J. Math. Soc. Japan
Vol. 19, No. 4, 1967

Nonlinear semigroups and evolution equations

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(Received May 2, 1967)

Introduction

This paper has been motivated by a recent paper by Y. Kōmura [3], in which a general theory of semigroups of nonlinear contraction operators in a Hilbert space is developed. Owing to the generality of the problem, Kōmura is led to consider multi-valued operators as the infinitesimal generators of such semigroups, which makes his theory appear somewhat complicated.

The object of the present paper is to restrict ourselves to single-valued operators in a Banach space X and to construct the semigroups generated by them in a more elementary fashion. Furthermore, we are able to treat, without essential modifications, time-dependent nonlinear equations of the form

$$du/dt+A(t)u=0, \qquad 0\leqq t\leqq T, \tag{E}$$

where the unknown $u(t)$ is an X-valued function and where $\{A(t)\}$ is a family of nonlinear operators with domains and ranges in X. In particular we shall prove existence and uniqueness of the solution of (E) for a given initial condition.

The basic assumptions we make for (E) are that the adjoint space X^* is uniformly convex and that the $A(t)$ are m-monotonic operators (see below), together with some smoothness condition for $A(t)$ as a function of t. We make no explicit assumptions on the continuity of the operators $A(t)$.

Here an operator A with domain $D(A)$ and range $R(A)$ in an arbitrary Banach space X is said to be *monotonic* if

$$\|u-v+\alpha(Au-Av)\|\geqq\|u-v\| \qquad \text{for every } u, v\in D(A) \text{ and } \alpha>0. \tag{M}$$

This implies that $(1+\alpha A)^{-1}$ exists and is Lipschitz continuous provided $\alpha>0$, where $1+\alpha A$ is the operator with domain $D(A)$ which sends u into $u+\alpha Au$. It can be shown (see Lemma 2.1) that $(1+\alpha A)^{-1}$ has domain X either for every $\alpha>0$ or for no $\alpha>0$; in the former case we say that A is *m-monotonic.*

The monotonicity thus defined can also be expressed in terms of the *duality*

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* This work represents part of the results obtained while the author held a Miller Research Professorship.

map F from X to X^*. (Here X^* is defined to be the set of all bounded semilinear forms on X, and the pairing between $x \in X$ and $f \in X^*$ is denoted by (x, f), which is thus linear in x and semilinear in f. If X is a Hilbert space, X^* is identified with X and $(\,,\,)$ with the inner product in X.) F is in general a multi-valued operator; for each $x \in X$, Fx is by definition the (nonempty) set of all $f \in X^*$ such that $(x, f) = \|x\|^2 = \|f\|^2$. (Thus we employ a special "gauge function" for F.)

(M) is now equivalent to the following condition (see Lemma 1.1):

(M′) For each $u, v \in D(A)$, there is $f \in F(u-v)$ such that

$$\mathrm{Re}\,(Au - Av, f) \geqq 0\,.$$

Note that the inequality is not required to hold for every $f \in F(u-v)$. If X is a Hilbert space, (M′) is equivalent to the monotonicity of A in the sense of Minty [4] and Browder [1].

The main results of this paper are stated in § 3 and the proofs are given in § 4. §§ 1 and 2 contain some preliminary results for the duality map F and for m-monotonic operators.

The crucial step in our existence proof is the proof of convergence for the approximate solutions $u_n(t)$ of (E), which is a straightforward generalization of an ingenious proof given in [3]. The author is indebted to Professor Y. Kōmura for having a chance to see his paper before publication and to Professor K. Yosida for many stimulating conversations.

1. The duality map

We first consider an arbitrary Banach space X. The duality map F from X to X^* was defined in Introduction.

LEMMA 1.1. *Let $x, y \in X$. Then $\|x\| \leqq \|x + \alpha y\|$ for every $\alpha > 0$ if and only if there is $f \in Fx$ such that* $\mathrm{Re}\,(y, f) \geqq 0$.

PROOF. The assertion is trivial if $x = 0$. So we shall assume $x \neq 0$ in the following. If $\mathrm{Re}\,(y, f) \geqq 0$ for some $f \in Fx$, then $\|x\|^2 = (x, f) = \mathrm{Re}\,(x, f) \leqq \mathrm{Re}(x + \alpha y, f) \leqq \|x + \alpha y\| \|f\|$ for $\alpha > 0$. Since $\|f\| = \|x\|$, we obtain $\|x\| \leqq \|x + \alpha y\|$.

Suppose, conversely, that $\|x\| \leqq \|x + \alpha y\|$ for $\alpha > 0$. For each $\alpha > 0$ let $f_\alpha \in F(x + \alpha y)$ and $g_\alpha = f_\alpha / \|f_\alpha\|$ so that $\|g_\alpha\| = 1$. Then $\|x\| \leqq \|x + \alpha y\| = (x + \alpha y, g_\alpha) = \mathrm{Re}\,(x, g_\alpha) + \alpha\, \mathrm{Re}\,(y, g_\alpha) \leqq \|x\| + \alpha\, \mathrm{Re}\,(y, g_\alpha)$. Thus

$$\liminf_{\alpha \downarrow 0} \mathrm{Re}\,(x, g_\alpha) \geqq \|x\| \qquad \text{and } \mathrm{Re}\,(y, g_\alpha) \geqq 0\,. \tag{1.1}$$

Since the closed unit ball of X^* is compact in the weak* topology, the net $\{g_\alpha\}$ (with the index set $\{\alpha\}$ directed as $\alpha \downarrow 0$) has a cluster point $g \in X^*$ with $\|g\| \leqq 1$. In view of (1.1), however, g satisfies $\mathrm{Re}\,(x, g) \geqq \|x\|$ and $\mathrm{Re}\,(y, g) \geqq 0$.

Hence we must have $\|g\|=1$ and $(x, g)=\|x\|$. On setting $f=\|x\|g$, we see that $f\in Fx$ and $\operatorname{Re}(y, f)\geqq 0$.

It is known (and is easy to prove) that F is single-valued if X^* is strictly convex. One would need somewhat stronger condition to ensure that F is continuous. A convenient sufficient condition is given by

LEMMA 1.2. *If X^* is uniformly convex, F is single-valued and is uniformly continuous on any bounded set of X. In other words, for each $\varepsilon>0$ and $M>0$, there is $\delta>0$ such that $\|x\|<M$ and $\|x-y\|<\delta$ imply $\|Fx-Fy\|<\varepsilon$.*

PROOF. It suffices to show that the assumptions

$$\|x_n\|<M,\ \|x_n-y_n\|\to 0,\ \|Fx_n-Fy_n\|\geqq\varepsilon_0>0,\ n=1, 2, \cdots,$$

lead to a contradiction. If $x_n\to 0$ (we denote by $\to$ strong convergence), then $y_n\to 0$ and so $\|Fx_n\|=\|x_n\|\to 0$ and similarly $\|Fy_n\|\to 0$, hence $\|Fx_n-Fy_n\|\to 0$, a contradiction. Thus we may assume that $\|x_n\|\geqq\alpha>0$, replacing the given sequences by suitable subsequences if necessary. Then $\|y_n\|\geqq\alpha/2$ for sufficiently large n. Set $u_n=x_n/\|x_n\|$ and $v_n=y_n/\|y_n\|$. Then $\|u_n\|=\|v_n\|=1$ and $u_n-v_n=(x_n-y_n)/\|x_n\|+(\|x_n\|^{-1}-\|y_n\|^{-1})y_n$ so that $\|u_n-v_n\|\leqq 2\|x_n-y_n\|/\|x_n\|\to 0$.

Since $\|Fu_n\|=\|u_n\|=1$ and similarly $\|Fv_n\|=1$, we thus obtain $\operatorname{Re}(u_n, Fu_n+Fv_n)=(u_n, Fu_n)+(v_n, Fv_n)+\operatorname{Re}(u_n-v_n, Fv_n)\geqq 1+1-\|u_n-v_n\|\to 2$. Hence $\liminf\|Fu_n+Fv_n\|\geqq\liminf\operatorname{Re}(u_n, Fu_n+Fv_n)\geqq 2$. Since $\|Fu_n\|=\|Fv_n\|=1$ and X^* is uniformly convex, it follows that $Fu_n-Fv_n\to 0$.

Since $Fx_n=F(\|x_n\|u_n)=\|x_n\|Fu_n$ and similarly $Fy_n=\|y_n\|Fv_n$, we now obtain $Fx_n-Fy_n=\|x_n\|(Fu_n-Fv_n)+(\|x_n\|-\|y_n\|)Fv_n\to 0$ by $\|x_n\|<M$. Thus we have arrived at a contradiction again.

In this paper the usefulness of the duality map depends mainly on the following lemma.

LEMMA 1.3. *Let $x(t)$ be an X-valued function on an interval of real numbers. Suppose $x(t)$ has a weak derivative $x'(s)\in X$ at $t=s$ (that is, $d(x(t), g)/dt$ exists at $t=s$ and equals $(x'(s), g)$ for every $g\in X^*$). If $\|x(t)\|$ is also differentiable at $t=s$, then*

$$\|x(s)\|(d/ds)\|x(s)\|=\operatorname{Re}(x'(s), f) \tag{1.2}$$

for every $f\in Fx(s)$.

PROOF. Since $\operatorname{Re}(x(t), f)\leqq\|x(t)\|\|f\|=\|x(t)\|\|x(s)\|$ and $\operatorname{Re}(x(s), f)=\|x(s)\|^2$, we have

$$\operatorname{Re}(x(t)-x(s), f)\leqq\|x(s)\|(\|x(t)\|-\|x(s)\|).$$

Dividing both sides by $t-s$ and letting $t\to s$ from above and from below, we obtain $\operatorname{Re}(x'(s), f)\lesseqqgtr\|x(s)\|(d/ds)\|x(s)\|$. Thus we must have the equality (1.2).

2. Monotonic operators in X

Monotonic operators A in X have been defined by the equivalent conditions (M) and (M′) given in Introduction. Their equivalence follows immediately from Lemma 1.1.

If X is a Hilbert space, the inverse of an invertible monotonic operator is also monotonic, but it might not be true in the general case.

If A is monotonic, $1+\alpha A$ is invertible for $\alpha>0$ and the inverse operator $(1+\alpha A)^{-1}$ is Lipschitz continuous:

$$\|(1+\alpha A)^{-1}x-(1+\alpha A)^{-1}y\| \leqq \|x-y\|, \qquad x, y\in D((1+\alpha A)^{-1}). \tag{2.1}$$

This follows directly from (M).

LEMMA 2.1. *Let A be monotonic. If $D((1+\alpha A)^{-1})=R(1+\alpha A)$ is the whole of X for some $\alpha>0$, the same is true for all $\alpha>0$.*

PROOF. $R(1+\alpha A)=X$ is equivalent to $R(A+\lambda)=X$ where $\lambda=1/\alpha$. Thus it suffices to show that $R(A+\lambda)=X$ for all $\lambda>0$ if it is true for some $\lambda>0$. But this is proved essentially in [3].

As stated in Introduction, we say that A is m-monotonic if the conditions of Lemma 2.1 are satisfied. Here we do not assume that $D(A)$ is dense in X. If A is a *linear* operator in a Hilbert space, the m-monotonicity of A implies that $D(A)$ is dense, but we do not know whether or not the same is true in the general case.

For an m-monotonic operator A, we introduce the following sequences of operators $(n=1, 2, \cdots)$:

$$J_n=(1+n^{-1}A)^{-1}, \tag{2.2}$$

$$A_n=AJ_n=n(1-J_n), \tag{2.3}$$

where AJ_n denotes the composition of the two maps A and J_n. The J_n and A_n are defined everywhere on X. The identity given by (2,3), which is easy to verify, is rather important in the following arguments.

LEMMA 2.2. *Let A be m-monotonic. J_n and A_n are uniformly Lipschitz continuous, with*

$$\|J_nx-J_ny\| \leqq \|x-y\|, \qquad \|A_nx-A_ny\| \leqq 2n\|x-y\|, \tag{2.4}$$

where $2n$ may be replaced by n if X is a Hilbert space.

PROOF. The first inequality of (2.4) is a special case of (2.1). The second then follows from (2.3). The assertion about the case of X a Hilbert space is easy to prove and the proof is omitted (it is not used in the following).

LEMMA 2.3. *Let A be m-monotonic. The A_n are also monotonic. Furthermore, we have*

$$\|A_n u\| \leqq \|Au\| \qquad \text{for } u \in D(A). \tag{2.5}$$

PROOF. Let $x, y \in X$ and $f \in F(x-y)$. Then

$$\begin{aligned}\operatorname{Re}(A_n x - A_n y, f) &= n \operatorname{Re}(x-y, f) - n \operatorname{Re}(J_n x - J_n y, f) \\ &\geqq n\|x-y\|^2 - n\|J_n x - J_n y\|\|f\| \geqq n\|x-y\|^2 - n\|x-y\|^2 = 0,\end{aligned}$$

where we have used (2.3) and (2.4). Thus A_n is monotonic by (M′). If $u \in D(A)$, we have $A_n u = n(u - J_n u) = n[J_n(1+n^{-1}A)u - J_n u]$ by (2.3) and so $\|A_n u\| \leqq n\|u + n^{-1}Au - u\| = \|Au\|$ by (2.4).

LEMMA 2.4. *If $u \in [D(A)]$ (the closure of $D(A)$ in X), $J_n u \to u$ as $n \to \infty$.*

PROOF. If $u \in D(A)$, then $u - J_n u = n^{-1}A_n u \to 0$ since the $\|A_n u\|$ are bounded by (2.5). The result is extended to all $u \in [D(A)]$ since the J_n are Lipschitz continuous uniformly in n.

LEMMA 2.5. *Let X^* be uniformly convex and let A be m-monotonic in X.*

(a) *If $u_n \in D(A)$, $n = 1, 2, \cdots$, $u_n \to u \in X$ and if the $\|Au_n\|$ are bounded, then $u \in D(A)$ and $Au_n \rightharpoonup Au$ (we denote by $\rightharpoonup$ weak convergence).*

(b) *If $x_n \in X$, $n = 1, 2, \cdots$, $x_n \to u \in X$ and if the $\|A_n x_n\|$ are bounded, then $u \in D(A)$ and $A_n x_n \rightharpoonup Au$.*

(c) *$A_n u \rightharpoonup Au$ if $u \in D(A)$.*

PROOF. In this case the duality map F is single-valued and is continuous (see Lemma 1.2).

(a) The monotonicity condition (M′) gives

$$\operatorname{Re}(Av - Au_n, F(v - u_n)) \geqq 0 \tag{2.6}$$

for any $v \in D(A)$. Since X is reflexive with X^* and the $\|Au_n\|$ are bounded, there is a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $Au_{n'} \rightharpoonup x \in X$. Since $v - u_{n'} \to v - u$ and hence $F(v - u_{n'}) \to F(v-u)$ by the continuity of F, we obtain from (2.6) the inequality $\operatorname{Re}(Av - x, F(v-u)) \geqq 0$.

Using Lemma 1.1 with $\alpha = 1$, we then have $\|v - u + Av - x\| \geqq \|v-u\|$. On setting $v = J_1(u+x)$ so that $v \in D(A)$ and $v + Av = u + x$, we see that $\|v-u\| \leqq 0$, hence $u = v$ and $Au = x$. Thus $Au_{n'} \rightharpoonup x = Au$.

Since we could have started with any subsequence of $\{u_n\}$ instead of $\{u_n\}$ itself, the result obtained shows that Au_n converges weakly to Au.

(b) Set $u_n = J_n x_n \in D(A)$. Then $Au_n = A_n x_n$ and the $\|Au_n\|$ are bounded. Also $x_n - u_n = (1 - J_n)x_n = n^{-1}A_n x_n \to 0$ so that $u_n \to u$. Thus the result of (a) is applicable, with the result that $u \in D(A)$ and $A_n x_n = Au_n \rightharpoonup Au$.

(c) It suffices to set $x_n = u$ in (b); note that $\|A_n u\| \leqq \|Au\|$ by Lemma 2.3.

3. The theorems

We now consider the Cauchy problem for the nonlinear evolution equation (E). We introduce the following conditions for the family $\{A(t)\}$.

I. The domain D of $A(t)$ is independent of t.

II. There is a constant L such that for all $v \in D$ and $s, t \in [0, T]$,

$$\|A(t)v - A(s)v\| \leqq L|t-s|(1+\|v\|+\|A(s)v\|). \tag{3.1}$$

III. For each t, $A(t)$ is m-monotonic.

(3.1) implies that $A(t)v$ is continuous in t and hence is bounded. Then (3.1) shows that $A(t)v$ is uniformly Lipschitz continuous in t. It further shows that the Lipschitz continuity is uniform for $v \in D$ in a certain metric.

On the other hand, we do not make any assumptions on the continuity of the maps $v \rightarrow A(t)v$, except those implicitly contained in the m-monotonicity.

The main results of this paper are given by the following theorems.

THEOREM 1 (*existence theorem*). *Assume that X^* is uniformly convex and that the conditions* I, II, III *are satisfied. For each $a \in D$, there exists an X-valued function $u(t)$ on $[0, T]$ which satisfies* (E) *and the initial condition $u(0) = a$ in the following sense.* (a) *$u(t)$ is uniformly Lipschitz continuous on $[0, T]$, with $u(0) = a$.* (b) *$u(t) \in D$ for each $t \in [0, T]$ and $A(t)u(t)$ is weakly continuous on $[0, T]$.* (c) *The weak derivative of $u(t)$ exists for all $t \in [0, T]$ and equals $-A(t)u(t)$.* (d) *$u(t)$ is an indefinite integral of $-A(t)u(t)$, which is Bochner integrable, so that the strong derivative of $u(t)$ exists almost everywhere and equals $-A(t)u(t)$.*

THEOREM 2 (*uniqueness and continuous dependence on the initial value*). *Under the assumptions of Theorem* 1, *let $u(t)$ and $v(t)$ satisfy conditions* (a), (b), (c) *with the initial conditions $u(0) = a$ and $v(0) = b$, where $a, b \in D$. Then $\|u(t) - v(t)\| \leqq \|a - b\|$ for all $t \in [0, T]$.*

THEOREM 3. *In addition to the assumptions of Theorem* 1, *assume that X is uniformly convex. Then the strong derivative $du/dt = -A(t)u(t)$ exists and is strongly continuous except at a countable number of values t.*

REMARKS. 1. Conditions (a) to (d) in Theorem 1 are not all independent. (a) follows from (b) and (c) (except, of course, $u(0) = a$).

2. When $A(t) = A$ is independent of t, these results give a partial generalization of the Hille-Yosida theorem to semigroups of nonlinear operators. Suppose X^* is uniformly convex and A is m-monotonic in X. Since $T > 0$ is arbitrary in this case, on setting $u(t) = U(t)a$ we obtain a family $\{U(t)\}$, $0 \leqq t < \infty$, of nonlinear operators $U(t)$ on $D(A)$ to itself. Obviously $\{U(t)\}$ forms a semigroup generated by $-A$. It is a *contraction semigroup* on $D(A)$, for $\|U(t)a - U(t)b\| \leqq \|a - b\|$, and it can be extended by continuity to a contraction semi-

group on $[D(A)]$ (the closure of $D(A)$ in X). It should be noted, however, that we have not been able to prove the strong differentiability of $U(t)a$ at $t=0$ for all $a \in D(A)$.

3. If the $A(t)$ are linear operators, the above theorems contain very little that is new. But their proofs are independent of the earlier ones, such as are given by [2], and are even simpler (of course under the restriction on X).

4. The assumptions I to III could be weakened to some extent. For example, it would suffice to assume, instead of III, that for each $t \in [0, T]$ there is a norm $\| \ \|_t$, equivalent to the given norm of X and depending on t "smoothly", with respect to which $A(t)$ is m-monotonic. I and II could be replaced by the condition that there is a function $Q(t)$, depending on t "smoothly", such that $Q(t)$ and $Q(t)^{-1}$ are bounded linear operators with domain and range X and that $\bar{A}(t) = Q(t)^{-1}A(t)Q(t)$ satisfies I and II. We want to deal with such generalizations in later publications.

5. III could also be weakened to the condition that $A(t)+\lambda$ be m-monotonic for some $\lambda > 0$. It should be noted that this is not a trivial generalization. If $A(t)$ were linear, the transformation $u(t) = e^{\lambda t}v(t)$ would change (E) into $dv/dt + (A(t)+\lambda)v = 0$. But the same transformation does not always work in the nonlinear case, for the transformed equation involves the operator $e^{-\lambda t}[A(t)+\lambda]e^{\lambda t}$, the domain of which may depend on t when $D(A(t))$ does not.

4. Proofs of the theorems

To construct a solution of (E), we introduce the operators

$$J_n(t) = (1+n^{-1}A(t))^{-1}, \qquad A_n(t) = A(t)\,J_n(t), \ n = 1, 2, \cdots, \tag{4.1}$$

for which the results of § 2 are available, and consider the approximate equations

$$du_n/dt + A_n(t)u_n = 0, \qquad u_n(0) = a\,. \tag{E_n}$$

To solve (E_n) and prove the convergence of $\{u_n(t)\}$, we need some estimates for the $A_n(t)$.

LEMMA 4.1. *For all n and $v \in D$, we have*

$$\|A_n(t)v - A_n(s)v\| \leqq L\,|t-s|\,(1+\|v\| + (1+n^{-1})\|A_n(s)v\|)\,. \tag{4.2}$$

PROOF. Since $A_n(t) = n(1-J_n(t))$ by (2.3), we have

$$\begin{aligned} A_n(t)v - A_n(s)v &= nJ_n(s)v - nJ_n(t)v \\ &= nJ_n(t)[1+n^{-1}A(t)]J_n(s)v - nJ_n(t)[1+n^{-1}A(s)]J_n(s)v\,. \end{aligned}$$

Using the Lipschitz continuity (2.4) of the operator $J_n(t)$, we obtain

$$\|A_n(t)v - A_n(s)v\| \leqq n\|[1+n^{-1}A(t)]J_n(s)v - [1+n^{-1}A(s)]J_n(s)v\|$$
$$= \|[A(t)-A(s)]J_n(s)v\|\,,$$

and using (3.1),

$$\|A_n(t)v - A_n(s)v\| \leqq L|t-s|(1+\|J_n(s)v\|+\|A(s)J_n(s)v\|)\,. \tag{4.3}$$

Here $\|J_n(s)v\|$ is estimated by (2.3) as $\|J_n(s)v\| \leqq \|v\|+n^{-1}\|A_n(s)v\|$. Since $A(s)J_n(s) = A_n(s)$, (4.3) gives (4.2).

(4.2) shows that $A_n(t)v$ is Lipschitz continuous in t for each $v \in X$. On the other hand, the map $v \rightarrow A_n(t)v$ is Lipschitz continuous for fixed t, uniformly in v and t (see (2.4)). Thus (E_n) has a unique solution $u_n(t)$ for $t \in [0, T]$, for any initial condition $u_n(0) = a \in X$. We shall now deduce some estimates for $u_n(t)$.

LEMMA 4.2. *Let $a \in D$. Then there is a constant K such that $\|u_n(t)\| \leqq K$, $\|u_n'(t)\| = \|A_n(t)u_n(t)\| \leqq K$, for all $n = 1, 2, \cdots$ and $t \in [0, T]$.* (We write $du_n/dt = u_n'$.)

PROOF. We apply Lemma 1.3 to $x_n(t) = u_n(t+h) - u_n(t)$, where $0 < h < T$. Since $x_n(t)$ is differentiable with $x_n'(t) = -[A_n(t+h)u_n(t+h) - A_n(t)u_n(t)]$, (1.2) gives

$$\|x_n(t)\|(d/dt)\|x_n(t)\| = -\mathrm{Re}\,(A_n(t+h)u_n(t+h) - A_n(t)u_n(t), Fx_n(t)) \tag{4.4}$$

for each t where $\|x_n(t)\|$ is differentiable; note that the duality map F is single-valued because X^* is uniformly convex (see Lemma 1.2).

The first factor in the scalar product on the right of (4.4) can be written

$$[A_n(t+h)u_n(t+h) - A_n(t+h)u_n(t)] + [A_n(t+h)u_n(t) - A_n(t)u_n(t)]\,,$$

of which the first term contributes to (4.4) a nonpositive value by the monotonicity of $A_n(t+h)$ (see Lemma 2.3). The second term can be estimated by (4.2); it is thus majorized in norm by $Lh(1+\|u_n(t)\|+(1+n^{-1})\|u_n'(t)\|)$, where we have used $A_n(t)u_n(t) = -u_n'(t)$. In this way we obtain from (4.4), using the Schwarz inequality and the norm-preserving property of F,

$$\|x_n(t)\|(d/dt)\|x_n(t)\| \leqq Lh(1+\|u_n(t)\|+(1+n^{-1})\|u_n'(t)\|)\|x_n(t)\|\,. \tag{4.5}$$

Since $\|x_n(t)\|$ is Lipschitz continuous with $x_n(t)$, it is differentiable almost everywhere, where (4.5) is true as shown above. Let N be the set of t for which $x_n(t) = 0$. If t is not in N, we can cancel $\|x_n(t)\|$ in (4.5) to obtain

$$(d/dt)\|x_n(t)\| \leqq Lh(1+\|u_n(t)\|+(1+n^{-1})\|u_n'(t)\|)\,. \tag{4.6}$$

If t is a cluster point of N, then $(d/dt)\|x_n(t)\| = 0$ as long as it exists, so that (4.6) is still true. Since there are only a countable number of isolated points of N, it follows that (4.6) is true almost everywhere. Since $\|x_n(t)\|$ is absolutely continuous, we obtain finally

$$\|x_n(t)\| \leqq \|x_n(0)\| + Lh\int_0^t (1+\|u_n(s)\|+(1+n^{-1})\|u_n'(s)\|)ds\,. \tag{4.7}$$

Since $x_n(t)=u_n(t+h)-u_n(t)$, by dividing (4.7) by h and letting $h\downarrow 0$ we obtain

$$\|u_n'(t)\| \leqq \|u_n'(0)\| + Lt + L\int_0^t (\|u_n(s)\|+(1+n^{-1})\|u_n'(s)\|)ds\,. \tag{4.8}$$

Since $\|u_n'(0)\| = \|A_n(0)a\| \leqq \|A(0)a\|$ by (2.5), we have

$$\|u_n'(t)\| \leqq K + 2L\int_0^t (\|u_n(s)\|+\|u_n'(s)\|)ds\,,$$

where K is a constant independent of n. On the other hand, $u_n(t)=a+\int_0^t u_n'(s)d$ so that

$$\|u_n(t)\| \leqq \|a\| + \int_0^t \|u_n'(s)\|ds\,.$$

Adding the two inequalities, we obtain

$$\|u_n(t)\|+\|u_n'(t)\| \leqq K+(2L+1)\int_0^t (\|u_n(s)\|+\|u_n'(s)\|)ds$$

with a different constant K. Solving this integral inequality, we see that $\|u_n(t)\|+\|u_n'(t)\|$ is bounded for all n and t.

LEMMA 4.3. *The strong limit $u(t)=\lim_{n\to\infty} u_n(t)$ exists uniformly for $t\in[0, T]$. $u(t)$ is Lipschitz continuous with $u(0)=a$.*

PROOF. We apply Lemma 1.3 to $x_{mn}(t)=u_m(t)-u_n(t)$. As above we obtain for almost all t

$$\frac{1}{2}(d/dt)\|x_{mn}(t)\|^2 = -\mathrm{Re}\,(A_m(t)u_m(t)-A_n(t)u_n(t), Fx_{mn}(t))\,. \tag{4.9}$$

Since $A_m(t)u_m(t)=A(t)J_m(t)u_m(t)$ etc. and since $A(t)$ is monotonic, we have

$$0 \leqq \mathrm{Re}\,(A_m(t)u_m(t)-A_n(t)u_n(t), Fy_{mn}(t))\,, \tag{4.10}$$

where $y_{mn}(t)=J_m(t)u_m(t)-J_n(t)u_n(t)$. Addition of (4.9) and (4.10) gives

$$\begin{aligned}\frac{1}{2}(d/dt)\|x_{mn}(t)\|^2 &\leqq \mathrm{Re}\,(A_m(t)u_m(t)-A_n(t)u_n(t), Fy_{mn}(t)-Fx_{mn}(t))\\ &\leqq 2K\|Fy_{mn}(t)-Fx_{mn}(t)\| \qquad \text{for almost all } t,\end{aligned}$$

where we have used Lemma 4.2.

Since $\|x_{mn}(t)\|^2$ is absolutely continuous and $x_{mn}(0)=a-a=0$, we obtain

$$\|x_{mn}(t)\|^2 \leqq 4K\int_0^t \|Fy_{mn}(s)-Fx_{mn}(s)\|ds\,. \tag{4.11}$$

We want to prove that $\|x_{mn}(t)\|\to 0$ uniformly in t, by showing that the integrand in (4.11) tends to zero uniformly in s. Now $\|x_{mn}(s)\| = \|u_m(s)-u_n(s)\| \leqq 2K$ by Lemma 4.2. Also

$$\|y_{mn}(s)-x_{mn}(s)\| \leqq J_m(s)u_m(s)-u_m(s)\| + \|J_n(s)u_n(s)-u_n(s)\|$$
$$\leqq m^{-1}\|A_m(s)u_m(s)\| + n^{-1}\|A_n(s)u_n(s)\| \leqq (m^{-1}+n^{-1})K \to 0$$

as $m, n \to \infty$, where we have used (2.3) and Lemma 4.2. It follows from Lemma 1.2 that for any $\varepsilon > 0$, we have $\|Fy_{mn}(s)-Fx_{mn}(s)\| < \varepsilon$, $0 \leqq s \leqq T$, for sufficiently large m, n, as we wished to show. Thus $u(t) = \lim u_n(t)$ exists uniformly in t.

Since $u_n(t)$ is Lipschitz continuous uniformly in t and n by $\|u_n'(t)\| \leqq K$, the limit $u(t)$ is also Lipschitz continuous uniformly in t, with $u(0) = a$.

LEMMA 4.4. *$u(t) \in D$ for all $t \in [0, T]$, and $A(t)u(t)$ is bounded and is weakly continuous.*

PROOF. For each t we have $u_n(t) \to u(t)$ and $\|A_n(t)u_n(t)\| \leqq K$. It follows from Lemma 2.5, (b), that $u(t) \in D(A(t)) = D$ and $A_n(t)u_n(t) \rightharpoonup A(t)u(t)$. Thus $\|A(t)u(t)\| \leqq K$, too.

To prove the weak continuity of $A(t)u(t)$, let $t_k \to t$: we have to show that $A(t_k)u(t_k) \rightharpoonup A(t)u(t)$. Now

$$\|[A(t)-A(t_k)]u(t_k)\| \leqq L|t-t_k|(1+\|u(t_k)\|+\|A(t_k)u(t_k)\|) \tag{4.12}$$
$$\leqq L|t-t_k|(1+2K) \to 0 .$$

This implies, in particular, that $\limsup \|A(t)u(t_k)\| = \limsup \|A(t_k)u(t_k)\| \leqq K$. Since $u(t_k) \to u(t)$, it follows from Lemma 2.5, (a), that $A(t)u(t_k) \rightharpoonup A(t)u(t)$. Using (4.12) once more, we see that $A(t_k)u(t_k) \rightharpoonup A(t)u(t)$.

LEMMA 4.5. *For each $f \in X^*$, $(u(t), f)$ is continuously differentiable on $[0, T]$, with $d(u(t), f)/dt = -(A(t)u(t), f)$.*

PROOF. Since $u_n(t)$ satisfies (E_n), we have

$$(u_n(t), f) = (a, f) - \int_0^t (A_n(s)u_n(s), f)ds .$$

Since $u_n(t) \to u(t)$, $A_n(s)u_n(s) \rightharpoonup A(s)u(s)$, and $|(A_n(s)u_n(s), f)| \leqq K\|f\|$ by Lemma 4.2, we obtain

$$(u(t), f) = (a, f) - \int_0^t (A(s)u(s), f)ds \tag{4.13}$$

by bounded convergence. Since the integrand is continuous in s by Lemma 4.4, the assertion follows.

LEMMA 4.6. *$A(t)u(t)$ is Bochner integrable, and $u(t)$ is an indefinite integral of $-A(t)u(t)$. The strong derivative $du(t)/dt$ exists almost everywhere and equals $-A(t)u(t)$.*

PROOF. Let X_0 be the smallest closed linear subspace of X containing all the values of the $A_n(t)u_n(t)$ for $t \in [0, T]$ and $n = 1, 2, \cdots$. Since the $A_n(t)u_n(t)$

are continuous, X_0 is separable. Since $A_n(t)u_n(t) \rightharpoonup A(t)u(t)$ as shown above in the proof of Lemma 4.4 and since X_0 is weakly closed, $A(t)u(t) \in X_0$ too. Thus $A(t)u(t)$ is separably-valued. Since it is weakly continuous, it is strongly measurable (see e. g. Yosida [5], p. 131) and, being bounded, it is Bochner integrable (see [5], p. 133). Then (4.13) shows that $u(t)$ is an indefinite integral of $-A(t)u(t)$. The last statement of the lemma is a well-known result for Bochner integrals (see [5], p. 134).

LEMMA 4.7. *Let $u(t)$ and $v(t)$ be any functions satisfying the conditions of Lemma 4.5 and the initial conditions $u(0)=a \in D$, $v(0)=b \in D$. Then $\|u(t)-v(t)\| \leqq \|a-b\|$.*

PROOF. $x(t)=u(t)-v(t)$ has weak derivative $-A(t)u(t)+A(t)v(t)$, which is weakly continuous and hence bounded. Thus $x(t)$ is Lipschitz continuous and so $\|x(t)\|$ is differentiable almost everywhere. It follows from Lemma 1.3 that

$$\frac{1}{2}(d/dt)\|x(t)\|^2 = -\mathrm{Re}\,(A(t)u(t)-A(t)v(t), Fx(t)) \leqq 0$$

almost everywhere. Since $\|x(t)\|^2$ is absolutely continuous, it follows that $\|x(t)\| \leqq \|x(0)\| = \|a-b\|$.

The lemmas proved above give complete proof to Theorems 1 and 2. In particular we note that the solution $u(t)$ of the Cauchy problem is unique.

LEMMA 4.8. *For sufficiently large $M>0$, $\|A(t)u(t)\|-Mt$ is monotonically decreasing in t. (Hence $\|A(t)u(t)\|$ is continuous except possibly at a countable number of points t.)*

PROOF. Returning to (4.8) and noting that the integrand is uniformly bounded by Lemma 4.2, we obtain

$$\|u_n'(t)\| \leqq \|A(0)a\|+Mt\,, \tag{4.14}$$

where M is a constant independent of t and n (note that $\|u_n'(0)\| = \|A_n(0)a\| \leqq \|A(0)a\|$ as shown before). Since $u_n'(t) = -A_n(t)u_n(t) \rightharpoonup -A(t)u(t)$, going to the limit $n \to \infty$ in (4.14) gives

$$\|A(t)u(t)\| \leqq \|A(0)u(0)\|+Mt\,. \tag{4.15}$$

If we consider (E) on the interval $[s, T]$ with the initial value $u(s)$, the solution must coincide with our $u(t)$ on $[s, T]$ owing to the uniqueness of the solution. If we apply (4.15) to the new initial value problem, we see that $\|A(t)u(t)\| \leqq \|A(s)u(s)\|+M(t-s)$ for $t>s$. Thus $\|A(t)u(t)\|-Mt$ is monotonically nonincreasing.

LEMMA 4.9. *If X is uniformly convex, then $A(t)u(t)$ is strongly continuous except possibly at a countable number of points t.*

PROOF. Since $A(t)u(t)$ is weakly continuous, it is strongly continuous at each point t where $\|A(t)u(t)\|$ is continuous. Thus the assertion follows from

Lemma 4.8.

Lemma 4.9 immediately leads to Theorem 3.

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(Notes added in proof) 1. In a recent paper by F. E. Browder, *Nonlinear accretive operators in Banach spaces*, Bull. Amer. Math. Soc., **73** (1967), 470–476, the notion of accretive operators A in a Banach space X is introduced, which is almost identical with that of monotonic operators defined in the present paper. There is a slight difference that he requires $\mathrm{Re}\,(Au-Av, f)\geqq 0$ for every $f\in F(u-v)$ whereas we require it only for some $f\in F(u-v)$. Of course the two definitions coincide if F is single-valued.

2. Browder has called the attention of the writer to a paper by S. Ôharu, *Note on the representation of semi-groups of non-linear operators*, Proc. Japan Acad., **42** (1967), 1149–1154, which contains, among others, a proof of Lemma 2.1.

3. Browder remarked also that the condition (3.1) can be weakened to

$$\|A(t)v-A(s)v\|\leqq|t-s|L(\|v\|)(1+\|A(s)v\|)\,, \tag{3.1$'$}$$

where $L(r)$ is a positive, nondecreasing function of $r>0$. In this case the proof of the theorems needs a slight modification. First, it is easily seen that we have, instead of (4.2),

$$\|A_n(t)v-A_n(s)v\|\leqq|t-s|L_1(\|v\|)(1+\|A_n(s)v\|)\,, \tag{4.2$'$}$$

where $L_1(r)=L(r+K_1)$ for some constant $K_1>0$ (we may choose $K_1=2\|a\|+\sup\limits_{0\leqq t\leqq T}\|A(t)a\|$). Lemma 4.2 is seen to remain true, but to prove it we first prove the uniform boundedness for $\|u_n(t)\|$, independently of $\|u_n'(t)\|$. This can be done easily by estimating $(d/dt)\|u_n(t)-a\|^2$ in the manner similar to the estimate for $\|x_n(t)\|$, with the result

$$\|u_n(t)-a\|\leqq\int_0^t\|A_n(s)a\|ds\leqq\int_0^t\|A(s)a\|ds\leqq K_2\,.$$

Then the estimate for $\|u_n'(t)\|$ can be obtained from (the analogue of) (4.8) by solving an integral inequality for $\|u_n'(t)\|$. The proof of the remaining lemmas are unchanged.

4. The proof of Lemma 4.6 was unnecessarily long. It is sufficient to notice that a weakly continuous function of t is separably-valued.

5. Our theorems are rather weak when applied to *regular equations* (E), in which

the $A(t)$ are continuous operators defined everywhere on X, for it is known that we then need much less continuity of $A(t)$ as a function of t. The theorems could be strengthened by writing $A(t)=A_0(t)+B(t)$ in which $A_0(t)$ is assumed to satisfy Conditions I to III and $B(t)$ to be "regular" with a milder continuity condition as a function of t.