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Similarity principle of the generalized Cauchy-Riemann equations for several complex variables

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§ 0. Introduction.

Newlander and Nirenberg [12] proved that a sufficiently differentiable almost complex manifold satisfying the complete integrability conditions is a complex manifold. From a different point of view we may consider this Newlander-Nirenberg theorem as an extension of the Beltrami equation to several complex variables. Vekua [15], [16] established the theory of generalized analytic functions which are solutions of the so-called generalized Cauchy-Riemann equation $\partial_{\bar{z}} f=a(z, \bar{z}) f+b(z, \bar{z}) \bar{f}$, where $a(z, \bar{z})$, $b(z, \bar{z})$ are defined in a domain in z-plane and satisfy some conditions there, where the differential operator $\partial_{\bar{z}}$ is used in the sense of distributions and is defined by $(1/2)(\partial_x+i\partial_y)$, $z=x+iy$, in the case of ordinary derivations, and where we denote the conjugate of a complex number by a bar. The generalized analytic functions preserve a number of fundamental topological properties of analytic functions of one complex variable (the identity theorem, the argument principle, etc.). Moreover such analytic facts as the Taylor and Laurent expansions, the Cauchy integral formula, etc. remain valid.

In the theory of generalized analytic functions the representation formulas of the first (the reciprocal formula) and second (the generalized Cauchy integral formula) kinds play important roles. Bers [1] called the representation formula of the first kind the similarity principle.

Accordingly from the above-mentioned it is seen that in order to extend the Vekua's theory to several complex variables we shall need the analogous representation formulas for several complex variables.

We can ask the following question: Under what conditions can we have such formulas? Since the system of first order differential equations, with which we want to deal in the present paper, has the conjugate $\bar{f}$ of the unknown function f, so long as we have the compatibility conditions, we are compelled to take the additional first order differential equations. Therefore for the purpose of making full use of such additional differential equations we shall impose some integrability conditions on the coefficients of our system and then there exists a change of variables such that our system is reduced to a single differential equation of one variable. Though the additional differential equations still remain, the above problem is solved locally by reducing to the case of one complex variable. Moreover for the purpose of showing that our discussion in the present paper is not empty we shall show examples satisfying the above integrability conditions.

We should remark that if the coefficients of our system are anti-holomorphic, we have a necessary and sufficient condition in order that a non-trivial solution may exist, and remark that from this we obtain that any solution of our system is written in the composed form $\hat{f}\{\sigma(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)\}$, where $\hat{f}$ is the solution of a generalized Cauchy-Riemann equation of one complex variable and σ is determined by the coefficients and is holomorphic. Furthermore we shall remark that, when a domain of the definition of coefficients is a simply connected domain, we obtain global solutions and the above problem is solved.

On the contrary if the coefficients of our system have a smoothness only, we must have several integrability conditions which are local properties. Therefore, so long as we globally consider our system, we must check the compatibility of conditions, but such a problem is solved. And we obtain locally the same result as the case of the coefficients being anti-holomorphic.

Here we explain the notations used in this paper.

$C^\infty(U)$ is the set of all functions defined in U whose partial derivatives of all orders exist.

$\boldsymbol{C}^n = \boldsymbol{C} \times \cdots \times \boldsymbol{C}$ is the Cartesian product of n copies of the complex plane $\boldsymbol{C}$ which we identify with $\boldsymbol{R}^{2n}$.

For the points of $\boldsymbol{C}^n$ we shall use the notation $z = (z_1, \cdots, z_n)$, where $z_j = x_j + iy_j \in \boldsymbol{C}$ and x_j, y_j are real numbers. However, since we shall mainly think of the functions of class C^∞, the function of z will be denoted by $f(z, \bar{z})$ and hence in case of holomorphic functions we shall use the notation $f(z)$.

$\Delta(a, r)$ is a polydisc defined by a subset $\{z \in \boldsymbol{C}^n \mid |z_j - a_j| < r,\ 1 \leqq j \leqq n\}$; the point $a \in \boldsymbol{C}^n$ is called the center of the polydisc, and r is called the polyradius.

As a convenient notation we shall introduce the first order partial operators

$$\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j}) \quad \text{and} \quad \partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j}),$$

where $z_j = x_j + iy_j$.

By ∂_j, $\bar{\partial}_j$ we shall denote the notations ∂_{z_j}, $\partial_{\bar{z}_j}$ respectively.

Whenever we speak of a neighborhood in $\boldsymbol{C}^n$ we mean that it is homeomorphic to a polydisc.

§ 1. Reduction to the normal form.

We first consider the following system of first order partial differential equations (overdetermined system for $n \neq 1$):

$$L_j f \equiv \bar{\partial}_j f - \sum_{\mu=1}^{n} a_{j\mu}\partial_\mu f = a_j f + \bar{b}_j \bar{f}, \qquad 1 \leqq j \leqq n, \tag{1.1}$$

where all the coefficients are, for simplicity, of class C^∞ in a neighborhood of the origin in $\boldsymbol{C}^n$, and where $a_{pq}(0)=0$, $1 \leqq p$, $q \leqq n$.

REMARK 1.1. Suppose that the $n \times n$ matrix $\boldsymbol{A}(0)=(a_{pq}(0))$ is not the zero one. We put

$$\gamma_{\mu\nu}(z, \bar{z}) = \sum_{s=1}^{n} \overline{a_{s\nu}(0)}\, a_{\mu s}(z, \bar{z}), \qquad 1 \leqq \mu, \nu \leqq n.$$

We define two $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{\Gamma}$ as follows:

$$\boldsymbol{A} = (a_{pq}(z, \bar{z})) \quad \text{and} \quad \boldsymbol{\Gamma} = (\gamma_{pq}(z, \bar{z})).$$

We denote the transposed matrix of $\boldsymbol{A}(0)$ by $\boldsymbol{A}(0)^*$. Suppose that $\det(\boldsymbol{E}-\boldsymbol{\Gamma}) \neq 0$ at the origin and that

$$\det\begin{pmatrix} \boldsymbol{E} & \boldsymbol{A}(0)^* \\ \overline{\boldsymbol{A}(0)^*} & \boldsymbol{E} \end{pmatrix} \neq 0,$$

where $\boldsymbol{E}$ is the $n \times n$ unit matrix. For example, if $|a_{pq}(0)|$, $1 \leqq p$, $q \leqq n$, are sufficiently small, then such conditions are fulfilled.

Then, by introducing new independent variables

$$\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{A}(0)^* \bar{\boldsymbol{z}},$$

we may reduce the equations (1.1) to the case $a_{pq}(0)=0$, $1 \leqq p$, $q \leqq n$, where $\boldsymbol{w}$, $\boldsymbol{z}$ are n-dimensional column vectors and $\bar{\boldsymbol{z}}$ is the conjugate vector of $\boldsymbol{z}$.

We want to find a solution of (1.1) of the following form

$$f = g \exp \Omega.$$

Inserting this into (1.1), we obtain

$$L_j(g) = \bar{b}_j\{\exp(\bar{\Omega}-\Omega)\}\bar{g} + \{L_j(\Omega) - a_j\}g, \qquad 1 \leqq j \leqq n.$$

If we may take a solution Ω of the system of differential equations

$$L_j(\Omega) = a_j, \qquad 1 \leqq j \leqq n, \tag{1.2}$$

then we may reduce the system (1.1) to the system of equations

$$L_j(f) = \bar{b}_j \bar{f}, \qquad 1 \leqq j \leqq n. \tag{1.3}$$

We first require the hypothesis: the commutators vanish, that is,

$$[L_j, L_k] = 0,$$

and hence

$$(C_1) \qquad \boldsymbol{L}_j(a_{kt}) = \boldsymbol{L}_k(a_{jt}), \qquad 1 \leqq j, k, t \leqq n.$$

Then we obtain the compatibility conditions

$$(C_2) \qquad \boldsymbol{L}_j(a_k) = \boldsymbol{L}_k(a_j), \qquad 1 \leqq j, k \leqq n.$$

Changing the variables z into the variables ζ, with the aid of the Newlander-Nirenberg theorem [12], we may transform (1.2) into

$$(1.4) \qquad \partial_{\bar{\zeta}_j}\hat{\Omega} = \alpha_j, \qquad 1 \leqq j \leqq n,$$

where

$$\alpha_j = \sum_{\mu=1}^{n} (\overline{\partial_{\zeta_j} z_\mu}) \hat{a}_\mu,$$

$\hat{\Omega} = \Omega\{z(\zeta, \bar{\zeta}), \overline{z(\zeta, \bar{\zeta})}\}$ and $\hat{a} = a\{z(\zeta, \bar{\zeta}), \overline{z(\zeta, \bar{\zeta})}\}$.

Then it is verified by some computations that

$$(1.5) \qquad \partial_{\bar{\zeta}_k}\alpha_j = \partial_{\bar{\zeta}_j}\alpha_k, \qquad 1 \leqq j, k \leqq n.$$

In fact, using the relations

$$\partial_{\bar{\zeta}_j} z_\mu + \sum_{\nu=1}^{n} a_{\nu\mu}(\partial_{\zeta_j} z_\nu) = 0, \qquad 1 \leqq j, \mu \leqq n \text{ (see [12])},$$

we obtain

$$\begin{aligned} (1.6) \qquad \partial_{\bar{\zeta}_k}\hat{a}_j &= \sum_{\nu=1}^{n} (\bar{\partial}_\nu a_j - \sum_{\mu=1}^{n} a_{\nu\mu}\partial_\mu a_j)(\overline{\partial_{\zeta_k} z_\nu}) \\ &= \sum_{\nu} \boldsymbol{L}_\nu(a_j)(\overline{\partial_{\zeta_k} z_\nu}) \\ &= \sum_{\nu} \boldsymbol{L}_j(a_\nu)(\overline{\partial_{\zeta_k} z_\nu}) \qquad (\text{by } (C_2)). \end{aligned}$$

On the other hand, using (1.6), we have

$$(1.7) \qquad \partial_{\bar{\zeta}_k}\alpha_j = \sum_{\mu} (\partial_{\bar{\zeta}_k\bar{\zeta}_j}\bar{z}_\mu)\hat{a}_\mu + \sum_{\mu,\nu} \boldsymbol{L}_\mu(a_\nu)\partial_{\bar{\zeta}_j}\bar{z}_\mu\partial_{\bar{\zeta}_k}\bar{z}_\nu.$$

Since the right-hand side of (1.7) is symmetric with respect to j, k, we obtain (1.5).

Without loss of generality, we may assume that the neighborhood of the origin into which is transformed by the above change of variables is a polydisc $\{\zeta \in \boldsymbol{C}^n \mid |\zeta_j| < r, 1 \leqq j \leqq n\}$. Therefore it is seen that a solution of (1.4), under the compatibility conditions (1.5), is given explicitly (see [12]). Thus we have reached the following

LEMMA 1.1. *Under the assumptions* (C_1), (C_2) *there exists a solution of* (1.2) *in a neighborhood of the origin.*

REMARK 1.2. This lemma is contained in Hörmander's theorem (see [9]).

It is seen from Lemma 1.1 that under the assumptions (C_1), (C_2) the system of equations (1.1) is reduced locally to the system (1.3).

Furthermore, by using the change of variables in the proof of Lemma 1.1, we may reduce the system (1.3) into the following normal form:

$$\bar{\partial}_j f = \bar{b}_j \bar{f}, \qquad 1 \leqq j \leqq n. \tag{1.8}$$

In the case of $\boldsymbol{L}_j = \bar{\partial}_j$, that is, all a_{pq} of the system (1.1) being identically zero, our reduction to the normal form is carried out globally:

Let D be a polydisc $\Delta(z^0, r)$. General solutions of the equations

$$\bar{\partial}_j f = a_j f, \qquad 1 \leqq j \leqq n, \tag{1.9}$$

are given under the assumption (C_2) in the following form (see [**10**], [**12**])

$$f(z, \bar{z}) = \varphi(z) \exp \Omega(z, \bar{z}), \tag{1.10}$$

where $\varphi(z)$ is a holomorphic function in D and $\Omega(z, \bar{z})$ is defined as follows:

$$\Omega(z, \bar{z}) = \sum_{s=0}^{n-1} \frac{(-1)^s}{(s+1)!} \sum{}' \boldsymbol{T}_{j_1} \bar{\partial}_{j_1} \cdots \boldsymbol{T}_{j_s} \bar{\partial}_{j_s} \boldsymbol{T}_k(a_k) \tag{1.11}$$

where $\boldsymbol{T}_j$ denotes the integral operator

$$(\boldsymbol{T}_j a)(z, \bar{z}) = \frac{1}{2\pi i} \iint_{|\zeta_j - \zeta_j^0| < r} a(\cdots, z_{j-1}, \zeta, \cdots, \bar{z}_{j-1}, \bar{\zeta}, \cdots) \frac{d\zeta d\bar{\zeta}}{\zeta - z_j}$$

and $\sum'$ denotes the summation over all $(s+1)$-tuples with $j_1, \cdots, j_s, k$ mutually distinct.

Let D be a domain of holomorphy. With the aid of Hörmander's theorem [**9**], it is seen that (1.9) has a solution in D under the assumption (C_2). Therefore the system of equations (1.1) $(\boldsymbol{L}_j = \bar{\partial}_j)$ can be reduced to the type (1.8).

REMARK 1.3. We note that in case of D being a polydisc solutions of the system (1.9) are given explicitly in the form (1.10).

REMARK 1.4. We may consider a polydomain instead of a polydisc in the formula (1.11).

§ 2. Definitions, and properties of solutions.

Let G be a domain (connected open subset) in $\boldsymbol{C}^n$ and the coefficients b_j be of class $C^\infty(G)$.

We shall call the system (1.8) the "generalized Cauchy-Riemann equations".

Let D be any subdomain of G whose closure is contained in G (we denote by $D \Subset G$). We call a function f, which is not identically zero and is of class $C^1(D)$, a regular solution (or merely solution) of (1.8) in D, when f satisfies the system of equations (1.8) at every point of D.

Unless otherwise stated, it will be assumed that *all the functions under discussion are of class C^{∞} in a domain considered.*

Let D be a bounded domain. Following the definition of one variable, we call two functions $f(z, \bar{z})$, $g(z, \bar{z})$, defined in D, *similar* if the ratio f/g is bounded, bounded away from zero and continuous on the closure of D (see [1], [7]).

Let $\mathcal{M}(D)$ denote a class consisting of all functions which satisfy a certain condition on D and let $\mathcal{M}_0(D)$ denote a subclass of $\mathcal{M}(D)$. We say that a given system of the differential equations defined in G satisfies *the similarity principle with respect to the class* $\mathcal{M}_0(D)$, when every solution of the system of differential equations considered is similar to a function in $\mathcal{M}_0(D)$ and, conversely, to correspond a given function in $\mathcal{M}_0(D)$ there exists a solution of the system of differential equations such that two functions are similar.

If D is a neighborhood of a point, then we shall use the following terminology: *local* similarity principle.

Let $\mathcal{A}(D)$ be the class consisting of all holomorphic functions in D and $\mathcal{A}_0(D)$ be a subclass of $\mathcal{A}(D)$ such that $\mathcal{A}_0(D)$ satisfies a certain condition given later on. Of course, we may consider $\mathcal{A}_0(D)=\mathcal{A}(D)$. Then, under what assumptions will the system of differential equations (1.8) satisfy the similarity principle with respect to $\mathcal{A}_0(D)$? The author [10] proved the following

THEOREM 2.1. *Let all b_j be holomorphic functions in G and not zero at a point z^0 in G. Assume that in a neighborhood U of z^0, $U\subset G$,*

$$X_k(b_n^{-1}b_j)-X_j(b_n^{-1}b_k)=0\,, \qquad 1\leqq j,\, k\leqq n-1\,,$$

where we put $X_s h=b_n\partial_s h-b_s\partial_n h$. Then (1.8) *has a solution in a neighborhood V of z^0, $V\subset U$, if and only if $\partial_k b_j=\partial_j b_k$ in G. And the system* (1.8) *satisfies the local similarity principle with respect to $\mathcal{A}_0(V)$ which is defined as follows: Let ϕ be a holomorphic function in V which is a solution of the differential equations*

$$b_n\partial_j\phi-b_j\partial_n\phi=0\,, \qquad 1\leqq j\leqq n-1\,.$$

The class $\mathcal{A}_0(V)$ consists of all the composite functions $\varphi\circ\phi=\varphi\{\phi(z)\}$, where φ is a regular analytic function of a complex variable defined in $\phi(V)$.

In section 10 we shall give another formulation and proof of this theorem.

Let the class $\tilde{Q}(K)$ denote the set of all functions which are defined in an open disc K and satisfy except, possibly, for isolated singularities, the Beltrami equation

$$\partial_{\bar{z}}g+\alpha\partial_z g=0\,, \qquad |\alpha|\leqq c_0<1\,.$$

We shall denote by $Q(K)$ a subclass of $\tilde{Q}(K)$, whose elements do not have singularities.

Here we state the following lemma for a complex variable, which we shall use later. The conditions on coefficients are, for convenience, very strong (see [**1**], [**2**], [**3**], [**4**], [**7**] and [**16**]).

LEMMA 2.1. *Let K be an open disc with center the origin and radius r in the complex plane. Suppose that the functions α, β and γ are defined in a neighborhood of the closure of K and that $|\alpha|\leqq c_0<1$ in K, where c_0 is a positive number. Then the differential equation*

$$\partial_{\bar z} f+\alpha\partial_z f=\beta f+\gamma\bar f$$

satisfies the similarity principle with respect to the class $\tilde{Q}(K)$.

In case of $\alpha=0$ we have the so-called similarity principle. In this case $\tilde{Q}(K)$ is identical with the class $\tilde{\mathcal{A}}(K)$ which is composed of all regular analytic functions in K except, possibly, for singular points.

In order that we may obtain a sufficient condition under which the system (1.8) has a solution having not zero points, we have the following lemma.

LEMMA 2.2. *Let $f(z, \bar z)$ be a function of a complex variable z defined in an open disc K and satisfy the system of differential equations*

$$\partial_{\bar z} f=af+b\bar f\,, \qquad \partial_z f=Af+B\bar f\,,$$

where a, b, A and B are defined in a neighborhood of the closure of K. Then f has no null point in K.

In fact, assume that f has a null point z_0 in K. By Lemma 2.1 we have

$$\begin{aligned} f &=(z-z_0)^m \exp g \\ &=(\bar z-\bar z_0)^m \exp h \end{aligned}$$

in a sufficiently small neighborhood of z_0, where m is a non-negative integer and g, h are defined and of class C in that neighborhood. By considering $\arg(z-z_0)$ we obtain $m=0$.

Putting

$$\partial=(\partial_1, \cdots, \partial_n)\,, \qquad \bar\partial=(\bar\partial_1, \cdots, \bar\partial_n)\,,$$

$$\partial f=(\partial_1 f, \cdots, \partial_n f)\,, \qquad \bar\partial f=(\bar\partial_1 f, \cdots, \bar\partial_n f)\,,$$

we now define two matrices $\boldsymbol{B}$, $\boldsymbol{C}$ by

$$\boldsymbol{B}=\begin{pmatrix}\partial f\\ \partial\bar f\end{pmatrix}, \qquad \boldsymbol{C}=\begin{pmatrix}\bar\partial f\\ \bar\partial\bar f\end{pmatrix}.$$

Let f be of class $C^1(D)$. We may assume, removing, if necessary, a suitable relatively closed and nowhere dense set from D, that rank$(\boldsymbol{B}, \boldsymbol{C})$ is a constant in D. So, let rank$(\boldsymbol{B}, \boldsymbol{C})$ equal 1 in D. Then, in a neighborhood U of every point of D there exist two functions λ, μ of class $C^1(U)$ such that they satisfy $|\lambda|+|\mu|\neq 0$ and fulfill

$$\left\{\begin{array}{l}\lambda\partial_j f+\mu\partial_j\bar{f}=0,\\ \lambda\bar{\partial}_j f+\mu\bar{\partial}_j\bar{f}=0,\end{array}\right.\quad 1\leqq j\leqq n. \tag{2.1}$$

Let $\lambda\neq 0$ at a point $z_0\in U$. If f satisfies the system of differential equations (1.8), then we obtain from the first part of (2.1) and (1.8) that

$$\partial_j f=-\lambda^{-1}\mu b_j f,\qquad 1\leqq j\leqq n,$$

in a still smaller neighborhood V of z_0. With the aid of Lemma 2.2 it is seen that f has no null point in V. Thus we obtain the following

THEOREM 2.2. *Let f be a solution of* (1.8) *such that* $\operatorname{rank}(\boldsymbol{B},\boldsymbol{C})=1$ *in D. Then f has no null point in D.*

We next shall consider the case that a solution f of (1.8) has null points in D.

Let M_0 be the set $\{z\in D \mid f(z,\bar{z})=0\}$ and let M_0 be not empty. Suppose that M_0 has an interior point z^0. Since D is connected, without loss of generality we may assume that D is a polydisc $\Delta(z^0,r)$. Then there exists a positive number ε such that the polydisc $\Delta(z^0,\varepsilon)$ is contained in M_0. Now, since for any fixed $(z_1,\cdots,z_{n-1})$ the function f satisfies the n-th equation of (1.8), so, by the identity theorem for one complex variable, f is identically zero in the open disc $|z_n-z_n^0|<r$ for any fixed $z_1,\cdots,z_{n-1}$ such that $|z_j-z_j^0|<\varepsilon$, $1\leqq j\leqq n-1$, and hence f is identically zero in the polydisc $\{z\in\boldsymbol{C}^n \mid |z_j-z_j^0|<\varepsilon, |z_n-z_n^0|<r, 1\leqq j\leqq n-1\}$. By repeating this procedure we shall attain a contradiction that f is identically zero in D. Thus we obtain the following

THEOREM 2.3 [**10**]. *M_0 has no interior point.*

It is known that in the case of $n=1$ the null points of solutions of the differential equation in Lemma 2.1 are isolated points (see [**1**], [**3**], [**4**] and [**16**]). For $n\geqq 2$ we obtain the better result than Theorem 2.3.

We suppose that f is a solution of (1.8) such that $\operatorname{rank}(\boldsymbol{B},\boldsymbol{C})=2$ in D. Then we know that the set M_0 defined above is a real $(2n-2)$-dimensional differentiable regularly embedded submanifold of $\boldsymbol{C}^n$. On M_0 we have $\operatorname{rank}\boldsymbol{B}=1$ and hence, with the aid of the Levi-Civita-Sommer theorem (see [**14**]), we obtain the following

THEOREM 2.4. *Let f be a solution of* (1.8) *such that f satisfies* $\operatorname{rank}(\boldsymbol{B},\boldsymbol{C})=2$ *in D. If M_0 is not empty, then M_0 is a complex pure $(n-1)$-dimensional analytic manifold.*

REMARK 2.1. Let E be the nowhere dense set mentioned above, that is, such that for every point of $D-E$ there exists a neighborhood in which $\operatorname{rank}(\boldsymbol{B},\boldsymbol{C})$ is constant. If for a point of $D-E$ $\operatorname{rank}(\boldsymbol{B},\boldsymbol{C})=0$, then f is identically zero in D by Theorem 2.3.

In case of holomorphic functions of one or several complex variables, since D is connected, the set $D-E$ is also connected [8]. So if we assume that

$D-E$ is connected, then the alternatives are rank $(\boldsymbol{B}, \boldsymbol{C})=1$ and rank $(\boldsymbol{B}, \boldsymbol{C})=2$ in $D-E$. And hence we obtain the following

THEOREM 2.5. *Let f be a solution of* (1.8) *such that* rank $(\boldsymbol{B}, \boldsymbol{C})=2$ *in $D-E$. If M_0 is not empty, then M_0-E is a complex pure $(n-1)$-dimensional analytic manifold.*

REMARK 2.2. Let f be any function defined in D (it is not always a solution of the system of equations (1.8)) such that rank $(\boldsymbol{B}, \boldsymbol{C}) \leqq 1$ in D. Then the image $f(D)$ is nowhere dense [**11**].

§ 3. Assumptions on the coefficients.

Let f be a solution of (1.8) in D. Note that the coefficients of (1.8) are defined in a domain G in $\boldsymbol{C}^n$ and solutions of (1.8) are defined in a subdomain D such that $D \Subset G$. Unless stated to the contrary, from now on we shall mean that the domain of definition of solutions $\Subset$ the domain of definition of the coefficients. If f is identically zero for some coordinate, say z_n, then f is identically zero in D by Theorem 2.3. Hence we shall also assume that f is not identically zero for any coordinate.

As it is readily seen, f must satisfy the additional first order differential equations

$$\left\{\begin{array}{l} \boldsymbol{X}_{(j,k)} f=-b_{jk} f, \\ \bar{\boldsymbol{X}}_{(j,k)} f=0, \end{array}\right. \qquad 1 \leqq j, k \leqq n, \tag{3.1}$$

where

$$\boldsymbol{X}_{(j,k)}=b_k \partial_j-b_j \partial_k, \qquad \bar{\boldsymbol{X}}_{(j,k)}=\bar{b}_k \bar{\partial}_j-\bar{b}_j \bar{\partial}_k,$$
$$b_{jk}=\partial_j b_k-\partial_k b_j, \qquad 1 \leqq j, k \leqq n.$$

We can obtain the first part of (3.1) in another way:

We set $\alpha_j=(\bar{b}_j \bar{f}/f)$ for $z \in D-M_0$, $1 \leqq j \leqq n$. Then we obtain the differential equations of the type (1.9) in $D-M_0$, that is,

$$\bar{\partial}_j f=\alpha_j f, \qquad 1 \leqq j \leqq n.$$

Since the integrability conditions are $\bar{\partial}_j \alpha_k=\bar{\partial}_k \alpha_j$, $1 \leqq j, k \leqq n$, we have the additional relations $\boldsymbol{X}_{(j,k)} f=-b_{jk} f$, $1 \leqq j \leqq n$. Then by virtue of Theorem 2.3 the first part of (3.1) is obtained.

Let a solution f of (1.8) be defined in a polydomain D. Then f is written in the form of (1.10) in $D-M_0$ and hence it is seen by simple computations that the terms $\partial_j f/f$, $1 \leqq j \leqq n$, appear in the integrals of the right-hand side of (1.11). Accordingly we shall have to consider the existence of the integrals of functions $\partial_j f/f$. Even though such a problem is solved, we could not expect the continuity of $\Omega(z, \bar{z})$ on M_0 and the existence of the continuation of the holomorphic function $\varphi(z)$ into D without additional conditions.

In this section we shall consider the following special infinitesimal transformations defined in G:

$$X_{(j,k)}=b_k\partial_j-b_j\partial_k\,,\qquad \bar{X}_{(j,k)}=\bar{b}_k\bar{\partial}_j-\bar{b}_j\bar{\partial}_k\,,\qquad 1\leqq j,\,k\leqq n\,,$$

where all b_s are of class $C^\infty(G)$. $X_{(j,k)}$ and $\bar{X}_{(j,k)}$ are of type (1, 0) and (0, 1) respectively, and $\bar{X}_{(j,k)}$ is the conjugate one of $X_{(j,k)}$.

Now we consider a solution f such that rank $(\boldsymbol{B},\boldsymbol{C})=2$ in D, where D is as given at the beginning of this section. When M_0 is not empty, then it is found from the additional equations (3.1) that

$$X_{(j,k)}f=0\,,\qquad \bar{X}_{(j,k)}f=0\,,\qquad 1\leqq j,\,k\leqq n$$

at every point of M_0. Or, putting $u=\mathrm{Re}\,f$ and $v=\mathrm{Im}\,f$, we have

$$X_{(j,k)}*=0\,,\qquad \bar{X}_{(j,k)}*=0\,,\qquad 1\leqq j,\,k\leqq n\,, \tag{3.2}$$

where $*$ denotes u or v. Hence it follows from (3.2) that at every point z of M_0 the contravariant vectors

$$X_{(j,k)}(z,\bar{z})=b_k(z,\bar{z})\partial_j-b_j(z,\bar{z})\partial_k\,,$$
$$\bar{X}_{(j,k)}(z,\bar{z})=\bar{b}_k(z,\bar{z})\bar{\partial}_j-\bar{b}_j(z,\bar{z})\bar{\partial}_k\,,\qquad 1\leqq j,\,k\leqq n$$

are tangential to the real manifold M_0 and that such vectors build the $(2n-2)$-dimensional complex vector space V_C^{2n-2}. The tangent vector $X(z,\bar{z})$ to M_0 and the contravariant vector of M_0 at z correspond 1—1 each other. So, consider the following real infinitesimal transformations defined in G:

$$\begin{cases} Y_{(j,k)}=\dfrac{1}{2}\{X_{(j,k)}+\bar{X}_{(j,k)}\}\,, \\[2mm] Z_{(j,k)}=\dfrac{1}{2i}\{X_{(j,k)}-\bar{X}_{(j,k)}\}\,. \end{cases} \tag{3.3}$$

And let $Y^*_{(j,k)}$, $Z^*_{(j,k)}$ be the restrictions of $Y_{(j,k)}$, $Z_{(j,k)}$ to M_0 respectively. Then we may consider that $Y^*_{(j,k)}(z,\bar{z})$, $Z^*_{(j,k)}(z,\bar{z})$ belong to the real tangent vector space V^{2n-2} to M_0 at the point z considered.

Now we suppose that $(b_1,\cdots,b_n)\neq(0,\cdots,0)$ at every point of G. Let $b_n\neq 0$ at a point z_0 of M_0 and hence $b_n\neq 0$ in some neighborhood U of z_0, $U\subset G$. Since we have

$$\begin{cases} X_{(j,k)}(P)=b_n^{-1}(P)\{b_k(P)X_{(j,n)}(P)-b_j(P)X_{(k,n)}(P)\}\,, \\ \bar{X}_{(j,k)}(P)=\bar{b}_n^{-1}(P)\{\bar{b}_k(P)\bar{X}_{(j,n)}(P)-\bar{b}_j(P)\bar{X}_{(k,n)}(P)\}\,, \end{cases} \tag{3.4}$$

where $P\in U$ and $P=(z,\bar{z})$, the real tangent vectors $Y^*_{(j,n)}(P)$, $Z^*_{(j,n)}(P)$, $1\leqq j\leqq n$, form a base of V^{2n-2}, $P\in U\cap M_0$. With the aid of Theorem 2.4 we see that V^{2n-2} is the complex $(n-1)$-dimensional vector space V_C^{n-1} whose base is $X^*_{(j,n)}(P)$, $1\leqq j\leqq n-1$.

Hence we see that M_0 is an integral manifold of a system of $2n-2$ infinitesimal transformations $Y^*_{(j,n)}$, $Z^*_{(j,n)}$, $1 \leqq j \leqq n-1$, which we denote by $\mathfrak{M}^*$. Accordingly the commutators $[Y^*_{(j,n)}, Y^*_{(k,n)}]$, $[Y^*_{(j,n)}, Z^*_{(k,n)}]$ and $[Z^*_{(j,n)}, Z^*_{(k,n)}]$ belong to $\mathfrak{M}^*$, each of which is denoted by the linear combination of elements in $\mathfrak{M}^*$ whose coefficients are of class $C^\infty(U \cap M_0)$. From this fact we have reached the following result: on $M_0 \cap U$ there exists necessarily the following relationship between the coefficients b_s (see [**14**], Satz 4):

$$b_j X^*_{(k,n)} b_n - b_n X^*_{(k,n)} b_j = b_k X^*_{(j,n)} b_n - b_n X^*_{(j,n)} b_k ,$$

$$b_j \bar{X}^*_{(k,n)} b_n - b_n \bar{X}^*_{(k,n)} b_j = 0 , \qquad 1 \leqq j,\, k \leqq n-1 .$$

If $b_l(z_0, \bar{z}_0) \neq 0$ for $l \neq n$, then by considering a neighborhood U_l instead of U, we also obtain the same relationship as the case of n, that is, the above relationship holds with n replaced by l and with $1 \leqq j,\, k \leqq n$.

We define the sets S and T as follows: for all j, k, l

$$S = \{z \in G \mid b_j X_{(k,l)} b_l - b_l X_{(k,l)} b_j = b_k X_{(j,l)} b_l - b_l X_{(j,l)} b_k\} ,$$

$$T = \{z \in G \mid b_j \bar{X}_{(k,l)} b_l - b_l \bar{X}_{(k,l)} b_j = 0\} .$$

Then it follows that $M_0 \subset S \cap T$.

Thus we require, first, that at every point of G

$$(H_0) \qquad (b_1, \cdots, b_n) \neq (0, \cdots, 0) .$$

By (H_0), corresponding to every point of G there exist a number l $(1 \leqq l \leqq n)$ and a neighborhood U of the point such that $b_l \neq 0$ everywhere in U. Without loss of generality we can take $l = n$. Since our argument is local, we fix the above neighborhood U, that is, we shall discuss about a fixed point of G. Thus we shall require that at every point of U

$$(H_1)_n \qquad b_j X_{(k,n)} b_n - b_n X_{(k,n)} b_j = b_k X_{(j,n)} b_n - b_n X_{(j,n)} b_k ,$$

$$(H_2)_n \qquad b_j \bar{X}_{(k,n)} b_n - b_n \bar{X}_{(k,n)} b_j = 0 , \qquad 1 \leqq j,\, k \leqq n-1 .$$

It is easily verified that the assumption $(H_1)_n$ is equivalent to

$$(H_1)'_n \qquad b_j b_{kn} + b_k b_{nj} + b_n b_{jk} = 0 , \qquad 1 \leqq j,\, k \leqq n-1 .$$

For convenience we set

$$X_j = X_{(j,n)} , \qquad \bar{X}_j = \bar{X}_{(j,n)} , \qquad 1 \leqq j \leqq n-1 .$$

If f satisfies

$$(3.5) \qquad X_j f = -b_{jn} f , \qquad \bar{X}_j f = 0 , \qquad 1 \leqq j \leqq n-1 ,$$

then it is derived by (3.4) that in U

$$X_{(j,k)} f = b_n^{-1}(b_k X_j f - b_j X_k f)$$

$$= b_n^{-1}(-b_k b_{jn} + b_j b_{kn}) f$$
$$= b_n^{-1}(b_j b_{kn} + b_k b_{nj}) f \qquad (\text{by } b_{nj} = -b_{jn})$$
$$= -b_{jk} f \qquad (\text{by } (H_1)'_n)$$

and similarly

$$\bar{X}_{(j,k)} f = 0 .$$

Consequently we see from the assumption $(H_1)_n$ that it is sufficient to consider the system of equations (3.5) instead of the system of equations (3.1) in U.

The equations (3.5) allow us to obtain the additional first order equations by forming brackets:

$$\begin{aligned}[X_j, X_k] f &= X_k X_j f - X_j X_k f \\ &= -X_k(b_{jn} f) + X_j(b_{kn} f) \\ &= (-X_k b_{jn} + X_j b_{kn}) f ,\end{aligned} \tag{3.6}$$

$$\begin{aligned}[X_j, \bar{X}_k] f &= \bar{X}_k X_j f - X_j \bar{X}_k f \\ &= -(\bar{X}_k b_{jn}) f .\end{aligned} \tag{3.7}$$

On the other side, on account of $(H_1)_n$, $(H_2)_n$, we have the following relationship in U:

$$\begin{aligned}[X_j, X_k] f &= b_n^{-1}(X_k b_n X_j f - X_j b_n X_k f) \\ &= b_n^{-1}(-b_{jn} X_k b_n + b_{kn} X_j b_n) f ,\end{aligned} \tag{3.8}$$

$$\begin{aligned}[X_j, \bar{X}_k] f &= b_n^{-1} \bar{X}_k b_n X_j f - \bar{b}_n^{-1} X_j \bar{b}_n \bar{X}_k f \\ &= -(b_n^{-1} b_{jn} \bar{X}_k b_n) f .\end{aligned} \tag{3.9}$$

Combining (3.8), (3.9) with (3.6), (3.7) and using that M_0 has no interior point, we find that at every point of U there exist the following compatibility conditions:

$$b_n X_j b_{kn} - b_{kn} X_j b_n = b_n X_k b_{jn} - b_{jn} X_k b_n , \tag*{$(H_3)_n$}$$

$$b_n \bar{X}_k b_{jn} - b_{jn} \bar{X}_k b_n = 0 , \qquad 1 \leqq j,\ k \leqq n-1 . \tag*{$(H_4)_n$}$$

REMARK 3.1. As seen in section 4, the condition $(H_3)_n$ is derived from $(H_1)_n$ and $(H_2)_n$, and hence $(H_3)_n$ is not the assumption.

§ 4. Main lemma and a consideration about assumptions on the coefficients.

In this and the following sections, without loss of generality, we shall assume that G contains the origin and $b_n \neq 0$ at every point in a neighborhood

U of the origin, $U \subset G$. Let $\mathfrak{M}$ denote a system of real infinitesimal transformations $\boldsymbol{Y}_{(j,k)}$, $\boldsymbol{Z}_{(j,k)}$ defined in G ((3.3)). It is seen from $(H_1)_n$, $(H_2)_n$ that the system $\mathfrak{M}$ has the dimension $(2n-2)$ and is involutive in U. Therefore, by virtue of the Frobenius lemma [5], [13], $\mathfrak{M}$ is locally integrable, and hence there exist two real valued functions u, v of class C^∞ in a neighborhood V of the origin, $V \subset U$, such that

$$\operatorname{rank}\begin{pmatrix} \partial u & \bar{\partial} u \\ \partial v & \bar{\partial} v \end{pmatrix} = 2$$

and they are solutions of the system of differential equations

$$\boldsymbol{Y}_j \tau = 0\,, \qquad \boldsymbol{Z}_j \tau = 0\,, \qquad 1 \leqq j \leqq n-1\,, \tag{4.1}$$

where we put $\boldsymbol{Y}_j = \boldsymbol{Y}_{(j,n)}$, $\boldsymbol{Z}_j = \boldsymbol{Z}_{(j,n)}$.

$\boldsymbol{Y}_j$, $\boldsymbol{Z}_j$, $1 \leqq j \leqq n-1$, build a base of $\mathfrak{M}$ around the origin. For every solution g of the equations (4.1) there exists a real valued function F of a complex variable w such that F is defined in $\varDelta$ and $g = F(u, v)$, where $\varDelta$ is the image of V by $f = u + iv$. The function g also is written in the form

$$g = \tilde{F}(f, \bar{f}) = F\left\{\frac{1}{2}(f+\bar{f}),\ \frac{1}{2i}(f-\bar{f})\right\}$$

and for another solution h we also see that $h = \tilde{H}(f, \bar{f})$. Setting $\lambda = \tilde{F} + i\tilde{H}$, we see that the function λ is a solution of the system of differential equations

$$\begin{cases} \boldsymbol{X}_j \lambda = 0\,, \\ \bar{\boldsymbol{X}}_j \lambda = 0\,, \end{cases} \qquad 1 \leqq j \leqq n-1\,. \tag{4.2}$$

Since for every solution λ of (4.2) $\operatorname{Re}\lambda$, $\operatorname{Im}\lambda$ are solutions of (4.1) respectively, it is seen that λ is written in the form

$$\lambda = \varphi \circ f = \varphi(f, \bar{f})\,,$$

where φ is a complex valued function defined in $\varDelta$ and f is a solution of (4.2) such that it satisfies $\operatorname{rank}(\boldsymbol{B}, \boldsymbol{C}) = 2$ in V ($\varDelta = \text{image } f(V)$).

Conversely, if a function σ is the solution of (4.2) in some neighborhood V of any point z^0 of U such that $\operatorname{rank}(\boldsymbol{B}, \boldsymbol{C}) = 2$ in V, where we consider the matrices $\boldsymbol{B}, \boldsymbol{C}$ with σ instead of f, then the $(2n-2)$-dimensional real manifold defined by $\sigma = t$ is the $(n-1)$-dimensional complex manifold, where t is any point of the image $\sigma(V)$. For, from that σ satisfies (4.2) and that b_n is not zero at any point of U, we obtain that in V

$$\frac{\partial_j \sigma}{\partial_j \bar{\sigma}} = \frac{\partial_n \sigma}{\partial_n \bar{\sigma}}\,, \qquad 1 \leqq j \leqq n-1\,,$$

and hence $\operatorname{rank} \boldsymbol{B} = 1$ in V. Therefore the infinitesimal transformations $\boldsymbol{X}_j^*$, $\bar{\boldsymbol{X}}_j^*$, $1 \leqq j \leqq n-1$, satisfy the conditions $(H_1)_n$, $(H_2)_n$ on the manifold considered (see section 3). Noting that t is any point of $\sigma(V)$ and the point z^0 is any point of U, we see that the conditions $(H_1)_n$, $(H_2)_n$ are satisfied in U. Thus we have reached the following

LEMMA 4.1. *A necessary and sufficient condition in order that the system of first order partial differential equations* (4.2) *may have a solution σ in some neighborhood V of any point in U such that*

$$\operatorname{rank}(\boldsymbol{B}, \boldsymbol{C}) = 2 \quad in\ V \tag{4.3}$$

is that $(H_1)_n$, $(H_2)_n$ are satisfied in U. Furthermore, for such a solution

$$\operatorname{rank} \boldsymbol{B} = 1 \quad in\ V, \tag{4.3$'$}$$

and every solution of (4.2) *is given in the form*

$$(\varphi \circ \sigma)(z, \bar{z}) = \varphi\{\sigma(z, \bar{z}), \overline{\sigma(z, \bar{z})}\} .$$

Suppose that we impose the assumptions $(H_1)_n$, $(H_2)_n$, $(H_3)_n$ and $(H_4)_n$ upon the coefficients b_j in U. Let $b_l \neq 0$ in a neighborhood U_l of the origin, $U_l \subset G$ $(l \neq n)$. By Lemma 4.1 we see that the system $\mathfrak{M}$ of $2n-2$ infinitesimal transformations $\boldsymbol{X}_j$, $\bar{\boldsymbol{X}}_j$, $1 \leqq j \leqq n-1$, defined in U is involutive and hence the manifold defined by $\sigma(z, \bar{z}) = t$ is an integral manifold of $\mathfrak{M}$. The infinitesimal transformations $\boldsymbol{X}_{(j,l)}$, $\bar{\boldsymbol{X}}_{(j,l)}$, $1 \leqq j \leqq n$, $j \neq l$, are linearly independent in U_l. Hence $\boldsymbol{X}_{(j,l)}$, $\bar{\boldsymbol{X}}_{(j,l)}$, $1 \leqq j \leqq n$, form a base of $\mathfrak{M}$ around the origin. Hence it is seen that there are the conditions $(H_1)_l$, $(H_2)_l$ in $U_l \cap U$. As a consequence, we must assume $(H_3)_l$, $(H_4)_l$ in $U_l \cap U$, where $(H_3)_l$, $(H_4)_l$ denote $(H_3)_n$, $(H_4)_n$ with n replaced by l respectively (see Remark 3.1).

REMARK 4.1. In section 8 we shall see that $(H_4)_l$ is derived from $(H_j)_n$, $j = 1, 2, 4, 5, 6, 7$.

From (4.2) it follows that in V

$$b_j = b(z, \bar{z})\partial_j\sigma, \qquad 1 \leqq j \leqq n \quad \text{and} \quad b \neq 0 .$$

Then $\boldsymbol{X}_{(j,k)}$, $\bar{\boldsymbol{X}}_{(j,k)}$ are written in the form

$$\begin{cases} \boldsymbol{X}_{(j,k)} = b(\partial_k\sigma\partial_j - \partial_j\sigma\partial_k) , \\ \bar{\boldsymbol{X}}_{(j,k)} = \bar{b}(\bar{\partial}_k\bar{\sigma}\bar{\partial}_j - \bar{\partial}_j\bar{\sigma}\bar{\partial}_k) . \end{cases} \tag{4.4}$$

Now we set

$$\begin{cases} \boldsymbol{Y}_{(j,k)} = \partial_k\sigma\partial_j - \partial_j\sigma\partial_k , \\ \bar{\boldsymbol{Y}}_{(j,k)} = \bar{\partial}_k\bar{\sigma}\,\bar{\partial}_j - \bar{\partial}_j\bar{\sigma}\bar{\partial}_k . \end{cases} \tag{4.5}$$

Remark that these infinitesimal transformations differ from those in section 3 and in the proof of Lemma 4.1. From now on, unless stated to the contrary, the infinitesimal transformations $\boldsymbol{Y}$ are those defined by (4.5).

Then we have

$$X_{(j,k)} = bY_{(j,k)}\,, \qquad \bar{X}_{(j,k)} = \bar{b}\bar{Y}_{(j,k)}\,, \tag{4.6}$$

$$\begin{aligned} b_{jk} &= \partial_j(b\partial_k\sigma) - \partial_k(b\partial_j\sigma) \\ &= (\partial_k\sigma)(\partial_j b) - (\partial_j\sigma)(\partial_k b) \\ &= Y_{(j,k)}b\,, \end{aligned} \tag{4.7}$$

$$\bar{b}_{jk} = \bar{Y}_{(j,k)}\bar{b}\,. \tag{4.8}$$

Since $b_l \neq 0$ in U_l and $Y_{(j,l)}\sigma = 0$, $\bar{Y}_{(j,l)}\sigma = 0$, $1 \leqq j \leqq n$, so it is obtained that the infinitesimal transformations $Y_{(j,l)}$, $\bar{Y}_{(j,l)}$ also form a base of $\mathfrak{M}$ around the origin. Hence we obtain

$$[Y_{(j,l)}, Y_{(k,l)}] = \frac{1}{\partial_l\sigma}\{(Y_{(k,l)}\partial_l\sigma)Y_{(j,l)} - (Y_{(j,l)}\partial_l\sigma)Y_{(k,l)}\}\,, \qquad 1 \leqq j, k \leqq n\,. \tag{4.9}$$

REMARK 4.2. We shall see in section 8 that $[Y_{(j,l)}, \bar{Y}_{(k,l)}] = 0$.

Now we want to express $(H_3)_n$ by means of Y_j, where $Y_j = Y_{(j,n)}$, $1 \leqq j \leqq n-1$. Observe that

$$\begin{cases} b_n X_j b_{kn} - b_n X_k b_{jn} = (b^2\partial_n\sigma)[Y_k, Y_j]b \\ b_{kn}X_j b_n - b_{jn}X_k b_n = b^2\{(Y_j\partial_n\sigma)Y_k b - (Y_k\partial_n\sigma)Y_j b\}\,. \end{cases} \tag{4.10}$$

Then, with the aid of (4.9), (4.10), we can assert that the assumption $(H_3)_n$ follows from $(H_1)_n$, $(H_2)_n$. Similarly $(H_3)_l$ follows from $(H_1)_l$, $(H_2)_l$ and hence from $(H_1)_n$, $(H_2)_n$ in $U_l \cap U$.

At the end of this section we shall have the following corollary as one of the properties of solutions of (4.2).

COROLLARY 4.1. *Let ϕ be defined in the image $\sigma(V)$ such that*

$$\operatorname{rank}\begin{pmatrix} \partial_w\phi & \partial_{\bar{w}}\phi \\ \partial_w\bar{\phi} & \partial_{\bar{w}}\bar{\phi} \end{pmatrix} = 2 \quad in \quad \sigma(V)\,.$$

Put $M_t = \{z \in V \mid (\phi\circ\sigma)(z, \bar{z}) = t\}$. Then if M_t is not empty, the real $(2n-2)$-dimensional manifold M_t is the complex pure $(n-1)$-dimensional analytic manifold.

In fact, since ϕ has the isolated t-points, on account of (4.3) and (4.3)$'$ it is sufficient to note that

$$\begin{pmatrix} \partial_w\phi & \partial_{\bar{w}}\phi \\ \partial_w\bar{\phi} & \partial_{\bar{w}}\bar{\phi} \end{pmatrix}\begin{pmatrix} \partial\sigma & \bar{\partial}\sigma \\ \partial\bar{\sigma} & \bar{\partial}\bar{\sigma} \end{pmatrix} = \begin{pmatrix} \partial\phi^* & \bar{\partial}\phi^* \\ \partial\bar{\phi}^* & \bar{\partial}\bar{\phi}^* \end{pmatrix},$$

where $\phi^*(z, \bar{z}) = (\phi\circ\sigma)(z, \bar{z})$.

§ 5. Reduction to a single differential equation.

In this section we shall see that if the system of differential equations (1.8) satisfies the assumptions considered in section 3, the system (1.8) is reduced to a single differential equation of a complex variable by a change of variables. However, since we have the additional differential equations, our system (i. e. (1.8) and (3.5)) is not reduced to a single equation. As we shall see in section 7, by adding suitable assumptions our system is essentially reduced to a differential equation of a complex variable.

We now introduce a non-singular C^∞-transformation of variables $w_j=\psi_j(z,\bar{z})$, $1\leqq j\leqq n$, such that ψ_j are defined in V and its Jacobian is not zero in V. Then the system (1.8), (3.5):

$$\bar{\partial}_j f=\bar{b}_j\bar{f}, \qquad X_j f=-b_{jn}f, \qquad \bar{X}_j f=0, \qquad 1\leqq j\leqq n$$

(X_n denotes the zero infinitesimal transformation) are transformed into (5.1), (5.2) and (5.3) respectively:

$$\sum_{s=1}^{n} d_s\hat{f}\,\bar{\partial}_j w_s+\sum_{s=1}^{n}\bar{d}_s\hat{f}\,\bar{\partial}_j\bar{w}_s=\bar{\hat{b}}_j\bar{\hat{f}}, \tag{5.1}$$

$$\sum_{s=1}^{n} X_j w_s d_s\hat{f}+\sum_{s=1}^{n} X_j\bar{w}_s\bar{d}_s\hat{f}=-\hat{b}_{jn}\hat{f}, \tag{5.2}$$

$$\sum_{s=1}^{n}\bar{X}_j w_s d_s\hat{f}+\sum_{s=1}^{n}\bar{X}_j\bar{w}_s\bar{d}_s\hat{f}=0, \qquad 1\leqq j\leqq n, \tag{5.3}$$

where we set $d_s=\partial_{w_s}$, $\bar{d}_s=\partial_{\bar{w}_s}$ and where $\hat{f}$ will denote the function into which f is transformed. From now on we intend to use the symbol ^ in such a sense. Here, if we choose ψ_j as follows:

$$\psi_j=z_j, \qquad \psi_n=\sigma(z,\bar{z}), \qquad 1\leqq j\leqq n-1, \tag{5.4}$$

where $\sigma(z,\bar{z})$ is a solution of (4.2) which satisfies (4.3) in V and is zero at the origin, then, observing that

$$X_s z_j=b_n\partial_s z_j-b_s\partial_n z_j=b_n\delta_{sj},$$
$$X_s\bar{z}_j=\bar{X}_s z_j=0,$$
$$\bar{X}_s\bar{z}_j=\bar{b}_n\delta_{sj}, \qquad (\delta_{sj} \text{ is Kronecker delta.})$$

it follows that the system (5.1), (5.2) and (5.3) are transformed into (5.5), (5.6) and (5.7) respectively:

$$d_n\hat{f}\,\bar{\partial}_j\sigma+\bar{d}_n\hat{f}\,\bar{\partial}_j\bar{\sigma}=\bar{\hat{b}}_j\bar{\hat{f}}, \qquad 1\leqq j\leqq n, \tag{5.5}$$

$$d_j\hat{f}=-\hat{b}_n^{-1}\hat{b}_{jn}\hat{f}, \qquad 1\leqq j\leqq n-1, \tag{5.6}$$

$$\bar{d}_j\hat{f}=0, \qquad 1\leqq j\leqq n-1. \tag{5.7}$$

The assumptions $(H_1)_n$, $(H_2)_n$ and Lemma 4.1 assure the existence of such a transformation.

Without loss of generality we may assume that

$$|\partial_n\sigma|^2-|\bar{\partial}_n\sigma|^2>0 \quad \text{in} \quad V. \tag{5.8}$$

In fact, from (4.2) there exists a pair of numbers j_0 and j_1 such that

$$\partial_{j_0}\sigma\,\bar{\partial}_{j_1}\bar{\sigma}-\bar{\partial}_{j_1}\sigma\,\partial_{j_0}\bar{\sigma}\neq 0\,,$$

where we consider, if necessary, furthermore smaller neighborhood of the origin than V. Hence, using (4.2) and considering, if necessary, $\bar{\sigma}$ instead of σ, we obtain (5.8).

Again, using (4.2) and considering sufficiently small neighborhood W of the origin, $W\subset V$, we see in turn that we can set

$$a=\frac{\bar{\partial}_j\sigma}{\bar{\partial}_j\bar{\sigma}} \tag{5.9}$$

and can assume that $1-|a|^2$ is bounded away from zero.

Thus, because of $b_j=b\partial_j\sigma$, it is seen that the system (5.5) is transformed into a single equation

$$\bar{d}_n\hat{f}+\hat{a}\,d_n\hat{f}=\bar{\hat{b}}\bar{\hat{f}}, \qquad |\hat{a}|\leqq c_0<1\,, \tag{5.10}$$

where $\hat{a}$, $\hat{b}$ are defined in $\hat{W}$, where $\hat{W}$ is the image of W under the transformation (5.4).

§ 6. Properties of the transformed differential equations.

In order to obtain the local similarity principle we shall add another assumptions on the coefficients to the assumptions (H_j), $0\leqq j\leqq 4$.

We shall consider the following system of differential equations

$$\begin{cases} \bar{d}_jF=A_jF\,, \quad d_jF=B_jF\,, \qquad 1\leqq j\leqq n-1\,,\\ \bar{d}_nF+\alpha\,d_nF=\beta F+\gamma\bar{F}\,. \end{cases} \tag{6.1}$$

Now we shall impose a number of assumptions upon the coefficients of (6.1). Let all the coefficients be defined in a polydisc $\hat{P}=\{w\in \boldsymbol{C}^n\,|\,|w_j|<R,\ 1\leqq j\leqq n\}$.

(i) α is a function of w_n and $\bar{w}_n$ only such that $|\alpha|\leqq c_0<1$ for a positive constant c_0,

(ii) $\bar{d}_j\beta=\bar{d}_nA_j+\alpha d_nA_j$, $d_j\beta=\bar{d}_nB_j+\alpha d_nB_j$,

(iii) $\bar{d}_j\gamma=(A_j-\bar{B}_j)\gamma$, $d_j\gamma=(B_j-\bar{A}_j)\gamma$,

(iv) $\bar{d}_jA_k=\bar{d}_kA_j$, $d_jA_k=\bar{d}_kB_j$, $d_jB_k=d_kB_j$, $1\leqq j,\ k\leqq n-1$.

Now we introduce a function $\omega(w,\bar{w})$ defined in $\hat{P}$:

$$\omega(w, \bar{w}) = \sum_{l=1}^{n-1} \phi_l(w_l, w_{l+1}, \cdots, w_n, \bar{w}_{l+1}, \cdots, \bar{w}_n)$$

$$+ \sum_{l=1}^{n-1} (\boldsymbol{T}_l \tilde{A}_l)(w_l, \cdots, w_n, \bar{w}_l, \cdots, \bar{w}_n),$$

where we put

$$\phi_l(w_l, w_{l+1}, \cdots, w_n, \bar{w}_{l+1}, \cdots, \bar{w}_n) = \int_0^{w_l} (\tilde{B}_l - d_l \boldsymbol{T}_l \tilde{A}_l) d\zeta_l, \qquad |w_l| < R, \quad (1 \leqq l \leqq n-1)$$

$$\tilde{A}_l = A_l - \bar{d}_l \sum_{\mu=1}^{l-1} (\phi_\mu + \boldsymbol{T}_\mu \tilde{A}_\mu), \qquad \tilde{A}_1 = A_1,$$

$$(2 \leqq l \leqq n-1)$$

$$\tilde{B}_l = B_l - d_l \sum_{\mu=1}^{l-1} (\phi_\mu + \boldsymbol{T}_\mu \tilde{A}_\mu), \qquad \tilde{B}_1 = B_1.$$

For each l, $2 \leqq l \leqq n-1$, $\tilde{A}_l$ and $\tilde{B}_l$ do not include $w_1, \cdots, w_{l-1}, \bar{w}_1, \cdots, \bar{w}_{l-1}$.

It follows from the integrability conditions (iv) that

$$\tilde{B}_l - d_l \boldsymbol{T}_l \tilde{A}_l \qquad (2 \leqq l \leqq n-1)$$

is a holomorphic function of a variable w_l in the disc $|w_l| < R$ when fixing other variables. From this it is seen that $\phi_l(w_l, \cdots)$ is holomorphic with respect to w_l.

The integral operator $\boldsymbol{T}_l$ has been defined in section 1, but it should be noted that the variables z and $\bar{z}$ are replaced by the variables w and $\bar{w}$ respectively.

LEMMA 6.1. $\bar{d}_j \omega = A_j$, $d_j \omega = B_j$, $1 \leqq j \leqq n-1$.

In fact,

$$\begin{aligned}
\bar{d}_j \omega &= \sum_{l=1}^{n-1} \bar{d}_j \phi_l + \sum_{l=1}^{n-1} \bar{d}_j \boldsymbol{T}_l \tilde{A}_l \\
&= \sum_{l=1}^{j-1} \bar{d}_j \phi_l + \sum_{l=1}^{j-1} \bar{d}_j \boldsymbol{T}_l \tilde{A}_l + \tilde{A}_j \\
&= A_j, \\
d_j \omega &= d_j \sum_{l=1}^{j-1} \phi_l + (\tilde{B}_j - d_j \boldsymbol{T}_j \tilde{A}_j) + d_j \sum_{l=1}^{j} \boldsymbol{T}_l \tilde{A}_l \\
&= B_j.
\end{aligned}$$

From this lemma, (i), (ii) and (iii) we obtain the following

LEMMA 6.2. *We set*

$$\rho = \beta - \bar{d}_n \omega - \alpha d_n \omega,$$

$$\tau = \gamma \exp(\bar{\omega} - \omega).$$

Then we have in $\hat{P}$

$$\bar{d}_j \rho = d_j \rho = \bar{d}_j \tau = d_j \tau = 0, \qquad 1 \leqq j \leqq n-1.$$

That is, ρ *and* τ *do not include* $2n-2$ *variables* $w_j, \bar{w}_j$, $1 \leqq j \leqq n-1$.

Let F be a regular solution of the system of differential equations (6.1) in a subpolydisc $\hat{P}_0=\{w\in \boldsymbol{C}^n \mid |w_j|<r<R,\ 1\leqq j\leqq n\}$. Then, setting

$$g=F(w,\bar{w})\exp\{-\omega(w,\bar{w})\},$$

we obtain from Lemma 6.1, 6.2 that g does not include the variables $w_j, \bar{w}_j$, $1\leqq j\leqq n-1$, and satisfies the following differential equation in the disc K: $|w_n|<r$ in the complex plane w_n

$$\bar{d}_n g+\alpha d_n g=\rho g+\tau\bar{g}\,,\qquad |\alpha|\leqq c_0<1\,. \tag{6.2}$$

It follows with the aid of Lemma 2.1 that the function g is similar to a function of $Q(K)$ and vice versa.

Regarding all functions of a complex variable as functions of several complex variables, we shall denote the class $Q(K)$ by $Q(\hat{P}_0)$.

Thus we are now in a position to state the

THEOREM 6.1. *The system of differential equations* (6.1) *satisfies the similarity principle with respect to the class* $Q(\hat{P}_0)$.

§ 7. Additional assumptions on the coefficients and main theorem.

In this section we return to the system of equations (5.6), (5.7) and (5.10):

$$d_j\hat{f}=-\hat{b}_n^{-1}\hat{b}_{jn}\hat{f}\,, \tag{5.6}$$

$$\bar{d}_j\hat{f}=0\,, \qquad (1\leqq j\leqq n-1) \tag{5.7}$$

$$\bar{d}_n\hat{f}+\hat{a}\,d_n\hat{f}=\bar{\hat{b}}\bar{\hat{f}},\qquad |\hat{a}|\leqq c_0<1\,. \tag{5.10}$$

It does not always follow that the function a defined by (5.9) is written in the composite form $\alpha\circ\sigma$, where α is defined in the image $\sigma(W)$. We remark the following fact: Let G be a domain in $\boldsymbol{C}^n$ and let a function $\sigma(z,\bar{z})$ be defined such that $|\partial_n\sigma|^2-|\bar{\partial}_n\sigma|^2\neq 0$ in G. Then the image $\sigma(G)$ is also the domain in the complex plane.

We impose the following condition:

$(H_5)_n$ There exists a function α defined in the image $\sigma(W)$ such that a is written in the form $\alpha\circ\sigma$ and $|\alpha|\leqq c_0<1$ in W.

Furthermore we shall assume that for j, $1\leqq j\leqq n-1$,

$$\bar{X}_j b=-\bar{b}_{jn}b\,,\qquad (b_j=b\partial_j\sigma) \tag*{$(H_6)_n$}$$

$$b_n\bar{\partial}_n b_{jn}-b_{jn}\bar{\partial}_n b_n=0\,. \tag*{$(H_7)_n$}$$

It is seen that the assumption $(H_5)_n$ is the condition which σ must moreover satisfy. However, giving α in a neighborhood of the origin in the complex plane such that $|\alpha|\leqq c_0<1$, we assume $(H_5)'_n$:

$(H_5)'_n$ There exist a neighborhood W of the origin in C^n and a function $\sigma(z, \bar{z})$ such that $\sigma(0)=0$, $\partial_n\sigma \neq 0$ in W and σ satisfies the system of first order non-linear differential equations $\bar{\partial}_j\sigma=(\alpha\circ\sigma)\bar{\partial}_j\bar{\sigma}$, $1\leqq j\leqq n$. Moreover σ has the property that there exists a function b defined in W such that $b_j=b\partial_j\sigma$, $1\leqq j\leqq n$.

Then, noting that σ in $(H_5)'_n$ satisfies (4.2), Lemma 4.1 shows that the assumptions $(H_1)_n$, $(H_2)_n$ and $(H_5)_n$ can be replaced by $(H_5)'_n$.

To sum up, in order to assure the existence of σ which completes the change of variables we assume $(H_1)_n$ and $(H_2)_n$ and in order that we may reduce essentially the system of equation (1.8) and the additional equations (3.1) to a single equation of one complex variable we assume $(H_j)_n$, $4\leqq j\leqq 7$.

In making examples of our system which satisfies the above six assumptions we shall use the assumption $(H_5)'_n$.

Without loss of generality we can assume that W corresponds homeomorphically to $\hat{P}$ and hence that $\hat{P}=\hat{W}$. We put

$$\left\{\begin{array}{l} A_j=0\,, \quad B_j=-(\widehat{b_{jn}b_n^{-1}})\,, \\ \alpha=\hat{a}\,, \quad \beta=0\,, \quad \gamma=\bar{\hat{b}}\,. \end{array}\right. \tag{7.1}$$

We want to check that (7.1) satisfies the conditions (i), (ii), (iii) and (iv) in section 6 under the assumptions $(H_j)_n$, $4\leqq j\leqq 7$.

First, (i) is obvious from $(H_5)_n$. Observe that

$$X_j=b_nd_j\,, \qquad \bar{X}_j=\bar{b}_n\bar{d}_j\,, \qquad 1\leqq j\leqq n-1\,. \tag{7.2}$$

Then (iii) is equivalent to

$$\bar{X}_j\bar{b}=\bar{b}_{jn}\bar{b}\,, \qquad X_j\bar{b}=-b_{jn}\bar{b}\,, \qquad 1\leqq j\leqq n-1\,. \tag{7.3}$$

The second part of (7.3) is $(H_6)_n$ itself.

From (4.6) and (4.7) we see that the first part of (7.3) is trivial.

The condition (iv) is transformed by the inverse transformation of (5.4) into

$$\bar{X}_j(b_n^{-1}b_{jn})=0\,, \tag{7.4}$$

$$X_j(b_n^{-1}b_{kn})=X_k(b_n^{-1}b_{jn})\,, \qquad 1\leqq j\leqq n-1\,. \tag{7.5}$$

It is seen that (7.4), (7.5) are nothing but $(H_4)_n$, $(H_3)_n$ respectively.

It remains to verify that the condition (ii) is satisfied. To this end, we must state the following

LEMMA 7.1. *$\bar{d}_nB_j+\alpha d_nB_j=0$ is equivalent to $(H_7)_n$.*

PROOF. Observe first that

$$dB_j=\partial_n\check{B}_jdz_n+\bar{\partial}_n\check{B}_jd\bar{z}_n\,, \tag{7.6}$$

where the symbol d denotes either of the differential operators $\bar{d}_n$, d_n and the symbol $\check{}$ denotes the function into which a function under $\check{}$ is transformed by the inverse change of variables of (5.4).

From (7.6) we obtain that

$$\bar{d}_n B_j + \alpha d_n B_j = \partial_n B_j^*(\bar{d}_n z_n + a d_n z_n) + \bar{\partial}_n B_j^*(\bar{d}_n \bar{z}_n + a d_n \bar{z}_n)\,, \tag{7.7}$$

where $B_j^* = b_{jn}/b_n$ and $a = \alpha \circ \sigma$.

By differentiating the both sides of $w_n = \sigma(z, \bar{z})$ with respect to $w_n, \bar{w}_n$, we may derive that

$$\begin{cases} d_n z_n = J^{-1}\bar{\partial}_n \bar{\sigma}\,, \\ d_n \bar{z}_n = -J^{-1}\partial_n \bar{\sigma}\,, \end{cases} \tag{7.8}$$

where $J = |\partial_n \sigma|^2 - |\bar{\partial}_n \sigma|^2$.

Observe secondly that by (7.8)

$$\begin{cases} \bar{d}_n z_n + a\, d_n z_n = -J^{-1}(\bar{\partial}_n \sigma - a\, \bar{\partial}_n \bar{\sigma}) \\ \qquad\qquad = 0\,, \\ \bar{d}_n \bar{z}_n + a\, d_n \bar{z}_n = J^{-1}(\partial_n \sigma - a\, \partial_n \bar{\sigma}) \\ \qquad\qquad = (\bar{\partial}_n \bar{\sigma})^{-1}\,. \end{cases} \tag{7.9}$$

Inserting (7.9) into the right-hand side of (7.7), we find

$$\bar{d}_n B_j + \alpha\, d_n B_j = (\bar{\partial}_n \bar{\sigma})^{-1} \bar{\partial}_n B_j^*$$

which completes the proof.

The following lemma is easily derived.

LEMMA 7.2. *Let g is in $Q(K)$. Then the composite function $g \circ \sigma$ is holomorphic in W_0 which corresponds to the polydisc $\hat{P}_0$ under the change of variables* (5.4).

We shall consider a class $\mathcal{A}(W_0)$ and a subclass $\mathcal{A}_0(W_0)$ of the class $\mathcal{A}(W_0)$ which *is composed of functions $g \circ \sigma$, g being in $Q(K)$.*

Remark that, giving the system of equations (1.8) in G, we consider the assumptions on the coefficients of (1.8) which are restricted to a neighborhood of a point, say, the origin in G.

We are now in a position to state the following

THEOREM 7.1 (Main Theorem). *Suppose that in a neighborhood W of the origin in $\boldsymbol{C}^n$ the coefficients of the system* (1.8) *satisfy the conditions $(H_j)_n$, $j = 1, 2, 4, 5, 6, 7$. Then there exists a neighborhood W_0 of the origin such that $W_0 \subset W$ and the system of equations* (1.8) *satisfies the similarity principle with respect to the class $\mathcal{A}_0(W_0)$.*

§ 8. Lemmas for an analysis of the coefficients, and an example.

In this and following sections we shall show the existence of the system of equations (1.8) which satisfies the assumptions of Theorem 7.1. Viewing from a different angle, we analyse the coefficients of (1.8).

It follows from (4.5), (4.6) and (4.8) that the assumption $(H_6)_n$ is turned into

$$Y_j|b|^2=0\,,\qquad 1\leqq j\leqq n-1\,, \tag{8.1}$$

that is, $(H_6)_n$ is equivalent to (8.1).

From this and with the aid of Lemma 4.1, it turns out that

$$|b|=r\circ\sigma\,, \tag{8.2}$$

where r is a non-negative valued function of a complex variable which is defined in the image $\sigma(W)$.

Similarly, inserting (4.5), (4.6) and (4.7) into $(H_4)_n$ and deforming them, we shall obtain

$$b\alpha_n\bar{Y}_kY_jb-\alpha_nY_jb\bar{Y}_kb-b\bar{Y}_k\alpha_nY_jb=0\,, \tag{8.3}$$

where we set $\alpha_n=\partial_n\sigma$.

Before considering (8.3) we must state the

LEMMA 8.1. (1) $\bar{Y}_{(j,k)}\partial_j\sigma=0$ *for* $j,k=1,2,\cdots,n$.

(2) *For any number* l *such that* $\partial_l\sigma\neq 0$ *in* W *and for* $j,k=1,2,\cdots,n$,

$$\bar{Y}_{(j,k)}\partial_l\sigma=0\,,$$

$$Y_{(j,k)}\bar{\partial}_l\sigma=0\,.$$

PROOF. It follows from $(H_5)_n$ that

$$\bar{\partial}_s\sigma=(\alpha\circ\sigma)\bar{\partial}_s\bar{\sigma}\,.$$

For $s=j,k$, by differentiating the above both sides with respect to z_j respectively, and by eliminating $\partial_j\alpha\circ\sigma$ from them, we obtain

$$\bar{\partial}_k\bar{\sigma}\partial_j\bar{\partial}_j\sigma-\bar{\partial}_j\bar{\sigma}\partial_j\bar{\partial}_k\sigma=(\alpha\circ\sigma)(\bar{\partial}_k\bar{\sigma}\partial_j\bar{\partial}_j\bar{\sigma}-\bar{\partial}_j\bar{\sigma}\partial_j\bar{\partial}_k\bar{\sigma})\,.$$

Expressing this in terms of $\bar{Y}_{(j,k)}$, we have

$$\begin{aligned}\bar{Y}_{(j,k)}\partial_j\sigma&=(\alpha\circ\sigma)\bar{Y}_{(j,k)}\partial_j\bar{\sigma}\\&=(\alpha\circ\sigma)\bar{Y}_{(j,k)}\{(\overline{\alpha\circ\sigma})\partial_j\sigma\}\\&=|\alpha\circ\sigma|^2\bar{Y}_{(j,k)}\partial_j\sigma\\&\qquad(\text{by } Y_{(j,k)}\alpha\circ\sigma=0 \text{ and Lemma 4.1})\,.\end{aligned}$$

Because of $|\alpha\circ\sigma|<1$, we obtain the required result for either $l=j$ or k.

To complete the proof of the first part of (2), it is sufficient to remark that we may have

$$Y_{(j,k)}\bar{\partial}_l\bar{\sigma}=(\alpha_l)^{-1}(\alpha_k Y_{(j,l)}\bar{\alpha}_l-\alpha_j Y_{(k,l)}\bar{\alpha}_l)\,,$$

where we set $\alpha_s=\partial_s\sigma$, $1\leqq s\leqq n$.

Next we have

$$\begin{aligned} Y_{(j,k)}\bar{\partial}_l\sigma &= Y_{(j,k)}\{(\alpha\circ\sigma)\bar{\partial}_l\bar{\sigma}\} \\ &=(\alpha\circ\sigma)Y_{(j,k)}\bar{\partial}_l\bar{\sigma} \\ &=0\,. \end{aligned}$$

This completes the proof.

From this lemma and Lemma 4.1 the following lemma is immediately obtained.

LEMMA 8.2. *For any function h defined in the image $\sigma(W)$*

$$\bar{Y}_{(j,k)}\partial_l h\circ\sigma=0\,,$$

$$Y_{(j,k)}\bar{\partial}_l h\circ\sigma=0\,.$$

With the aid of Lemma 8.1, we see that (8.3) is turned into

$$b\bar{Y}_k Y_j b-Y_j b\bar{Y}_k b=0\,,\qquad 1\leqq j,\,k\leqq n-1\,. \tag{8.4}$$

By (8.2) b is written in the form

$$b=(r\circ\sigma)\exp\{i\theta(z,\bar{z})\}\,, \tag{8.5}$$

where θ is a real valued function defined in W.

Substituting (8.5) in (8.4) and using Lemma 4.1, we obtain the system of differential equations with respect to θ

$$\bar{Y}_k Y_j\theta=0\,,\qquad 1\leqq j,\,k\leqq n-1\,. \tag{8.6}$$

The following lemma is readily obtained from Lemma 8.1.

LEMMA 8.3. $[Y_j,\bar{Y}_k]=0$, *that is, Y_j and $\bar{Y}_k$ commute.*

COROLLARY 8.1. $[Y_{(j,l)},\bar{Y}_{(k,l)}]=0$.

Observe that we have, by the change of variables (5.4) and using (7.2), (4.6),

$$Y_j=\alpha_n d_j\,,\qquad \bar{Y}_j=\bar{\alpha}_n\bar{d}_j\,,\qquad 1\leqq j\leqq n-1\,. \tag{8.7}$$

Note that

(8.8) $\bar{Y}_j\alpha_k=0$, $Y_j\bar{\partial}_k\sigma=0$ are equivalent to $\bar{d}_j\hat{\alpha}_k=0$, $d_j\widehat{\bar{\partial}_k\sigma}=0$, respectively.

Then it follows that (8.4) is equivalent to

$$\bar{d}_j d_k\hat{\theta}=0\,,\qquad 1\leqq j,\,k\leqq n-1\,. \tag{8.9}$$

Thus it is seen that $\hat{\theta}$ is, for each fixed w_n, a pluriharmonic function with respect to $w_j, \bar{w}_j$, $1 \leqq j \leqq n-1$. Since we can assume that the given neighborhood W of the origin corresponds to the polydisc $\hat{P}$ under the change of variables considered, $\hat{\theta}$ is represented in the form $\mathrm{Re}\,\phi$, where ϕ is a holomorphic function with respect to $w_1, \cdots, w_{n-1}$ in $\hat{P}$. And hence θ is written in the form

$$\mathrm{Re}\,\phi\{z_1, \cdots, z_{n-1}, \sigma(z, \bar{z}), \bar{z}_1, \cdots, \bar{z}_{n-1}, \overline{\sigma(z, \bar{z})}\}\,. \tag{8.10}$$

Using (4.7), (8.5), it is derived from Lemma 4.1 that

$$\begin{aligned} \frac{b_{jk}}{b_k} &= \frac{Y_{(j,k)}b}{b\partial_k\sigma} \\ &= i\frac{Y_{(j,k)}\theta}{\partial_k\sigma}\,. \end{aligned} \tag{8.11}$$

Differentiating the both sides of (8.11) with respect to $\bar{z}_l$, we obtain

$$\bar{\partial}_l\frac{b_{jk}}{b_k} = \frac{i}{\alpha_k^2}\{\alpha_k Y_{(j,k)}\bar{\partial}_l\theta - (\partial_k\theta)Y_{(j,k)}\bar{\partial}_l\sigma\}\,.$$

By virtue of these and Lemma 8.1, we obtain the

LEMMA 8.4.

$$\bar{\partial}_l\frac{b_{jk}}{b_k} = 0 \textit{ is equivalent to } Y_{(j,l)}\bar{\partial}_k\theta = 0\,, \qquad 1 \leqq j, k, l \leqq n\,.$$

We shall again remark that the symbol $\hat{}$ denotes the functions into which our functions are transformed by the change of variables (5.4).

Note that

$$Y_{(j,k)} = \alpha_k d_j - \alpha_j d_k\,, \qquad 1 \leqq j, k \leqq n-1\,,$$

$$\bar{\partial}_l = \bar{\partial}_l\sigma d_n + \bar{d}_l + \bar{\alpha}_l\bar{d}_n \qquad \text{for} \quad l \neq n\,,$$

and

$$\bar{\partial}_n = \bar{\partial}_n\sigma d_n + \bar{\alpha}_n\bar{d}_n\,.$$

The following lemma is readily obtained from Lemma 7.1 (also Lemma 8.4).

LEMMA 8.5. *$(H_7)_n$ is equivalent to*

$$d_j(\bar{d}_n\hat{\theta} + \alpha d_n\hat{\theta}) = 0\,, \qquad 1 \leqq j \leqq n-1\,. \tag{8.12}$$

LEMMA 8.6. *Under the assumption $(H_7)_n$ we have*

(1) $$Y_{(j,k)}\bar{\partial}_l\theta = 0 \rightleftarrows \alpha_k d_j\bar{d}_l\hat{\theta} - \alpha_j d_k\bar{d}_l\hat{\theta} = 0\,, \qquad 1 \leqq j, k \leqq n-1\,;$$

(2) $$Y_k\bar{\partial}_k\theta = 0 \rightleftarrows d_k\bar{d}_k\hat{\theta} = 0\,, \qquad 1 \leqq k \leqq n-1\,.$$

PROOF. From $\bar{\partial}_l\sigma = (\alpha \circ \sigma)\bar{\partial}_l\bar{\sigma}$, we see that

$$\bar{\partial}_l = \bar{\alpha}_l(\bar{d}_n + \alpha d_n) + \bar{d}_l\,, \qquad 1 \leqq l \leqq n-1\,.$$

Observing that $Y_l\bar{\alpha}_m = 0 \rightleftarrows d_l\hat{\bar{\alpha}}_m = 0$, $1 \leqq l, m \leqq n$, then (1), (2) are easily obtained from Lemmas 8.1, 8.5.

COROLLARY 8.2. *Under the assumption* $(H_7)_n$ *we have*

$$Y_{(j,k)}\bar{\partial}_l\theta=0 \rightleftarrows \alpha_k Y_j \bar{Y}_l\theta-\alpha_j Y_k \bar{Y}_l\theta=0\,; \tag{1}$$

$$Y_k\bar{\partial}_k\theta=0 \rightleftarrows Y_k\bar{Y}_k\theta=0\,, \qquad 1\leqq j,\, k,\, l\leqq n-1\,. \tag{2}$$

REMARK 8.1. The assumption $(H_7)_l$, $l\neq n$, that is, $\bar{\partial}_l(b_{jl}/b_l)=0$, $1\leqq j\leqq n$, follows from the assumptions $(H_j)_n$, $1\leqq j\leqq 7, j\neq 3$ and with the aid of Lemmas 8.4, 8.6. Similarly the assumption $(H_4)_l$, $l\neq n$, does so. In fact, it is sufficient to observe the following relations which are obtained from (8.11) and with the aid of Lemma 8.1:

$$\bar{Y}_{(k,l)}Y_{(j,l)}=|\alpha_n|^{-2}\{\bar{\alpha}_l\alpha_l\bar{Y}_kY_j-\bar{\alpha}_k\alpha_l\bar{Y}_lY_j-\bar{\alpha}_l\alpha_j\bar{Y}_kY_l+\bar{\alpha}_k\alpha_j\bar{Y}_lY_l\}\,, \qquad (j,\, k\neq n)$$

$$\bar{Y}_{(n,l)}Y_{(j,l)}=(\alpha_n)^{-1}(\alpha_j\bar{Y}_lY_l-\alpha_l\bar{Y}_lY_j)\,, \qquad (j\neq n)$$

$$\bar{Y}_{(k,l)}Y_{(n,l)}=(\bar{\alpha}_n)^{-1}(\bar{\alpha}_k\bar{Y}_lY_l-\bar{\alpha}_l\bar{Y}_kY_l)\,, \qquad (k\neq n)\,.$$

REMARK 8.2. In section 11 we shall discuss about the assumptions $(H_5)_l$, $(H_6)_l$.

It is natural for the following question to arise: Will there exist a non-trivial solution of the system of equations (8.9)? Such a solution surely exists. That is, we have the following example:

$$\hat{\theta}=\frac{1}{2}(\hat{\phi}+\bar{\hat{\phi}})+\hat{\Theta}(w_n,\bar{w}_n)\,, \tag{8.13}$$

where $\hat{\phi}$ is a holomorphic function with respect to $w_1,\cdots,w_{n-1}$ and includes neither w_n nor $\bar{w}_n$ and where $\hat{\Theta}$ is a real valued function defined in the open disc $|w_n|<R$. Then (8.10) is described as follows:

$$\theta=\operatorname{Re}\phi(z_1,\cdots,z_{n-1})+(\Theta\circ\sigma)(z,\bar{z})\,, \tag{8.14}$$

where $\phi(z_1,\cdots,z_{n-1})$ is holomorphic.

To sum up, we have carried out in two stages in order to make examples which fulfill a number of assumptions $(H_j)_n$, $1\leqq j\leqq 7$, $j\neq 3$: we find σ such that $\bar{\partial}_j\sigma=(\alpha\circ\sigma)\bar{\partial}_j\bar{\sigma}$, $1\leqq j\leqq n$ (see the following section), and find the function b such that $(H_4)_n$, $(H_6)_n$ and $(H_7)_n$ are fulfilled. Thus the desired examples are obtained by putting $b_j=b\partial_j\sigma$.

As immediately seen, these examples show that the assumption $(H_5)'_n$ is satisfied.

§ 9. Non-emptiness of the assumption $(H_5)'_n$.

In this section we shall show that there exists a function σ which satisfies the first part of the assumption $(H_5)'_n$. Nirenberg [13] discussed more general system of non-linear first order partial differential equations than our system. Though our system is a special case of those in [12], [13], it should be noted that we need no integrability conditions. Our technique of the proof is the same as Newlander-Nirenberg's one [12] and hence we give the outline of the proof.

LEMMA 9.1. *Let Δ be an open disc with center at the origin in t-plane. Let $\alpha(t, \bar{t})$ be a complex valued function defined in Δ such that $\alpha(0)=0$. Then the system of first order non-linear differential equations*

$$\bar{\partial}_j g=(\alpha\circ g)\bar{\partial}_j \bar{g}\,, \qquad 1\leqq j\leqq n\,, \tag{9.1}$$

admits a solution defined in a sufficiently small neighborhood N of the origin such that $g(0)=0$, $\partial_j g\neq 0$, $1\leqq j\leqq n$ in N.

For such a solution g, we set $M_t=\{z\in N\mid g(z, \bar{z})=t\}$. Then M_t is a complex pure $(n-1)$-dimensional analytic manifold.

Set

$$\boldsymbol{F}=(a\bar{\partial}_1\bar{g},\, a\bar{\partial}_2\bar{g},\, \cdots,\, a\bar{\partial}_n\bar{g})\,, \qquad a=\alpha\circ g$$

and

$$\boldsymbol{TF}g=\sum_{s=0}^{n-1}\frac{(-1)^s}{(s+1)!}\sum{}'\,\boldsymbol{T}_{j_1}\bar{\partial}_{j_1}\cdots\boldsymbol{T}_{j_s}\bar{\partial}_{j_s}\boldsymbol{T}_k((\alpha\circ g)\bar{\partial}_k\bar{g}) \qquad \text{(see (1.11))}.$$

Assume that σ is a solution of (9.1) in some neighborhood of the origin. From (9.1) we obtain

$$\sigma(z, \bar{z})=\varphi(z)+(\boldsymbol{TF}\sigma)(z, \bar{z})\,, \tag{9.2}$$

where $\varphi(z)$ is a holomorphic function in that neighborhood.

Observe that without any conditions we have

$$\bar{\partial}_j\{(\alpha\circ\sigma)\bar{\partial}_k\bar{\sigma}\}=\bar{\partial}_k\{(\alpha\circ\sigma)\bar{\partial}_j\bar{\sigma}\}\,, \qquad 1\leqq j,\, k\leqq n\,.$$

Conversely, let $\varphi(z)$ be a given holomorphic function and $\sigma(z, \bar{z})$ be a solution of (9.2). Then we have the following relations:

$$\bar{\partial}_j\sigma-a\bar{\partial}_j\bar{\sigma}=\sum_{s=0}^{n-2}\frac{(-1)^s}{(s+2)!}\sum_{(j)}\boldsymbol{T}_{j_1}\bar{\partial}_{j_1}\cdots\boldsymbol{T}_{j_s}\bar{\partial}_{j_s}\boldsymbol{T}_k q_{jk}\,, \tag{9.3}$$

$$q_{jk}=\bar{\partial}_j(a\bar{\partial}_k\bar{\sigma})-\bar{\partial}_k(a\bar{\partial}_j\bar{\sigma})\,,$$

where $\sum_{(j)}$ denotes the summation over all $(s+1)$-tuples with $j_1, \cdots, j_s, k$ mutually distinct and different from j.

Setting the left-hand side of (9.3) p_j, we obtain the system of the linear integro-differential equations with respect to p_j, $1\leqq j\leqq n$

$$p_j=\sum_{s=0}^{n-2}\frac{(-1)^s}{(s+2)!}\sum_{(j)}\boldsymbol{T}_{j_1}\bar{\partial}_{j_1}\cdots\boldsymbol{T}_{j_s}\bar{\partial}_{j_s}\boldsymbol{T}_k(\bar{\partial}_k p_j-\bar{\partial}_j p_k) \tag{9.4}$$

which admits only null solution for sufficiently small neighborhood of the origin.

Now, in order to obtain the desired solution we consider the equation of the following type:

$$g=\sum_{j=1}^{n} z_j+\boldsymbol{TF}g-(\boldsymbol{TF}g)_0\,, \tag{9.5}$$

where $(\boldsymbol{TF}g)_0$ is the value of $\boldsymbol{TF}g$ at the origin.

The function space in which we shall solve (9.5) is the Banach space $\tilde{C}^{m+\delta}$, which is introduced in [**12**], where m is any integer $\geqq n$ and $0<\delta<1$. Each element in $\tilde{C}^{m+\delta}$ is defined in an open polydisc with center the origin and polyradius r, $r\leqq(1/4)$. Making use of the principle of the contraction mapping and taking, if necessary, a sufficiently small r, it follows that the integral equation (9.5) admits a unique solution σ and the system of integro-differential equations (9.4) has only zero solution.

From the definition of the norm of $\tilde{C}^{m+\delta}$ and for r possibly restricted still further it is seen that in the polydisc considered

$$\partial_j\sigma\neq 0\,,\qquad 1\leqq j\leqq n \tag{9.6}$$

and

$$|\alpha\circ\sigma|\leqq c_0<1\,.$$

Because of

$$\frac{\bar\partial_1\sigma}{\bar\partial_1\bar\sigma}=\frac{\bar\partial_2\sigma}{\bar\partial_2\bar\sigma}=\cdots=\frac{\bar\partial_n\sigma}{\bar\partial_n\bar\sigma}\,,$$

the second part of lemma is obvious.

§ 10. Special cases.

In this section we shall consider the system of equations (1.8) whose coefficients fulfill some conditions.

It is readily seen from (8.11) that

$$b_{jk}=0 \text{ is equivalent to } Y_{(j,k)}\theta=0\,,\qquad 1\leqq j,\,k\leqq n\,.$$

Since θ is a real valued function, we have also

$$\bar{Y}_{(j,k)}\theta=0\,,\qquad 1\leqq j,\,k\leqq n\,.$$

Hence, by Lemma 4.1 and (8.5), we can state the

LEMMA 10.1. *$b_{jk}=0$ is equivalent to that b is written in the form $\nu\circ\sigma$.*

Here we consider the case in which the coefficients are constants, that is,

$$\bar\partial_j f=\bar c_{1j}f+\bar c_{2j}\bar f\,,\qquad 1\leqq j\leqq n\,. \tag{10.1}$$

The domain G considered is a polydisc with center the origin and polyradius R. Then we obtain

$$\boldsymbol{T}_j\bar c_{1j}=\bar c_{1j}\bar z_j\,,\qquad |z_j|\leqq R\,,\quad 1\leqq j\leqq n$$

and hence

$$\Omega(z, \bar{z}) = \sum_{j=1}^{n} \bar{c}_{1j}\bar{z}_j\,, \qquad |z_j| \leqq R\,, \quad 1 \leqq j \leqq n\,.$$

Observing that

$$b_j = c_{2j} \exp(\Omega - \bar{\Omega})\,, \qquad 1 \leqq j \leqq n\,,$$

$$\theta = \frac{i}{2}(\bar{\Omega} - \Omega)\,,$$

$$b = \exp(\Omega - \bar{\Omega})\,,$$

we find from Lemma 10.1 that

$$b_{jk} = 0,\ 1 \leqq j,\ k \leqq n \rightleftarrows \operatorname{rank}\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \end{pmatrix} = 1\,.$$

Using $\sigma = c_{21}z_1 + c_{22}z_2 + \cdots + c_{2n}z_n$, we have

$$\bar{\Omega} = \left\{ \sum_{\mu=1}^{n-1} (c_{1\mu}c_{2n} - c_{1n}c_{2\mu})z_\mu + c_{1n}\sigma \right\} \frac{1}{c_{2n}}\,.$$

Setting

$$\phi = \frac{i}{c_{2n}} \sum_{\mu=1}^{n-1} (c_{1\mu}c_{2n} - c_{1n}c_{2\mu})z_\mu\,,$$

$$\Theta(t, \bar{t}) = \frac{i}{2}\left(\frac{c_{1n}}{c_{2n}}t - \frac{\bar{c}_{1n}}{\bar{c}_{2n}}\bar{t} \right),$$

we know the existence of solutions of equations (10.1) by (8.14).

Assume that $b_{jk} = 0$, $1 \leqq j, k \leqq n$, in U. If the system (1.8) has a solution f which fulfills $\operatorname{rank}(\boldsymbol{B}, \boldsymbol{C}) = 2$, then (3.1) turns into

$$\boldsymbol{X}_{(j,k)}f = 0\,, \qquad \bar{\boldsymbol{X}}_{(j,k)}f = 0\,, \qquad 1 \leqq j,\ k \leqq n\,,$$

and with the aid of Lemma 4.1 there must exist the relation $(H_2)_n$ and hence the relations $(H_2)_k$, $1 \leqq k \leqq n-1$, between the coefficients b_s, $1 \leqq s \leqq n$, and the solution f must take the form $g \circ \lambda$ where λ is the function given in Lemma 4.1.

REMARK 10.1. In the above case $(H_1)_k$, $1 \leqq k \leqq n$, are free (see $(H_1)'_n$).

In case the coefficients b_j are holomorphic, the assumption $(H_1)_n$ alone assures the existence of solutions of (1.8).

Let all b_j, $1 \leqq j \leqq n$ $(n \geqq 2)$, be holomorphic functions in a domain G which contains the origin and let $b_n \neq 0$ at the origin. Then there exists a neighborhood U of the origin such that $b_n \neq 0$ there. With the aid of Lemma 4.1 it follows that for a suitable neighborhood V of the origin, $V \subset U$, we have a holomorphic solution σ of the system of equations (4.2) in V such that $\sigma(0) = 0$. Hence b_j is written in the form

$$b_j = b\partial_j\sigma\,, \qquad 1 \leqq j \leqq n\,, \tag{10.2}$$

where b is a holomorphic function in V.

It is obvious that b_j, being given by (10.2), fulfills the assumption $(H_1)_n$.

Here we may consider a non-singular holomorphic transformation (5.4). We may assume, taking, if necessary, still smaller neighborhood, that V corresponds homeomorphically to a polydisc $\hat{W}$ with center the origin and polyradius r. Then, since ω is holomorphic in $\hat{W}$ and α vanishes (see section 6), it is derived that ρ vanishes. Observe that

$$X_j\check{\tau}=-\check{\tau}X_j\check{\omega}\,, \tag{10.3}$$

where the symbol $\check{}$ denotes the function into which a function under is transformed by the inverse transformation of (5.4) and we set

$$\tau=\overline{\hat{b}}\exp(\overline{\omega}-\omega)\,. \tag{10.4}$$

From (6.2) it is seen that τ is the function of $w_n, \overline{w}_n$ only. Hence

$$\check{\tau}=\kappa\circ\sigma\,,$$

where κ is a function defined in the image $\sigma(V)$. In other words, the function $\check{\tau}$ is the solution of equations

$$X_jh=0\,,\qquad 1\leqq j\leqq n\,.$$

And hence we obtain from (10.3) that $\check{\omega}$ satisfies the equations $X_j\check{\omega}=0$, that is, $\check{\omega}$ is written in the form $\check{\omega}=\nu\circ\sigma$. From this and (10.4) it follows that

$$b=\xi\circ\sigma\,, \tag{10.5}$$

where ν and ξ are holomorphic in the image $\sigma(V)$.

Thus we derive with the aid of Lemma 10.1 that $b_{jk}=0$, $1\leqq j, k\leqq n$. By virtue of the identity theorem, we obtain the following

THEOREM 10.1. *Let the coefficients b_j of the system* (1.8) *be holomorphic in a domain G, let V be a neighborhood of the point z^0 of G, and let b, σ be holomorphic in V such that $b_j=b\partial_j\sigma$, $1\leqq j\leqq n$, $b\neq 0$ and $\partial_n\sigma\neq 0$ in V. Then a necessary and sufficient condition in order that the system* (1.8) *may have a solution in V is that b be written in the form* (10.5) *in V and hence that $b_{jk}=0$ everywhere in G.*

Another formulation of Theorem 10.1 is Theorem 2.1.

REMARK 10.2. In case of $n=2$ the assumption $(H_1)_n$ is vacuous and hence the system (1.8) has locally a solution if and only if $b_{12}=0$, that is, $\partial_1b_2-\partial_2b_1=0$ everywhere in G.

In case that the coefficients b_j of the system (1.8) are holomorphic in a domain G, we may regard the condition $(H_1)_n$ as the necessary condition, so long as we deal with solutions with the null points.

We can scribe the commutators $[X_j, X_k]$, $1\leqq j, k\leqq n-1$, in the following forms:

$$[X_j, X_k]=\sum_{s=1}^{n-1}\alpha_{jks}X_s+\beta_{jkn}\partial_n\,,$$

where α_{jks}, β_{jkn}, $1 \leqq j, k \leqq n-1$, are holomorphic in G. We denote the analytic variety $\{z \in G \mid \beta_{jkn}(z)=0,\ 1 \leqq j, k \leqq n-1\}$ by S^*. Then we have $S \subset S^*$, where the set S is as defined in section 3. Let f be any solution in a neighborhood V, $V \subset G$. Remark that the set of null points of f is contained in S. We may consider that the system (1.8) has an infinite number of solutions as the case of one complex variable. And hence the above remark shows that S^* may have an interior point, that is, S^* is identical with G. We see readily that the relations $\beta_{jkn}=0$, $1 \leqq j, k \leqq n-1$, are $(H_1)_n$ itself. For that reason, in case that b_j, $1 \leqq j \leqq n$, are holomorphic in G, we may consider that the condition $b_{jn}=0$, $1 \leqq j \leqq n-1$, is a necessary and sufficient one in order that there may exist a local solution of the system (1.8).

§ 11. Compatibility of the assumptions.

We supposed a number of conditions $(H_j)_n$, $j \neq 3$, $1 \leqq j \leqq 7$ around a point of G and obtained Theorem 7.1 (Main Theorem). However, even if we suppose the conditions $(H_j)_l$, $j \neq 3$, $1 \leqq j \leqq 7$, for a number l such that $1 \leqq l \leqq n$, we shall also obtain the same result. Then it is natural for the following question to arise: Are the assumptions $(H_j)_n$ compatible with $(H_j)_l$, $j \neq 3$, $1 \leqq j \leqq 7$, around the same point? In this section we shall deal with this problem.

Assuming that G contains the origin, as usual we shall discuss around the origin and hence the notations follow those which are used until now.

Let l be a number different from n such that $b_l \neq 0$ in a neighborhood U_l of the origin. Assume that the coefficients b_j of the system (1.8) satisfy $(H_j)_l$, $j \neq 3$, $1 \leqq j \leqq 7$ in U_l.

It is seen from (3.4) that in $U \cap U_l$ the conditions $(H_j)_l$, $j=1, 2$, is derived from $(H_j)_n$, $j=1, 2$, and vice versa.

Let τ be a solution in a neighborhood $W_l \subset U_l$ of the origin of the following system of equations

$$X_{(j,l)}\tau = 0\,, \qquad \bar{X}_{(j,l)}\tau = 0\,, \qquad 1 \leqq j \leqq n\,,$$

such that in W_l

$$\operatorname{rank}\begin{pmatrix} \partial\tau & \bar{\partial}\tau \\ \partial\bar{\tau} & \bar{\partial}\bar{\tau} \end{pmatrix} = 2\,.$$

σ^* and τ^* will denote the restrictions of σ and τ to $W \cap W_l$ respectively. Then by virtue of Lemma 4.1 there exist functions g and $\tilde{g}$ such that g and $\tilde{g}$ are defined in $\sigma^*(W \cap W_l)$ and $\tau^*(W \cap W_l)$ respectively, and such that $\tau^* = g \circ \sigma^*$ and $\sigma^* = \tilde{g} \circ \tau^*$. And hence we obtain that $\tilde{g} = g^{-1}$ in $\tau^*(W \cap W_l)$.

From now on, for simplicity we shall use the notation σ and τ instead of σ^*, τ^* respectively.

In order to establish the relationship between $(H_5)_n$ and $(H_5)_l$ we shall have the following

LEMMA 11.1. *Let σ and τ be defined in a neighborhood W_0, let σ be a solution of the system* (4.2) *such that σ fulfills $(H_5)_n$ and $\partial_n\sigma\neq 0$ in W_0, and let τ be also a solution of* (4.2) *such that $|\tilde{a}|\leqq\tilde{c}_0<1$, where $\tilde{c}_0$ is a positive constant and $\tilde{a}$ is defined by $\tilde{a}=\bar{\partial}_j\tau/\bar{\partial}_j\bar{\tau}$, $1\leqq j\leqq n$. Then τ fulfills the following conditions:*

i) *There exists a function g defined in $\sigma(W_0)$ such that $\tau=g\circ\sigma$, where g satisfies the condition that $|\partial_w g|^2-|\partial_{\bar{w}}g|^2>0$,*

ii) *$\partial_n\tau\neq 0$ in W_0,*

iii) *there exists a function $\tilde{\alpha}$ defined in $\tau(W_0)$ such that $\tilde{a}$ is written in the form $\tilde{a}=\tilde{\alpha}\circ\tau$.*

PROOF. Since, by virtue of Lemma 4.1, there exists g such that $\tau=g\circ\sigma$, we have

$$\begin{pmatrix} \partial\tau & \bar{\partial}\tau \\ \partial\bar{\tau} & \bar{\partial}\bar{\tau} \end{pmatrix}=\begin{pmatrix} \partial_w g & \partial_{\bar{w}} g \\ \partial_w\bar{g} & \partial_{\bar{w}}\bar{g} \end{pmatrix}\begin{pmatrix} \partial\sigma & \bar{\partial}\sigma \\ \partial\bar{\sigma} & \bar{\partial}\bar{\sigma} \end{pmatrix}. \tag{11.1}$$

From that both rank $\begin{pmatrix} \partial\tau & \bar{\partial}\tau \\ \partial\bar{\tau} & \bar{\partial}\bar{\tau} \end{pmatrix}$ and rank $\begin{pmatrix} \partial\sigma & \bar{\partial}\sigma \\ \partial\bar{\sigma} & \bar{\partial}\bar{\sigma} \end{pmatrix}$ equal 2 in W_0, we have that $|\partial_w g|^2-|\partial_{\bar{w}}g|^2\neq 0$ in W_0. From (11.1) we obtain

$$|\partial_n\tau|^2-|\bar{\partial}_n\tau|^2=(|\partial_w g|^2-|\partial_{\bar{w}}g|^2)(|\partial_n\sigma|^2-|\bar{\partial}_n\sigma|^2). \tag{11.2}$$

Hence we have

$$(1-|\tilde{a}|^2)|\partial_n\tau|^2=(|\partial_w g|^2-|\partial_{\bar{w}}g|^2)(1-|a|^2)|\partial_n\sigma|^2,$$

where a denotes $\alpha\circ\sigma$. This follows i) and ii).

On the other hand, we have

$$\begin{aligned} \bar{\partial}_n\tau &=(\alpha\partial_w g+\partial_{\bar{w}}g)\bar{\partial}_n\bar{\sigma}, \\ \bar{\partial}_n\bar{\tau} &=(\partial_{\bar{w}}\bar{g}+\alpha\partial_w\bar{g})\bar{\partial}_n\bar{\sigma}. \end{aligned} \tag{11.3}$$

From (11.3) we obtain iii).

By virtue of Lemma 11.1 it is seen that $(H_5)_l$ is derived from $(H_5)_n$ in $W\cap W_l$.

Next we shall demonstrate that the assumption $(H_6)_l$ is also derived from $(H_6)_n$ in $W\cap W_l$.

The assumption $(H_6)_l$ is as follows:

$$X_{(j,l)}\bar{\bar{b}}=-b_{jl}\bar{\bar{b}}, \qquad 1\leqq j\leqq n, \tag*{$(H_6)_l$}$$

where $b_j=\tilde{b}\partial_j\tau$, $1\leqq j\leqq n$.

Since $b_j=b\partial_j\sigma=\tilde{b}\partial_j\tau$ in $W\cap W_l$, we have that

$$\tilde{b}=b\frac{\partial_n\sigma}{\partial_n\tau} \quad \text{in} \quad W\cap W_l. \tag{11.3}$$

We change the complex conjugate of the left-hand side of $(H_6)_l$ as follows:

(11.4)
$$\begin{aligned}\bar{X}_{(j,l)}\bar{b} &= \bar{\bar{b}}(\bar{\partial}_l\bar{\tau}\bar{\partial}_j\bar{b}-\bar{\partial}_j\bar{\tau}\bar{\partial}_l\bar{b}) \\ &= \bar{b}(\bar{\partial}_l\bar{\sigma}\,\bar{\partial}_j\bar{b}-\bar{\partial}_j\bar{\sigma}\,\bar{\partial}_l\bar{b}) \\ &= \bar{b}\,\bar{Y}_{(j,l)}\left(b\frac{\partial_n\sigma}{\partial_n\tau}\right) \qquad \text{(by (11.3))} \\ &= \bar{b}\,\bar{Y}_{(j,l)}\left(b\frac{\partial_n\sigma}{\partial_n g\circ\sigma}\right) \\ &= \bar{b}\frac{\partial_n\sigma}{\partial_n g\circ\sigma}\cdot\bar{Y}_{(j,l)}b \qquad \text{(by Lemmas 8.1, 8.2)} \\ &= \frac{\bar{b}}{b}\bar{X}_{(j,l)}b \qquad \text{(by (11.3))}.\end{aligned}$$

On the other hand, we have that

$$X_{(j,l)}=\frac{1}{b_n}(b_lX_j-b_jX_l), \qquad 1\leqq j\leqq n .$$

Hence, by a simple computation we see that $(H_6)_n$ follows

(11.5)
$$\bar{X}_{(j,l)}b=-\bar{b}_{jl}b,$$

where we use $(H_1)'_n$.

From (11.4) and (11.5) we obtain that

$$\bar{X}_{(j,l)}\bar{b}=-\bar{b}_{jl}\bar{b}, \qquad 1\leqq j\leqq n .$$

This completes the proof.

Thus, by virtue of the above mentioned and Remark 8.1, we obtain the following

PROPOSITION 11.1. *The assumptions $(H_j)_l$, $j\neq 3$, $1\leqq j\leqq 7$ are derived from $(H_j)_n$, $j\neq 3$, $1\leqq j\leqq 7$ and vice versa, equivalently that $(H_j)_l$, $j\neq 3$, $1\leqq j\leqq 7$ are compatible with $(H_j)_n$, $j\neq 3$, $1\leqq j\leqq 7$.*

Since, if $b_l\neq 0$ $(l\neq n)$ at a point of G, we may change the coordinate numbers, without restricting the generality of the argument we can suppose that at every point of G

i) $b_n\neq 0$,

ii) $(H_j)_n$, $j=1, 2$, are satisfied.

With the aid of Lemma 4.1, corresponding to each point z^0 of G there exists a neighborhood W of z^0 $(W\subset G)$ such that the system of equations (4.2) has a solution σ. We shall say that *a triple (n, W, σ) is assigned to each point z^0 of G.*

We furthermore assume that σ satisfies the conditions $(H_j)_n$, $4\leqq j\leqq 7$.

In short, we suppose the existence of a solution σ of (4.2) with the properties $(H_j)_n$, $4 \leqq j \leqq 7$ in W. Then by Theorem 7.1 we are now in a position to state the following

THEOREM 11.1. *Under the assumptions stated above the system of equations* (1.8) *satisfies the similarity principle with respect to the class* $\mathcal{A}_0(W_0)$, *where* W_0 *is a subneighborhood of* z^0, $W_0 \subset W$, *simply speaking, the local similarity principle at each point of* G.

REMARK 11.1. The Cauchy-Riemann equations $\bar{\partial}_j f = 0$, $1 \leqq j \leqq n$, has a solution (i. e. holomorphic function) in any neighborhood of each point of G. Let W_j, $j = 1, 2$, be neighborhoods such that $W_1 \cap W_2$ is not empty and let φ_j, $j = 1, 2$, be the solutions in W_j, $j = 1, 2$, respectively. We know that, in general, no relation between φ_j, $j = 1, 2$, exists.

It is natural for the following question to arise: Will there exist the system (1.8) such that it satisfies the assumptions of Theorem 11.1. We shall answer this problem in the subsequent section.

§ 12. Examples.

In this section we shall give examples which satisfy the conditions of Theorem 11.1. In case that the coefficients b_j are holomorphic, we can readily find desired examples, because we may assume the condition $(H_1)_n$ alone. On the contrary, in case that all b_j are of class C^∞, we make use of (8.2), (8.14) and σ, which is given in Lemma 9.1. We give examples of the case $n = 2$.

EXAMPLE 12.1. Δ_z and Δ_w denote the following discs: $\Delta_z = \{z \in \boldsymbol{C} \mid |z| < 1\}$, $\Delta_w = \{w \in \boldsymbol{C} \mid |w| < 1\}$. Let G be the polydisc with the origin deleted: $\Delta_z \times \Delta_w - \{(0, 0)\}$. Let $b_1 = z$, $b_2 = w - (1/2)$. Then we have a solution $\sigma = z^2 + (w - 1/2)^2$ such that $b_2 \partial_z \sigma - b_1 \partial_w \sigma = 0$ in G. Let N_0 denote the set of all non-positive real numbers in the z-plane, $P_{1/2}$ the set of all real numbers which are not smaller than 1/2 in the w-plane, and I_0 the set of all pure imaginary numbers whose imaginary parts are non-positive in the z-plane. We define $\tilde{\Delta}_z^1$, $\tilde{\Delta}_z^2$ and $\tilde{\Delta}_w$ as follows: $\tilde{\Delta}_z^1 = \Delta_z - N_0$, $\tilde{\Delta}_z^2 = \Delta_z - I_0$ and $\tilde{\Delta}_w = \Delta_w - P_{1/2}$. Moreover we define W_1, W_2 and W_3 as follows: $W_1 = \tilde{\Delta}_z^1 \times \Delta_w$, $W_2 = \Delta_z \times \tilde{\Delta}_w$ and $W_3 = \tilde{\Delta}_z^2 \times \Delta_w$. Let σ^*, σ^{**} and σ^{***} denote the restrictions of σ to W_1, W_2 and W_3 respectively. We assign a triple to each point of G as follows: Let the same triple $(1, W_3, \nu)$ assign to each point $(z, w) \in G$ such that $z \in N_0$ and $w \in P_{1/2}$, where $\nu = \exp\{(1/2)\sigma^{***}\}$. From now on we denote the above as follows: $(z, w) \in G$: $z \in N_0$, $w \in P_{1/2} \to (1, W_3, \nu)$. For other points of G,

$$(z, w) \in G:\ z \in N_0,\ w \notin P_{1/2} \longrightarrow (2, W_2, \tau)\,,$$

$$(z, w) \in G:\ z \notin N_0,\ w \in P_{1/2} \longrightarrow (1, W_1, \sigma^*)\,,$$

$$(z, w) \in G:\ z \notin N_0,\ w \notin P_{1/2} \longrightarrow (1, W_1, \sigma^*) \quad (\text{or } (2, W_2, \tau))\,,$$

where $\tau=(\sigma^{**}+4)^2$. Then we see that

$$b_1=\frac{1}{2}\partial_z\sigma^*\,,\qquad b_2=\frac{1}{2}\partial_w\sigma^*\qquad \text{in } W_1\,,$$

$$b_1=\frac{1}{4(\sigma^{**}+4)}\partial_z\tau\,,\qquad b_2=\frac{1}{4(\sigma^{**}+4)}\partial_w\tau\qquad \text{in } W_2\,,$$

$$b_1=\left\{\exp\left(-\frac{1}{2}\right)\sigma^{***}\right\}\partial_z\nu\,,\qquad b_2=\left\{\exp\left(-\frac{1}{2}\right)\sigma^{***}\right\}\partial_w\nu\qquad \text{in } W_3\,.$$

EXAMPLE 12.2. Let $\Delta_z=\{z\in \boldsymbol{C}\,|\,|z|<r\}$, $\Delta_w=\{w\in \boldsymbol{C}\,|\,|w|<r\}$, where r is sufficiently small such that $r<(1/4)$ (see § 9). Let $G=\Delta_z\times\Delta_w$. We define W_1 and W_2 as follows:

$$W_1=\left\{z\in \boldsymbol{C}\,|\,|z|<r,\ \mathrm{Im}\, z>-\frac{1}{2}r\right\}\times\Delta_w\,,$$

$$W_2=\left\{z\in \boldsymbol{C}\,|\,|z|<r,\ \mathrm{Im}\, z<\frac{1}{2}r\right\}\times\Delta_w\,.$$

Let c be a complex number such that $\sup_G|\sigma|<|c|$. We define b_j, $j=1, 2$, by $b\partial_j\sigma$, $j=1, 2$, respectively, where σ is given by Lemma 9.1 and $b=|\sigma+c|$ $\exp\{i(\mathrm{Re}\, z+|\sigma|)\}$ (see (8.2), (8.14)). Let b^* and b^{**} be the restrictions of b to W_1 and W_2 respectively. Let $\tau=\sigma^*+c$ and $\nu=(\sigma^{**}+c)^2$. We assign a triple to each point of G as follows:

$$(z, w)\in G:\ \mathrm{Im}\, z>-\frac{1}{2}r\longrightarrow(1, W_1, \tau)\qquad(\text{or } (2, W_1, \tau))\,,$$

$$(z, w)\in G:\ \mathrm{Im}\, z<\frac{1}{2}r\longrightarrow(2, W_2, \nu)\qquad(\text{or } (1, W_2, \nu))\,.$$

We define $\tilde{b}$ by $b/\{2(\sigma+c)\}$. We obtain that

$$b_1=b^*\partial_z\tau\,,\qquad b_2=b^*\partial_w\tau\qquad \text{in } W_1\,,$$
$$b_1=\tilde{b}^{**}\partial_z\nu\,,\qquad b_2=\tilde{b}^{**}\partial_w\nu\qquad \text{in } W_2\,,$$

where $\tilde{b}^{**}$ is the restriction of $\tilde{b}$ to W_2. Thus, putting $\tilde{\alpha}(\xi, \bar{\xi})=\alpha(\xi-c, \bar{\xi}-\bar{c})$, we see that

$$\frac{\partial_z\tau}{\partial_z\bar{\tau}}=\frac{\partial_w\tau}{\partial_w\bar{\tau}}=\tilde{\alpha}\circ\tau\qquad \text{in } W_1\,,$$

$$\frac{\partial_z\nu}{\partial_z\bar{\nu}}=\frac{\partial_w\nu}{\partial_w\bar{\nu}}=\tilde{\beta}\circ\nu\qquad \text{in } W_2\,,$$

where $\tilde{\beta}(\xi, \bar{\xi})$ is defined by

$$\frac{\xi^{1/2}}{\overline{\xi^{1/2}}}\cdot\alpha(\xi^{1/2}-c, \overline{\xi^{1/2}-c})$$

and $\xi^{1/2}$ denotes the branch such that $(c^2)^{1/2}=c$.

EXAMPLE 12.3. Δ_z and Δ_w are the same as those in Example 12.1. Let $\tilde{\Delta}_z^0 = \Delta_z - \{0\}$. We define G by $\tilde{\Delta}_z^0 \times \Delta_w$, and define W_1 and W_2 as follows: $W_1 = (\Delta_z - N_0) \times \Delta_w$, $W_2 = (\Delta_z - I_0) \times \Delta_w$, where N_0 and I_0 are as defined in Example 12.1. Let $b_1 = 1/z$ and $b_2 = w$. We assign a triple to each point of G as follows:

$$(z, w) \in G: z \in I_0 \longrightarrow \left(1, W_1, \operatorname{Log} z + \frac{1}{2} w^2\right),$$

$$(z, w) \in G: z \in N_0 \longrightarrow \left(1, W_2, \operatorname{Log} z + \frac{1}{2} \pi i + \frac{1}{2} w^2\right),$$

$$(z, w) \in G: z \notin I_0 \cup N_0 \longrightarrow \left(1, W_1, \operatorname{Log} z + \frac{1}{2} w^2\right)$$

$$\left(\text{or } \left(1, W_2, \operatorname{Log} z + \frac{1}{2} \pi i + \frac{1}{2} w^2\right)\right),$$

where $\operatorname{Log} z$ denotes the principal value.

As seen from Example 12.3, since $\log z$ is not well-defined in G, in general we cannot expect a global solution of the system (1.8). However, for the case in which all b_j are holomorphic in G which is a simply connected domain, from that $b_{jn} = 0$ in G, we have

$$\sigma(z) = \sigma(z^0) + \int_{z^0}^{z} \sum_{k=1}^{n} b_k(\zeta) d\zeta_k, \qquad z^0 \in G,$$

where the integral takes along a polygonal line contained in G which joins z^0 and $z \in G$.

It is obvious that $\sigma(z)$ satisfies the system of equations $b_n \partial_j \sigma - b_j \partial_n \sigma = 0$, $1 \leqq j \leqq n$. We immediately obtain a solution of the following form: $(\hat{f} \circ \sigma)(z, \bar{z})$, where $\hat{f}$ satisfies the equation

$$\partial_{\bar{w}} \hat{f} = \bar{\hat{b}} \bar{\hat{f}}$$

in the image $\sigma(D)$, $D \Subset G$ and b is given by $b_j / \partial_j \sigma$. Remark that $b = \hat{b} \circ \sigma$, where $\hat{b}$ is a holomorphic function in $\sigma(G)$.

Thus the system (1.8) satisfies the similarity principle with respect to the class $\mathcal{A}_0(D)$.

In short, we always have the global result whenever G is a bounded simply connected domain and all coefficients b_j are holomorphic there.

On the contrary, when G is merely a domain, the global solutions of the system (1.8) are closely related with the analytic continuation of $\sigma(z)$.

For the case in which the coefficients b_j are of class $C^\infty(G)$, we also see that the existence of a global solution of the system of equations (1.8) is derived from that of the equations (4.2) which satisfies the conditions $(H_j)_n$, $1 \leqq j \leqq n$, in G.

§ 13. Solutions with singularities.

Until now we have considered regular solutions of the system (1.8). As it is seen from Theorem 7.1, it is possible to extend the definition of solutions of the system (1.8).

Suppose that the coefficients b_j satisfy the assumptions in Theorem 7.1. Let f be defined and of class C^1 in W_0-C, where W_0 is a neighborhood of the origin such that $W_0 \subset W$, and where C is as follows:

1) A subset E of the image $\sigma(W_0)$, which depends on f, are composed of isolated points. E may be the empty set.

2) $C=\{\text{complex analytic manifold } M_t \mid \sigma(z, \bar{z})=t,\ t \in E\}$, that is, C is composed of pairwise disjoint complex $(n-1)$-dimensional analytic manifolds.

We shall say that f is a solution of the system (1.8), when f satisfies (1.8) at every point of W_0-C.

We extend the class $\mathcal{A}(W_0)$, $\mathcal{A}_0(W_0)$ considered in section 7 to the following class $\tilde{\mathcal{A}}(W_0)$, $\tilde{\mathcal{A}}_0(W_0)$ respectively:

$\tilde{\mathcal{A}}(W_0)$ is composed of all holomorphic functions in W_0 except, perhaps, for singular points.

$\tilde{\mathcal{A}}_0(W_0)$ is composed of all composite functions $g \circ \sigma$ such that g is in the class $\tilde{Q}(K)$.

Thus we are in a position to state the following

THEOREM 13.1. *Under the assumptions of Theorem* 7.1 *the system of equations* (1.8) *satisfies the* (*local*) *similarity principle with respect to the class* $\tilde{\mathcal{A}}_0(W_0)$.

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References

[1] L. Bers, Theory of Pseudo-analytic Functions, New York Univ., 1953.
[2] L. Bers, Riemann surfaces, Lecture Notes, New York Univ., 1957–58.
[3] L. Bers and L. Nirenberg, On a representation theorem for linear elliptic system with discontinuous coefficients and applications, Convegno Internazionale sulle Equazioni lineari alle derivate parziali, Rome, (1955), 111–140.
[4] B. V. Boyarskii, Generalized solutions of system of differential equations of first order and elliptic type with discontinuous coefficients (in Russian), Mat. Sb., **43**(83) (1957), 451–503.
[5] S. S. Chern, Differentiable manifolds, Lecture Notes, Chicago Univ., 1959.
[6] C. Chevalley, Theory of Lie Groups I, Princeton, 1946.
[7] R. Courant and D. Hilbert, Methods of mathematical physics, Vol. II, 1961.
[8] R. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice-Hall, 1965.
[9] L. Hörmander, The Frobenius-Nirenberg theorem, Ark. Mat., **5** (1964), 425–432.
[10] A. Koohara, Properties of Solutions of Overdetermined System (in Japanese),

Reports of Himeji Inst. Tech., **19A** (1966).
[11] M. Kneser, Abhängigkeit von Funktionen, Math. Z., **54** (1951), 34–51.
[12] A. Newlander and L. Nirenberg, Complex analytic coordinates in almost complex manifolds, Ann. of Math., **65** (1957), 391–404.
[13] L. Nirenberg, A complex Frobenius theorem, Seminars on analytic functions I, Princeton, (1957), 172–189.
[14] F. Sommer, Komplex-analytische Blätterung reeller Mannig-faltigkeiten im C^n, Math. Ann., **136** (1958), 111–133; **137** (1959), 392–411.
[15] I. N. Vekua, Systeme von Differentialgleichungen erster Ordnung vom elliptischen Typus und Randwertaufgaben, Math. Forschungsberichte II, Berlin, 1956.
[16] I. N. Vekua, Generalized analytic functions, Pergamon, 1962.