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A remark on the principal ideal theorem

Dedicated to Professor S. Iyanaga on his sixtieth birthday

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1. The principal ideal theorem was conjectured by D. Hilbert and was proved by Ph. Furtwängler [**3**]. The proof was simplified by S. Iyanaga [**4**], H. G. Schumann (and W. Franz) [**8**] and E. Witt [**11**]. The purpose of this short note is to give a cohomology-theoretic interpretation of these proofs.

The problem can be formulated in any class formation. We use the same notations as in [**5**]. Let $\mathfrak{K}$ be the given family of fields. To each $k \in \mathfrak{K}$ an abelian group $E(k)$ is attached such that (i) for each extension K/k ($k, K \in \mathfrak{K}$) there is an injection $\varphi_{k/K}: E(k) \to E(K)$, (ii) for each Galois extension K/k ($k, K \in \mathfrak{K}$) the Galois group $G = G(K/k)$ operates on $E(K)$ and $\varphi_{k/K}E(k) = E(K)^G$ [1]. We assume, furthermore, that $\{E(K); K \in \mathfrak{K}\}$ satisfies the axioms of a class formation in [**5**].

Let $k, K, L \in \mathfrak{K}$, $k \subset K \subset L$ such that K/k and L/k are both Galois extensions, and let K/k be the maximal abelian extension in L/k. Hence the Galois group $H = G(L/K)$ is the commutator subgroup of the Galois group $G = G(L/k)$: $H = [G, G]$.

PRINCIPAL IDEAL THEOREM. *Under the above assumptions*

$$\text{(I)} \qquad \varphi_{k/K}E(k) \subset N_H E(L) \qquad \textit{for } H = [G, G].$$

Since $H^0(G, E(L)) \cong E(k)/N_G E(L)$, $H^0(H, E(L)) \cong E(K)/N_H E(L)$ and the restriction mapping $\mathrm{res}_{G/H}: H^0(G, E(L)) \to H^0(H, E(L))$ is the canonical mapping: $\alpha \bmod N_G E(L) \to \alpha \bmod N_H E(L)$, the above proposition (I) is equivalent to

$$\text{(I)*} \qquad \mathrm{res}_{G/H} H^0(G, E(L)) = 1 \qquad \textit{for } H = [G, G].$$

2. Let $\xi_{L/k} \in H^2(G, E(L))$ and $\xi_{L/K} \in H^2(H, E(L))$ be the canonical cohomology classes. By the fundamental theorem of J. Tate [**9**] there are isomorphisms: $H^{-2}(G, \boldsymbol{Z}) \cong H^0(G, E(L))$ by $\eta_{L/k} \to \xi_{L/k} \cup \eta_{L/k}$ and $H^{-2}(H, \boldsymbol{Z}) \cong H^0(H, E(L))$ by $\zeta_{L/K} \to \xi_{L/K} \cup \zeta_{L/K}$, where $\boldsymbol{Z}$ denotes the additive group of integers on which

1) For a G-group A we denote by A^G the set of all G-invariant elements of A, by $I_G A$ the set $\sum_{\sigma \in G} (\sigma - 1)A$, and by ${}_G A$ the set of all $\alpha \in A$ such that $N_G \alpha = 0$. Here N_G means the norm operation with respect to the group G.

G and H operate trivially. By the formulas $\mathrm{res}_{G/H}(\alpha \cup \beta) = (\mathrm{res}_{G/H}\alpha) \cup (\mathrm{res}_{G/H}\beta)$ and $\mathrm{res}_{G/H}\xi_{L/k} = \xi_{L/K}$ the proposition (I)* is equivalent to

(II)* $$\mathrm{res}_{G/H} H^{-2}(G, \boldsymbol{Z}) = 1 \qquad \textit{for } H = [G, G].$$

Since $H^{-2}(G, \boldsymbol{Z}) \cong G/[G, G]$, $H^{-2}(H, \boldsymbol{Z}) \cong H/[H, H]$, and $\mathrm{res}_{G/H} : H^{-2}(G, \boldsymbol{Z}) \rightarrow H^{-2}(H, \boldsymbol{Z})$ is given by the transfer mapping: $\sigma \bmod [G, G] \rightarrow (V_{G\rightarrow H}\sigma) \bmod [H, H]$, the proposition (II)* is equivalent to

(II) $$V_{G\rightarrow H} G \subset [H, H] \qquad \textit{for } H = [G, G].$$

This is the group-theoretical formulation of the principal ideal theorem due to E. Artin [1].

3. Let H be a normal subgroup of G. Let $Z(G) = \sum_{\sigma \in G} \boldsymbol{Z}\sigma$ be the group algebra of G over $\boldsymbol{Z}$ and $I(G) = I_G(Z(G)) = \sum_{\sigma \neq 1} \boldsymbol{Z}(\sigma - 1)$. We denote $\delta\sigma = \sigma - 1$ for $\sigma \in G$. Then we have an exact sequence of G-groups and G-homomorphisms

(1) $$0 \longrightarrow I(G) \xrightarrow{\iota} Z(G) \xrightarrow{T} \boldsymbol{Z} \longrightarrow 0 \qquad \text{(exact)},$$

where ι is the injection and $T(\sum a_\sigma \sigma) = \sum a_\sigma$ $(a_\sigma \in \boldsymbol{Z})$. Since $Z(G)$ has trivial cohomologies with respect to both G and H we have the isomorphisms: $\delta_G^\# : H^{-2}(G, \boldsymbol{Z}) \cong H^{-1}(G, I(G)) \cong I(G)/I_G I(G)$ and $\delta_H^\# : H^{-2}(H, \boldsymbol{Z}) \cong H^{-1}(H, I(G)) \cong {}_H I(G)/I_H I(G) = I_H Z(G)/I_H I(G)$. Here $\delta_G^\#$ and $\delta_H^\#$ are given by $\sigma \bmod [G, G] \rightarrow \delta\sigma \bmod I_G I(G)$ and $\rho \bmod [H, H] \rightarrow \delta\rho \bmod I_H I(G)$, respectively. $H^{-1}(G, I(G))$ and $H^{-1}(H, I(G))$ are both G-groups and H operates trivially on them. Hence we can consider them as G/H-groups. The restriction mapping $\mathrm{res}_{G/H} : H^{-1}(G, I(G)) \rightarrow H^{-1}(H, I(G))^G$ is a G/H-homomorphism, and is given by $\alpha \rightarrow N_{G/H}\alpha$ for $\alpha \in H^{-1}(G, I(G))$. Since $\delta_H^\# \cdot \mathrm{res}_{G/H} = \mathrm{res}_{G/H} \cdot \delta_G^\#$ holds, the proposition (II)* is equivalent to

(III)* $$\mathrm{res}_{G/H} H^{-1}(G, I(G)) = 1 \qquad \textit{for } H = [G, G].$$

Let $G = \bigcup_{i=1}^{n} H\tau_i$. Then by the above considerations (III)* is equivalent to

(III) $$\left(\sum_{i=1}^{n} \tau_i\right) I(G) \subset I_H I(G) \qquad \textit{for } H = [G, G].$$

This is the additive formulation of the principal ideal theorem due to E. Witt [11].

4. Let H be a normal subgroup of G as above. Consider the G/H-group $I(G)/I_H I(G)$ as the group extension:

(2) $$0 \rightarrow I_H Z(G)/I_H I(G) \rightarrow I(G)/I_H I(G) \rightarrow I(G)/I_H Z(G) \rightarrow 0 \qquad \text{(exact)}$$

where the first term is isomorphic to $H/[H, H]$ and the third term is isomorphic to $I(G/H)$, (see [6], p. 420).

PROPOSITION. *The G/H-group $I(G)/I_H I(G)$ is isomorphic to the G/H-group $\bar{\mathfrak{U}}$ defined by Iyanaga* ([4], p. 353) *for the group $\mathfrak{U}=H/[H, H]$. Namely, the G/H-group $I(G)/I_H I(G)$ is essentially the same as the splitting group of $G/[H, H]$ considered by Iyanaga.*

PROOF. Let $G=\sum_{i=1}^{n} H\tau_i$ and $\rho\in H$. Since $\delta(\rho\tau_i)=\delta\rho+\delta\tau_i+(\delta\rho)(\delta\tau_i)$ $\equiv\delta\rho+\delta\tau_i \bmod I_H I(G)$, we can express $\alpha\in I(G)$ by $\alpha\equiv\sum_{\rho\in H} a_\rho\delta\rho+\sum_{i=2}^{n} b_i\delta\tau_i \bmod I_H I(G)$, a_ρ, $b_i\in \boldsymbol{Z}$. Let $\dot{\mathfrak{U}}=(\prod_{i=2}^{n} A_i^{Z})\mathfrak{U}$ and consider the mapping $\Phi: I(G)/I_H I(G)\rightarrow\bar{\mathfrak{U}}$ defined by $\Phi(\alpha)=(\prod_{i=2}^{n} A_i^{b_i})(\prod_{\rho\in H}\rho^{a_\rho})$. Let $\tau_j\cdot\tau_i=\rho\cdot\tau_k$, $\rho\in H$. Since $\tau_j(\delta\tau_i)$ $\equiv\delta\tau_k-\delta\tau_j+\delta\rho \bmod I_H I(G)$, we have $\Phi(\tau_j\cdot\delta\tau_i)=A_k\cdot A_j^{-1}\cdot\rho$. Comparing this with the definition of $\bar{\mathfrak{U}}$ in [4] we see that Φ is a G/H-isomorphism, Q. E. D.

5. By using this Proposition we can translate the proof by Iyanaga to a proof of (III). This was done by Witt [11]. For the sake of completeness we shall indicate the outline of the proof of (III).

Let $\lambda\in Z(G)$ and consider the mapping $\Psi_\lambda: I(G)\rightarrow I(G)$ defined by $\Psi_\lambda(\alpha)$ $=\lambda\alpha$ for $\alpha\in I(G)$.

(IV) In order that Ψ_λ maps $I(G)/I_G I(G)\rightarrow I_H Z(G)/I_H I(G)$ it is necessary and sufficient that

$$\lambda\equiv r(\sum_{i=1}^{n}\tau_i) \bmod I_H Z(G).$$

Moreover, r is given by $r=T(\lambda)/n$, where $n=[G:H]$.

PROOF. (i) Let $\tau_j\cdot\sigma=\rho_j\cdot\tau_{j'}(\rho_j\in H)$ for $\sigma\in G$. Then we can see easily $(\sum\tau_j)\delta\sigma\equiv\sum\delta\rho_j \bmod I_H I(G)$.

(ii) Let $\Psi_\lambda(I(G))\subset I_H Z(G)$. Put $\lambda=\sum_{j=1}^{n}(\sum_{\rho\in H} a_{\rho j}\rho)\tau_j\cdot a_{\rho j}\in\boldsymbol{Z}$. Then $\lambda\delta\sigma$ $=\sum_{j'}\alpha_{j'}\tau_{j'}$, $\alpha_{j'}=\sum_\rho a_{\rho j}\rho\cdot\rho_j-\sum_\rho a_{\rho j'}\rho\cdot\lambda\delta\sigma$ belongs to $I_H Z(G)$ if and only if all $\alpha_{j'}$ belong to $I(H)$. This is equivalent to $T(\alpha_{j'})=0$ $(j=1,\cdots,n)$, i.e. $\sum_{\rho\in H} a_{\rho j}=r$ $(j=1,\cdots,n)$. This implies $\lambda\equiv r(\sum\tau_j) \bmod I_H Z(G)$, Q. E. D.

REMARK. Since H operates trivially both on $I(G)/I_G I(G)$ and on $I_H Z(G)$ $/I_H I(G)$, we can consider $\Psi_{\bar\lambda}$ for $\bar\lambda\in Z(G/H)$.

(V) In order that (III) holds it is necessary and sufficient that there exists an element $\lambda\in Z(G)$ (or $\in Z(G/H)$) with the property $T(\lambda)=n$ and $\Psi_\lambda(I(G))\subset I_H I(G)$.

This follows immediately from (IV). We see also

(VI) $\delta(\sigma_1^{m_1},\cdots,\sigma_N^{m_N})=\sum_{i=1}^{N}\mu_i(\delta\sigma_i)$, $\mu_i\in Z(G)$ with $T(\mu_i)=m_i$ $(i=1,\cdots,N)$.

The last step is the most important part of the proof.

(VII) (Witt) Assume that $[H, H]=1$. We can choose generators $\sigma_1, \cdots, \sigma_N$ of G with the relations

$$\prod_k \sigma_k^{m_{ik}} \cdot \varphi_i = 1, \qquad i=1, \cdots, N, \tag{3}$$

where $\varphi_i \in [G, G]$ and $\det(m_{ik})=n$, $(n=[G:H])$. Apply the homomorphism $\delta: G/[G, G] \rightarrow I(G)$ to (3). Then we have

$$\sum_{k=1}^{N} \mu_{ik}\delta\sigma_k = 0 \qquad (i=1, \cdots, N)$$

with $T(\mu_{ik})=m_{ik}$. Consider $\bar{\mu}_{ik}$ in $Z(G/H)$ and take $\bar{\lambda}=\det(\bar{\mu}_{ik})$ in $Z(G/H)$. Then $\bar{\lambda}$ satisfies the condition in (V).

PROOF. Take the cofactor $\bar{\lambda}_{ik}$ of $\bar{\mu}_{ik}$ in $Z(G/H)$. Then we have $0=\sum_{k,i}\bar{\lambda}_{hi}\bar{\mu}_{ik}\delta\sigma_k \equiv \det(\bar{\mu}_{ik})\delta\sigma_h \mod I_H I(G)$ $(h=1, \cdots, N)$ and $T(\det(\bar{\mu}_{ik}))=\det(m_{ik})=n$. Hence, $\bar{\lambda}=\det(\bar{\mu}_{ik})$ satisfies the condition in (V) (cf. also Artin-Tate [2], Chap. 13). Q. E. D.

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