s; \varphi)$ as in Theorem 8.3, we will essentially use the method of Arakawa, Makino, and Sato, [**AMS**]. Let us begin with the following lemma.

Lemma 8.2. *For every cuspidal Maass form ϕ and $\varphi \in \mathscr{H}_d$, the twisted Mellin transform $\mathscr{M}(\phi, s; \varphi)$ is equal to*

$$\int_{\Gamma_1 \backslash \mathfrak{H}_1} \int_{K_M^1} \left(\int_1^\infty (t^s + t^{-s}) E^{(2)}(g_{\tau,t} r; \varphi) d^\times t + I(\tau, r, s; \varphi) \right) \phi(g_\tau r) dr d\mu(\tau),$$

where $I(\tau, r, s; \varphi)$ is defined by

$$\begin{aligned} \int_1^\infty t^{-s} \big(& E^{(1)}(g_{\tau,t} r; \varphi) + E_M(g_{\tau,t} r)(\varphi) + \Lambda(g_{\tau,t} r)(\varphi) \\ & - E^{(1)}(g_{\tau,t^{-1}} r; \varphi) - E_M(g_{\tau,t^{-1}} r)(\varphi) - \Lambda(g_{\tau,t^{-1}} r)(\varphi) \big) d^\times t. \end{aligned}$$

Proof. By Lemma 8.1 it suffices to prove the formula for every $\varphi = \varphi_k$ with k even. We compute the inner integral in the definition (8.1) as

$$\begin{aligned} \int_0^\infty t^s E^{(2)}(g_{\tau,t}; \varphi_k) d^\times t &= \int_1^\infty t^{-s} E^{(2)}(g_{\tau,t^{-1}}; \varphi_k) + t^s E^{(2)}(g_{\tau,t}; \varphi_k) d^\times t \\ &= \int_1^\infty (t^s + t^{-s}) E^{(2)}(g_{\tau,t}; \varphi_k) d^\times t \\ &\quad + \int_1^\infty t^{-s} \big(E^{(2)}(g_{\tau,t^{-1}}; \varphi_k) - E^{(2)}(g_{\tau,t}; \varphi_k) \big) d^\times t. \end{aligned}$$

On the other hand, the modularity of the Eisenstein series implies that

$$E(g_{\tau,t}, s_d)(\varphi_k) = E\left(\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} g_{\tau,t}, s_d\right)(\varphi_k) = E(g_{\tau,t^{-1}}, s_d)(\varphi_k) \qquad (k: \text{ even}).$$

These combined together conclude the formula in the lemma. □

THEOREM 8.3. *Let ϕ be any cuspidal Maass wave form. Then the twisted Mellin transform $\mathscr{M}(\phi, s; \varphi)$ has an entire continuation in s for all $\varphi \in \mathscr{H}_d$. Moreover it satisfies the functional equation*

$$\mathscr{M}(\phi, -s; \varphi) = \mathscr{M}(\phi, s; \varphi).$$

PROOF. By Lemma 8.2, it is sufficient to show that the integral

$$\int_{\Gamma_1\backslash\mathfrak{H}_1} \int_{K_M^1} I(\tau, r, s; \varphi)\phi(g_\tau r) dr d\mu(\tau)$$

has the analytic continuation in s. In fact, we prove that it vanishes for all $s \in \boldsymbol{C}$.

To show the vanishing, it suffices to check that

$$\int_{\Gamma_1\backslash\mathfrak{H}_1} \int_1^\infty t^{-s}\big[\Lambda(g_{\tau,t})(\varphi_0) - \Lambda(g_{\tau,t^{-1}})(\varphi_0)\big]\phi(g_\tau) d^\times t d\mu(\tau) = 0 \quad \text{and}$$

$$\int_{\Gamma_1\backslash\mathfrak{H}_1} \int_1^\infty t^{-s}\big[E^{(1)}(g_{\tau,t}; \varphi_k) + E_M(g_{\tau,t})(\varphi_k) - E^{(1)}(g_{\tau,t^{-1}}; \varphi_k) - E_M(g_{\tau,t^{-1}})(\varphi_k)\big]\phi(g_\tau) d^\times t d\mu(\tau) = 0.$$

If $\mathrm{Re}(s) > (d+1)/2$, the former integral is equal to

$$\left(\frac{1}{s - \dfrac{d+1}{2}} - \frac{1}{s + \dfrac{d+1}{2}}\right) \int_{\Gamma_1\backslash\mathfrak{H}_1} \phi(g_\tau) d\mu(\tau),$$

which is obviously meromorphic in s on the whole $\boldsymbol{C}$. It is, indeed, equal to 0, since ϕ is assumed to be cuspidal.

On the other hand, by (7.2) and (7.6), we unfold the second integral, provided $\mathrm{Re}(s) > d/2$, to obtain

$$\int_{F_\infty}\biggl(\int_1^\infty t^{-s}\bigl[G(g_{\tau,t})(\varphi_k)+\varLambda_M(g_{\tau,t})(\varphi) \\ -G(g_{\tau,t^{-1}})(\varphi_k)-\varLambda_M(g_{\tau,t^{-1}})(\varphi_k)\bigr]d^\times t\biggr)v^{-1}\phi(\tau)d\mu_\infty,$$

where we put $F_\infty=\{\tau\in\mathfrak{H}_1 \mid -1/2<u\le 1/2,\ 0<v<\infty\}$ with the measure $d\mu_\infty=dud^\times v$. Since $\varLambda(w_1g_{iv,t})(\varphi)=\varLambda(w_1g_{iv,t^{-1}})(\varphi)$, this can also be written as

$$\biggl(\int_1^\infty t^{-s+(d/2)}d^\times t-\int_1^\infty t^{-s-(d/2)}d^\times t\biggr) \\ \times\int_{F_\infty}v^{(d-1)/2}\bigl[G(m'(v^{-1/2}))(\varphi_k)+\varLambda_M(m'(v^{-1/2}))(\varphi_k) \\ -\varLambda(w_1m'(v^{-1/2}))(\varphi_k)\bigr]\phi(\tau)d\mu_\infty \tag{8.2}$$

using the notation $m'(c)=m'\bigl(\bigl(\begin{smallmatrix}c&0\\0&c^{-1}\end{smallmatrix}\bigr)\bigr)$ with $c>0$. Now we check the following.

Lemma 8.4. *For every cuspidal ϕ, the integral*

$$\int_{F_\infty}v^{(d-1)/2}\bigl[G(m'(v^{-1/2}))(\varphi)+\varLambda_M(m'(v^{-1/2}))(\varphi) \\ -\varLambda(w_1m'(v^{-1/2}))(\varphi)\bigr]\phi(\tau)d\mu_\infty \tag{8.3}$$

absolutely converges.

Proof. Recall (7.7) and the last paragraph of Section 7. Then we know that $E_{M'}(m')(\varphi)=G(m')(\varphi)+\varLambda_M(m')(\varphi)+\varLambda(m')(\varphi)$ gives an automorphic form on $M'_{\boldsymbol{A}}$. In particular, it is equal to $E_{M'}(w_1m')(\varphi)$. Let us put $F_1=\{\tau\in F_\infty \mid 0<v\le 1\}$. Then we split the integral (8.3) into the sum of

$$\int_{F_1}v^{(d-1)/2}\bigl(G(m'(v^{-1/2}))(\varphi)+\varLambda_M(m'(v^{-1/2}))(\varphi)-\varLambda(m'(v^{1/2})w_1)(\varphi)\bigr)\phi(\tau)d\mu_\infty$$

and

$$\int_{F_\infty-F_1}v^{(d-1)/2}\bigl(G(m'(v^{1/2})w_1)(\varphi)+\varLambda_M(m'(v^{1/2})w_1)(\varphi) \\ -\varLambda(m'(v^{-1/2}))(\varphi)\bigr)\phi(\tau)d\mu_\infty.$$

Now we have the following estimates for the functions in the integrands; namely,

$$\left|G(m'(v^{-1/2}))(\varphi)\right| \leq M_1 v^{(d-1)/2}, \quad \left|G(m'(v^{1/2})w_1)(\varphi)\right| \leq M_2 v^{-(d-1)/2},$$
$$\Lambda_M(m'(v^{-1/2}))(\varphi) = v^{(d-1)/2}\Lambda_M(e_2)(\varphi),$$
$$\Lambda_M(m'(v^{1/2})w_1)(\varphi) = v^{-(d-1)/2}\Lambda_M(w_1)(\varphi),$$
$$\Lambda(m'(v^{1/2})w_1)(\varphi) = v^{(d+1)/2}\Lambda(w_1)(\varphi), \quad \Lambda(m'(v^{-1/2}))(\varphi) = v^{-(d+1)/2}\Lambda(e_2)(\varphi)$$

with some constants M_1, M_2. These bounds and the cuspidality of ϕ concludes the assertion. $\square$

The absolute convergence in the above lemma justifies to write (8.3) as

$$\int_0^\infty v^{(d-1)/2}\big(G(m'(v^{-1/2}))(\varphi_k) + \Lambda_M(m'(v^{-1/2}))(\varphi) \\ - \Lambda(m'(v^{1/2})w_1)(\varphi)\big)\phi_0(v)d^\times v,$$

where $\phi_0(v) = \int_{-1/2}^{1/2}\phi(\tau)du$ is the constant term of ϕ. We have that $\phi_0(v) = 0$ by the cuspidal assumption. Therefore, (8.2) vanishes everywhere in $\boldsymbol{C}$, which completes the proof of Theorem 8.3. $\square$

9. Dirichlet series.

Let us recall Cohen's function $H(d,N)$, [**C**], for an even integer d and every integer $N \geq 0$. If $N \equiv 0$ or 1 mod 4, then we write $N = Df^2 > 0$ with D being the discriminant of the real quadratic algebra $\boldsymbol{Q}(\sqrt{N})$, and define

$$H(d,N) = L(1-d,\chi_D)\sum_{c|f}\mu(c)\chi_D(c)c^{d-1}\sigma_{d-1}\left(\frac{f}{c}\right), \tag{9.1}$$

where $\mu(\cdot)$ is Möbius function and $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ is Kronecker symbol. We define $H(d,N) = 0$ if $N \equiv 2$ or 3 mod 4, and $H(d,0) = \zeta(1-2d)$. Then Cohen proved that the Fourier series

$$H_{d+(1/2)}(\tau) = \sum_{N=0}^{\infty} H(d,N)e^{2\pi i N\tau} \quad (\tau \in \mathfrak{H}_1)$$

provides a holomorphic Eisenstein series of weight $d + (1/2)$ for the group $\Gamma_0(4)$.

Now we take a normalization of $E(g, s_d)(\varphi)$ as

$$\widetilde{E}(g)(\varphi) = \kappa_d \cdot \zeta(1+d)H(d,0)E(g,s_d)(\varphi).$$

The regular part of its Fourier expansion is given by

$$\widetilde{E}^{(2)}(g_{\tau,t})(\varphi) = \frac{1}{2}\sum_{B \in L_0} A(B) D_B^{(1/2)-d} W_{B,\infty}^{(2)}(m(a_{\tau,t}), s_d)(\varphi) \tag{9.2}$$

with the coefficients

$$A(B) = L(1-d, \chi_B) F_B(2-d),$$

where we set $L_0 = \{B \in L \mid D_B > 0\}$. Also the equality

$$A(B) = \sum_{l \mid e_B} l^d H\left(d, \frac{D_B}{l^2}\right) \tag{9.3}$$

is well known.

We will relate the Mellin transform of (9.2):

$$\widetilde{\mathscr{M}}(\phi, s; \varphi) = i^d \Gamma(d+1)\zeta(d+1)\zeta(1-2d)\mathscr{M}(\phi, s; \varphi),$$

twisted by a cuspidal Maass form ϕ, to some familiar Dirichlet series of the convolution type.

For each $B \in L_0$, $\tau = u + iv \in \mathfrak{H}_1$, $r \in SO(2)$, and $\varphi \in \mathscr{H}_d$ we define an archimedean integral by

$$\widehat{W}_B(s, a_\tau r; \varphi) = \frac{1}{2}\int_0^\infty t^s W_{B,\infty}^{(2)}(m(a_{\tau,t} r), s_d)(\varphi) d^\times t, \quad s \in \boldsymbol{C}.$$

Let $\Gamma_B \subset \Gamma_1$ be the stabilizer of $B \in L_0$ with respect to the Γ_1-action on L_0. It is easy to see that Γ_B is isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}$, if B is anisotropic over $\boldsymbol{Q}$; or to $\boldsymbol{Z}/2\boldsymbol{Z}$, if B is isotropic. Also let $\overline{L}_0$ denote the complete set of representatives of all Γ_1-classes in L_0.

Given a cuspidal Maass form ϕ we define

$$P_B(\phi, s; \varphi) = \int_{\Gamma_B \backslash \mathfrak{H}_1} \left(\int_{SO(2)} \widehat{W}_B(s, a_\tau r; \varphi)\phi(a_\tau r) dr \right) v^{-2} du dv.$$

Using this, we can write the Mellin transform as

$$\widetilde{\mathscr{M}}(\phi, s; \varphi) = \sum_{[B]\in\overline{L}_0} A(B) D_B^{(1/2)-d} P_B(\phi, s; \varphi).$$

We compute $P_B(\phi, s; \varphi)$ for every $B \in L_0$ as follows. Firstly, we may assume without a loss of generality that the $SO(2)$-types of φ and ϕ are matching together, hence,

$$P_B(\phi, s; \varphi) = \int_{\Gamma_B\backslash\mathfrak{H}_1} \widehat{W}_B(s, a_\tau; \varphi)\phi(a_\tau)v^{-2}dudv.$$

Choose a matrix $h_B \in SL_2(\boldsymbol{R})$ so that one has $B[h_B] = D_B^{1/2} B_0$ with $B_0 = \left(\begin{smallmatrix} & 1/2 \\ 1/2 & \end{smallmatrix}\right)$. Then we find that

$$h_B^{-1}\Gamma_B h_B = \left\{ \pm \begin{pmatrix} c_B & \\ & c_B^{-1} \end{pmatrix}^n \middle| n \in \boldsymbol{Z} \right\}$$

with some number $c_B \geq 1$. Let us write $h_B^{-1}a_\tau = a_{h_B^{-1}\tau} r_1$, where $r_1 \in SO(2)$. Using these notations, we can compute $P_B(\phi, s; \varphi)$ as

$$\begin{aligned} &D_B^{-s+(d/2)-1} \int_{h_B^{-1}\Gamma_B h_B\backslash\mathfrak{H}_1} \widehat{W}_{B_0}(s, a_\tau; \varphi)\phi(h_B a_\tau)v^{-2}dudv \\ &= D_B^{-s+(d/2)-1} \int_{h_B^{-1}\Gamma_B h_B\backslash\mathfrak{H}_1} \widehat{W}_{B_0}\left(s, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}; \varphi\right) \\ &\qquad\qquad \times \phi\left(h_B \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) v^{-1}dudv. \end{aligned}$$

Also using the c_B above, this is further equal to

$$\begin{aligned} &D_B^{-s+(d/2)-1} \int_{-\infty}^{\infty} \widehat{W}_{B_0}\left(s, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}; \varphi_k\right) \\ &\qquad \times \left(\int_1^{c_B^2} \phi\left(h_B \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) d^\times v \right) du. \end{aligned} \tag{9.4}$$

To state the results we need some more notations. If $B \in L_0$ is anisotropic over $\boldsymbol{Q}$, then we define a number $p_B(\phi)$ by

$$p_B(\phi) = \int_1^{c_B^2} \phi\left(h_B \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix} \right) d^\times v \tag{9.5}$$

for a cuspidal Maass form. This definition does not depend on the choice of h_B. On the other hand, if B is isotropic, that is, if B is equivalent to $\left(\begin{smallmatrix} 0 & r/2 \\ r/2 & n \end{smallmatrix}\right)$ by the Γ_1-action, then we put

$$p_B(\phi) = \int_0^\infty \phi\left(\begin{pmatrix} 1 & \dfrac{n}{r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & \\ & \sqrt{v}^{-1} \end{pmatrix} \right) d^\times v. \tag{9.6}$$

These values $p_B(\phi)$ in the both cases provide the integrals of ϕ over the geodesics in $\mathfrak{H}_1$ attached to $B \in L_0$.

Proposition 9.1. *Let k, $0 \le k \le d$, be an even integer. Let $-4\nu(\nu+1)$ denote the Casimir eigenvalue of a cuspidal Maass form ϕ of weight $-2k$. Then there exists a function $\Gamma(s;k,\nu)$, which depends only on s, k, ν, so that the identity*

$$P_B(\phi, s; \varphi_k) = \Gamma(s;k,\nu) p_B(\phi) D_B^{-s-1+(d/2)}$$

holds.

Proof. The proof is given by following the arguments of Maass, [**Ma**], and Katok-Sarnak, [**KS**]. We treat the case B being anisotropic. Within the expression (9.4) of $P_B(\phi, s; \varphi_k)$ we observe that

$$\widehat{W}_{B_0}\left(s, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}; \varphi_k \right) = (u^2+1)^{-k/2}(-u+i)^k \widehat{W}_{D(u)}(s, e_2; \varphi_k)$$

by putting

$$D(u) = \frac{1}{2} \begin{pmatrix} u + \sqrt{u^2+1} & 0 \\ 0 & u - \sqrt{u^2+1} \end{pmatrix}.$$

Therefore, $P_B(\phi, s; \varphi_k)$ is equal to

$$D_B^{-s+(d/2)-1} \int_{-\infty}^\infty (u^2+1)^{-k/2} \widehat{W}_{D(u)}(s, e_2; \varphi_k)(u-i)^k \phi_B\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) du, \tag{9.7}$$

where $\phi_B(g)$ is a function on $SL_2(\boldsymbol{R})$ defined by

$$\phi_B(g) = \int_1^{c_B^2} \phi\left(h_B \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix}\right) d^\times v.$$

We also note that

$$\widehat{W}_{D(-u)}(s, e_2; \varphi_k) = \widehat{W}_{D(u)}(s, e_2; \varphi_k),$$

since k is even.

The function $\phi_B(g)$ is left-invariant by the action of the subgroup $\left\{\left(\begin{smallmatrix} c & \\ & c^{-1} \end{smallmatrix}\right)\right\}$ in $SL_2(\boldsymbol{R})$, and also has the specified right $SO(2)$-action. Therefore ϕ_B is determined by the restrictions $\phi_B\left(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\right) = \phi_B(u)$ for $u \in \boldsymbol{R}$.

Let $C = H^2 + 2EF + 2FE$ with $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, be the Casimir element in the enveloping algebra of $\mathfrak{sl}_2$. Since it is assumed to act on ϕ by the scalar $-4\nu(\nu+1)$, we obtain an ordinary differential equation of second order satisfied by ϕ_B. Indeed, the differential equation satisfied by $j_{B,\phi}(u) = (u-i)^k \phi_B(u)$ is explicitly given by

$$(1+u^2) j''_{B,\phi} - 2(k-1)u j'_{B,\phi} + \big(k(k-1) - \nu(\nu+1)\big) j_{B,\phi} = 0.$$

We write its solutions in the form

$$j_{B,\phi}(u) = j_{B,\phi}(0)\psi_0(u) + j'_{B,\phi}(0)\psi_1(u)$$

where ψ_0, ψ_1 are the solutions satisfying $\psi_0(-u) = \psi_0(u)$, $\psi_0(0) = 1$ and $\psi'_0(0) = 0$; or $\psi_1(-u) = -\psi_1(u)$ and $\psi'_1(0) = 1$. In fact, we find

$$\psi_0(u) = {}_2F_1\left(-\frac{\nu+k}{2}, \frac{\nu-k+1}{2}; \frac{1}{2}; -u^2\right).$$

We see that only the even function $j_{B,\phi}(0)\psi_0(u)$ contributes to the integral (9.7), and also $j_{B,\phi}(0) = (-1)^{k/2} p_B(\phi)$. Therefore we get the expression of $P_B(\phi, s; \varphi_k)$ by putting

$$\Gamma(s; k, \nu) = (-1)^{k/2} \int_{-\infty}^{\infty} (u^2+1)^{-k/2} \psi_0(u) \widehat{W}_{D(u)}(s, e_2; \varphi_k) du. \tag{9.8}$$

This obviously depends only on s, k, ν, but not on B, ϕ. Hence the proof is completed for every anisotropic B. A similar argument works also for isotropic B, for which we omit the details here. $\square$

Define $\overline{L}_0^N = \{[B] \in \overline{L}_0 \mid D_B = N\}$, which is a finite set, and let us put

$$b(N) = N^{-3/4} \sum_{[B] \in \overline{L}_0^N} p_B(\phi). \tag{9.9}$$

Theorem 9.2. *With the same assumption as in Proposition* 9.1 *one has*

$$\widetilde{\mathscr{M}}(\phi, s; \varphi_k) = \Gamma(s; k, \nu)\zeta(2s+1)D_k(s, \phi),$$

where $D_k(s, \phi)$ is a Dirichlet series given by

$$D_k(s, \phi) = \sum_{N=1}^{\infty} \frac{H(d, N)b(N)}{N^{s+(d/2)-(1/4)}}.$$

Proof. Proposition 9.1 implies that $\widetilde{\mathscr{M}}(\phi, s; \varphi_k)$ is equal to the sum

$$\Gamma(s; k, \nu) \sum_{[B] \in \overline{L}_0} A(B) p_B(\phi) D_B^{-s-(d+1)/2}.$$

Note that one has $p_{eB}(\phi) = p_B(\phi)$ by taking $h_{eB} = h_B$. Let $\overline{L}_0^* = \{[C] \in \overline{L}_0 \mid e_C = 1\}$ denote the subset of all primitive classes in $\overline{L}_0$. Then we can rewrite the above summation over $\overline{L}_0$ as

$$\begin{aligned}
&\sum_{[C] \in \overline{L}_0^*} p_C(\phi) \sum_{e=1}^{\infty} (e^2 D_C)^{-s-(d+1)/2} \sum_{l \mid e} l^d H\left(d, \frac{e^2 D_C}{l^2}\right) \\
&= \sum_{[C] \in \overline{L}_0^*} p_C(\phi) D_C^{-s-(d+1)/2} \sum_{m,l=1}^{\infty} \frac{H(d, D_{mC})}{m^{2s+d+1} l^{2s+1}} \\
&= \zeta(2s+1) \sum_{[B] \in \overline{L}_0} H(d, D_B) p_B(\phi) D_B^{-s-(d+1)/2}.
\end{aligned}$$

This provides the desired formula. □

The Dirichlet series $D_k(s, \phi)$ can be understood as the convolution product attached to $H_{k+1/2}(\tau)$ and ϕ. To explain this, we recall some facts on the theta correspondence studied by Shintani [**Sh**], Niwa [**N**], and [**KS**].

Let us take

$$Q = \begin{pmatrix} & & 4 \\ & -2 & \\ 4 & & \end{pmatrix},$$

which has 1 positive and 2 negative eigenvalues, and define

$$(x, y) = {}^t y Q x = 2(2x_1 y_3 + 2x_3 y_1 - x_2 y_2) \quad \text{for} \quad x, y \in \boldsymbol{R}^3.$$

For every even integer k, $0 \leq k \leq d$, we define a function $f_k(x_1, x_2, x_3)$ on $\boldsymbol{R}^3$ by

$$f_k(x_1, x_2, x_3) = (ix_1 - x_2 - ix_3)^k e^{-2\pi(2x_1^2 + x_2^2 + 2x_3^2)}.$$

For any $g \in SL_2(\boldsymbol{R})/\{\pm e_2\} \simeq SO(Q)$ and $\tau = u + iv \in \mathfrak{H}_1$ we define

$$\theta(\tau, g) = v^{3/4} \sum_{x = (x_1, x_2, x_3) \in \boldsymbol{Z}^3} e^{-2\pi i u (x_2^2 - 4x_1 x_3)} f_k\left(\sqrt{v} g^{-1} x\right).$$

It satisfies $\theta(\tau, g r_\theta) = e^{2ki\theta}\theta(\tau, g)$ for all the rotation matrices $r_\theta \in SO(2)$.

Take a cuspidal Maass form ϕ of weight $-2k$ for the group Γ_1. We define the lifting of ϕ by

$$\theta_\phi(\tau) = (4\pi)^{(k/2)+(1/4)} \int_{\Gamma_1 \backslash SL_2(\boldsymbol{R})} \theta(\tau, g) \phi(g) dg.$$

This lifting satisfies

$$\theta_\phi(\gamma\tau) = \overline{j}(\gamma, \tau)^{2k+1} \theta_\phi(\tau) \quad \text{for every} \quad \gamma \in \Gamma_0(4)$$

and provides a cuspidal Maass form for the group $\Gamma_0(4)$. Here we put

$$\overline{j}(\gamma, \tau) = \varepsilon_d \left(\frac{c}{d}\right) (c\overline{\tau} + d)^{1/2} |c\tau + d|^{-1/2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

by setting $\varepsilon_d = 1$ or i according to $d \equiv 1$ or $3 \bmod 4$, and using the quadratic residue symbol $\left(\frac{\cdot}{\cdot}\right)$ defined in [**S3**].

As for the Fourier expansion

$$\theta_\phi(\tau) = \sum_{n \neq 0} c(n, v) e^{2\pi i n u},$$

we get that $c(n,v)=0$ for every $n\equiv 1$ or $2 \bmod 4$. One can compute the coefficients $c(n,v)$ for any negative $n=-N<0$ using the methods of [**Ma**] and [**KS**]. Let $-4\nu(\nu+1)$ be the Casimir eigenvalue of ϕ as before. Then we find that

$$c(-N,v)=\begin{cases}2^{-1}b(N)W_{(k/2)+(1/4),-(\nu/2)-(1/4)}(4\pi Nv), & \text{if } N\equiv 0 \text{ or } 1 \bmod 4,\\ 0, & \text{otherwise,}\end{cases}$$

where $W_{\kappa,\nu}(z)$ denotes the Whittaker function, see [**MOS**, Chapter VII].

We introduce an Eisenstein series

$$E_\infty^{-l}(\tau,s)=\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}\left(\frac{c\tau+d}{|c\tau+d|}\right)^l \operatorname{Im}(\gamma\tau)^s,$$

for the group $\Gamma_0(4)$ of even weight $-l$, where $\Gamma_\infty=\left\{\gamma=\pm\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\mid b\in\boldsymbol{Z}\right\}$. Also one sets

$$\widetilde{E}_\infty^{-l}(\tau,s)=\Gamma\left(s+\frac{l}{2}\right)\pi^{-s}\zeta(2s)E_\infty^{-l}(\tau,s).$$

Define a normalization of the Dirichlet series $D_k(s,\phi)$ by

$$\Lambda_k(s,\phi)=\pi^{-2s}\Gamma\left(s+\frac{d+\nu+1}{2}\right)\Gamma\left(s+\frac{d-\nu}{2}\right)\zeta(2s+1)D_k(s,\phi)$$

for every even k, $0\le k\le d$. Then we have the following result.

PROPOSITION 9.3. *The normalized Dirichlet series $\Lambda_k(s,\phi)$ has the integral expression*

$$\Lambda_k(s,\phi)=2^{2s+1}\int_{\Gamma_0(4)\backslash\mathfrak{H}_1}(4\pi v)^{(d/2)+(1/4)}H_{d+(1/2)}(\tau)\theta_\phi(\tau)\widetilde{E}_\infty^{k-d}\left(\tau,s+\frac{1}{2}\right)d\mu(\tau).$$

In particular, it is continued to an entire function in s, and satisfies the functional equation

$$\Lambda_k(-s,\phi)=\Lambda_k(s,\phi).$$

PROOF. We can prove the integral expression by the standard unfolding argument combined with the formulas of Fourier coefficients of $\theta_\phi(\tau)$ and

$H_{d+(1/2)}(\tau)$. Also the idea of the proof of [**DI**, Theorem 6] (and [**KZ**, Corollary 5]), can be applied successfully to our situation, which gives the holomorphic continuation and the functional equation of the integral expression. We omit the details here. □

COROLLARY 9.4. *For each k, $0 \leq k \leq d$, the quotient*

$$\pi^{2s}\Gamma\left(s+\frac{d+\nu+1}{2}\right)^{-1}\Gamma\left(s+\frac{d-\nu}{2}\right)^{-1}\Gamma(s;k,\nu)$$

gives an entire function of $s \in \boldsymbol{C}$, which is invariant under the exchange of variables between s and $-s$.

PROOF. This is a direct consequence of Theorems 8.3, 9.2 and Proposition 9.3. □

Using Proposition 5.5, one can compute explicitly the above quotient for $k = 0$, which yields the constant function $2^d/(\pi(2d)!)$.

10. Concluding remarks.

We keep the previous notations. Take a holomorphic elliptic cusp form $f = f(\tau)$ for the group $SL_2(\boldsymbol{Z})$ of weight $2d$, and let $h(\tau)$ be the holomorphic cusp form for $\Gamma_0(4)$ of weight $d+(1/2)$ attached to f by Shimura correspondence. Let $c(N)$ denote the N-th Fourier coefficient of $h(\tau)$ for every $N \geq 1$. Let us put

$$F_{0_2}(g_{\tau,t})(\varphi_k) = \begin{cases} (4i)^d\kappa_d^2 \cdot t^{1/2}v^d \displaystyle\sum_{N\geq 1}\left(\sum_{l|N} l^{d-1}c\left(\frac{N^2}{l^2}\right)\right)q^N & (k=d), \\ 0 & (0 \leq k < d) \end{cases}$$

and $F_{0_2}(g_{\tau,t})(\varphi_{-k})$ be its complex conjugation. Also, for every $B \in L_0$ and $\varphi \in \mathscr{H}_d$, we set

$$F_B(g_{\tau,t})(\varphi) = \frac{1}{2}\left(\sum_{l|e_B} l^d c\left(\frac{D_B}{l^2}\right)\right)D_B^{(1/2)-d}W_{B,\infty}^{(2)}(m(a_{\tau,t}),s_d)(\varphi).$$

Using these functions, we define a function $F(g)(\varphi)$ on KG_∞ for all $\varphi \in \mathscr{H}_d$ by the Fourier series

$$F(n(u)g)(\varphi) = F_{0_2}(g)(\varphi) + \sum_{B\in L_0} e^{2\pi i\sigma(Bu)}F_B(g)(\varphi), \tag{10.1}$$

where $u \in S_\infty$ and $g \in KG_\infty$. We understand $F(gr)(\varphi)$ translates according to

$$F(gr)(\varphi) = \delta(\overline{\mu_r(\boldsymbol{i})})^{-d} F(g)\big(\tau_{2d}(\overline{\mu_r(\boldsymbol{i})})\varphi\big)$$

for every $r \in K_\infty$.

We define a twisted Mellin transform $\mathscr{M}_F(\phi, s; \varphi)$ of $F(g)(\varphi)$ by the integral

$$\mathscr{M}_F(\phi, s; \varphi) = \int_{\Gamma_1\backslash\mathfrak{H}_1} \int_{K_M^1} \left(\int_0^\infty t^s F^{(2)}(g_{\tau,t} r; \varphi) d^\times t\right) \phi(g_\tau r) dr d\mu(\tau), \qquad (10.2)$$

which is similar to (8.1) but $E^{(2)}(g;\varphi)$ is replaced by $F^{(2)}(g;\varphi) = \sum_{B\in L_0} F_B(g)(\varphi)$. For this integral transform we claim the following.

Proposition 10.1. *Suppose that ϕ is a cuspidal Maass form of weight $-2k$, $0 \le k \le d$, and has Casimir eigenvalue $-4\nu(\nu+1)$. Then $\mathscr{M}_F(\phi, s; \varphi_k)$ has a meromorphic continuation in s to the whole $\boldsymbol{C}$ for each base polynomial $\varphi_k \in \mathscr{H}_d$, which, furthermore, is entire if $k \neq d$. Also the functional equation*

$$\mathscr{M}_F(\phi, -s; \varphi) = \mathscr{M}_F(\phi, s; \varphi)$$

is satisfied for every $\varphi \in \mathscr{H}_d$.

Proof. Our discussion in the proofs of Proposition 9.1 and Theorem 9.2 continues to work in the current situation. Then, we can write $\mathscr{M}_F(\phi, s; \varphi_k)$ for $\varphi = \varphi_k$ as

$$\Gamma(s; k, \nu)\zeta(2s+1) \sum_{N=1}^\infty \frac{c(N)b(N)}{N^{s+(d/2)-(1/4)}},$$

up to a constant multiple, with the same notations as before. Also the completed Dirichlet series

$$\pi^{-2s}\Gamma\left(s + \frac{d+\nu+1}{2}\right)\Gamma\left(s + \frac{d-\nu}{2}\right)\zeta(2s+1) \sum_{N=1}^\infty \frac{c(N)b(N)}{N^{s+(d/2)-(1/4)}}$$

can be expressed by the integral

$$2^{2s+1} \int_{\Gamma_0(4)\backslash\mathfrak{H}_1} (4\pi v)^{(d/2)+(1/4)} h(\tau)\theta_\phi(\tau) \widetilde{E}_\infty^{k-d}\left(\tau, s+\frac{1}{2}\right) d\mu.$$

This has the meromorphic continuation to $\boldsymbol{C}$, and satisfies the functional equation for changing s with $-s$. It is entire if $\theta_\phi(\tau)$ is linearly independent of $h(\tau)$. Then, Corollary 9.4 completes the assertion. $\square$

If the Fourier series (10.1) was known to give an automorphic form, then Proposition 10.1 could be justified by repeating the arguments in the proof of Lemma 8.2 and Theorem 8.3. Lastly, we remark that the θ-lift $h(\tau)$ to the orthogonal group $SO(3,2) \simeq PGSp(2)$ has the Fourier expansion whose appearance is very close to the one in (10.1), see [**Mi**].

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Takuya Miyazaki
Department of Mathematics
Keio University
3-14-1 Hiyoshi, Kohoku
Yokohama 223-8522, Japan
E-mail: miyazaki@math.keio.ac.jp