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On regularity of solutions of abstract differential equations of parabolic type in Banach space

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The object of the present paper is to establish some estimates for the derivatives of the solution $u(t)$ of the abstract differential equation

$$du(t)/dt+A(t)u(t)=f(t) \tag{0.1}$$

of parabolic type in a Banach space X. Let $\{M_k\}$ be a sequence of positive numbers which has the properties specified later (cf. [2], [5], [6]; e.g. $M_k=(k!)^\sigma$, $\sigma\geqq 1$). Assuming that $A(t)$ belongs to the class $\{M_k\}$ ([5], [6]) as a function of t in some sense, we shall prove that $u(t)$ is also a function of t of the class $\{M_k\}$ provided that $f(t)$ is of the same class (see (0.3), (0.4) below). This sort of problem was investigated in [5] and [6] for wide classes of equations in Hilbert spaces including equations of parabolic type, of hyperbolic type, of Schrödinger type, etc.; all of these equations discussed there are associated with certain sesquilinear forms defined in some dense subspace. The authors of these papers investigate solutions belonging to the spaces $\mathfrak{D}_+(X)$, $\mathfrak{D}'_+(X)$, etc. ($\mathfrak{D}_+(X)$ and $\mathfrak{D}'_+(X)$ are the set of X-valued C^∞ functions and distributions respectively vanishing identically in $t<a$ for some $a\in(-\infty, \infty)$; consequently smooth solutions investigated there satisfy so many initial conditions $u^{(k)}(t_0)=0$, $k=1, 2, \cdots$, at a certain time t_0. For this reason quasi-analytic cases were naturally excluded so that the space $(\mathfrak{D}_{+,M_k}(X))$, the set of all functions of the class $\{M_k\}$ and belonging to $\mathfrak{D}_+(X)$, might contain sufficiently many elements. In the present paper only equations of parabolic type are concerned; however, the basic space X may be an arbitrary Banach space and furthermore quasi-analytic cases are equally treated since we investigate solutions satisfying (0.1) in the ordinary sense imposing upon them only the ordinary initial condition at an initial time.

The greater part of the paper is occupied by estimating the derivatives of the evolution operator $U(t, s)$ which is a bounded-operator-valued function satisfying

$$(\partial/\partial t)U(t, s)+A(t)U(t, s)=0\,, \qquad 0\leqq s<t\leqq T\,,$$

$$U(s, s)=I\,, \qquad 0\leqq s\leqq T\,,$$

and whose existence was established in [3]. The main result is that for some constants L_0, L and for all triplet of non-negative integers n, m, l we have

(0.2) $$\left\| \left(\frac{\partial}{\partial t} \right)^n \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^m \left(\frac{\partial}{\partial s} \right)^l U(t, s) \right\| \leqq L_0 L^{n+m+l} M_{n+m+l} (t-s)^{-n-l} .$$

With the aid of this result we shall establish the estimates for the solution $u(t)$ of the inhomogeneous equation (0.1) taking a prescribed initial value at $t=s$; the result is that for some constants C_0, C and for all integers $n \geqq 0$ we have

(0.3) $$\| d^n u(t)/dt^n \| \leqq C_0 C^n M_n (t-s)^{-n} , \qquad s < t \leqq T ,$$

provided that there exist constants F_0 and F such that for all integers $n \geqq 0$

(0.4) $$\| d^n f(t)/dt^n \| \leqq F_0 F^n M_n , \qquad s \leqq t \leqq T .$$

In case $\{M_k\} = \{k!\}$ these are refinements of the results of [3] and [4] where the analyticity of the solution was proved without the establishment of the estimates of the form (0.2) and (0.3). In the last section we shall show that the results stated above are applicable to the initial-boundary value problems for parabolic differential equations using S. Agmon's result on general elliptic boundary value problems ([1]).

1. Assumptions and consequences.

Let $\{M_k\}$, $k=0, 1, 2, \cdots$, be a sequence of positive numbers which has the properties listed in [2], [5] and [6], i.e., for some positive constants d_0, d_1 and d_2

(1.1) $$M_{k+1} \leqq d_0{}^k M_k \qquad \text{for all } k \geqq 0 ,$$

(1.2) $$\binom{k}{j} M_{k-j} M_j \leqq d_1 M_k \qquad \text{for all } k \text{ and } j \text{ such that } 0 \leqq j \leqq k ,$$

(1.3) $$M_k \leqq M_{k+1} \qquad \text{for all } k \geqq 0 ,$$

(1.4) $$M_{j+k} \leqq d_2{}^{j+k} M_j M_k \qquad \text{for all } j \text{ and } k \geqq 0 .$$

From (1.2) we easily deduce

(1.5) $$k M_{k-1} \leqq \frac{d_1}{M_1} M_k \qquad \text{for all } k \geqq 1 ,$$

(1.6) $$M_{k-p} \leqq \left(\frac{d_1}{M_1} \right)^p \frac{(k-p)!}{k!} M_k \qquad \text{for all } p \text{ and } k \text{ satisfying } p \leqq k.$$

Taking $p=k$ in (1.6) we get

(1.7) $$k! \leqq \left(\frac{d_1}{M_1} \right)^k \frac{M_k}{M_0} \qquad \text{for all } k \geqq 0 .$$

As was indicated in [6], (1.1), $\cdots$, (1.4) hold in case $M_k=(k!)^\sigma$ for some $\sigma\geqq 1$.

Let X be a complex Banach space and T be a positive number.

ASSUMPTIONS (i). For each $t\in[0, T]$, $A(t)$ is a densely defined linear closed operator in X. The resolvent set of $A(t)$ contains a fixed closed sector $\Sigma=\{\lambda: \theta\leqq\arg\lambda\leqq 2\pi-\theta\}$, $0<\theta<\pi/2$.

(ii) $A(t)^{-1}$, which is a bounded operator for each $t\in[0, T]$ on account of the preceding assumption, is infinitely differentiable in t.

(iii) There exist constants K_0 and K such that for all $\lambda\in\Sigma$, $t\in[0, T]$ and non-negative integers n

$$\left\|\left(\frac{\partial}{\partial t}\right)^n(\lambda-A(t))^{-1}\right\|\leqq\frac{K_0K^nM_n}{|\lambda|}. \tag{1.8}$$

REMARK 1. If $A(t)$ has a continuation to a complex neighbourhood Δ of $[0, T]$ so that the assumptions (i) and (iii) for $n=0$ are satisfied in Δ and $A(t)^{-1}$ is holomorphic there, then the assumption (iii) holds with $\{M_n\}=\{n!\}$. This is a simple consequence of the Cauchy integral formula

$$(\lambda-A(t))^{-1}=\frac{1}{2\pi i}\int_\gamma(\lambda-A(\tau t))^{-1}(\tau-t)^{-1}d\tau, \qquad 0\leqq t\leqq T,$$

where γ is a smooth closed curve in Δ enclosing $[0, T]$.

REMARK 2. Consider the case in which the domain $D(A(t))$ of $A(t)$ is independent of t. Suppose the assumptions (i) and (iii) for $n=0$ are satisfied and $A(t)A(0)^{-1}$, which is a bounded operator, is infinitely differentiable. Then the assumptions (ii) and (iii) hold provided that there exist constants B_0 and B such that for any $n\geqq 0$

$$\|(d/dt)^nA(t)A(0)^{-1}\|=\|A^{(n)}(t)A(0)^{-1}\|\leqq B_0B^nM_n, \tag{1.9}$$

where $A^{(n)}(t)u=(d/dt)^nA(t)u$ for $u\in D\equiv D(A(t))$ (which does not depend on t by assumption).

PROOF. For each fixed s $A(t)A(s)^{-1}$ is infinitely differentiable since $A(t)A(s)^{-1}=A(t)A(0)^{-1}A(0)A(s)^{-1}$. Consequently in view of (1.9) we may suppose that

$$\|A^{(n)}(t)A(t)^{-1}\|\leqq B_0B^nM_n,\ 0\leqq t\leqq T, \tag{1.10}$$

for each n replacing B_0 and B by some other constants if necessary. As is easily seen

$$(\partial/\partial t)^{-1}(\lambda-A(t))^{-1}=(\lambda-A(t))^{-1}A'(t)(\lambda-A(t))^{-1}. \tag{1.11}$$

Let us prove that (1.8) as well as

$$\|A(t)(\partial/\partial t)^n(\lambda-A(t))^{-1}\|\leqq K_0K^nM_n \tag{1.12}$$

holds for any $n\geqq 0$ if K_0 and K are suitably chosen. It is clear that (1.8) and (1.12) are true for $n=0$ if K_0 is sufficiently large. Suppose that they are valid for $0, 1, \cdots, n$. Differentiating both sides of (1.11) n times we get

$$\left(\frac{\partial}{\partial t}\right)^{n+1}(\lambda-A(t))^{-1}$$

$$(1.13)\qquad =\sum_{j=0}^{n}\binom{n}{j}\left(\frac{\partial}{\partial t}\right)^{n-j}(\lambda-A(t))^{-1}\sum_{k=0}^{j}\binom{j}{k}A^{(j-k+1)}(t)\left(\frac{\partial}{\partial t}\right)^{k}(\lambda-A(t))^{-1}.$$

In view of (1.10) and the induction hypothesis we get

$$\left\|\left(\frac{\partial}{\partial t}\right)^{n+1}(\lambda-A(t))^{-1}\right\|$$

$$(1.14)\qquad \leqq\sum_{j=0}^{n}\binom{n}{j}K_0K^{n-j}M_{n-j}|\lambda|^{-1}\sum_{k=0}^{j}\binom{j}{k}B_0B^{j-k+1}M_{j-k+1}K_0K^kM_k.$$

Since in virtue of (1.4) and (1.2)

$$\binom{n}{j}M_{n-j}\binom{j}{k}M_{j-k+1}M_k\leqq\binom{n}{j}M_{n-j}\binom{j}{k}d_2^{\,j-k+1}M_{j-k}M_1M_k$$

$$\leqq d_1M_1d_2^{\,j-k+1}\binom{n}{j}M_{n-j}M_j\leqq d_1^2M_1d_2^{\,j-k+1}M_n,$$

it follows from (1.14) that

$$(1.15)\qquad \begin{aligned}&\left\|\left(\frac{\partial}{\partial t}\right)^{n+1}(\lambda-A(t))^{-1}\right\|\\ &\leqq d_1^2B_0K_0^2M_1M_n|\lambda|^{-1}\sum_{j=0}^{n}K^{n-j}(d_2B)^{j+1}\sum_{k=0}^{j}(K/d_2B)^k.\end{aligned}$$

If $K\geqq 2d_2B$, the right member of (1.15) is bounded by

$$2d_1^2B_0K_0^2M_1M_n|\lambda|^{-1}\sum_{j=0}^{n}K^{n-j}(d_2B)^{j+1}(K/d_2B)^j\leqq\bar{B}K_0^2K^nnM_n|\lambda|^{-1}.$$

where $\bar{B}=2d_1^2d_2B_0BM_1$. Thus we obtain

$$\|(\partial/\partial t)^{n+1}(\lambda-A(t))^{-1}\|\leqq\bar{B}K_0^2K^nnM_n|\lambda|^{-1},$$

and similarly

$$\|A(t)(\partial/\partial t)^{n+1}(\lambda-A(t))^{-1}\|\leqq\bar{B}K_0^2K^nnM_n.$$

Hence noting (1.5) we conclude that (1.8) and (1.12) are true for $n+1$ provided that K is so large that

$$K\geqq\max(2Bd_2,\ \bar{B}K_0d_1/M_1).$$

Let Γ be a smooth path running in Σ from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$. In view of the assumption (i) $-A(t)$ generates an analytic semigroup

$$(1.16)\qquad \exp(-\sigma A(t))=-\frac{1}{2\pi i}\int_{\Gamma}e^{-\lambda\sigma}(\lambda-A(t))^{-1}d\lambda,\ \sigma>0.$$

A bounded-operator-valued function $U(t, s)$ is called the evolution operator associated with (0.1) if it satisfies

$$(\partial/\partial t)U(t, s)+A(t)U(t, s)=0, \qquad 0\leqq s<t\leqq T,$$
$$U(s, s)=I, \qquad 0\leqq s\leqq T.$$

According to [3] the evolution operator can be constructed as follows:

(1.17) $$U(t, s)=\exp(-(t-s)A(t))+W(t, s)$$

(1.18) $$W(t, s)=\int_s^t \exp(-(t-\tau)A(t))R(\tau, s)d\tau,$$

(1.19) $$R(t, s)=\sum_{l=1}^{\infty} R_l(t, s),$$

(1.20) $$R_1(t, s)=-(\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(t)),$$

(1.21) $$R_l(t, s)=\int_s^t R_1(t, \tau)R_{l-1}(\tau, s)d\tau,\ l=2, 3, \cdots.$$

In view of [3; pp. 115-116] $U(t, s)$ has another expression

(1.22) $$U(t, s)=\exp(-(t-s)A(s))+Z(t, s),$$

(1.23) $$Z(t, s)=\int_s^t Q(t, \tau)\exp(-(\tau-s)A(s)),$$

(1.24) $$Q(t, s)=\sum_{l=1}^{\infty} Q_l(t, s),$$

(1.25) $$Q_1(t, s)=(\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(s)),$$

(1.26) $$Q_l(t, s)=\int_s^t Q_{l-1}(t, \tau)Q_1(\tau, s)d\tau, \qquad l=2, 3, \cdots.$$

$R(t, s)$ and $Q(t, s)$ are the solutions of the following integral equations

(1.27) $$R(t, s)-\int_s^t R_1(t, \tau)R(\tau, s)d\tau=R_1(t, s),$$

(1.28) $$Q(t, s)-\int_s^t Q(t, \tau)Q_1(\tau, s)d\tau=Q_1(t, s)$$

respectively. Set

$$R_{0,n,m}(t, s)=-(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^m\exp(-(t-s)A(t)),$$
$$R_{l,n,m}(t, s)=(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^m R_l(t, s),$$
$$Q_{0,n,m}(t, s)=(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^m\exp(-(t-s)A(s)),$$
$$Q_{l,n,m}(t, s)=(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^m Q_l(t, s)$$

for all non-negative integers l, n, m. Clearly

(1.29) $$R_{1,n,m}(t,s)=(\partial/\partial t+\partial/\partial s)R_{0,n,m}(t,s),$$

(1.30) $$Q_{1,n,m}(t,s)=(\partial/\partial t+\partial/\partial s)Q_{0,n,m}(t,s).$$

LEMMA 1.1. *There exist constants N_0 and N such that for $l=0,1$ and all non-negative integers n, m*

(1.31) $$\|R_{l,n,m}(t,s)\|\leqq N_0N^{n+m}M_nM_m(t-s)^{-n},$$

(1.32) $$\|Q_{l,n,m}(t,s)\|\leqq N_0N^{n+m}M_nM_m(t-s)^{-n}.$$

PROOF. In view of (1.16) we get

$$R_{0,n,m}(t,s)=I+II,$$

where

(1.33) $$I=\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}\Big(\frac{\partial}{\partial t}\Big)^{n+m}(\lambda-A(t))^{-1}d\lambda,$$

(1.34) $$II=\frac{1}{2\pi i}\int_\Gamma\sum_{k=0}^{n-1}\binom{n}{k}(-\lambda)^{n-k}e^{-\lambda(t-s)}\Big(\frac{\partial}{\partial t}\Big)^{m+k}(\lambda-A(t))^{-1}d\lambda.$$

All the integrands in the above are holomorphic functions of λ in Σ. On the right of (1.33) we take $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$, where Γ_1 is the half line connecting $\infty e^{-i\theta}$ and $(t-s)^{-1}e^{-i\theta}$, Γ_2 is the arc $\{(t-s)^{-1}e^{i\varphi};\ 2\pi-\theta\leqq\varphi\leqq\theta\}$, and Γ_3 is the half line connecting $(t-s)^{-1}e^{i\theta}$ and $\infty e^{i\theta}$. In view of (1.8)

$$\Big\|\frac{1}{2\pi i}\int_{\Gamma_1}e^{-\lambda(t-s)}\Big(\frac{\partial}{\partial t}\Big)^{n+m}(\lambda-A(t))^{-1}d\lambda\Big\|$$

$$\leqq\frac{1}{2\pi}\int_{(t-s)^{-1}}^{\infty}e^{-\lambda(t-s)r\cos\theta}K_0K^{n+m}M_{n+m}r^{-1}dr$$

$$=\frac{1}{2\pi}K_0K^{n+m}M_{n+m}\int_{\cos\theta}^{\infty}e^{-\xi}\frac{d\xi}{\xi},$$

and similarly for the integral along Γ_3. The integral of the same function along Γ_2 is dominated in norm by

$$\frac{1}{2\pi}K_0K^{n+m}M_{n+m}\int_\theta^{2\pi-\theta}e^{-\cos\varphi}d\varphi.$$

Collecting these results and using (1.4) we obtain

$$\|I\|\leqq c_0K_0(d_2K)^{n+m}M_nM_m\leqq c_0K_0(d_2KT)^n(d_2K)^mM_nM_m(t-s)^{-n},$$

where

$$c_0=\frac{1}{\pi}\int_{\cos\theta}^{\infty}e^{-\xi}\frac{d\xi}{\xi}+\frac{1}{2\pi}\int_\theta^{2\pi-\theta}e^{-\cos\varphi}d\varphi.$$

On the right of (1.34) we take for Γ the boundary of Σ. Then putting $\lambda=re^{i\theta}$ or $re^{-i\theta}$ according as $\mathrm{Re}\,\lambda>0$ or $\mathrm{Re}\,\lambda<0$ for $\lambda\in\Gamma$ and using (1.8) we see that II is dominated in norm by

$$\frac{1}{\pi}\int_0^\infty \sum_{k=0}^{n-1}\binom{n}{k} r^{n-k}e^{-(t-s)r\cos\theta}K_0K^{m+k}M_{m+k}r^{-1}dr \tag{1.35}$$

$$=\frac{K_0}{\pi}\sum_{k=0}^{n-1}\binom{n}{k}K^{m+k}M_{m+k}\int_0^\infty r^{n-k-1}e^{-(t-s)r\cos\theta}dr$$

$$=\frac{K_0}{\pi}\sum_{k=0}^{n-1}\binom{n}{k}K^{m+k}M_{m+k}\frac{(n-k-1)!}{((t-s)\cos\theta)^{n-k}}.$$

In virtue of (1.4) and (1.6)

$$M_{m+k}\leqq d_2{}^{m+k}M_mM_k\leqq d_2{}^{m+k}M_m\Big(\frac{d_1}{M_1}\Big)^{n-k}\frac{k!}{n!}M_n. \tag{1.36}$$

Inserting this into the last member of (1.35) and putting

$$c_1=\max(1,\ d_1{}^{-1}M_1d_2NT\cos\theta),$$

we get

$$\|II\|\leqq\frac{K_0}{\pi}\Big(\frac{d_1}{M_1}\Big)^n\frac{(d_2K)^mM_nM_m}{((t-s)\cos\theta)^n}\sum_{k=0}^{n-1}(d_1{}^{-1}M_1d_2N(t-s)\cos\theta)^k$$

$$\leqq\frac{K_0}{\pi}\frac{n}{c_1}\Big(\frac{c_1d_1}{M_1}\Big)^n\frac{(d_2K)^mM_nM_m}{((t-s)\cos\theta)^n}$$

$$\leqq\frac{K_0}{\pi c_1}\Big(\frac{c_1d_1e}{M_1}\Big)^n\frac{(d_2K)^mM_nM_m}{((t-s)\cos\theta)^n}.$$

Thus if we put

$$N_0=\frac{K_0}{\pi c_1}+c_0K_0,\quad N=\max\Big(\frac{c_1d_1e}{M_1\cos\theta},\ d_2K,\ d_2KT\Big),$$

we get (1.31) for $l=0$. Noting (1.29) and (1.1) we can establish (1.31) for $l=1$ replacing N_0 and N by some other constants if necessary. The proof of (1.32) is similar.

2. Differentiability of $U(t, s)$.

In this section we investigate the differentiability of $U(t, s)$ and the estimates for its derivatives replacing (iii) by the following weaker assumption:

(iii′) there exists a set of positive numbers $\{B_n\}$ such that for all integers $n\geqq 0$

$$\Big\|\Big(\frac{\partial}{\partial t}\Big)^n(\lambda-A(t))^{-1}\Big\|\leqq\frac{B_n}{|\lambda|},\ \lambda\in\Sigma,\qquad 0\leqq t\leqq T. \tag{2.1}$$

LEMMA 2.1. *Let $F(t, s)$ and $G(t, s)$ be bounded-operator-valued functions m times continuously differentiable in $0\leqq s<t\leqq T$. Suppose that*

$$(\partial/\partial t+\partial/\partial s)^jF(t, s),\ 0\leqq j\leqq m,$$

are uniformly bounded in $0\leqq s<t\leqq T$. Then for $s<r<t$

$$(2.2)\qquad \left(\frac{\partial}{\partial t}\right)^m \int_r^t F(t,\tau)G(\tau,s)d\tau$$

$$=\sum_{k=0}^{m-1}\sum_{j=0}^{m-1-k}\binom{m-1-k}{j}\left(\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)^{m-1-k-j}F(t,r)\left(\frac{\partial}{\partial r}\right)^j G(r,s)$$

$$+\int_r^t \sum_{k=0}^m \binom{m}{k}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)^{m-k}F(t,\tau)\cdot\left(\frac{\partial}{\partial \tau}\right)^k G(\tau,s)d\tau\,;$$

and for $s<\rho<r<t$

$$(2.3)\qquad \left(\frac{\partial}{\partial t}\right)^m \int_\rho^r F(t,\tau)G(\tau,s)d\tau$$

$$=-\sum_{k=0}^{m-1}\sum_{j=1}^{m-1-k}\binom{m-1-k}{j}\left[\left(\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)^{m-1-k-j}F(t,\tau)\cdot\left(\frac{\partial}{\partial \tau}\right)^j G(\tau,s)\right]_{\tau=\rho}^{\tau=r}$$

$$+\int_\rho^r \sum_{k=0}^m \binom{m}{k}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)^{m-k}F(t,\tau)\cdot\left(\frac{\partial}{\partial \tau}\right)^k G(\tau,s)d\tau\,,$$

PROOF. If $\varepsilon>0$ is sufficiently small, then

$$\frac{\partial}{\partial t}\int_r^{t-\varepsilon}F(t,\tau)G(\tau,s)d\tau$$

$$=F(t,t-\varepsilon)G(t-\varepsilon,s)+\int_r^{t-\varepsilon}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)F(t,\tau)\cdot G(\tau,s)d\tau$$

$$-\int_r^{t-\varepsilon}\frac{\partial}{\partial \tau}F(t,\tau)\cdot G(\tau,s)d\tau\,.$$

Integrating by part in the last integral

$$\frac{\partial}{\partial t}\int_r^{t-\varepsilon}F(t,\tau)G(\tau,s)d\tau$$

$$=F(t,r)G(r,s)+\int_r^{t-\varepsilon}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)F(t,\tau)\cdot G(\tau,s)d\tau$$

$$+\int_r^{t-\varepsilon}F(t,\tau)\cdot\frac{\partial}{\partial \tau}G(\tau,s)d\tau\,.$$

Letting $\varepsilon\to 0$ we get

$$\frac{\partial}{\partial t}\int_r^{t}F(t,\tau)G(\tau,s)d\tau$$

$$=F(t,r)G(r,s)+\int_r^{t}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)F(t,\tau)\cdot G(\tau,s)d\tau$$

$$+\int_r^{t}F(t,\tau)\frac{\partial}{\partial \tau}G(\tau,s)d\tau\,.$$

Repeating this argument we obtain (2.2). (2.3) can be established integrating by part in a similar manner.

LEMMA 2.2. *For* $l=2, 3, \cdots$

$$R_l(t, s)=\int_s^t R_{l-1}(t, \tau)R_1(\tau, s)d\tau . \tag{2.4}$$

PROOF. The lemma can be proved easily by induction.

LEMMA 2.3. *For any integer* $l \geqq 1$ $R_l(t, s)$ *is infinitely differentiable function of* (t, s). *There exists a set of positive numbers* $\{C_{l,n,m}\}$ *such that for all integers* $l, n, m \geqq 0$

$$\|R_{l,n,m}(t, s)\| \leqq C_{l,n,m}(t-s)^{l-n-1} . \tag{2.5}$$

PROOF. The statement in case $l=1$ can be proved just as Lemma 1.1. We prove the general case by induction with respect to l, and suppose that (2.5) is true for $l-1$ in place of l. In virtue of Lemma 2.2 and the argument in the proof of Lemma 2.1 we get

$$R_{l,0,m}(t, s)=\int_s^t \sum_{i=0}^{m} \binom{m}{i} R_{l-1,0,m-i}(t, \tau) \cdot R_{1,0,i}(\tau, s)d\tau . \tag{2.6}$$

Let r be any real number such that $s<r<t$. Applying Lemma 2.1 to the integral over (r, t) on the right of (2.6) we get

$$\begin{aligned} &R_{l,n,m}(t, s) \\ &=\sum_{i=0}^{m} \binom{m}{i} \Big\{ \sum_{k=0}^{n-1} \sum_{j=0}^{n-1-k} \binom{n-1-k}{j} R_{l-1,k,n-1-k-j+m-i}(t, r) \cdot R_{1,j,i}(r, s) \\ &\quad +\int_r^t \sum_{k=0}^{n} \binom{n}{k} R_{l-1,0,n-k+m-i}(t, \tau) \cdot R_{1,k,i}(\tau, s)d\tau \\ &\quad +\int_s^r R_{l-1,n,m-i}(t, \tau)R_{1,0,i}(\tau, s)d\tau \Big\} . \end{aligned} \tag{2.7}$$

Taking $r=(t+s)/2$ we can establish (2.5) without any difficulty.

LEMMA 2.4. *There exists a sequence of positive numbers* $\{B_{n,m}\}$ *such that for any integers* n, m, l *with* $l \geqq n+1$

$$\|R_{l,n,m}(t, s)\| \leqq B_{n,m}^{l-n}(t-s)^{l-n-1}/(l-n-1)! \tag{2.8}$$

PROOF. If $l=n+1$, (2.8) holds with $B_{n,m} \geqq C_{n+1,n,m}$ in view of Lemma 2.3. Suppose (2.8) is true for $l-1$ in place of l. Differentiating in t both sides of (2.6) n times we get, on noting $n \leqq l-2$ and (2.5),

$$R_{l,n,m}(t, s)=\int_s^t \sum_{i=0}^{m} \binom{m}{i} R_{l-1,n,m-i}(t, \tau) \cdot R_{1,0,i}(\tau, s)d\tau .$$

Using the induction hypothesis we get

$$\|R_{l,n,m}(t,s)\|\leqq\int_s^t\sum_{i=0}^m\binom{m}{i}B_{n,m-i}^{l-1-n}(t-\tau)^{l-2-n}\{(l-2-n)!\}C_{1,0,i}d\tau$$
$$\leqq 2^m\max_{0\leqq i\leqq m}B_{n,m-i}^{l-1-n}\cdot\max_{0\leqq i\leqq m}C_{1,0,i}\frac{(t-s)^{l-1-n}}{(l-1-n)!}.$$

Thus if $\{B_{n,m}\}$ is a sequence which is increasing with respect to m and satisfies

$$C_{n+1,n,m}\leqq B_{n,m},\quad 2^m\max_{0\leqq i\leqq m}C_{1,0,i}\leqq B_{n,m}$$

for any n and m, then we can proceed by induction to show (2.8) for all n, m. with $l\geqq n+1$.

LEMMA 2.5. *$R(t, s)$ is infinitely differentiable in $0\leqq s<t\leqq T$. There exists a sequence of positive numbers $\{C_{n,m}\}$ such that for any $n, m\geqq 0$*

$$\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m R(t,s)\right\|\leqq C_{n,m}(t-s)^{-n},\ 0\leqq s<t\leqq T. \tag{2.9}$$

PROOF. In virtue of Lemma 2.3 and 2.4

$$\sum_{l=1}^\infty\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m R_l(t,s)\right\|=\sum_{l=1}^\infty\|R_{l,n,m}(t,s)\|$$
$$\leqq\sum_{l=1}^n C_{l,n,m}(t-s)^{l-n-1}+\sum_{l=n+1}^\infty B_{n,m}^{l-n}(t-s)^{l-n-1}/(l-n-1)!$$
$$\leqq\sum_{l=1}^n C_{l,n,m}(t-s)^{l-n-1}+B_{n,m}\exp(B_{n,m}(t-s)),$$

from which the conclusions of the lemma follow easily.

LEMMA 2.6. *$U(t, s)$ is infinitely differentiable in $0\leqq s<t\leqq T$. There exists a sequence of positive number $\{\bar C_{n,m}\}$ such that for any $n, m\geqq 0$*

$$\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m U(t,s)\right\|\leqq \bar C_{n,m}(t-s)^{-n},\ 0\leqq s<t\leqq T. \tag{2.10}$$

PROOF. By the same argument as the proof of Lemma 1.1 we get for all $n, m\geqq 0$

$$\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m\exp(-(t-s)A(t))\right\| \tag{2.11}$$
$$=\|R_{0,n,m}(t,s)\|\leqq C'_{n,m}(t-s)^{-n},\ 0\leqq s<t\leqq T,$$

where $\{C'_{n,m}\}$ is some sequence of positive numbers. In the same manner as we deduced (2.6) we get

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m W(t,s) \tag{2.12}$$
$$=\int_s^t\sum_{i=0}^m\binom{m}{i}R_{0,0,m-i}(t,\tau)\cdot\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial s}\right)^i R(\tau,s)d\tau.$$

Using the argument by which we obtained (2.7) we deduce the similar formula for $(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^m W(t, s)$ from (2.12). With the aid of that formula and (2.9) we get for all $n, m \geqq 0$

$$\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m W(t, s)\right\| \leqq C''_{n,m}(t-s)^{1-n}, \tag{2.13}$$

where $\{C''_{n,m}\}$ is another sequence of positive numbers. Combining (2.12) and (2.13) we conclude (2.10).

3. Proof of the main result.

In what follows the notations $C_1, C_2, \cdots$ will be used to denote constants depending only on θ, K_0, K, T and the system $\{M_k\}$ all of which appeared in the assumptions (i), (ii) and (iii). In order to establish the main result it is convenient to use the expression (1.22) of $U(t, s)$.

LEMMA 3.1.

$$Z(t, s)=\int_s^t Q_1(t, \tau)U(\tau, s)d\tau . \tag{3.1}$$

PROOF. By induction we can easily show

$$Q_l(t, s)=\int_s^t Q_1(t, \tau)Q_{l-1}(\tau, s)d\tau$$

for any $l \geqq 2$. In view of (1.24) we get

$$Q(t, s)=Q_1(t, s)+\int_s^t Q_1(t, \tau)Q(\tau, s)d\tau . \tag{3.2}$$

Inserting this into (1.23) we easily get (3.1).

In order to prove the main theorem we need more careful treatment for the derivatives of $U(t, s)$ than that in the previous section. To begin with we prepare the following lemma.

LEMMA 3.2. *If* $s=r_0<r_1<\cdots<r_n<r_{n+1}=t$, *then*

$$\left(\frac{\partial}{\partial t}\right)^n Z(t, s)=\sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{i-1}{j}Q_{1,n-i,i-1-j}(t, r_i)\cdot\left(\frac{\partial}{\partial r_i}\right)^j U(r_i, s)$$

$$+\sum_{i=0}^{n}\int_{r_i}^{r_{i+1}}\sum_{j=0}^{i}\binom{i}{j}Q_{1,n-i,i-j}(t, \tau)\left(\frac{\partial}{\partial \tau}\right)^j U(\tau, s)d\tau .$$

PROOF. In view of (3.1)

$$\left(\frac{\partial}{\partial t}\right)^n Z(t, s)=\left(\frac{\partial}{\partial t}\right)^n \sum_{i=0}^{n}\int_{r_i}^{r_{i+1}}Q_1(t, \tau)U(\tau, s)d\tau$$

$$=\left(\frac{\partial}{\partial t}\right)^n\int_{r_n}^{t}Q_1(t, \tau)U(\tau, s)d\tau$$

$$+\sum_{i=1}^{n-1}\Big(\frac{\partial}{\partial t}\Big)^{n-i}\Big\{\Big(\frac{\partial}{\partial t}\Big)^{i}\int_{r_i}^{r_{i+1}}Q_1(t,\tau)\cdot U(\tau,s)d\tau\Big\}$$

$$+\Big(\frac{\partial}{\partial t}\Big)^{n}\int_{s}^{r_1}Q_1(t,\tau)U(\tau,s)d\tau\,.$$

Applying (2.2) to the first term and (2.3) to the inside of the bracket { } in the middle on the right member of the above equality, we get

$$(\partial/\partial t)^n Z(t,s)$$

$$=\sum_{k=0}^{n-1}\sum_{j=0}^{n-1-k}\binom{n-1-k}{j}Q_{1,k,n-1-k-j}(t,r_n)\cdot\Big(\frac{\partial}{\partial r_n}\Big)^{j}U(r_n,s)$$

$$+\int_{r_n}^{t}\sum_{k=0}^{n}\binom{n}{k}Q_{1,0,n-k}(t,\tau)\Big(\frac{\partial}{\partial\tau}\Big)^{k}U(\tau,s)d\tau$$

$$+\sum_{i=1}^{n-1}\Big(\frac{\partial}{\partial t}\Big)^{n-i}\Big\{-\sum_{k=0}^{i-1}\sum_{j=0}^{i-1-k}\binom{i-1-k}{j}\Big[Q_{1,k,i-1-k-j}(t,\tau)\Big(\frac{\partial}{\partial\tau}\Big)^{j}U(\tau,s)\Big]_{\tau=r_i}^{\tau=r_i+1}$$

$$+\int_{r_i}^{r_i+1}\sum_{k=0}^{i}\binom{i}{k}Q_{1,0,i-k}(t,\tau)\Big(\frac{\partial}{\partial\tau}\Big)^{k}U(\tau,s)d\tau\Big\}$$

$$+\int_{s}^{r_1}\Big(\frac{\partial}{\partial t}\Big)_{s}^{n}Q_1(t,\tau)U(\tau,s)d\tau\,.$$

Putting the right member of the above equality in order, we get

$$(\partial/\partial t)^n Z(t,s)=I+II+III\,,$$

where

$$I=\sum_{i=1}^{n}\sum_{k=0}^{i-1}\sum_{j=0}^{i-1-k}\binom{i-1-k}{j}Q_{1,n-i+k,i-1-k-j}(t,r_i)\cdot\Big(\frac{\partial}{\partial r_i}\Big)^{j}U(r_i,s)\,,$$

$$II=-\sum_{i=1}^{n-1}\sum_{k=0}^{i-1}\sum_{j=0}^{i-1-k}\binom{i-1-k}{j}Q_{1,n-i+k,i-1-k-j}(t,r_{i+1})\Big(\frac{\partial}{\partial r_{i+1}}\Big)^{j}U(r_{i+1},s)\,,$$

$$III=\sum_{i=0}^{n}\int_{r_i}^{r_{i+1}}\sum_{k=0}^{i}\binom{i}{k}Q_{1,n-i,i-k}(t,\tau)\cdot\Big(\frac{\partial}{\partial t}\Big)^{k}U(\tau,s)d\tau\,.$$

Replacing i by $i-1$ and then k by $k-1$ in II, we get

$$II=-\sum_{i=2}^{n}\sum_{k=1}^{i-1}P_{i,k}\,,$$

where

$$P_{i,k}=\sum_{j=0}^{i-1-k}\binom{i-1-k}{j}Q_{1,n-i+k,i-1-k-j}(t,r_i)\Big(\frac{\partial}{\partial r_i}\Big)^{j}U(r_i,s)\,.$$

Hence

$$I+II=\sum_{i=1}^{n}\sum_{k=0}^{i-1}P_{i,k}-\sum_{i=2}^{n}\sum_{k=1}^{i-1}P_{i,k}$$

$$=P_{1,0}+\sum_{i=2}^{n}\sum_{k=0}^{i-1}P_{i,k}-\sum_{i=2}^{n}\sum_{k=1}^{i-1}P_{i,k}$$

$$=P_{1,0}+\sum_{i=2}^{n}P_{i,0}=\sum_{i=1}^{n}P_{i,0}$$

$$=\sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{i-1}{j}Q_{1,n-i,i-1-j}(t, r_i)\left(\frac{\partial}{\partial r_i}\right)^j U(r_i, s).$$

Thus the proof of Lemma 3.2 is completed.

According to Stering's formula there exists a positive constant ω such that for any $n\geqq 1$

$$\omega^{-1}n^n e^{-n}\sqrt{n}\leqq n!\leqq \omega n^n e^{-n}\sqrt{n}. \tag{3.3}$$

THEOREM 3.1. *There exist constants H_0 and H such that for any $n\geqq 0$*

$$\|(\partial/\partial t)^n U(t, s)\|\leqq H_0 H^n M_n(t-s)^{-n}. \tag{3.4}$$

PROOF. For $n=0$ (3.4) holds for some $H_0\geqq \bar{C}_{0,0}$ (see (2.10)). Let us prove (3.4) for arbitrary n by induction, and suppose that it is true for $0, 1, \cdots, n-1$. Suppose

$$H\geqq \max\{2N, 2NT\}. \tag{3.5}$$

With the aid of Lemma 3.2 we get

$$(\partial/\partial t)^n U(t, s)=I+II+III+IV+V, \tag{3.6}$$

where

$$I=(\partial/\partial t)^n \exp(-(t-s)A(s)),$$

$$II=\sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{i-1}{j}Q_{1,n-i,\,i-1-j}(t, r_i)\left(\frac{\partial}{\partial r_i}\right)^j U(r_i, s),$$

$$III=\sum_{i=0}^{n-1}\int_{r_i}^{r_i+1}\sum_{j=0}^{i}\binom{i}{j}Q_{1,n-i,i-j}(t, \tau)\cdot\left(\frac{\partial}{\partial \tau}\right)^j U(\tau, s)d\tau,$$

$$IV=\int_{r_n}^{t}\sum_{j=0}^{n-1}\binom{n}{j}Q_{1,0,n-j}(t, \tau)\cdot\left(\frac{\partial}{\partial \tau}\right)^j U(\tau, s)d\tau$$

$$V=\int_{r_n}^{t}Q_1(t, \tau)\left(\frac{\partial}{\partial \tau}\right)^n U(\tau, s)d\tau.$$

According to Lemma 1.1

$$\|I\|\leqq N_0 M_0 N^n M_n(t-s)^{-n}. \tag{3.7}$$

In virtue of (1.32), (1.2), (3.5) and the induction hypothesis we get

$$
(3.8)\quad
\begin{aligned}
\|II\| &\leqq \sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{i-1}{j}\frac{N_0N^{n-1-j}M_{n-i}M_{i-1-j}}{(t-r_i)^{n-i}}\frac{H_0H^jM_j}{(r_i-s)^j}\\
&\leqq d_1N_0H_0N^{n-1}\sum_{i=1}^{n}\frac{M_{i-1}M_{n-i}}{(t-r_i)^{n-i}}\sum_{j=0}^{i-1}\Big(\frac{H}{N}\Big)^j\frac{1}{(r_i-s)^j}\\
&\leqq 2d_1N_0H_0N^{n-1}\sum_{i=1}^{n}\frac{M_{i-1}M_{n-i}}{(t-r_i)^{n-i}}\Big(\frac{H}{N}\Big)^{i-1}\frac{1}{(r_i-s)^{i-1}}\\
&\leqq 2d_1^2N_0H_0H^{n-1}M_{n-1}\sum_{i=1}^{n}\frac{(i-1)\,!(n-i)\,!}{(n-1)\,!}(t-r_i)^{i-n}(r_i-s)^{1-i}\Big(\frac{N}{H}\Big)^{n-i}.
\end{aligned}
$$

If we choose $r_i=s+i(t-s)/(n+1)$, $i=1,\cdots,n$, then according to (3.3) the sum in the last member of (3.8) is dominated by

$$
\begin{aligned}
&\Big(\frac{n+1}{n}\Big)^{n-1}(t-s)^{1-n}\Big(\frac{N}{H}\Big)^{n-1}+\Big(\frac{n+1}{n}\Big)^{n-1}(t-s)^{1-n}\\
&+\sum_{i=2}^{n-1}\omega^3\Big(\frac{n+1}{n-1}\Big)^{n-1}\Big(\frac{i-1}{i}\Big)^{i-1}\Big(\frac{n-i}{n+1-i}\Big)^{n-i}\Big\{\frac{(i-1)(n-i)}{n-1}\Big\}^{\frac{1}{2}}\Big(\frac{N}{H}\Big)^{n-i}(t-s)^{1-n}\\
&\leqq e(t-s)^{1-n}(N/H)^{n-1}+e(t-s)^{1-n}\\
&+\omega^3e^2(t-s)^{1-n}\sum_{i=2}^{n-1}\Big\{\frac{(i-1)(n-i)}{n-1}\Big\}^{\frac{1}{2}}\Big(\frac{N}{H}\Big)^{n-i}.
\end{aligned}
$$

Noting (3.5) and $i-1\leqq(n-1)/2$, $n-i\leqq(n-1)/2$ for $i=2,\cdots,n-1$, we see that the last member is dominated by

$$
\begin{aligned}
&e(t-s)^{1-n}2^{1-n}+e(t-s)^{1-n}+\omega^3e^2(t-s)^{1-n}2^{-1}\sqrt{n-1}\ \sum_{i=2}^{n-1}2^{i-n}\\
&\leqq\omega^3e^2(t-s)^{1-n}2^{-1}\sqrt{n-1}\ \sum_{i=1}^{n}2^{i-n}\leqq\omega^3e^2\sqrt{n-1}\,(t-s)^{1-n}
\end{aligned}
$$

Inserting this into (3.8) and using (1.5) we get

$$
(3.9)\quad
\begin{aligned}
\|II\| &\leqq 2\omega^3e^2d_1^3M_1^{-1}N_0H_0H^{n-1}(t-s)^{1-n}n^{-1/2}M_n\\
&\leqq C_1H_0H^{n-1}M_n(t-s)^{1-n}.
\end{aligned}
$$

With the aid of the similar argument

$$
(3.10)\quad
\begin{aligned}
\|III\| &\leqq d_1N_0H_0M_0N^nM_n\Big\{(t-r_1)^{-n}(r_1-s)\\
&\quad+\sum_{i=1}^{n-1}\frac{i\,!(n-i)\,!}{n\,!}\frac{r_{i+1}-r_i}{(t-r_i+1)^{n-i}(r_i-s)^i}\Big(\frac{H}{N}\Big)^i\Big\}\\
&\leqq C_2H_0H^{n-1}M_n(t-s)^{1-n},
\end{aligned}
$$

and

$$\|IV\| \leqq C_3 H_0 H^{n-1} M_n (t-s)^{1-n}. \tag{3.11}$$

Combining (3.6), (3.7), (3.9), (3.10) and (3.11), we obtain

$$\|(\partial/\partial t)^n U(t, s)\| \leqq N_0 M_0 N^n M_n (t-s)^{-n}$$
$$+ C_4 H_0 H^{n-1} M_n (t-s)^{1-n} + N_0 M_0^2 \int_{r_n}^t \|(\partial/\partial\tau)^n U(\tau, s)\| d\tau .$$

Thus if H is so large that

$$H \geqq \max\{2N, 2NT\}, \ H_0 H \geqq 2M_0 N_0 N J, \ H \geqq 2C_4 J T ,$$

where $J = \exp(N_0 M_0^2 e T)$, then

$$\|(\partial/\partial t)^n U(t, s)\| \leqq J^{-1} H_0 H^n M_n (t-s)^{-n} + N_0 M_0^2 \int_{r_n}^t \|(\partial/\partial\tau)^n U(\tau, s)\| d\tau . \tag{3.12}$$

Multiplying both members of (3.12) by $(t-s)^n$ and noting that $r_n < \tau < t$ implies

$$(t-s)^n < (1+n^{-1})^n (\tau-s)^n < e(\tau-s)^n ,$$

we see that $Y(t, s) = (t-s)^n \|(\partial/\partial t)^n U(t, s)\|$ satisfies the following differential inequality

$$Y(t, s) \leqq J^{-1} H_0 H^n M_n + N_0 M_0^2 e \int_s^t Y(\tau, s) d\tau , \tag{3.13}$$

where we invoked Lemma 2.6 to ensure that the integral on the right of (3.13) is finite. Integrating (3.13) we get

$$Y(t, s) \leqq \exp(N_0 M_0^2 e(t-s)) J^{-1} H_0 H^n M_n \leqq H_0 H^n M_n ,$$

completing the proof of the theorem.

THEOREM 3.2. *There exist constants L_0 and L such that for any non-negative integers n, m and l*

$$\left\| \left(\frac{\partial}{\partial t}\right)^n \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^m \left(\frac{\partial}{\partial s}\right)^l U(t, s) \right\| \leqq L_0 L^{n+m+l} M_{n+m+l} (t-s)^{-n-l} , \tag{3.14}$$

$$\left\| \left(\frac{\partial}{\partial t}\right)^n \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^m \left(\frac{\partial}{\partial s}\right)^l W(t, s) \right\| \leqq L_0 L^{n+m+l} M_{n+m+l} (t-s)^{1-n-l} , \tag{3.15}$$

$$\left\| \left(\frac{\partial}{\partial t}\right)^n \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^m \left(\frac{\partial}{\partial s}\right)^l Z(t, s) \right\| \leqq L_0 L^{n+m+l} M_{n+m+l} (t-s)^{1-n-l} . \tag{3.16}$$

PROOF. For $n = l = 0$, (3.14) may be proved by induction with the aid of the formula

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^m U(t, s) = Q_{0,0,m}(t, s) + \sum_{k=0}^m \binom{m}{k} \int_s^t Q_{1,0,m-k}(t, \tau) \left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial s}\right)^k U(\tau, s) d\tau , \tag{3.17}$$

(3.14) for $l=0$ may be established by induction with respect to $n+m$ following the proof of Theorem 3.1 to each summand on the right of (3.17). In the general case with the aid of the result established already and (1.2) we get

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial t}\right)^n\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^m\left(\frac{\partial}{\partial s}\right)^l U(t,s)\right\| \\ &=\left\|\sum_{k=0}^{l}(-1)^k\binom{l}{k}\left(\frac{\partial}{\partial t}\right)^{n+k}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^{m+l-k}U(t,s)\right\| \\ &\leqq\sum_{k=0}^{l}\binom{l}{k}L_0L^{n+m+l}M_{n+k}M_{m+l-k}(t-s)^{-n-k} \\ &\leqq d_1L_0L^{n+m+l}M_{n+m+l}\sum_{k=0}^{l}\binom{l}{k}\left\{\binom{n+m+l}{n+k}\right\}^{-1}(t-s)^{-k-n}.\end{aligned}$$

Since

$$\binom{l}{k}\left\{\binom{n+m+l}{n+k}\right\}^{-1}\leqq\binom{l}{k}\left\{\binom{n+l}{n+k}\right\}^{-1}=\frac{l!}{(n+l)!}\frac{(n+k)!}{k!}\leqq 1,$$

the last member does not exceed

$$\begin{aligned}&d_1L_0L^{n+m+l}M_{n+m+l}\sum_{k=0}^{l}(t-s)^{-k-n} \\ &\qquad\leqq d_1L_0L^{n+m+l}M_{n+m+l}(t-s)^{-n-l}\sum_{k=0}^{l}\{\max(T,1)\}^{l-k} \\ &\qquad\leqq d_1L_0lL^{n+m+l}M_{n+m+l}(t-s)^{-n-l}\{\max(T,1)\}^{l}.\end{aligned}$$

From this (3.14) follows if we replace L_0 and L by some other constants if necessary. (3.15) and (3.16) can be proved with the aid of (3.14), (3.1) and

$$W(t,s)=\int_s^t U(t,\tau)R_1(\tau,s)d\tau$$

following the argument used to establish (3.14).

THEOREM 3.3. *Suppose $f(t)$ is an infinitely differentiable function with values in X and there exist constants F_0 and F such that for any $n\geqq 0$*

$$\|d^nf(t)/dt^n\|\leqq F_0F^nM_n,\quad s\leqq t\leqq T. \tag{3.18}$$

If $v(t)$ is the solution of (0.1) *satisfying the initial condition $v(s)=0$, then for some constants $\bar{F}_0$ and $\bar{F}$ and for any integer $n=0$ we have*

$$\|d^nv(t)/dt^n\|\leqq \bar{F}_0\bar{F}^nM_n(t-s)^{1-n},\quad s<t\leqq T. \tag{3.19}$$

If $u(t)$ is the solution of (0.1) *satisfying the initial condition $u(s)=u_0$, then for any integer $n\geqq 0$ we have*

$$\|d^nu(t)/dt^n\|\leqq H_0H^nM_n\|u_0\|(t-s)^{-n}+\bar{F}_0\bar{F}^nM_n(t-s)^{1-n}. \tag{3.20}$$

PROOF. First note

$$v(t)=\int_s^t U(t, \sigma)f(\sigma)d\sigma ,$$

$$u(t)=U(t, s)u_0+\int_s^t U(t, \sigma)f(\sigma)d\sigma .$$

As is easily seen

$$(3.21) \qquad \left(\frac{d}{dt}\right)^n \int_s^t U(t, \sigma)f(\sigma)d\sigma$$

$$=\sum_{i=1}^n \sum_{j=0}^{i-1}\binom{i-1}{j}\left(\frac{\partial}{\partial t}\right)^{n-i}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)^{i-1-j} U(t, s)\cdot f^{(j)}(s)$$

$$+\int_s^t \sum_{j=0}^n \binom{n}{j}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \sigma}\right)^{n-j} U(t, \sigma)\cdot f^{(j)}(\sigma)d\sigma .$$

Hence in virtue of theorem 2.2 we have

$$(3.22) \qquad \|d^n v(t)/dt^n\| \leqq \sum_{i=1}^n \sum_{j=0}^{i-1}\binom{i-1}{j} L_0 L^{n-1-j} M_{n-1-j}(t-s)^{j-n} F_0 F^j M_j$$

$$+\int_s^t \sum_{j=0}^n \binom{n}{j} L_0 L^{n-j} M_{n-j} F_0 F^j M_j d\sigma .$$

Since in virtue of (1.2)

$$\binom{i-1}{j} M_{n-1-j} M_j \leqq d_1 M_{n-1}\binom{i-1}{j}\left\{\binom{n-1}{j}\right\}^{-1} \leqq d_1 M_{n-1}$$

for $j<i\leqq n$, the right member of (3.22) does not exceed

$$(3.23) \qquad d_1 L_0 F_0 L^{n-1} M_{n-1} \sum_{i=1}^n \sum_{j=0}^{i-1} (F/L)^j (t-s)^{i-n}$$

$$+d_1 L_0 F_0 L^n M_n \sum_{j=0}^n (F/L)^j (t-s) .$$

Thus (3.19) holds provided that

$$\bar{F}_0=6d_1 L_0 F_0,\ \bar{F}=\max\{1, FT, 2LT, 2L\} .$$

(3.20) is a simple consequence of (3.19) and Theorem 3.1.

4. Application.

In this section using S. Agmon's result on general elliptic boundary value problems we show that the results of the preceding sections are applicable to a general class of well posed initial-boundary value problems for parabolic differential equations. We denote by Ω a bounded domain in $n(\geqq 2)$-space E_n with boundary $\partial\Omega$ and closure $\bar{\Omega}$. We let $x=(x_1, \cdots, x_n)$ be the generic point of E_n and use the notations

$$D_x=(D_1, \cdots, D_n)=(\partial/\partial x_1, \cdots, \partial/\partial x_n),$$

denoting by

$$D_x^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$$

a general derivative in x. Here α stands for the multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ whose length $\alpha_1+\cdots+\alpha_n$ is denoted by $|\alpha|$. $W_p^j(\Omega)$, $1<p<\infty$, is to be the set of all complex valued functions whose distribution derivatives up to order j belong to $L^p(\Omega)$. In this set of functions we introduce the norm

$$\|u\|_{j,p,\Omega}=\Big(\sum_{|\alpha|\leqq j}\int_\Omega |D_x^\alpha u|^p dx\Big)^{1/p}.$$

$W_p^0(\Omega)=L^p(\Omega)$ and $\|u\|_{0,p,\Omega}$ is the usual L^p norm. For $k>0$ $W_p^{k-1/p}(\partial\Omega)$ is to be the class of functions ϕ which are the boundary values of functions v belonging to $W_p^k(\Omega)$. In this class we introduce the norm

$$\|\phi\|_{k-1/p,p,\partial\Omega}=\inf \|v\|_{j,p,\Omega},$$

where the infimum is taken over all functions v in $W_p^k(\Omega)$ which equal ϕ on the boundary.

For each $t\in[0, T]$ let $A(x, t, D_x)$ be an elliptic linear differential operator in $\bar\Omega$ of order $2m$, and $B_j(x, t, D_x)$, $j=1, \cdots, m$, be a linear differential operator of order $m_j(<2m)$ with coefficients defined on the boundary, their highest order parts being denoted by $A^\#$ and $B_j^\#$, $j=1, \cdots, m$, respectively. We consider the following initial-boundary value problem

(4.1) $$\partial u(x, t)/\partial t+A(x, t, D_x)u(x, t)=f(x, t),\ x\in\Omega,\ 0<t\leqq T,$$

(4.2) $$u(x, 0)=u_0(x), \qquad x\in\Omega,$$

(4.3) $$B_j(x, t, D_x)u(x, t)=0,\ j=1, \cdots, m,\ x\in\partial\Omega,\ 0<t\leqq T.$$

In what follows we assume that the coefficients of $\{B_j\}$ are defined in the whole of $\bar\Omega\times[0, T]$.

We suppose the following conditions are satisfied (cf. [**1**]):

(a) For every pair of linearly independent real vectors ξ, η and $(x, t)\in\bar\Omega\times[0, T]$ the polynomial in τ, $A^\#(x, t, \xi+\tau\eta)$ has exactly m roots with positive imaginary parts;

(b) $\{B_j\}$ is a normal system, i.e., (i) $m_j\neq m_k$ if $j\neq k$, (ii) $\partial\Omega$ is nowhere characteristic with respect to B_j, $j=1, \cdots, m$;

(c) $$(-1)^m A^\#(x, t, \xi)/|A^\#(x, t, \xi)|\neq e^{i\varphi}$$

for all real vectors $\xi\neq 0$, all $(x, t)\in\bar\Omega\times[0, T]$ and $\varphi\in[\pi/2, 3\pi/2]$;

(d) let (x, t) be any point on $\partial\Omega\times[0, T]$. Let ν be the normal vector and $\xi\neq 0$ be any real vector parallel to the boundary $\partial\Omega$ at x. Denote by $\tau_k^+(\xi, \lambda; x, t)$ the m roots with positive imaginary parts of the polynomial in τ

$$(-1)^m A^\#(x, t, \xi+\tau\nu)-\lambda,$$

where λ is any complex number with non-positive real part (it follows from (a) and (c) that this polynomial has no real root and has exactly m roots with positive imaginary parts). Then the polynomials in τ

$$B_j^\#(x, t, \xi+\tau\nu),\ j=1, \cdots, m,$$

are linearly independent modulo the polynomial

$$\prod_{k=1}^{m}(\tau-\tau_k^+(\xi, \lambda; x, t));$$

(e) Ω is of class C^{2m};

(f) the coefficients of A and the x-derivatives of those of $B_j, j=1, \cdots, m$, up to order $2m-m_j$ have derivatives in t of all orders which are continuous in $\bar{\Omega}\times[0, T]$. If a stands for any of the coefficients of A or the x-derivatives of the coefficients of $B_j, j=1, \cdots, m$, up to order $2m-m_j$, then for all $(x, t)\in\bar{\Omega}\times[0, T]$ and all integers $l\geqq 0$

$$|D_t{}^l a(x, t)|\leqq B_0 B^l M_l$$

for some constants B_0 and B independent of x, t and l.

According to the uniform continuity of the coefficients of A and $\{B_j\}$ there exists a constant $\theta\in(0, \pi/2)$ such that (c) and (d) remain valid for $\theta\leqq\varphi\leqq 2\pi-\theta$ and $\theta\leqq\arg\lambda\leqq 2\pi-\theta$ respectively.

For each $t\in[0, T]$ the operator $A(t)$ is defined as follows:

(i) $D(A(t))=\{u\in L^p(\Omega): B_j(x, t, D_x)u(x)=0, j=1, \cdots, m, x\in\partial\Omega\}$,

(ii) for $u\in D(A(t))$, $(A(t)u)(x)=A(x, t, D_x)u(x)$.

We write (4.1)—(4.2)—(4.3) in the following form of the abstract differential equation in $L^p(\Omega)$:

$$du(t)/dt+A(t)u(t)=f(t), \qquad 0<t\leqq T,$$
$$u(0)=u_0.$$

THEOREM 4.1. *Under the conditions* (a), $\cdots$, (f) *the assumptions* (i), (ii), (iii) *of section* 1 *are satisfied by the family* $\{A(t)\}$ *defined above if we replace* $A(x, t, D_x)$ *by* $A(x, t, D_x)+k$ *with some real number* k *if necessary.*

As preparation let us consider the following boundary value problem

$$(\lambda-A(x, D_x))u(x)=f(x), \qquad x\in\Omega, \tag{4.4}$$

$$B_j(x, D_x)u(x)=g_j(x),\ j=1, \cdots, m, \qquad x\in\partial\Omega \tag{4.5}$$

concerning the system $(A(x, D_x), \{B_j(x, D_x)\}_{j=1}^m)=(A(x, t_0, D_x), \{B_j(x, t_0, D_x)\}_{j=1}^m)$ for some $t_0\in[0, T]$. Here $f\in L^p(\Omega)$ and $g_j\in W_p^{2m-m_j}(\Omega), j=1, \cdots, m$, are known functions. (4.5) is assumed to be satisfied only on the boundary although the functions on both sides are defined also in the interior. In what follows we denote by $C_0, C_1, \cdots$ constants depending only on the conditions (a), $\cdots$, (f), and simply write $\|\ \|_k$, $\|\ \|$, $\|\ \|_{k-1/p}$ in place of $\|\ \|_{k,p,\Omega}$, $\|\ \|_{0,p,\Omega}$, $\|\ \|_{k-1/p,p,\partial\Omega}$ respectively if there is no fear of confusion.

LEMMA 4.1. *If λ is a complex number with $\theta \leqq \arg\lambda \leqq 2\pi-\theta$ and with sufficiently large absolute value, then for the solution $u \in W_p^{2m}(\Omega)$ of* (4.4)—(4.5)

$$\|u\|_{2m}+|\lambda|\,\|u\| \leqq C_0\Big\{\|f\|+\sum_{j=1}^{m}\|g_j\|_{2m-m_j}+\sum_{j=1}^{m}|\lambda|^{(2m-m_j)/2m}\|g_j\|\Big\}. \tag{4.6}$$

PROOF. This lemma is nothing other than a very slight extension of Theorem 2.1 of [1], and we follow the proof of that theorem. Let Q be the cylindrical domain $\{(x, y): x \in \Omega,\ -\infty < y < \infty\}$ in $(n+1)$-space. According to the assumptions of the present section, the following *a priori* estimate holds for functions $v \in W_p^{2m}(Q)$ vanishing for $|y| \geqq 1$:

$$\begin{aligned}\|v\|_{2m,p,Q} \leqq C_1\{&\|((-1)^m e^{i\varphi}D_y^{2m}-A(x, D_x))v\|_{0,p,Q}\\ &+\sum_{j=1}^{m}\|B_j(x, D_x)v\|_{2m-m_j-1/p,p,\partial Q}+\|v\|_{0,p,Q}\}\end{aligned} \tag{4.7}$$

when $\theta \leqq \varphi \leqq 2\pi-\theta$. Let μ be an arbitrary real number and $\zeta(y)$ be some fixed C^∞ function such that $\zeta(y)=0$ for $|y| \geqq 1$, $\zeta(y)=1$ for $|y| \leqq 1/2$. We apply (4.7) to the function

$$v_\mu(x, y)=\zeta(y)e^{i\mu y}u(x).$$

As is easily seen (cf. [1])

$$2\|v_\mu\|_{2m,p,Q} \geqq \|u\|_{2m}+|\mu|^{2m}\|u\|, \tag{4.8}$$

$$\begin{aligned}&\|((-1)^m e^{i\varphi}D_y^{2m}-A(x, D_x))v_\mu\|_{0,p,Q}\\ &\qquad\leqq \|(\mu^{2m}e^{i\varphi}-A(x, D_x))u\|+C_2|\mu|^{2m-1}\|u\|.\end{aligned} \tag{4.9}$$

Since on the boundary of Q

$$B_j(x, D_x)v_\mu(x, y)=\zeta(y)e^{i\mu y}g_j(x), \qquad j=1, \cdots, m,$$

we have

$$\begin{aligned}\|B_j(x, D_x)v_\mu\|_{2m-m_j-1/p,p,\partial Q} &\leqq \|\zeta e^{i\mu y}g_j\|_{2m-m_j,p,Q}\\ &\leqq C_3\sum_{k=0}^{2m-m}|\mu|^{2m-m_j-k}\|g_j\|_k\\ &\leqq C_4(\|g_j\|_{2m-m_j}+|\mu|^{2m-m_j}\|g_j\|),\end{aligned} \tag{4.10}$$

where we used the well known inequality

$$\|w\|_i \leqq c_{j,i}\|w\|_j^{i/j}\|w\|^{(j-i)/j} \tag{4.11}$$

for $0<i<j$. Setting $\lambda=\mu^{2m}e^{i\varphi}$ we obtain (4.6) in view of (4.7), (4.8), (4.9) and (4.10).

Replacing $A(x, D_x)$ by $A(x, D_x)+k$ with some real number k we may assume that (4.6) holds for any λ satisfying $\theta\leqq \arg\lambda\leqq 2\pi-\theta$. We return to the proof of Theorem 4.1. If $g_j=0$, $j=1,\cdots,m$, we get in virtue of (4.6)

$$|\lambda|\,\|u\|\leqq C_0\|(\lambda-A)u\|$$

for any $u\in D(A)$ $(A=A(t_0))$ and λ such that $\theta\leqq\arg\lambda\leqq 2\pi-\theta$. Hence the assumption (i) of section 1 is satisfied. Let f be an arbitrary element of $L^p(\Omega)$ and $\theta\leqq\arg\lambda\leqq 2\pi-\theta$. Then $u(t)=(\lambda-A(t))^{-1}f$ is the solution of the boundary value problem

$$(\lambda-A(x,t,D_x))u(x,t)=f(x),\ x\in .\quad ,\leqq t\leqq T, \tag{4.12}$$

$$B_j(x,t,D_x)u(x,t)=0,\ j=1,\cdots,m,\ x\in\partial\Omega,\ 0\leqq t\leqq T. \tag{4.13}$$

Differentiating both sides of (4.12) and (4.13) in tl times we get

$$\begin{aligned}&(\lambda-A(x,t,D_x))D_t^l u(x,t)\\&\quad=\sum_{k=0}^{l-1}\binom{l}{k}A^{(l-k)}(x,t,D_x)D_t^k u(x,t),\ x\in\Omega,\ 0\leqq t\leqq T,\end{aligned} \tag{4.14}$$

$$\begin{aligned}&B_j(x,t,D_x)D_t^l u(x,t)\\&\quad=-\sum_{k=0}^{l-1}\binom{l}{k}B_j^{(l-k)}(x,t,D_x)D_t^k u(x,t),\ x\in\partial\Omega,\ 0\leqq t\leqq T,\end{aligned} \tag{4.15}$$

where $A^{(l-k)}$ and $B_j^{(l-k)}$ are differential operators obtained by differentiating the corresponding coefficients of A and B_j $l-k$ times in t. If we apply (4.6) to $D_t^l u(t)$, we get in view of (4.14) and (4.15)

$$\begin{aligned}&\|D_t^l u(t)\|_{2m}+|\lambda|\,\|D_t^l u(t)\|\\&\leqq C_5\{\|\sum_{k=0}^{l-1}\binom{l}{k}A^{(l-k)}(x,t,D_x)D_t^k u(t)\|\\&+\sum_{j=1}^{m}\|\sum_{k=0}^{l-1}\binom{l}{k}B_j^{(l-k)}(x,t,D_x)D_t^k u(t)\|_{2m-m_j}\\&+\sum_{j=1}^{m}|\lambda|^{(2m-m_j)/2m}\|\sum_{k=0}^{l-1}\binom{l}{k}B_j^{(l-k)}(x,t,D_x)D_t^k u(t)\|\},\end{aligned}$$

(using the condition (f))

$$\leqq C_6\{\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\|D_t^k u(t)\|_{2m}$$

$$+\sum_{j=1}^{m}\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}|\lambda|^{(2m-m_j)/2m}\|D_t^k u(t)\|_{m_j}\}$$

(applying (4.11) to the last sum)

$$\leqq C_7\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}(\|D_t^k u(t)\|_{2m}+|\lambda|\,\|D_t^k u(t)\|_0).$$

We intend to prove that

$$\|D_t^l u(t)\|_{2m}+|\lambda|\,\|D_t^l u(t)\|_0\leqq K_0K^lM_l\|f\| \tag{4.16}$$

for all integers $l\geqq 0$ and for some constants K_0 and K. It is clear that (4.16) is true for $l=0$ if K_0 is suitably chosen. If K is so large that

$$K\geqq \max(2B, 2d_1C_7B_0B),$$

then we can proceed by induction to show that (4.16) is true for all $l=0$. Hence

$$|\lambda|\,\|D_t^l(\lambda-A(t))^{-1}f\|=|\lambda|\,\|D_t^l u(t)\|\leqq K_0K^lM_l\|f\|$$

which implies the desired result

$$\|D_t^l(\lambda-A(t))^{-1}\|\leqq K_0K^lM_l/|\lambda|.$$

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