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A remark on the unique factorization theorem.

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It is well known that the ring $K[x_1, x_2, \cdots, x_n]/(\sum_{i=1}^{n} x_i^2)$ is a unique factorization ring if K is a field of characteristic different from 2 and if $n \geqq 5$*). But it seems to the writer that the known proofs are not so simple. Theorems 1 and 2 in the present note cover the fact and our proof is simpler than the known proofs.

LEMMA 1. *Let x be a non-zero element of a Noetherian integral domain $\mathfrak{o}$. If $x\mathfrak{o}$ is a prime ideal and if $\mathfrak{o}[1/x]$ is a unique factorization ring, then $\mathfrak{o}$ is also a unique factorization ring.*

PROOF. We have only to show that every prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{o}$ is principal. If $x \in \mathfrak{p}$, then $\mathfrak{p} = x\mathfrak{o}$ and we assume that $x \notin \mathfrak{p}$. Let f be an element of $\mathfrak{p}$ such that $f\mathfrak{o}[1/x] = \mathfrak{p}\mathfrak{o}[1/x]$. Since $x \notin \mathfrak{p}$, we may assume that $f \notin x\mathfrak{o}$. Let p be an element of $\mathfrak{p}$. Let r be the smallest integer such that $x^r p \in f\mathfrak{o}$. If r is positive, then the element $y \in \mathfrak{o}$ such that $x^r p = fy$ must be in $x\mathfrak{o}$ (because $x\mathfrak{o}$ is a prime ideal) and $x^{r-1}p \in f\mathfrak{o}$, which is a contradiction. Thus we have $p \in f\mathfrak{o}$ and $\mathfrak{p} = f\mathfrak{o}$, which proves the assertion.

THEOREM 1. *Let K be a Noetherian unique factorization ring and let $x_1, \cdots, x_n$ be indeterminates. If $g_0, g_1 \cdots, g_r$ are in $K[x_3, \cdots, x_n]$ and if g_0 is irreducible, then the ring $\mathfrak{o} = K[x_1, \cdots, x_n]/(x_1^s x_2 - \sum_{i=0}^{r} g_i x_1^i)$ is a unique factorization ring.*

PROOF. $\mathfrak{o}/x_1\mathfrak{o} = K[x_2, \cdots, x_n]/(g_0)$ and g_0 is irreducible, which shows that $x_1\mathfrak{o}$ is a prime ideal. $\mathfrak{o}[1/x_1] = K[x_1, 1/x_1, x_3, x_4, \cdots, x_n]$, which is a ring of quotients of the polynomial ring $K[x_1, x_3, \cdots, x_n]$ and is a unique factorization ring. Thus $\mathfrak{o}$ is a unique factorization ring by Lemma 1.

If a field K is not of characteristic 2 and if $\sqrt{-1} \in K$, then $K[x_1, \cdots, x_n]/(\sum x_i^2) \cong K[x_1, \cdots, x_n]/(x_1 x_2 - \sum_{i=3}^{n} x_i^2)$. Therefore in order to prove the unique factorization in the ring $K[x_1, \cdots, x_n]/(\sum x_i^2)$ $(n \geqq 5)$, it will be sufficient to prove the following

*) See, for example, van der Waerden, Einführung in die algebraische Geometrie, Berlin, 1939.

THEOREM 2. *Let K be a field and let $x_1, \cdots, x_n$ be indeterminates. Let $\mathfrak{a}$ be a homogeneous ideal of $K[x_1, \cdots, x_n]$. If there exists a field L containing K such that $L[x_1, \cdots, x_n]/(\mathfrak{a})$ is a unique factorization ring, then $K[x_1, \cdots, x_n]/\mathfrak{a}$ is also a unique factorization ring.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of rank 1 in $K[x_1, \cdots, x_n]/\mathfrak{a}$. Then $\mathfrak{p}L[x_1, \cdots, x_n]/(\mathfrak{a})$ has no imbedded prime divisor and is purely of rank 1, hence it is a principal ideal. Let $\sum_{i=0}^{m} p_i a_i$ be a generator of $\mathfrak{p}L[x_1, \cdots, x_n]/(\mathfrak{a})$, where $p_i \in \mathfrak{p}$ and $a_0, \cdots, a_m$ are linearly independent over K. Let f_i be the element such that $p_i = (\sum p_j a_j) f_i$. Since $\mathfrak{a}$ is homogeneous, $\deg p_i = \deg(\sum p_j a_j) + \deg f_i$. Since $a_0, \cdots, a_m$ are linearly independent over K, $\deg(\sum p_j a_j) \geqq \max(\deg p_j)$. Therefore $\deg f_i = 0$, i. e., $f_i \in L$. Therefore $p_i/p_j \in L$ for every pair (i, j). Hence $p_i/p_j \in L$ and p_0 generates $\mathfrak{p}L[x_1, \cdots, x_n]/(\mathfrak{a})$. It follows that $\mathfrak{p}$ is generated by p_0.

REMARK. *We have proved here that if $\mathfrak{p}L[x_1, \cdots, x_n]/(\mathfrak{a})$ is principal, then $\mathfrak{p}$ is principal, without assuming that $\mathfrak{p}$ is prime or that $L[x_1, \cdots, x_n]/(\mathfrak{a})$ is a unique factorization ring (but assumed that $L[x_1, \cdots, x_n]/(\mathfrak{a})$ is an integral domain).*

By the way we shall give a remark that Lemma 1 stated above can be generalized as follows (by a similar proof):

LEMMA 2. *Let S be a multiplicatively closed subset of a Noetherian integral domain $\mathfrak{o}$. If every element of S is the product of a finite number of prime elements (=generators of principal prime ideals) and if $\mathfrak{o}_S$ is a unique factorization ring, then $\mathfrak{o}$ is also a unique factorization ring.*

If we apply the above Lemma 2 then Theorem 2 can be generalized as follows:

Let K be a Noetherian integral domain and let $x_1, \cdots, x_n$ be indeterminates. Let $\mathfrak{a}$ be a homogeneous prime ideal in $K[x_1, \cdots, x_n]$ and let L be a field containing K. Set $\mathfrak{o} = K[x_1, \cdots, x_n]/\mathfrak{a}$ and $\mathfrak{o}' = L[x_1, \cdots, x_n]/(\mathfrak{a})$. If every prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{o}$ containing elements of K is principal and if $\mathfrak{o}'$ is a unique factorization ring, then $\mathfrak{o}$ is also a unique factorization ring.

We shall give another remark that the assumption that $\mathfrak{a}$ is homogeneous in Theorem 2 is important.

For example, let K be the field of real numbers and let C be the field of complex numbers. Set $\mathfrak{o} = K[x, y]/(y^2 + x^2 - x)$, $\mathfrak{o}' = C[x, y]/(y^2 + x^2 - x)$. Then

$\mathfrak{o}'$ is a unique factorization ring. But $\mathfrak{o}$ is not a unique factorization ring.

PROOF. Set $x'=x+\sqrt{-1}\,y$, $y'=x-\sqrt{-1}\,y$. Then $\mathfrak{o}'=C[x',y']/(2x'y'-x'-y')$. Let $\mathfrak{p}'$ be a maximal ideal of $\mathfrak{o}'$. Then there exists a $c\in C$ such that $x'-c\in\mathfrak{p}'$. $\mathfrak{o}'/(x'-c)=L[y']/((2c-1)y'-c)$, which shows that $\mathfrak{p}'$ is generated by $x'-c$. Thus $\mathfrak{o}'$ is a unique factorization ring. Next we show that $\mathfrak{o}$ is not a unique factorization ring. (This is obvious if we make use of geometric intuition; for, $x^2+y^2=x$ defines a circle going through the origin. If a curve goes through the origin and if it intersects with the circle transversally, then there must be another common point.) The ideal $\mathfrak{p}=x\mathfrak{o}+y\mathfrak{o}$ is a prime ideal of rank 1. We shall show that $\mathfrak{p}$ is not principal. Assume the contrary. Then $\mathfrak{p}=f\mathfrak{o}$ with an $f\in\mathfrak{o}$. Every element of $\mathfrak{o}$ is expressed as $f_1(x)+f_2(x)y$ and therefore we assume that $f=f_1'+f_2y$ ($f_1', f_2\in K[x]$). Since $f\in\mathfrak{p}$, $y\in\mathfrak{p}$ we see that $f_1'\in\mathfrak{p}$ and therefore $f_1'=f_1x$ with $f_1\in K[x]$. Let v be a valuation whose valuation ring is $\mathfrak{o}_{\mathfrak{p}}$. Then $v(y)$ may be assumed to be 1. Then $v(x)=2$. Then $v(f)=1$ and $f_2(0)\neq 0$. Since $x\in\mathfrak{p}$, there must be a relation such that

$$x=(f_1x+f_2y)(h+ky) \quad (h, k\in K[x]).$$

Then $x=f_1hx+kf_2x(1-x)$, $hf_2+kf_1x=0$ because 1, y are linearly independent over $K[x]$. We have

$$1=hf_1+(1-x)kf_2 \tag{1}$$

Therefore f_1 and f_2 have no common factors and there exists $g\in K[x]$ such that

$$h=gf_1x, \qquad k=-gf_2$$

(because $hf_2=-kxf_1$ and $f_2(0)\neq 0$.)
Therefore (1) shows that

$$1=g(f_1^2x+(x-1)f_2^2) \tag{2}$$

Therefore g must be a non-zero element of K.
Setting $x=0$, we have from (2) that

$$1=-gf_2(0)^2 \text{ and therefore } g \text{ is a negative number.}$$

Setting $x=1$, we have from (2) that

$$1=gf_1(0)^2 \text{ and therefore } g \text{ is a positive number.}$$

Thus we have a contradiction and $\mathfrak{p}$ cannot be a principal ideal.

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