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On a certain cup product.

By Seiya SASAO

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Introduction. Let K be a complex of a form $S^q \cup e^n \cup e^{n+q}$, i.e. a complex obtained from a q-sphere S^q by attaching an n-cell e^n and then an $(n+q)$-cell e^{n+q} where $n-2 \geqq q \geqq 2$. It is clear that the integral cohomology ring of K is as follows:

$$H^0(K) \approx H^q(K) \approx H^n(K) \approx H^{n+q}(K) \approx Z,$$
$$H^i(K) = 0 \qquad i \neq 0, q, n, n+q,$$

where Z denotes the ring of integers.

Let x, y, z denote the cohomology classes carried by e^n, S^q, e^{n+q} respectively. Then there is an integer m determined by $mz = x \cup y$. Let $\alpha \in \pi_{n-1}(S^q)$ denote the homotopy class of a map, $S^{n-1} \rightarrow S^q$, by which e^n is attached to S^q. I. M. James [**5**] described then K as a complex of type (m, α) and proved the following theorem (Theorem (1.8) l.c.).

J. *Let $[\alpha, l_q] \in \pi_{n+q-2}(S^q)$ denote the Whitehead product of α and a generator $l_q \in \pi_q(S^q)$. Then there exists a complex of type (m, α), if and only if $m[\alpha, l_q]$ is contained in the image of the homomorphism $\alpha_* : \pi_{n+q-2}(S^{n-1}) \rightarrow \pi_{n+q-2}(S^q)$ which is induced by α.*

At the end of the introduction of [**5**], James remarks that it is possible to discuss this topic in term of the cohomology invariant of mappings which are defined in [**10**], although his discussion in [**5**] is based on different methods. We shall show in this paper that **J** can be indeed simply and mechanically proved by the cohomology invariant of mappings.

Let L be a complex of a form $S^q \cup e^n$ which is obtained by attaching e^n to S^q. Since the homotopy type of L depends only on the homotopy class of the attaching map, we denote by $L(\alpha)$ the complex L which has a map of the class $\alpha \in \pi_{n-1}(S^q)$ as the attaching map. Then all complexes of type (m, α) have $L(\alpha)$ as a subcomplex.

Now consider a relative functional cup product of a map $g: (E^{n+q-1}, \dot{E}^{n+q-1}) \rightarrow (L(\alpha), S^q)$, where E^{n+q-1} denotes an $(n+q-1)$-cell and $\dot{E}^{n+q-1}$ its boundary. If we denote by $\tilde{x}$ the generator of $H^n(L(\alpha), S^q)$ identified with the cohomology class of $H^n(L(\alpha))$ which is carried by e^n and denote by $\tilde{y}$ the cohomology class of $H^q(L(\alpha))$ which is carried by S^q, then we have $\tilde{x} \cup \tilde{y} = 0$

and $g^*(\tilde{x})=0$, where g^* is the homomorphism of the cohomology ring induced by g.

From the definition, $\tilde{x}\underset{g}{\cup}\tilde{y}$ is an element of $\boldsymbol{H}^{n+q-1}(\boldsymbol{E}^{n+q-1}, \dot{\boldsymbol{E}}^{n+q-1})$ which is isomorphic with $\boldsymbol{H}^{n+q-2}(\dot{\boldsymbol{E}}^{n+q-1})\approx \boldsymbol{Z}$, because $\boldsymbol{H}^{n-1}(\boldsymbol{E}^{n+q-1}, \dot{\boldsymbol{E}}^{n+q-1})=\boldsymbol{H}^{n+q-1}(\boldsymbol{L}(\alpha), S^q)\approx 0$ (see § 12 of [**10**]). Therefore, there is an integer m such that $\tilde{x}\underset{g}{\cup}\tilde{y}$ is m times a generator of $\boldsymbol{H}^{n+q-1}(\boldsymbol{E}^{n+q-1}, \dot{\boldsymbol{E}}^{n+q-1})$. Since it is clear that m is a homotopy invariant of g, we obtain the correspondence $\boldsymbol{T}: \pi_{n+q-1}(\boldsymbol{L}(\alpha), S^q)\rightarrow \boldsymbol{Z}$ with respect to a fixed generator of $\boldsymbol{H}^{n+q-1}(\boldsymbol{E}^{n+q-1}, \dot{\boldsymbol{E}}^{n+q-1})$.

Then we have

LEMMA 1. *The above correspondence* $\boldsymbol{T}$ *is a homomorphism.*

PROOF. Let $\boldsymbol{E}_1^{n+q-1}\cup\boldsymbol{E}_2^{n+q-1}$ denote the union of two copies of $\boldsymbol{E}^{n+q-1}$ and $\{g\}$ denote the homotopy class of g. If $\{g\}, \{h\}\in\pi_{n+q-1}(\boldsymbol{L}(\alpha), S^q)$, then we can easily construct a map F with the following properties

(I) $\qquad F: (\boldsymbol{E}^{n+q-1}, \dot{\boldsymbol{E}}^{n+q-1})\rightarrow(\boldsymbol{E}_1^{n-q-1}\cup\boldsymbol{E}_2^{n+q-1}, \dot{\boldsymbol{E}}_1^{n+q-1}\cup\dot{\boldsymbol{E}}_2^{n+q-1})\,,$

(II) $\qquad F^*\circ p_i^*: \boldsymbol{H}^{n+q-1}(\boldsymbol{E}_i^{n+q-1}, \dot{\boldsymbol{E}}_i^{n+q-1})\rightarrow\boldsymbol{H}^{n+q-1}(\boldsymbol{E}^{n+q-1}, \dot{\boldsymbol{E}}^{n+q-1})$

is an isomorphism and orientation preserving.

(III) If we define ϕ as follows:

$$\begin{aligned}\phi(p)&=g(p) && p\in\boldsymbol{E}_1^{n+q-1}\\ &=h(p) && p\in\boldsymbol{E}_2^{n+q-1}\,,\end{aligned}$$

then

$$\{\phi\circ F\}=\{g\}+\{h\}\,.$$

Therefore we have $\tilde{x}\underset{\phi\circ F}{\cup}\tilde{y}=F^*(\tilde{x}\underset{\phi}{\cup}\tilde{y})$ by the invariance of the functional cup product under transformations [**10**]. Hence $\tilde{x}\underset{\{g\}}{\cup}\tilde{y}+\tilde{x}\underset{\{h\}}{\cup}\tilde{y}=\tilde{x}\underset{\{g\}+\{h\}}{\cup}\tilde{y}$ by (II) and (III).

Let $[\ ,\ ]_r$ denote the relative Whitehead product and $\bar{\alpha}$ denote the map: $(\boldsymbol{E}^n, \dot{\boldsymbol{E}}^n)\rightarrow(\boldsymbol{L}(\alpha), S^q)$ such that $\bar{\alpha}$ maps homeomorphically the interior of $\boldsymbol{E}^n$ onto e^n and $\{\bar{\alpha}\,|\,\dot{\boldsymbol{E}}^n\}=\alpha$. Then James has proved in [**1**] that $\pi_{n+q-1}(\boldsymbol{L}(\alpha), S^q)$ is isomorphic to the direct sum of the infinite cyclic group generated by $[\bar{\alpha}, l_q]_r$ with $\bar{\alpha}\circ\pi_{n+q-1}(\boldsymbol{E}^n, \dot{\boldsymbol{E}}^n)$. Therefore, for any $\{g\}\in\pi_{n+q-1}(\boldsymbol{L}(\alpha), S^q)$, there exist an integer m, and an element $\rho\in\pi_{n+q-1}(\boldsymbol{E}^n, \dot{\boldsymbol{E}}^n)$ such that $\{g\}=m[\alpha, l_q]_r+\bar{\alpha}\circ\rho$. Then we have

LEMMA 2. $\boldsymbol{T}(\{g\})=\pm m$, *where the sign depends only on the choice of orientations.*

PROOF. We have only to show that $\boldsymbol{T}([\alpha, l_q])=1$ or -1 and $T(\bar{\alpha}\circ\rho)=0$. Then our Lemma 2 will follow from Lemma 1. We have

$$\tilde{x}\underset{\bar{\alpha}\circ\rho}{\cup}\tilde{y}=\bar{\alpha}^*(\tilde{x})\underset{\rho}{\cup}\bar{\alpha}^*(\tilde{y})=\bar{\alpha}^*(\tilde{x})\underset{\rho}{\cup}0=0$$

by the invariance of the functional cut product under transformations. Define a map $\psi:(\boldsymbol{E}^n\cup\dot{\boldsymbol{E}}^{q+1},\dot{\boldsymbol{E}}^n\cup\dot{\boldsymbol{E}}^{q+1})\to(\boldsymbol{L}(\alpha),S^q)$ as follows:

$$\begin{aligned}\psi(p)&=\bar{\alpha}(p) \qquad p\in\boldsymbol{E}^n\\ &=\ (p) \qquad p\in\boldsymbol{E}^{q+1}.\end{aligned}$$

On the other hand, we can easily construct a map φ with the following properties:

(I) $$\varphi:\ (\boldsymbol{E}^{n+q-1},\dot{\boldsymbol{E}}^{n+q-1})\to(\boldsymbol{E}^n\cup\dot{\boldsymbol{E}}^{q+1},\dot{\boldsymbol{E}}^n\cup\dot{\boldsymbol{E}}^{q+1}),$$

(II) $$[\bar{\alpha},l_q]_r=\{\psi\circ\varphi\},$$

(III) $$[\alpha,l_q]=\{\psi\circ\varphi\,|\,\dot{\boldsymbol{E}}^{n+q-1}\}.$$

Then we have $\tilde{x}\underset{\psi\circ\varphi}{\cup}\tilde{y}=\psi^*(\tilde{x})\underset{\varphi}{\cup}\psi^*(\tilde{y})$ by the invariance of the functional cup product under transformations. Let δ_1 denote the coboundary homomorphism: $\boldsymbol{H}^{n+q-2}(\dot{\boldsymbol{E}}^{n+q-1})\to\boldsymbol{H}^{n+q-1}(\boldsymbol{E}^{n+q-1},\dot{\boldsymbol{E}}^{n+q-1})$ and δ_2 the coboundary homomorphism: $\boldsymbol{H}^{n-1}(\dot{\boldsymbol{E}}^n\cup\dot{\boldsymbol{E}}^{q+1})\to\boldsymbol{H}^n(\boldsymbol{E}^n\cup\dot{\boldsymbol{E}}^{q+1},\dot{\boldsymbol{E}}^n\cup\dot{\boldsymbol{E}}^{q+1})$. Then there exists an element x' of $\boldsymbol{H}^{n-1}(\dot{\boldsymbol{E}}^n\cup\dot{\boldsymbol{E}}^{q+1})$ such that $\delta_2(x')=\psi^*(\tilde{x})$. Hence, by (13.2) of [**10**] we have

$$\delta_1(x'\underset{\varphi|\dot{\boldsymbol{E}}^{n+q-1}}{\cup}\psi^*(\tilde{y}))=-\{\delta_2(x')\underset{\varphi}{\cup}\psi^*(\tilde{y})\}=-\{\psi^*(\tilde{x})\underset{\varphi}{\cup}\psi^*(\tilde{y})\}.$$

It is clear that ψ^*,δ_1,δ_2 are isomorphisms and $x',\psi^*(\tilde{y})$ are generators of $\boldsymbol{H}^{n-1}(\dot{\boldsymbol{E}}^n\cup\dot{\boldsymbol{E}}^{q+1})$ and $\boldsymbol{H}^q\ (\dot{\boldsymbol{E}}^n\cup\dot{\boldsymbol{E}}^{q+1})$ respectively. Therefore, from (19.1) of [**10**], $x'\underset{\varphi|\dot{\boldsymbol{E}}^{n+q-1}}{\cup}\psi^*(\tilde{y})$ is a generator of $\boldsymbol{H}^{n+q-2}(\dot{\boldsymbol{E}}^{n+q-1})$. This completes the proof of the Lemma 2.

Now let j^* be the inclusion homomorphism, and η be the map: $\boldsymbol{E}^{n+q-1}\to S^{n+q-1}$, which carries $\dot{\boldsymbol{E}}^{n+q-1}$ to point s_0 and the interior of $\boldsymbol{E}^{n+q-1}$ homeomorphically onto $S^{n+q-1}-s_0$. Then, by the invariance of the functional cup product we have

LEMMA 3. *Let f be a map $(S^{n+q-1},s_0)\to(\boldsymbol{L}(\alpha),f(s_0))$ and j be a map $(\boldsymbol{L}(\alpha),f(s_0))\to(\boldsymbol{L}(\alpha),S^q)$, then*

$$\tilde{x}\underset{j\circ f\circ\eta}{\cup}\tilde{y}=\eta^*(j^*(\tilde{x})\underset{f}{\cup}\tilde{y}).$$

We notice that η^* is an isomorphism and orientation preserving. Now suppose that $\boldsymbol{K}$ is a complex which is obtained by attaching e^{n+q} to $\boldsymbol{L}(\alpha)$ and f is its attaching map. From the definition, we have $j^*(\tilde{x})\underset{f}{\cup}\tilde{y}\in\boldsymbol{H}^{n+q-1}(S^{n+q-1},s_0)\approx\boldsymbol{Z}$. Therefore we can identify $j^*(\tilde{x})\underset{f}{\cup}\tilde{y}$ with an integer.

LEMMA 4. $x\cup y=(j^*(\tilde{x})\underset{f}{\cup}\tilde{y})z$

PROOF. We can identify $\boldsymbol{K}$ with the space which is obtained from the mapping cylinder $\boldsymbol{L}_f$ of $f:S^{n+q-1}\to\boldsymbol{L}(\alpha)$ by shrinking S^{n+q-1} to a point $\boldsymbol{K}_0$. Let ψ be the identification map. Consider the following commutative diagram.

$$\begin{array}{ccccc} \boldsymbol{H}^n(\boldsymbol{L}_f) & \xleftarrow{} & \boldsymbol{H}^n(\boldsymbol{L}_f, S^{n+q-1}) & \xrightarrow{\cup \tilde{y}} & \boldsymbol{H}^{n+q}(\boldsymbol{L}_f, S^{n+q-1}) \\ \uparrow \psi^* & j^* & \uparrow \psi^* & & \uparrow \psi^* \\ \boldsymbol{H}^n(\boldsymbol{K}) & \xleftarrow[j^*]{} & \boldsymbol{H}^n(\boldsymbol{K}, \boldsymbol{K}_0) & \xrightarrow[\cup \tilde{y}]{} & \boldsymbol{H}^{n+q}(\boldsymbol{K}, \boldsymbol{K}_0) \end{array}$$

Then we have Lemma 4 by the naturality of cup product and the definition of the functional cup product.

Now we can easily obtain the following theorem from the Lemmas 2, 3 and 4.

THEOREM 1. *Let* $\boldsymbol{K}$ *be a complex of type* (m, α) *as above, then*

$$j^*(\{f\}) = \pm m[\bar{\alpha}, l_q]_r + \bar{\alpha} \circ \rho$$

for some $\rho \in \pi_{n+q-1}(\boldsymbol{E}^n, \dot{\boldsymbol{E}}^n)$.

Now, theorem **J** is an easy consequence of Theorem 1.

University of Tokyo.

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