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The word problem for free distributive lattices

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In his previous paper [2] the author gave a decision procedure for the word problem for free distributive lattices following Whitman's way for free lattices [3]. But its presentation and proof contained something inelegant and left much to be desired[1]. In the present paper the author gives a more natural decision procedure with a more acceptable proof which is also faithful to Whitman's.

We shall use the following notation. The symbol $\equiv$ means the equality between words identified under associativity and commutativity. (It is easily shown that to start with such an identification does not harm the effective decidability of our decision procedure.) Letters $x, y, \cdots$ stand for generators, $a, b, \cdots, \alpha, \beta, \cdots$ for non-empty words, while the Greek capitals Λ and Δ for words possibly empty. $|a|$ denotes the length of a, i.e., the number of generators appearing in a, counting repetitions.

THEOREM. *Let $\leqq$ denote the inclusion relation in a free distributive lattice. Then $a \leqq c$ if and only if one of the following conditions is satisfied:*

E: $\exists x_1, \cdots, x_m, y, z_1, \cdots, z_n \{a \equiv x_1 \cap \cdots \cap x_m \cap y \ \& \ c \equiv y \cup z_1 \cup \cdots \cup z_n\}$,[2]

A: $\exists \alpha, \beta \{a \equiv \alpha \cup \beta \ \& \ \alpha, \beta \leqq c\}$,[3]

B: $\exists \varepsilon, \varphi \{c \equiv \varepsilon \cap \varphi \ \& \ a \leqq \varepsilon, \varphi\}$,

C: $\exists \alpha, \beta, \gamma \{a \equiv (\alpha \cup \beta) \cap \gamma \ \& \ \alpha \cap \gamma, \beta \cap \gamma \leqq c\}$,[4] *or*

D: $\exists \delta, \varepsilon, \varphi \{c \equiv \delta \cup (\varepsilon \cap \varphi) \ \& \ a \leqq \delta \cup \varepsilon, \delta \cup \varphi\}$.

PROOF. 'If' is readily verified since our condition (E or A or B or C or D) in a distributive lattice implies $a \leqq c$. To prove 'only if' define a relation $\subset$,

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1) The author however has a particular preference in Bowden's distributive law ([1] p. 134) $(X \cup Y) \cap Z \leqq X \cup (Y \cap Z)$ because of its superior simplicity. It is worth noting that the corresponding modular law is $(X \cup Y) \cap Z \not> X \cup (Y \cap Z)$ *without any assumption*, thus improving a statement in [1] p. 65.

2) m and n are non-negative integers throughout this paper and $x_1, \cdots, z_n$ are generators not necessarily distinct.

3) '$a, b \leqq c, d$' means that '$a \leqq c$ & $a \leqq d$ & $b \leqq c$ & $b \leqq d$'. Similarly for '$a, b \subset c, d$' appearing later.

4) A and C can be put together into one condition:

$$\exists \alpha, \beta, \Lambda \{a \equiv (\alpha \cup \beta) \cap \Lambda \ \& \ \alpha \cap \Lambda, \beta \cap \Lambda \leqq c\}.$$

Similarly for B and D.

following Whitman's way [3], that $a \subset c$ if and only if one of the conditions E, A, B, C, D is true with $\leqq$ replaced by $\subset$. We have to show in the following that our system so obtained, though the exact meaning of which being indicated at the last stage of our proof, forms a distributive lattice.

We shall hereafter use the following notation. '$\xrightarrow{1} a \subset c$' means that we get $a \subset c$ by 1 or by the induction hypothesis of 1. '$\xrightarrow{A} a \subset c$' means that we get $a \subset c$ by using A with $\leqq$ replaced by $\subset$. '$A\{a \subset c\}$' means that $a \subset c$ is deduced immediately from A with $\leqq$ replaced by $\subset$.

1. $x \cap \Lambda \subset x \cup \Delta$.

PROOF. Induction on $|\Lambda| + |\Delta|$.

If $\Lambda \equiv y_1 \cap \cdots \cap y_m$ and $\Delta \equiv z_1 \cup \cdots \cup z_n$, $\xrightarrow{E} x \cap \Lambda \subset x \cup \Delta$.

If $\Lambda \equiv \alpha \cup \beta$ (or dually $\Delta \equiv \varepsilon \cap \varphi$), $\xrightarrow{1} x \cap \alpha,\ x \cap \beta \subset x \cup \Delta \xrightarrow{C} x \cap \Lambda \subset x \cup \Delta$.

If $\Lambda \equiv (\alpha \cup \beta) \cap \gamma$ (or dually $\Delta \equiv \delta \cup (\varepsilon \cap \varphi)$),

$$\xrightarrow{1} x \cap \alpha \cap \gamma,\ x \cap \beta \cap \gamma \subset x \cup \Delta \xrightarrow{C} x \cap \Lambda \subset x \cup \Delta\,.$$

I. $a \cap \Lambda \subset a \cup \Delta$.

PROOF. Induction on $|a|$.

If $a \equiv x$, $\xrightarrow{1} a \cap \Lambda \subset a \cup \Delta$.

If $a \equiv \alpha \cup \beta$ (or dually $a \equiv \varepsilon \cap \varphi$),

$$\xrightarrow{\mathrm{I}} \alpha \cap \Lambda,\ \beta \cap \Lambda \subset \alpha \cup \beta \cup \Delta \xrightarrow{A \text{ or } C} a \cap \Lambda \subset a \cup \Delta\,.$$

2. *$a \cup b \subset c$ implies $a, b \subset c$, and dually.*

PROOF. Induction on $|a| + |b| + |c|$.

If $A\{a \cup b \subset c\}$ in which $a \cup b \equiv \alpha \cup \beta$, then writing without loss of generality $a \equiv d \cup \Lambda$, $b \equiv e \cup \Delta$, $\alpha \equiv d \cup \Delta$ and $\beta \equiv e \cup \Lambda$, we have

$$d \cup \Delta,\ e \cup \Lambda \subset c \xrightarrow{2} d,\ \Delta,\ e,\ \Lambda \subset c \xrightarrow{A} d \cup \Lambda,\ e \cup \Delta \subset c\,, \qquad \text{i.e.,} \quad a, b \subset c\,.$$

If $B\{a \cup b \subset c\}$ in which $c \equiv \varepsilon \cap \varphi$, then

$$a \cup b \subset \varepsilon,\ \varphi \xrightarrow{2} a, b \subset \varepsilon,\ \varphi \xrightarrow{B} a, b \subset c\,.$$

If $D\{a \cup b \subset c\}$ in which $c \equiv \delta \cup (\varepsilon \cap \varphi)$, then

$$a \cup b \subset \delta \cup \varepsilon,\ \delta \cup \varphi \xrightarrow{2} a, b \subset \delta \cup \varepsilon,\ \delta \cup \varphi \xrightarrow{D} a, b \subset c\,.$$

3. *Let $a \equiv x_0 \cap \cdots \cap x_m$. Then $a \subset b \cup c$ implies $a \subset b$ or $a \subset c$. And dually.*

PROOF. Induction on $|b| + |c|$.

If $E\{a \subset b \cup c\}$, then there exists x_0, say, such that

$$b \cup c \equiv x_0 \cup z_1 \cup \cdots \cup z_n \xrightarrow{E} a \subset b \quad \text{or} \quad a \subset c\,.$$

If $D\{a \subset b \cup c\}$ in which $b \cup c \equiv \delta \cup (\varepsilon \cap \varphi)$, then writing without loss of generality $\delta \equiv b \cup \Lambda$ and $c \equiv \Lambda \cup (\varepsilon \cap \varphi)$, we have

$$a \subset \delta \cup \varepsilon, \delta \cup \varphi, \quad \text{i.e.,} \quad a \subset b \cup \Lambda \cup \varepsilon, b \cup \Lambda \cup \varphi$$

$$\xrightarrow{3} (a \subset b \text{ or } a \subset \Lambda \cup \varepsilon) \quad \& \quad (a \subset b \text{ or } a \subset \Lambda \cup \varphi)$$

$$\text{i.e.,} \quad a \subset b \text{ or } (a \subset \Lambda \cup \varepsilon, \Lambda \cup \varphi) \xrightarrow{D} a \subset b \text{ or } a \subset c.$$

4. *$(a \cup b) \cap c \subset d$ implies $a \cap c, b \cap c \subset d$, and dually.*

PROOF. Induction on $|a|+|b|+|c|+|d|$.
If $B\{(a \cup b) \cap c \subset d\}$ in which $d \equiv \varepsilon \cap \varphi$, then

$$(a \cup b) \cap c \subset \varepsilon, \varphi \xrightarrow{4} a \cap c, b \cap c \subset \varepsilon, \varphi \xrightarrow{B} a \cap c, b \cap c \subset d.$$

If $C\{(a \cup b) \cap c \subset d\}$ in which $(a \cup b) \cap c \equiv (\alpha \cup \beta) \cap \gamma$, then we consider two cases. When $c \equiv \gamma$, writing without loss of generality $a \equiv e \cup \Lambda$, $b \equiv f \cup \Delta$, $\alpha \equiv e \cup \Delta$ and $\beta \equiv f \cup \Lambda$, we have

$$(e \cup \Delta) \cap c, (f \cup \Lambda) \cap c \subset d \xrightarrow{4} e \cap c, \Delta \cap c, f \cap c, \Lambda \cap c \subset d$$

$$\xrightarrow{C} (e \cup \Lambda) \cap c, (f \cup \Delta) \cap c \subset d, \quad \text{i.e.,} \quad a \cap c, b \cap c \subset d.$$

When $c \not\equiv \gamma$, writing $c \equiv \Lambda \cap (\alpha \cup \beta)$ and $\gamma \equiv (\alpha \cup \beta) \cap \Lambda$, we have

$$(a \cup b) \cap \Lambda \cap \alpha, (a \cup b) \cap \Lambda \cap \beta \subset d$$

$$\xrightarrow{4} a \cap \Lambda \cap \alpha, a \cap \Lambda \cap \beta, b \cap \Lambda \cap \alpha, b \cap \Lambda \cap \beta \subset d$$

$$\xrightarrow{C} a \cap c, b \cap c \subset d.$$

If $D\{(a \cup b) \cap c \subset d\}$ in which $d \equiv \delta \cup (\varepsilon \cap \varphi)$, then

$$(a \cup b) \cap c \subset \delta \cup \varepsilon, \delta \cup \varphi \xrightarrow{4} a \cap c, b \cap c \subset \delta \cup \varepsilon, \delta \cup \varphi$$

$$\xrightarrow{D} a \cap c, b \cap c \subset d.$$

II. *$a \subset b$ and $b \subset c$ together imply $a \subset c$.*

PROOF. Induction on $|a|+|b|+|c|$.
If $a \equiv x_0 \cap \cdots \cap x_m$ and $c \equiv z_0 \cup \cdots \cup z_n$, then we consider two cases. When $b \equiv y$, we have $E\{a \subset b\}$ and $E\{b \subset c\}$, hence $a \subset c$. When $b \equiv \alpha \cup \beta$ (or dually $b \equiv \varepsilon \cap \varphi$), we have

$$a \subset b \subset c \xrightarrow{3,2} (a \subset \alpha \text{ or } a \subset \beta) \quad \& \quad (\alpha, \beta \subset c) \xrightarrow{\text{II}} a \subset c.$$

If $a \equiv \alpha \cup \beta$ (or dually $c \equiv \varepsilon \cap \varphi$), then

$$a \subset b \subset c \xrightarrow{2} \alpha, \beta \subset b \subset c \xrightarrow{\text{II}} \alpha, \beta \subset c \xrightarrow{A} a \subset c.$$

If $a \equiv (\alpha \cup \beta) \cap \gamma$ (or dually $c \equiv \delta \cup (\varepsilon \cap \varphi)$), then

$$a \subset b \subset c \xrightarrow{4} \alpha \cap \gamma, \beta \cap \gamma \subset b \subset c \xrightarrow{\text{II}} \alpha \cap \gamma, \beta \cap \gamma \subset c \xrightarrow{C} a \subset c.$$

The relation $\subset$ becomes a quasi-order by I and II above. After defining the equality relation $\sim$ as usual, i.e., as $a \sim c$ if and only if $a \subset c$ and $c \subset a$, we get a partially ordered set in which, by I and by A and B with $\leqq$ replaced by $\subset$, $\cup$ and $\cap$ come to denote the least upper bound and the greatest lower bound respectively. Thus we have a lattice, which is also distributive since

$$\xrightarrow{I} \alpha\cap\gamma,\ \beta\cap\gamma\subset(\alpha\cap\gamma)\cup(\beta\cap\gamma)\xrightarrow{C}(\alpha\cup\beta)\cap\gamma\subset(\alpha\cap\gamma)\cup(\beta\cap\gamma).$$

So the proof is completed.

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References

[1] G. Birkhoff, Lattice Theory, rev. ed., New York, 1948.
[2] K. Takeuchi, On free modular lattices II, Tôhoku Math. J., 11 (1959), 1-12.
[3] P. M. Whitman, Free lattices, Ann. of Math., 42 (1941), 325-330.