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Weak Harnack inequality for fully nonlinear uniformly elliptic PDE with unbounded ingredients

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Abstract. The weak Harnack inequality for L^p-viscosity solutions is shown for fully nonlinear, second order uniformly elliptic partial differential equations with unbounded coefficients and inhomogeneous terms. This result extends those of Trudinger for strong solutions [**21**] and Fok for L^p-viscosity solutions [**13**]. The proof is a modification of that of Caffarelli [**5**], [**6**]. We apply the weak Harnack inequality to obtain the strong maximum principle, boundary weak Harnack inequality, global C^α estimates for solutions of fully nonlinear equations, strong solvability of extremal equations with unbounded coefficients, and Aleksandrov-Bakelman-Pucci maximum principle in unbounded domains.

1. Introduction.

In this paper, we establish the weak Harnack inequality for L^p-viscosity supersolutions of fully nonlinear, second order uniformly elliptic partial differential equations (PDE) with unbounded coefficients and inhomogeneous terms. In fact this reduces to showing that the weak Harnack inequality holds for nonnegative L^p-viscosity supersolutions of Pucci extremal equations

$$\mathscr{P}^+(D^2u) + \mu(x)|Du| = f(x) \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \boldsymbol{R}^n$, and $\mu \in L^q(\Omega)$, $f \in L^p(\Omega)$ for some p, q. The Pucci operator $\mathscr{P}^+(X)$ is defined by $\mathscr{P}^+(X) = \max\{-\mathrm{tr}(AX) \mid A \in S^n,\ \lambda I \le A \le \Lambda I\}$ for $X \in S^n$, where S^n is the set of $n \times n$ symmetric matrices, and $0 < \lambda \le \Lambda$ are the ellipticity constants. The ellipticity constants will be fixed throughout this paper. We also have the Pucci operator $\mathscr{P}^-(X) := -\mathscr{P}^+(-X)$ for $X \in S^n$.

Trudinger showed in [**21**] that the weak Harnack inequality holds for strong solutions of linear PDE when the gradient coefficient is in $L^{2n}(\Omega)$. Afterwards,

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Fok in [**13**] verified this fact for viscosity supersolutions of (1.1) following the argument in Gilbarg-Trudinger's book [**14**]. In [**13**] and [**14**], in order to show the weak Harnack inequality one estimates $|\mu||Du| \leq 1/2(\mu^2/c + c|Du|^2)$ for an appropriate c and uses the exponential/logarithmic transformation to eliminate the quadratic term $c|Du|^2/2$. This is the place where one needs to suppose $\mu \in L^{2n}(\Omega)$ to apply the Aleksandrov-Bakelman-Pucci (ABP) maximum principle for the new inhomogeneous term $\mu^2/(2c)$. Thus, as long as one follows this argument, it seems hard to avoid the assumption $\mu \in L^{2n}(\Omega)$.

We will generalize this result in the current paper. More precisely, when $\mu \in L^q(\Omega)$ and $f \in L^p(\Omega)$, we will obtain the weak Harnack inequality for L^p-viscosity supersolutions of (1.1) if $q > n$ and $q \geq p > p_0$, where $p_0 = p_0(n, \lambda, \Lambda) \in [n/2, n)$ is the constant giving the range where the maximum principle holds (see [**12**], [**11**], [**10**]).

Since the ABP maximum principle holds for L^n-strong solutions when $\mu, f \in L^n(\Omega)$, we could hope that the weak Harnack inequality also holds for L^n-viscosity supersolutions when $p = q = n$. However, unfortunately, we do not know if the ABP maximum principle is true in this case. In fact, the difficulty comes from the lack of existence results for L^n-strong solutions of extremal PDE in [**18**] when $p = q = n$.

A direct consequence of the weak Harnack inequality is the strong maximum principle for L^p-viscosity solutions. We refer to [**2**], [**15**] for results on the strong maximum principle for viscosity solutions of possibly degenerate PDE without measurable terms. We also establish the boundary weak Harnack inequality which enables us to extend some qualitative properties of L^p-viscosity solutions. One consequence of it is the global Hölder continuity estimates for L^p-viscosity solutions. Similar and in fact more broad result for equations with quadratically growing gradient terms has been recently obtained by Sirakov [**20**] without using the weak Harnack inequality. When restricted to solutions of (1.1), Sirakov's result requires $q > n, p \geq n$ and in this case our result is a slight generalization of it as we allow $p > p_0$. Another consequence of the boundary weak Harnack inequality is the ABP type maximum principle in unbounded domains. The study of ABP maximum principle in unbounded domains was initiated by Cabré in [**3**]. For more results on this we refer to [**4**], [**22**] in the context of strong solutions, and [**8**] for viscosity solutions.

This paper is organized as follows. In section 2 we recall the definitions of L^p-viscosity solutions and L^p-strong solutions, and then list several preliminary results from our previous paper [**18**]. In section 3 we show that L^p-strong solutions are L^p-viscosity solutions for general PDE with unbounded ingredients. Therefore all the results of this paper also apply to strong solutions. Section 4 is devoted to the weak Harnack inequality for L^p-viscosity supersolutions of (1.1) when $q > n$,

$q \geq p > p_0$.

In section 5, we derive the strong maximum principle as a simple consequence of the weak Harnack inequality. We obtain the boundary weak Harnack inequality, and then the global Hölder continuity estimate in section 6. In section 7, as an application of the Hölder estimates of section 6, we show existence of strong solutions of extremal equations (1.1) when the support of μ is not necessary compact. Finally in section 8, we slightly improve a sufficient condition of [**8**] for an ABP type maximum principle in unbounded domains.

In the Appendix, following [**7**], we prove two important results in the theory of L^p-viscosity solutions. The first is the fact that if an L^p-viscosity subsolution (resp., supersolution) belongs to $W^{2,p}_{\mathrm{loc}}(\Omega)$, then it is an L^p-strong subsolution (resp., supersolution). The second is a stability result for L^p-viscosity solutions of general PDE, which is needed to prove the strong solvability of extremal equations in section 7.

2. Preliminaries.

Throughout the paper, unless specified otherwise, $\Omega \subset \boldsymbol{R}^n$ will always be a domain, *i.e.* an open and connected set. We remind that Ω is not necessary bounded unless stated. In particular, we notice that Ω can be unbounded in the strong maximum principle, Proposition 4.1 and Theorem 5.1.

We first recall the definition of L^p-viscosity solutions of general fully nonlinear PDE

$$F(x, u, Du, D^2 u) = f(x) \quad \text{in } \Omega, \tag{2.1}$$

where $F : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n \to \boldsymbol{R}$ and $f : \Omega \to \boldsymbol{R}$ are functions which are at least measurable. We will be using the standard notation of [**14**]. For $r \geq 1$ we denote by $L^r_+(\Omega)$ the set of nonnegative functions in $L^r(\Omega)$. We will often write $\|\zeta\|_r$ for $\|\zeta\|_{L^r(\Omega)}$ when the integration is taken over the whole domain of a function ζ.

Throughout this paper we always assume that

$$p > \frac{n}{2}.$$

Definition 2.1 ([**7**]). $u \in C(\Omega)$ is an L^p-viscosity subsolution (resp., supersolution) of (2.1) if

$$ess \liminf_{y \to x} \big(F(y, u(y), D\phi(y), D^2\phi(y)) - f(y) \big) \leq 0$$

$$\left(\text{resp., } ess \limsup_{y\to x}\big(F(y,u(y),D\phi(y),D^2\phi(y)) - f(y)\big) \geq 0\right)$$

provided that for $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$, $u-\phi$ attains its local maximum (resp., minimum) at $x \in \Omega$.

We call $u \in C(\Omega)$ an L^p-viscosity solution of (2.1) if it is both an L^p-viscosity sub- and supersolution of (2.1).

We remind that if u is an L^p-viscosity subsolution (resp., supersolution) of (2.1), then it is also an $L^{\hat{p}}$-viscosity subsolution (resp., supersolution) of (2.1) if $\hat{p} > p$.

We recall the definitions of L^p-strong sub- and supersolutions.

Definition 2.2. A function u is an L^p-strong subsolution (resp., supersolution) of (2.1) if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, and

$$F(x,u(x),Du(x),D^2u(x)) \leq f(x) \quad a.e. \text{ in } \Omega$$
$$\big(\text{resp., } F(x,u(x),Du(x),D^2u(x)) \geq f(x) \quad a.e. \text{ in } \Omega\big).$$

We call u an L^p-strong solution of (2.1) if it is both an L^p-strong sub- and supersolution of (2.1).

Contrary to L^p-viscosity solutions, if u is an L^p-strong subsolution (resp., supersolution) of (2.1), then it is an $L^{\hat{p}}$-strong subsolution (resp., supersolution) of (2.1) provided $p > \hat{p}$.

The upper-contact set $\Gamma[u;\Omega]$ of u over Ω is defined as

$$\Gamma[u;\Omega] = \{x \in \Omega \mid \exists p \in \boldsymbol{R}^n \text{ such that } u(y) \leq u(x) + \langle p, y-x\rangle \text{ for all } y \in \Omega\}.$$

We will write $B_r(x)$ for the open ball centered at $x \in \boldsymbol{R}^n$ with radius $r > 0$. For simplicity, B_r will mean $B_r(0)$.

In what follows we will often consider separately the case

$$q > n \quad \text{and} \quad q \geq p \geq n. \tag{2.2}$$

Notice that (2.2) is equivalent to $q \geq p > n$ or $q > p = n$.

We begin recalling a result on existence of L^p-strong sub- and supersolutions of extremal PDE. This is just a preliminary result which will be improved and generalized in Section 7. In particular we will then remove the condition supp $\mu \Subset \Omega$.

THEOREM 2.3 (cf. Proposition 2.6 and Remark 2.10 in [**18**]). *Let $\Omega \subset B_1$ satisfy the uniform exterior cone condition. Let $f \in L^p_+(\Omega)$ and $\mu \in L^q_+(\Omega)$, where*

$$q > n \quad \textit{and} \quad q \geq p > p_0, \tag{2.3}$$

and suppose that $\operatorname{supp} \mu \Subset \Omega$. *Then there exists an L^p-strong supersolution (resp., subsolution) $u \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of*

$$\mathscr{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \textit{in } \Omega \tag{2.4}$$

$$\big(\textit{resp., } \ \mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \textit{in } \Omega\big) \tag{2.5}$$

such that $u = 0$ on $\partial\Omega$. For $p \geq n$,

$$0 \leq u \leq C_1 \exp(C_2\|\mu\|_n^n)\|f\|_{L^n(\Gamma[u;\Omega])} \quad \textit{in } \Omega \tag{2.6}$$

$$\Big(\textit{resp., } \ 0 \geq u \geq -C_1 \exp(C_2\|\mu\|_n^n)\|f\|_{L^n(\Gamma[-u;\Omega])} \quad \textit{in } \Omega\Big), \tag{2.7}$$

where $C_k = C_k(n, \lambda, \Lambda)$ for $k = 1, 2$ are from Theorem 2.4, and for $n > p > p_0$,

$$0 \leq u \leq C_3\left\{\exp(C_2\|\mu\|_n^n)\|\mu\|_q^N + \sum_{k=0}^{N-1}\|\mu\|_q^k\right\}\|f\|_p \quad \textit{in } \Omega$$

$$\left(\textit{resp., } \ 0 \geq u \geq -C_3\left\{\exp(C_2\|\mu\|_n^n)\|\mu\|_q^N + \sum_{k=0}^{N-1}\|\mu\|_q^k\right\}\|f\|_p \quad \textit{in } \Omega\right),$$

where $C_3 > 0$ and $N \geq 1$ are from Theorem 2.5. Moreover for every $\Omega' \Subset \Omega$ we have

$$\|u\|_{W^{2,p}(\Omega')} \leq C_4\|f\|_p,$$

where $C_4 = C_4(n, p, \lambda, \Lambda, \|\mu\|_q, \operatorname{dist}(\Omega', \partial\Omega)) > 0$.

If $\mu, f \in L^n_+(\Omega)$, then for every $p_0 < p < n$ there exists an L^p-strong supersolution (resp., subsolution) $u \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of (2.4) *(resp.,* (2.5)*) satisfying* (2.6) *(resp.,* (2.7)*) such that for every $\Omega' \Subset \Omega$*

$$\|u\|_{W^{2,p}(\Omega')} \leq C_4\|f\|_n,$$

for some $C_4 = C_4(n, p, \lambda, \Lambda, \mu, \Omega, \operatorname{dist}(\Omega', \partial\Omega)) > 0$.

PROOF. Although we gave a complete proof in [**18**], we did not mention there that we may take the L^n-norm on the upper contact set in (2.6) and (2.7). We will only show (2.6).

Following the proof of Proposition 2.6 in [**18**], to find an L^p-strong supersolution of (2.4), we approximate μ and f by smooth functions μ_k and f_k such that *supp* $\mu_k \subset$ *supp* μ,

$$\|\mu - \mu_k\|_q + \|f - f_k\|_p \to 0 \quad \text{as } k \to \infty,$$

we find classical solutions $u_k \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} \mathscr{P}^-(D^2 u_k) - \mu_k(x)|Du_k| = f_k(x) & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

and obtain local $W^{2,p}$ estimates for the u_k. We then show that there is $u \in C(\overline{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega)$ such that, possibly along a subsequence,

$$\|u_k - u\|_\infty \to 0 \quad \text{as } k \to \infty.$$

However the ABP maximum principle applied to u_k yields

$$0 \le u_k \le C_1 \exp(C_2 \|\mu_k\|_n^n) \|f_k\|_{L^n(\Gamma[u_k;\Omega])} \quad \text{in } \Omega,$$

and according to Appendix A in [**7**], we may then replace $\|f_k\|_{L^n(\Gamma[u_k;\Omega])}$ by $\|f\|_{L^n(\Gamma[u;\Omega])}$ in the limit as $k \to \infty$.

The proof of the theorem in the case $n > p > p_0$ follows the lines of the proof of Proposition 2.6 in [**18**] when we now use the estimates of Theorem 2.5 (see Remark 2.10 in [**18**]). We pass to the limit with the u_k using the local $W^{2,p}$ estimates for the u_k (which are shown with only minor and rather straightforward technical differences) and the strong convergence, possibly along a subsequence, of the Du_k in $L^{p'}_{\text{loc}}$ for every $p' < p^* = np/(n-p)$.

For $\mu, f \in L^n_+(\Omega)$ the proof uses similar modifications. The dependence of C_4 on μ and Ω enters through the fact that for every $\varepsilon > 0$ there exists r_0 (depending on μ and Ω) such that if $r < r_0$ and $V \subset \Omega$ is a ball of radius r then $\|\mu\|_{L^n(V)} \le \varepsilon$ and this property can be assumed to be preserved by mollification. The smallness of $\|\mu\|_n$ is enough to obtain the uniform $W^{2,p}$ estimates for the u_k. $\square$

We next state the following version of the ABP maximum principle for L^p-viscosity solutions. It is a slight variation of Proposition 2.8 of [**18**].

THEOREM 2.4 (cf. Proposition 2.8 of [**18**]). *Let $\Omega \subset B_1$, and let* (2.2) *hold. There exists $C_k = C_k(n, \lambda, \Lambda) > 0$ $(k = 1, 2)$ such that if $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$, and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution (resp., supersolution) of*

$$\mathscr{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega_0 := \{x \in \Omega \mid u(x) > \sup_{\partial\Omega} u^+\}$$

$$\left(\text{resp., } \mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \text{in } \Omega_0 := \{x \in \Omega \mid u(x) < -\sup_{\partial\Omega} u^-\}\right),$$

then

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+ + C_1 \exp(C_2 \|\mu\|^n_{L^n(\Omega_0)}) \|f\|_{L^n(\Omega_0)}$$

$$\left(\text{resp., } \sup_\Omega (-u) \le \sup_{\partial\Omega} u^- + C_1 \exp(C_2 \|\mu\|^n_{L^n(\Omega_0)}) \|f\|_{L^n(\Omega_0)}\right).$$

The same conclusion also holds if $\mu, f \in L^n_+(\Omega)$ and u is an L^p-viscosity subsolution (resp., supersolution) of the above equations for some $p_0 < p < n$.

We could also give a corresponding result with Ω_0 when $n > p > p_0$ (see Theorem 2.9 in [**18**]). Moreover we could state the above estimates using the L^n norms of f over the upper contact set of u. We do not do it here since we will not need such results. We leave these easy versions and extensions to the interested readers. The proof for the case $\mu, f \in L^n_+(\Omega)$ is the same as the proof of Proposition 2.8 of [**18**] if we use Theorem 2.3.

Next we present the ABP maximum principle for L^p-strong solutions. It was proved in [**13**] however the constant p_0 there might have been different from ours. We sketch a different proof. We refer the reader to [**14**] or Proposition 2.3 in [**18**] for the case $p = q = n$.

THEOREM 2.5. *Let $\Omega \subset B_1$, and let $p_0 < p < n < q$. There exist an integer $N = N(n, p, q)$ and $C_3 = C_3(n, \lambda, \Lambda, p, q) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$, and $u \in C(\overline{\Omega})$ is an L^p-strong subsolution (respectively, supersolution) of*

$$\mathscr{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega \tag{2.8}$$

$$\big(\text{resp., } \mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \text{in } \Omega\big), \tag{2.9}$$

then

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C_3 \left\{ \exp(C_2 \|\mu\|^n_n) \|\mu\|^N_q + \sum_{k=0}^{N-1} \|\mu\|^k_q \right\} \|f\|_p \tag{2.10}$$

$$\left(resp., \ \inf_{\Omega} u \geq \inf_{\partial\Omega} u - C_3 \left\{ \exp(C_2 \|\mu\|_n^n) \|\mu\|_q^N + \sum_{k=0}^{N-1} \|\mu\|_q^k \right\} \|f\|_p \right), \tag{2.11}$$

where C_2 *is from Theorem* 2.4.

Proof. We will indicate how to show (2.10). First, we can choose $u_j \in C^2(\Omega)$ such that $u_j \to u$ in $C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$. Then

$$\mathscr{P}^-(D^2 u_j) - \mu(x)|Du_j| \to \mathscr{P}^-(D^2 u) - \mu(x)|Du|$$

in $L^p_{\mathrm{loc}}(\Omega)$. Therefore by considering slightly smaller domains we can assume without loss of generality that $f \in L^q_{\mathrm{loc}}(\Omega)$ as we can then recover the result in the limit. After this initial adjustment the proof then repeats the proof of Theorem 2.9 in [**18**]. The estimates on the size of the iteration functions v_k are controlled by $\|f\|_{L^p(\Omega)}$ however the functions v_k are now in $W^{2,q}$. Therefore in the final step we can use the classical ABP maximum principle in its nonlinear version, see [**14**] or Proposition 2.3 in [**18**]. □

Finally we state a slightly generalized version of the ABP maximum principle from [**18**] for equations with superlinear gradient terms.

Theorem 2.6. *Let* $\Omega \subset B_1$, $m > 1$, (2.2) *hold. For* $f \in L^p_+(\Omega)$, $\mu_1, \mu_m \in L^q_+(\Omega)$, *consider an* L^p*-viscosity subsolution* $u \in C(\overline{\Omega})$ *of*

$$\mathscr{P}^-(D^2 u) - \mu_1(x)|Du| - \mu_m(x)|Du|^m = f(x) \quad in\ \Omega. \tag{2.12}$$

Then:

(i) *Let* $q \geq p > n$. *There exist* $\delta = \delta(n, \lambda, \Lambda, m, p, \|\mu_1\|_q) > 0$ *and* $C_5 = C_5(n, \lambda, \Lambda, m, p, \|\mu_1\|_q) > 0$ *such that if*

$$\|f\|_p^{m-1} \|\mu_m\|_q < \delta,$$

then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C_5 \Big(\|f\|_n + \|f\|_p^m \|\mu_m\|_q \Big).$$

(ii) *Let* $p_0 < p \leq n < q$ *satisfy*

$$p > \frac{nq(m-1)}{mq-n}. \tag{2.13}$$

Denote $a_0 = 0$ *and* $a_k = 1 + m + \cdots + m^{k-1}$ *for* $k \geq 1$. *There exist an integer* $N = N(n, m, p, q) \geq 1$, $\delta = \delta(n, \lambda, \Lambda, m, p, q, \|\mu_1\|_q) > 0$ *and* $C_6 = C_6(n, \lambda, \Lambda, m, p, q, \|\mu_1\|_q) > 0$ *such that if*

$$\|f\|_p^{m^N(m-1)} \|\mu_m\|_q^{m^N} < \delta, \tag{2.14}$$

then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + C_6 \sum_{k=0}^{N+1} \|\mu_m\|_q^{a_k} \|f\|_p^{m^k}. \tag{2.15}$$

Remark 2.7. The constants C_5 and C_6 above are bounded if $\|\mu_1\|_q$ varies in a bounded set in $\boldsymbol{R}$. See [**19**] for the precise dependence.

The proof of this theorem is similar to those of Theorems 2.11 and 2.12 in [**18**], however in the iterative scheme there, we need to substitute solutions of extremal equations

$$\mathscr{P}^+(D^2 u_i) = -f_i(x)$$

(see the proofs of Theorems 2.11 and 2.12 in [**18**]) by solutions v_i of extremal inequalities

$$\mathscr{P}^+(D^2 v_i) + \mu_1(x)|Dv_i| \leq -f_i(x)$$

provided by Theorem 2.3. We refer to [**19**] for the details, particularly, a careful dependence on $\|\mu_1\|_q$ and $\|\mu_m\|_q$ in the constants C_k $(k = 5, 6)$ and δ.

3. Strong solutions are viscosity.

In [**18**] we presented various versions of the ABP maximum principle for L^p-viscosity solutions of extremal inequalities with possibly superlinear gradient terms and unbounded coefficients. However we did not mention there that L^p-strong solutions of such inequalities are L^p-viscosity solutions. We will show it here for general equations (2.1).

The function $F : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n \to \boldsymbol{R}$ will satisfy the following assumptions. First of all, without loss of generality we can always assume that

$$F(x,0,0,O)=0 \quad \text{in } \Omega. \tag{3.1}$$

This can be achieved by taking $F(x,r,p,X) := F(x,r,p,X) - F(x,0,0,O)$ and $f(x) := f(x) - F(x,0,0,O)$. Next we require that F is uniformly elliptic, i.e.

$$\mathscr{P}^-(X-Y) \le F(x,r,p,X) - F(x,r,p,Y) \le \mathscr{P}^+(X-Y) \tag{3.2}$$

for $(x,r,p,X,Y) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n \times S^n$. As regards the continuity in r,p, we assume that there are $m \ge 1$, $\mu \in L^q_+(\Omega)$, $c \in L^p_+(\Omega)$, and a nondecreasing function $\omega \in C([0,+\infty))$ satisfying $\omega(0)=0$ such that

$$|F(x,r,p,X) - F(x,s,q,X)| \le \mu(x)(|p|^{m-1} + |q|^{m-1} + 1)|p-q| + c(x)\omega(|r-s|) \tag{3.3}$$

for $(x,r,p,q,X) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^n \times S^n$. Note that (3.1) and (3.3) yield $|F(x,0,p,O)| \le \mu(x)(|p|^m + |p|)$ for $(x,p) \in \Omega \times \boldsymbol{R}^n$.

Theorem 3.1. *Let* (3.1)–(3.3) *hold. Suppose also that* $f \in L^p(\Omega)$ *and that one of the following conditions is satisfied*:

$$\begin{cases} (1) \quad q > n,\ q \ge p \ge n, \\ (2) \quad q > n > p > p_0,\ p((m-1)q + n(q-n)) > (m-1)qn, \\ (3) \quad q = n = p,\ m = 1. \end{cases}$$

If u is an L^p-strong subsolution (*resp., supersolution*) *of* (2.1), *then it is an L^p-viscosity subsolution* (*resp., supersolution*) *of* (2.1).

Remark 3.2. We remark that when $q > n > p > p_0$ and $m = 1$, the second condition in (2) automatically holds. Note that it is weaker than (2.13).

Proof. We only prove the claim for subsolutions. Suppose, contrary to the claim, that there are $\theta, r_0 > 0$ and $x \in \Omega$, $\phi \in W^{2,p}(B_{r_0}(x))$ such that $0 = (u-\phi)(x) \ge (u-\phi)(y)$ for $y \in B_{r_0}(x) \Subset \Omega$ and

$$F(y,u(y),D\phi(y),D^2\phi(y)) - f(y) \ge 2\theta \quad \textit{a.e.} \text{ in } B_{r_0}(x).$$

Since u is an L^p-strong subsolution, by (3.1)-(3.3), setting $v = u - \phi$, we have

$$\mathscr{P}^-(D^2 v) - \mu(y)(|Du(y)|^{m-1} + |D\phi(y)|^{m-1} + 1)|Dv| \le -2\theta \quad \textit{a.e.} \text{ in } B_{r_0}(x).$$

Setting $v_\varepsilon(y) = v(y) - \varepsilon|y-x|^2$ for $\varepsilon > 0$, we notice that v_ε achieves the strict maximum over $\overline{B}_{r_0}(x)$ at x. It is also easy to verify that for $0 < \varepsilon \le \theta/(2n\Lambda)$,

$$\mathscr{P}^-(D^2 v_\varepsilon) - \gamma(y)|Dv_\varepsilon| \le 2r\varepsilon\gamma(y) - \theta \quad \textit{a.e.} \text{ in } B_r(x), \text{ for all } r \le r_0,$$

where $\gamma(y) = \mu(y)(|Du(y)|^{m-1} + |D\phi(y)|^{m-1} + 1)$ in $B_{r_0}(x)$. Using Sobolev embeddings we have $\gamma \in L^{p'}(B_{r_0}(x))$ for some $q > p' > n$ in cases (1)-(2), and $\gamma = 3\mu \in L^n(B_{r_0}(x))$ in case (3). In cases (1) and (3) we can apply directly the maximum principle after scaling to get

$$0 = \sup_{B_r(x)} v_\varepsilon \le -\varepsilon r^2 + C_1 \exp(C_2\|\gamma\|_n^n)\varepsilon r^2 \|\gamma\|_{L^n(B_r(x))} \le -\varepsilon r^2 + C\varepsilon r^2\|\gamma\|_{L^n(B_r(x))}$$

for some constant $C > 0$ independent of r. Therefore, taking small $r > 0$, we obtain a contradiction.

In case (2) we need to justify that we can apply the maximum principle using the L^n norm of the right hand side even though we only have $p < n$. This is indeed a general fact which holds for both L^p-strong and viscosity solutions.

Define $\gamma_r(y) = \gamma(y)$ for $y \in B_r(x)$ and $\gamma_r(y) = 0$ for $y \in B_{r_0}(x) \setminus B_r(x)$. In view of Theorem 2.3, for $0 < r < r_0/2$, we can find L^p-strong subsolutions $\xi_r \in C(\overline{B}_{2r}(x)) \cap W^{2,p}_{\mathrm{loc}}(B_{2r}(x))$ of

$$\mathscr{P}^+(D^2\xi_r) + \gamma_r(y)|D\xi_r| = -2r\varepsilon\gamma_r(y) \quad \textit{a.e.} \text{ in } B_{2r}(x), \quad \xi_r = 0 \text{ on } \partial B_{2r}(x),$$

such that $0 \le -\xi_r \le C_1 \exp(C_2\|\gamma\|_n^n)\varepsilon r^2\|\gamma\|_{L^n(B_r(x))}$ in $B_{2r}(x)$. Then $w_\varepsilon = v_\varepsilon + \xi_r$ satisfies

$$\mathscr{P}^-(D^2 w_\varepsilon) - \gamma(y)|Dw_\varepsilon| \le 0 \quad \textit{a.e.} \text{ in } B_r(x).$$

Therefore, by Theorem 2.5,

$$-C\varepsilon r^2\|\gamma\|_{L^n(B_r(x))} \le \sup_{B_r(x)} w_\varepsilon \le \sup_{\partial B_r(x)} w_\varepsilon \le -\varepsilon r^2.$$

Again we obtain a contradiction by taking r sufficiently small. $\square$

REMARK 3.3. Thanks to Theorem 3.1, all the results for L^p-viscosity subsolutions (resp., supersolutions) in this paper hold true for L^p-strong subsolutions (resp., supersolutions).

4. Weak Harnack inequality.

In this section, we establish the weak Harnack inequality for nonnegative L^p-viscosity supersolutions of extremal PDE with L^q-coefficients.

We define $Q_1 := \prod_{k=1}^n [-1/2, 1/2]$, and then $Q_r = rQ_1$, and $Q_r(x) = x + Q_r$ for $r > 0$ and $x \in \boldsymbol{R}^n$.

We first recall the strong maximum principle for extremal equations without unbounded coefficients even though we will only need it for classical solutions in the proof of Lemma 4.2. Much more general strong maximum principle for viscosity solutions of fully nonlinear degenerate equations was shown in [**2**]. However we present here a simple proof for completeness. Results on strong maximum principle for classical solutions of fully nonlinear PDE can be found in [**14**].

PROPOSITION 4.1. *Let $u \in C(\overline{\Omega})$ be an L^p-viscosity subsolution (resp., supersolution) of*

$$\mathscr{P}^-(D^2u) = 0 \quad (\textit{resp.},\ \mathscr{P}^+(D^2u) = 0) \quad \textit{in } \Omega$$

such that $\sup_\Omega u < \infty$ (*resp.*, $\inf_\Omega u > -\infty$). *If u attains its maximum (resp., minimum) over $\overline{\Omega}$ at $x \in \Omega$, then u is constant in $\overline{\Omega}$.*

PROOF. We only show the assertion for L^p-viscosity subsolutions. Suppose that $u(x) = \sup_\Omega u =: K$ for some $x \in \Omega$. Then, for small $t > 0$, by setting $v := K - u$, the weak Harnack inequality (see for instance [**6**]) yields

$$\left(\int_{Q_t(x)} v^r dx \right)^{\frac{1}{r}} \leq C \inf_{Q_t(x)} v = 0 \quad \text{for some } C, r > 0,$$

which implies the claim by a standard argument. □

We now present a modification of Lemma 4.1 in [**6**]. This lemma gives an explicit construction of a barrier function and we could use it here as well. Our lemma uses classical solvability of extremal equations, together with the strong maximum principle.

Let U, V be fixed domains with smooth boundaries such that

$$Q_{\frac{1}{2}} \subset U \subset Q_{\frac{3}{4}}, \quad Q_{\frac{7}{2}} \subset V \subset Q_4.$$

LEMMA 4.2. *There exist $\phi \in C^2(\overline{V})$ and $\xi \in C(\overline{V})$ such that*

$$\begin{cases} \text{(i)} & \phi \le 0 \qquad \textit{in } V, \\ \text{(ii)} & \mathscr{P}^-(D^2\phi) = \xi \qquad \textit{in } V, \\ \text{(iii)} & \phi \le -2 \qquad \textit{in } Q_3, \\ \text{(iv)} & \phi = 0 \qquad \textit{in } \partial V, \\ \text{(v)} & \xi = 0 \qquad \textit{in } V \setminus U. \end{cases}$$

PROOF. Thanks to a result in Section 9 of [**6**], we can find a classical solution $\phi_0 \in C^2(\overline{V} \setminus U)$ of

$$\begin{cases} \mathscr{P}^-(D^2\phi_0) = 0 & \text{in } V \setminus \overline{U}, \\ \phi_0 = 0 & \text{on } \partial V, \\ \phi_0 = -1 & \text{on } \partial U. \end{cases}$$

Since this classical solution is an L^p-viscosity solution of the above, in view of Proposition 4.1, setting $\sigma = -\max_{\partial Q_3} \phi_0$, we see that $\sigma \in (0,1)$, and $\phi_0 \le -\sigma$ in $Q_3 \setminus U$. Hence, taking $\phi = 2\phi_0/\sigma$, and denoting by the same ϕ a smooth extension of ϕ to V such that $\phi \le -2$ in U, we obtain the required conclusion. $\square$

We can now show a preliminary version of the weak Harnack inequality for L^p-viscosity supersolutions.

LEMMA 4.3. *Suppose that* (2.2) *holds. There exist $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) \in (0, 1]$ satisfying the following property: for $\mu \in L^q_+(V)$ satisfying*

$$\|\mu\|_{L^n(V)} \le \varepsilon_0, \tag{4.1}$$

there are $r = r(n, \lambda, \Lambda) > 0$ and $C_7 = C_7(n, \lambda, \Lambda) > 0$ such that if $f \in L^p_+(V)$, and $u \in C(V)$ is a nonnegative L^p-viscosity supersolution of

$$\mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \textit{in } V, \tag{4.2}$$

then

$$\left(\int_{Q_1} u^r dx\right)^{\frac{1}{r}} \le C_7\left(\inf_{Q_1} u + \|f\|_{L^n(V)}\right). \tag{4.3}$$

Proof.

Step 1: By taking $u(x)(\inf_{Q_1} u + \delta + t\|f\|_n)^{-1}$, where $t, \delta > 0$, it is enough to find $r > 0$ and $C_7 > 0$ (independent of δ) such that

$$\left(\int_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq C_7. \tag{4.4}$$

Thus, we may suppose that $\inf_{Q_1} u \leq 1$ and $\|f\|_n \leq 1/t$. However, to use the cube decomposition argument, we will only need a weaker requirement $\inf_{Q_3} u \leq 1$.

Let ϕ be the function from Lemma 4.2. Setting $w = u + \phi$, we easily verify that w is an L^p-viscosity supersolution of

$$\mathscr{P}^+(D^2 w) + \mu(x)|Dw| = -f(x) - \xi(x) - \mu(x)|D\phi(x)| =: g(x) \quad \text{in } V.$$

Denote $\Omega_0 = \{x \in V \mid w(x) < 0\}$. We notice that $\Omega_0 \neq \emptyset$. Theorem 2.4 and (iii) of Lemma 4.2 imply

$$1 \leq \sup_{Q_3}(-w) \leq \sup_V(-w) = \sup_{\Omega_0}(-w) \leq C_1 \exp(C_2 \|\mu\|^n_{L^n(\Omega_0)}) \|g\|_{L^n(\Omega_0)}.$$

If $\varepsilon_0 \leq 1$, then we have

$$1 \leq C_1 e^{C_2} \|g\|_{L^n(\Omega_0)}$$

which, recalling (v) of Lemma 4.2, yields

$$\begin{aligned} \frac{e^{-C_2}}{C_1} &\leq \|f\|_{L^n(V)} + \|\xi\|_{L^\infty(Q_1)} |\Omega_1|^{1/n} + \|D\phi\|_{L^\infty(V)} \|\mu\|_{L^n(V)} \\ &\leq \frac{1}{t} + \|\xi\|_{L^\infty(Q_1)} |\Omega_1|^{1/n} + \|D\phi\|_{L^\infty(V)} \|\mu\|_{L^n(V)}, \end{aligned}$$

where $\Omega_1 = \{x \in Q_1 \mid w(x) < 0\}$. Hence, taking t sufficiently big, if ε_0 is small enough we can find $\theta = \theta(n, \lambda, \Lambda, \varepsilon_0) \in (0, 1)$ such that

$$\theta \leq |\Omega_1|.$$

We remind that θ is independent of δ. Putting $M := \sup_V(-\phi) > 1$, we have thus obtained

$$\theta \leq |\{x \in Q_1 \mid u(x) \leq M\}|. \tag{4.5}$$

Step 2: Following [**5**] (see also [**6**] and [**16**]) we will show that

$$|\{x \in Q_1 \mid u(x) > M^k\}| \leq (1-\theta)^k \quad \text{for integers } k \geq 1. \tag{4.6}$$

Inequality (4.6) is true for $k = 1$ by (4.5). Suppose it holds for $k-1$ for some $k \geq 2$. We will prove that it holds for k.

Setting $A = \{x \in Q_1 \mid u(x) > M^k\}$ and $B = \{x \in Q_1 \mid u(x) > M^{k-1}\}$, we observe that $A \subset B$ and $|A| \leq 1-\theta$. Therefore, in view of the Calderon-Zygmund decomposition lemma (Lemma 4.2 in [**6**]), letting $Q := Q_{1/2^j}(z)$ be a dyadic cube of $\hat{Q} := Q_{1/2^{j-1}}(\hat{z})$ for some $z, \hat{z} \in Q_1$ such that

$$|A \cap Q| > \frac{1-\theta}{2^{jn}}, \tag{4.7}$$

we only need to show that $\hat{Q} \subset B$.

Suppose, contrary to this, that there is $\hat{x} \in \hat{Q} \setminus B$, *i.e.* $u(\hat{x}) \leq M^{k-1}$. Setting $v(x) = u(z + 2^{-j}x)/M^{k-1}$ for $x \in V$, we see that $\inf_{Q_3} v \leq 1$, and v is an L^p-viscosity supersolution of

$$\mathscr{P}^-(D^2 v) + \hat{\mu}(x)|Dv| = -\hat{f}(x) \quad \text{in } V,$$

where $\hat{\mu}(x) = \mu(z + 2^{-j}x)/2^j$ and $\hat{f}(x) = f(z + 2^{-j}x)/(2^{2j}M^{k-1})$. Notice that $\|\hat{\mu}\|_{L^n(V)} \leq \varepsilon_0$. Therefore by Step 1 applied to v instead of u we obtain

$$|\{x \in Q_1 \mid v(x) \leq M\}| \geq \theta$$

which yields $|Q \setminus A| \geq \theta/2^{jn}$. This contradicts (4.7).

It is now standard to conclude that (4.6) implies that there exists $r > 0$ such that

$$\left(\int_{Q_1} u^r dx\right)^{\frac{1}{r}} \leq C_7$$

for some $C_7 > 0$. $\square$

REMARK 4.4. If u were an L^n-strong supersolution, then we could obtain Lemma 4.3 under the assumption $p = q = n$. This is due to the fact that the ABP maximum principle for L^n-strong solutions holds in this case. We do not know if such a result is true for L^n-viscosity solutions when $q = n$.

Theorem 4.5. *Let (2.2) hold, and $R > 1$. Let $\mu \in L^q_+(Q_R)$, $f \in L^p_+(Q_R)$ and let $r = r(n, \lambda, \Lambda) > 0$ be from Lemma 4.3. There exists $C_8 = C_8(n, \lambda, \Lambda, q, \|\mu\|_q, R) > 0$ such that $u \in C(Q_R)$ is a nonnegative L^p-viscosity supersolution of*

$$\mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \text{in } Q_R, \tag{4.8}$$

then

$$\left(\int_{Q_1} u^r dx\right)^{\frac{1}{r}} \leq C_8\left(\inf_{Q_1} u + \|f\|_{L^n(Q_R)}\right). \tag{4.9}$$

Proof. The theorem will follow from Lemma 4.3 and a covering argument of [**3**] once we know weak Harnack inequality in small cubes. To this end let $Q_{4t}(x) \subset Q_R$ and without loss of generality we can assume that $x = 0$, *i.e.* $Q_{4t}(x) = Q_{4t}$. Let ε_0 be from Lemma 4.3. We verify that $v(x) = u(tx)$ is an L^p-viscosity supersolution of

$$\mathscr{P}^+(D^2v) + \mu_t(x)|Dv| = -f_t(x) \quad \text{in } V,$$

where $\mu_t(x) = t\mu(tx)$ and $f_t(x) = t^2 f(tx)$. We notice that $\|f_t\|_{L^n(V)} = t\|f\|_{L^n(V_t)}$ and

$$\|\mu_t\|_{L^n(V)} \leq (4t)^{1-\frac{n}{q}}\|\mu\|_{L^q(Q_{4t})} \leq \varepsilon_0$$

if t is sufficiently small. Hence, it follows from Lemma 4.3 that for $t \leq \bar{t} = \bar{t}(n, q, \lambda, \Lambda, \|\mu\|_q, R)$

$$\left(\frac{1}{t^n}\int_{Q_t} u^r dx\right)^{\frac{1}{r}} \leq C_7\left(\inf_{Q_t} u + t\|f\|_{L^n(Q_{4t})}\right).$$

The result now follows from a covering argument of [**3**]. $\square$

Remark 4.6. A rather straightforward analysis of the proofs of Lemma 4.3 and Theorem 4.5, together with the use of Theorem 2.3, shows that (4.9) also holds for nonnegative L^p-viscosity supersolutions of (4.8) if $\mu, f \in L^n_+(Q_R)$ and $p_0 < p < n$. However then the constant $C_8 = C_8(n, \lambda, \Lambda, \mu, R)$ and it depends on μ and R in a way similar to the way C_4 depended on μ and Ω in Theorem 2.3.

In what follows, we will not make any distinction between a function and its zero-extension outside its domain.

THEOREM 4.7. *Let*

$$q > n > p > p_0 \tag{4.10}$$

and $r = r(n, \lambda, \Lambda) > 0$ *be from Theorem* 4.5. *Let* $1 < R \le 2$, $\mu \in L^q_+(Q_R)$ *and* $f \in L^p_+(Q_R)$. *There exists* $C_9 = C_9(n, \lambda, \Lambda, p, q, \|\mu\|_q, R) > 0$ *such that if* $u \in C(Q_R)$ *is a nonnegative* L^p*-viscosity supersolution of* (4.8) *in* Q_R, *then*

$$\left(\int_{Q_1} u^r dx\right)^{\frac{1}{r}} \le C_9\left[\inf_{Q_1} u + \left\{\exp(C_2\|\mu\|_n^n)\|\mu\|_q^N + \sum_{k=0}^{N-1}\|\mu\|_q^k\right\}\|f\|_p\right], \tag{4.11}$$

where N *and* C_2 *are the constants from Theorem* 2.3.

PROOF. In view of Theorem 2.3, we can find an L^p-strong supersolution $v \in C(\overline{Q}_R) \cap W^{2,p}_{\text{loc}}(Q_R)$ of

$$\mathscr{P}^-(D^2 v) - \mu(x)|Dv| = f(x) \quad \text{in } Q_{R+1}$$

such that $v = 0$ on ∂Q_{R+1}, and

$$0 \le v \le C_3\left\{\exp(C_2\|\mu\|_n^n)\|\mu\|_q^N + \sum_{k=0}^{N-1}\|\mu\|_q^k\right\}\|f\|_p \quad \text{in } Q_R. \tag{4.12}$$

Since $w := u + v$ is a nonnegative L^p-viscosity supersolution of

$$\mathscr{P}^+(D^2 w) + \mu(x)|Dw| = 0 \quad \text{in } Q_R,$$

Theorem 4.5 yields

$$\left(\int_{Q_1} u^r dx\right)^{\frac{1}{r}} \le \left(\int_{Q_1} w^r dx\right)^{\frac{1}{r}} \le C_8 \inf_{Q_1} w.$$

This, together with (4.12), gives (4.11) for some constant C_9. □

We now state scaled versions of Theorems 4.5 and 4.7 whose obvious proof is obtained by applying Theorems 4.3 and 4.7 to the function $v(x) = u(tx)$.

COROLLARY 4.8. *Let $1 < R \le 2$. There exist $r = r(n, \lambda, \Lambda) > 0$, $C_8 = C_8(n, \lambda, \Lambda, q, t^{1-\frac{n}{q}}\|\mu\|_q, R) > 0$ and $C_9 = C_9(n, \lambda, \Lambda, p, q, t^{1-\frac{n}{q}}\|\mu\|_q, R) > 0$ such that if $t \in (0, 1]$, $\mu \in L^q_+(Q_{tR})$, $f \in L^p_+(Q_{tR})$, and $u \in C(Q_{tR})$ is a nonnegative L^p-viscosity supersolution of*

$$\mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \text{in } Q_{tR},$$

then for $q > n$ and $q \ge p \ge n$,

$$\left(\frac{1}{t^n}\int_{Q_t} u^r dx\right)^{\frac{1}{r}} \le C_8\left(\inf_{Q_t} u + t\|f\|_{L^n(Q_{tR})}\right),$$

and for $q > n > p > p_0$,

$$\left(\frac{1}{t^n}\int_{Q_t} u^r dx\right)^{\frac{1}{r}} \le C_9\Bigg[\inf_{Q_t} u + \bigg\{\exp(C_2\|\mu\|^n_{L^n(Q_{tR})})\|\mu\|^N_{L^q(Q_{tR})} + \sum_{k=0}^{N-1}\|\mu\|^k_{L^q(Q_{tR})}\bigg\}t^{2-\frac{n}{p}}\|f\|_{L^p(Q_{tR})}\Bigg].$$

5. Strong maximum principle.

As an application of the weak Harnack inequality, we can now derive the strong maximum principle for L^p-viscosity solutions of PDE with the first derivative term, which extends Proposition 4.1.

THEOREM 5.1. *Let (2.3) hold. Let $\mu \in L^q_+(\Omega)$, and $u \in C(\overline{\Omega})$ be an L^p-viscosity subsolution (resp., supersolution) of*

$$\mathscr{P}^-(D^2u) - \mu(x)|Du| = 0 \quad (\text{resp., } \mathscr{P}^+(D^2u) + \mu(x)|Du| = 0) \quad \text{in } \Omega \tag{5.1}$$

such that $\sup_\Omega u < \infty$ (resp., $\inf_\Omega u > -\infty$). If u attains its maximum (resp., minimum) over $\overline{\Omega}$ at $x \in \Omega$, then u is constant in $\overline{\Omega}$.

PROOF. The proof is the same as the proof of Proposition 4.1. We only need to replace the standard weak Harnack inequality ([**6**]) by either Theorem 4.5 or Theorem 4.7. $\square$

REMARK 5.2. If $q = p = n$, then the strong maximum principle also holds for L^n-strong sub- and supersolutions of (5.1) since the ABP maximum principle holds in this case, and it implies the weak Harnack inequality (see Remark 4.4).

6. Boundary weak Harnack inequality and the global Hölder estimate.

We will first establish the boundary weak Harnack inequality. We recall again that, unless specified otherwise, all functions are extended by zero outside of their domains.

THEOREM 6.1. *Suppose that* (2.3) *holds, and* $\Omega \subset \boldsymbol{R}^n$ *is such that*

$$\Omega \cap Q_1 \neq \emptyset.$$

Let $R > 1$ *and let* r, C_k $(k = 8, 9)$ *and* N *be the constants from Theorems* 4.5 *and* 4.7. *Let* $\mu \in L^q_+(\Omega)$ *and* $f \in L^p_+(\Omega)$. *Let* $u \in C(\Omega)$ *be a nonnegative* L^p*-viscosity supersolution of*

$$\mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \text{in } \Omega. \tag{6.1}$$

Then for $q > n$ *and* $q \geq p \geq n$,

$$\left(\int_{Q_1} (u_m^-)^r dx\right)^{\frac{1}{r}} \leq C_8\left(\inf_{Q_1} u_m^- + \|f\|_{L^n(Q_R)}\right), \tag{6.2}$$

and for $q > n > p > p_0$,

$$\begin{aligned}\left(\int_{Q_1} (u_m^-)^r dx\right)^{\frac{1}{r}} \quad \leq \quad & C_9\Bigg[\inf_{Q_1} u_m^- + \Bigg\{\exp(C_2\|\mu\|^n_{L^n(Q_R)})\|\mu\|^N_{L^q(Q_R)} \\ & \qquad + \sum_{k=0}^{N-1}\|\mu\|^k_{L^q(Q_R)}\Bigg\}\|f\|_{L^p(Q_R)}\Bigg],\end{aligned} \tag{6.3}$$

where for $m := \inf\{u(x) \mid x \in \partial\Omega \cap Q_R\}$,

$$u_m^-(x) = \begin{cases} \min\{u(x), m\} & \text{in } \Omega \cap Q_R, \\ m & \text{in } Q_R \setminus \Omega. \end{cases}$$

(*The functions* μ *and* f *are equal to* 0 *outside of* Ω.)

PROOF. For $u \in C(\overline{\Omega})$, we define $u_m^- \in C(\overline{Q}_R)$ as above. Since constants are L^p-viscosity supersolutions of (6.1) in Q_R, it is easy to see that u_m^- is an L^p-viscosity supersolution of (6.1) in Q_R. Therefore, applying Theorem 4.5 or 4.7, we conclude the proof. □

We will now present the global Hölder estimate for L^p-viscosity solutions of (2.1). As we have mentioned in the introduction a similar result has been obtained before by Sirakov [**20**] by a different method. His result is a little more general as it applies to equations with quadratically growing gradient terms, however the difficulty is in proving it for solutions of extremal equations of type (1.1). Comparing it to our Theorem 6.2 Sirakov's result would require $q > n, p \geq n$. Our improvement is in allowing $n > p > p_0$ and in obtaining a Hölder exponent which does not depend on $\|\mu\|_q$.

We need an additional condition on Ω. We assume that there exists $\Theta > 0, t_0 > 0$, such that

$$|Q_t(x) \setminus \Omega| \geq \Theta t^n \quad \text{for } x \in \partial\Omega \text{ and } 0 < t \leq t_0. \tag{6.4}$$

THEOREM 6.2. *Let Ω be a bounded domain which satisfies* (6.4). *Let* (2.3), (3.1), (3.2) *and* (3.3) *with $m = 1$ hold. Let $g \in C^\beta(\partial\Omega)$ with $\beta \in (0,1)$, and $L > 0$. There exist $\alpha = \alpha(n,p,q,\lambda,\Lambda,\Theta,\beta) \in (0,1)$ and $C_{10} = C_{10}(n,p,q,\lambda,\Lambda,\Theta,\|\mu\|_q$, $\|f\|_p, \omega(L)\|c\|_p, \|g\|_{C^\beta(\partial\Omega)}, \mathrm{diam}\,(\Omega), L, t_0) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^p-viscosity solution of* (2.1) *such that*

$$|u| \leq L \quad \text{in } \Omega, \quad \text{and} \quad u = g \quad \text{on } \partial\Omega,$$

then

$$|u(x) - u(y)| \leq C_{10}|x - y|^\alpha \quad \text{for } x, y \in \overline{\Omega}. \tag{6.5}$$

PROOF. The proof follows rather standard arguments, however we present it here for completeness and to keep track of the dependence of various constants. We first notice that $\pm u$ are L^p-viscosity supersolutions of

$$\mathscr{P}^+(D^2(\pm u)) + \mu(x)|Du| = -|f(x)| - c(x)\omega(L).$$

This is the only information needed to show the weak Harnack and boundary weak Harnack inequalities. Thus, we may assume that $c(x)\omega(L) = 0$ regarding it as the inhomogeneous term.

We will only prove the assertion when $q > n > p > p_0$, since the other case is proved by the same argument.

We first show an estimate at a boundary point. By translation, we may suppose that $0 \in \partial\Omega$ and without loss of generality we can assume that $t_0 = 2$. We note that for any constant $K \in \boldsymbol{R}$, $K \pm u$ are L^p-viscosity supersolutions of

$$\mathscr{P}^+(D^2u) + \mu(x)|Du| = -|f(x)| \quad \text{in } \Omega.$$

Thus, by setting $M_t = \sup_{Q_t \cap \Omega} u$ and $m_t = \inf_{Q_t \cap \Omega} u$ for $0 < t \leq 2$, $M_2 - u$ and $u - m_2$ are nonnegative L^p-viscosity supersolutions of the above PDE in $Q_2 \cap \Omega$.

For a function $w : Q_2 \cap \overline{\Omega} :\to \boldsymbol{R}$, we define $m[w] := \inf\{w(x) \mid x \in Q_2 \cap \partial\Omega\}$, and

$$w_m^-(x) = \begin{cases} \min\{w(x), m[w]\} & \text{for } x \in Q_2 \cap \Omega, \\ m[w] & \text{for } x \in Q_2 \setminus \Omega. \end{cases}$$

Setting also $\partial M_t = \sup_{\partial\Omega \cap Q_t} u$ and $\partial m_t = \inf_{\partial\Omega \cap Q_t} u$, we observe that

$$\inf_{Q_1} (M_2 - u)_m^- \leq M_2 - M_1 \quad \text{and} \quad \inf_{Q_1} (u - m_2)_m^- \leq m_1 - m_2.$$

Hence, by (6.3) applied to the functions $(M_2 - u)_m^-$ and $(u - m_2)_m^-$, using (6.4) we have

$$\Theta(M_2 - \partial M_2) \leq C_9 \Big(M_2 - M_1 + A[\mu] \|f\|_p \Big)$$

and

$$\Theta(\partial m_2 - m_2) \leq C_9 \Big(m_1 - m_2 + A[\mu] \|f\|_p \Big),$$

where $A[\mu] = \Big\{ \exp(C_2 \|\mu\|_n^n) \|\mu\|_q^N + \sum_{k=0}^{N-1} \|\mu\|_q^k \Big\}$. Therefore, we find $\theta = \theta(n, \lambda, \Lambda, p, q, \Theta, \|\mu\|_q) \in (0,1)$ and $C > 0$ such that

$$M_1 - m_1 \leq \theta(M_2 - m_2) + C(\partial M_2 - \partial m_2) + CA[\mu] \|f\|_p. \tag{6.6}$$

Here and below C stands for various positive constants independent of u.

However we will only use a scaled version of (6.6). We then have

$$M_t - m_t \leq \theta(M_{2t} - m_{2t}) + Ct^\beta + CA[\mu_t] \|f_t\|_p$$

for $\theta = \theta(n, \lambda, \Lambda, p, q, \Theta, \|\mu_t\|_q) \in (0,1)$ where $\mu_t(x) = t\mu(tx)$ and $f_t(x) = t^2 f(tx)$ for $x \in Q_2$ and $0 < t \le 1$. We have $\|\mu_t\|_{L^q(Q_2)} = t^{1-\frac{n}{q}}\|\mu\|_{L^q(Q_{2t})} \le 1$ for $t \le \bar{t} = \bar{t}(n, q, \|\mu\|_q)$, and $\|f_t\|_{L^p(Q_2)} = t^{2-\frac{n}{p}}\|f\|_{L^p(Q_{2t})} \le t^{2-\frac{n}{p}}\|f\|_p$. Therefore if we take $t \le \bar{t}$ we obtain that $\theta = \theta(n, \lambda, \Lambda, p, q, \Theta) \in (0,1)$, *i.e.* it is independent of $\|\mu\|_q$. We can now follow a standard argument (see for instance [**14**]) to establish that

$$M_t - m_t \le Ct^{\alpha_1} \tag{6.7}$$

for all $t > 0$ for some $C > 0$ and $\alpha_1 \in (0, \min\{\beta, 2 - \frac{n}{p}\})$ (which also depends on $n, \lambda, \Lambda, p, q, \Theta$).

Next, we show a precise local estimate for u. Instead of (6.4), we will use the fact that for every region Ω'

$$\left(\int_{\Omega'} |\zeta + \eta|^r dx\right)^{\frac{1}{r}} \le 2^{\frac{1}{r}-1}\left\{\left(\int_{\Omega'} |\zeta|^r dx\right)^{\frac{1}{r}} + \left(\int_{\Omega'} |\eta|^r dx\right)^{\frac{1}{r}}\right\} \tag{6.8}$$

for $\zeta, \eta \in L^r(\Omega')$ when $0 < r < 1$, which, together with the weak Harnack inequality, will give a precise Hölder continuity estimate with no use of the local maximum principle.

We fix $x \in \Omega$ and let $d = \mathrm{dist}(x, \partial\Omega) > 0$. We notice that $Q_{2d/\sqrt{n}} \subset \overline{\Omega}$. We will suppose that $x = 0$ for the sake of simplicity.

For $t \in (0, d/\sqrt{n}\,]$, setting $v = M_{2t} - u$ and then $v = u - m_{2t}$, Corollary 4.8 yields

$$\left(\frac{1}{t^n}\int_{Q_t} v^r dx\right)^{\frac{1}{r}} \le C_9\left(\inf_{Q_t} v + A[\mu]t^{2-\frac{n}{p}}\|f\|_p\right),$$

where $A[\mu] = \exp(C_2\|\mu\|_n^n)\|\mu\|_q^N + \sum_{k=0}^{N-1}\|\mu\|_q^k$ for some integer N, where the norms are taken over Q_{2t}. Hence, in view of (6.8), we obtain

$$M_{2t} - m_{2t} \le C\left(M_{2t} - M_t + m_t - m_{2t} + t^{2-\frac{n}{p}}\right)$$

for some $C = C(n, \lambda, \Lambda, t^{1-\frac{n}{q}}\|\mu\|_q)$, which implies that for $t \le \bar{t} = \bar{t}(n, q, \|\mu\|_q)$

$$M_t - m_t \le \theta(M_{2t} - m_{2t}) + t^{2-\frac{n}{p}}$$

for some $\theta = \theta(n, \lambda, \Lambda, p, q) \in (0,1)$. It is then standard (see Lemma 8.23 of [**14**]) to verify that

$$M_t - m_t \le C\left\{\left(\frac{t}{\delta}\right)^{\alpha_2}(M_{2\delta} - m_{2\delta}) + t^{\alpha_2}\right\} \tag{6.9}$$

for some $\alpha_2 \in (0, 2 - \frac{n}{p})$ (depending only on $n, \lambda, \Lambda, p, q$), and $t \in (0, \delta)$, where $\delta = \min\{\bar{t}, d/\sqrt{n}\}$.

We now define $\alpha := \min\{\alpha_1, \alpha_2\}$. For $x, y \in \overline{\Omega}$, in order to show that

$$|u(x) - u(y)| \le C|x-y|^\alpha,$$

we only need to consider the case when $x, y \in \Omega$ and $|x - y| < \bar{t}$, because of (6.7), and $g \in C(\partial\Omega)$.

We may suppose that $\operatorname{dist}(x, \partial\Omega) \ge \operatorname{dist}(y, \partial\Omega) > 0$. Thus, we set $d := \operatorname{dist}(x, \partial\Omega)$ and suppose for simplicity that $x = 0$.

Case 1: $|x - y| \ge d/\sqrt{n}$

We choose $x_0, y_0 \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = |x - x_0|$ and $\operatorname{dist}(y, \partial\Omega) = |y - y_0|$. Noting that $|x - y| \ge |x - x_0|/(\sqrt{n}) \ge |y - y_0|/(\sqrt{n})$, in view of (6.7), we see that

$$|u(x) - u(y)| \le C(|x - x_0|^\alpha + |x_0 - y_0|^\beta + |y - y_0|^\alpha) \le C|x - y|^\alpha.$$

Case 2: $|x - y| \le d/\sqrt{n}$

Because of (6.9) we have

$$|u(x) - u(y)| \le C\left(\frac{|x-y|}{\delta}\right)^\alpha (M_{2\delta} - m_{2\delta}) + C|x - y|^\alpha. \tag{6.10}$$

If $\delta = \bar{t}$ we are done. Otherwise $\delta = d/\sqrt{n}$. Then, if $x_0 \in \partial\Omega$ is such that $|x - x_0| = \operatorname{dist}(x, \partial\Omega) = d$, (6.7) implies that

$$\max_{Q_{\frac{4d}{\sqrt{n}}}(x_0)\cap\overline{\Omega}} u - \min_{Q_{\frac{4d}{\sqrt{n}}}(x_0)\cap\overline{\Omega}} u \le Cd^\alpha$$

which, together with (6.10) gives

$$|u(x) - u(y)| \le C|x - y|^\alpha. \qquad \square$$

Remark 6.3. Theorems 6.1 and 6.2 also hold for nonnegative L^p-viscosity supersolutions of (6.1) and L^p-viscosity solutions of (2.1) if $\mu, f, c \in L^n_+(Q_R)$ and $p_0 < p < n$. The constant C_8 in (6.2) is then the one from Remark 4.6 and in Theorem 6.2 we have $\alpha = \alpha(n, \lambda, \Lambda, \Theta, \beta) \in (0,1)$ and $C_{10} = C_{10}(n, \lambda, \Lambda, \Theta, \|f\|_n$, $\omega(L)\|c\|_n, \|g\|_{C^\beta(\partial\Omega)}, \mu, \Omega, L, t_0) > 0$. We leave the details to the readers as the proofs are almost the same as those above if we carefully use Remark 4.6 and its small cube version.

7. Strong solvability of extremal equations.

As an application of the global continuity estimates of Theorem 6.2, we prove a result about strong solvability of general extremal equations. Fok [**13**] showed this result for $f \in L^p(\Omega) \cap L^{2n}(\Omega^\eta)$, where $\Omega^\eta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \eta\}$ for some $\eta > 0$, however his p_0 may be different from ours. (We incorrectly attributed the full Theorem 7.1 to [**13**] in [**17**].) When $p = n$, Theorem 7.1 can be also deduced from the results of [**20**].

THEOREM 7.1. *Let $\Omega \subset B_1$ be a domain satisfying the uniform exterior cone condition. Under (2.3), let $f \in L^p(\Omega)$ and $\mu \in L^q(\Omega)$, and let $g \in C(\partial\Omega)$. Then, there exist L^p-strong solutions u, $v \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of*

$$\mathscr{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega \tag{7.1}$$

and

$$\mathscr{P}^+(D^2v) + \mu(x)|Dv| = f(x) \quad \text{in } \Omega \tag{7.2}$$

such that $u = v = g$ on $\partial\Omega$. Moreover for every $\Omega' \Subset \Omega$, we have

$$\|w\|_{W^{2,p}(\Omega')} \le C(\|g\|_\infty + \|f\|_p), \tag{7.3}$$

for $w = u, v$, and $C = C(n, p, q, \lambda, \Lambda, \|\mu\|_q, \operatorname{dist}(\Omega', \partial\Omega))$.

REMARK 7.2. We notice that the uniform exterior cone condition for Ω implies (6.4).

PROOF. We will only consider (7.1). We first approximate g by $g_i \in C^\beta(\partial\Omega)$ for some $\beta \in (0,1)$, $i = 1, 2, \ldots$. Then, approximating f and μ by smooth functions f_k and μ_k such that $\|f - f_k\|_p + \|\mu - \mu_k\|_q \to 0$ as $k \to \infty$, we find $u_k \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\mathscr{P}^-(D^2u_k) - \mu_k(x)|Du_k| = f_k(x) \quad \text{in } \Omega,$$

under $u_k = g_i$ on $\partial\Omega$.

Thanks to Theorem 6.2, we may assume that u_k converges to a function $\bar{u}_i \in C(\overline{\Omega})$ uniformly in $\overline{\Omega}$. On the other hand, it is known (*e.g.* [**18**]) that for each $\Omega' \Subset \Omega$, there is $C = C(n, \lambda, \Lambda, \mathrm{dist}(\Omega', \partial\Omega), \|\mu_k\|_q) > 0$ such that

$$\|u_k\|_{W^{2,p}(\Omega')} \leq C(\|g_i\|_\infty + \|f_k\|_p).$$

Therefore, we may suppose that $w = \bar{u}_i \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ satisfies (7.3).

It is standard to show that $\bar{u}_i$ is an L^p-strong supersolution of (7.1) because of the concavity of $\mathscr{P}^-$ and the fact that we may suppose that $D^2 u_k$ converges weakly to $D^2 \bar{u}_i$. It remains to show that $\bar{u}_i$ is also an L^p-strong subsolution of (7.1). However, Proposition 9.4 in Appendix implies that $\bar{u}_i$ is an L^p-viscosity subsolution of (7.1). On the other hand, since $\bar{u}_i$ is twice differentiable for almost all $x \in \Omega$ (because $p > n/2$), Proposition 9.1 implies that $\bar{u}_i$ satisfies $\mathscr{P}^-(D^2\bar{u}_i(x)) - \mu(x)|D\bar{u}_i(x)| \leq f(x)$ for almost all $x \in \Omega$.

We can now pass to the limit as $i \to +\infty$. It follows from the maximum principle that the $\bar{u}_i$ converge uniformly on $\overline{\Omega}$ to some function $u \in C(\overline{\Omega})$, $u = g$ on $\partial\Omega$. Moreover the function $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ and it satisfies (7.3). The fact that u is a strong solution of (7.1) follows from the same arguments as these described above to establish this fact for the $\bar{u}_i$. $\square$

8. Maximum principle in unbounded domains.

An application of Theorem 6.1 is an ABP type maximum principle in unbounded domains. Following [**4**] (see also [**22**]), we say that Ω satisfies (wG) if there exist $0 < \tau$, $\sigma < 1$ such that

$$\forall x \in \Omega,\ \exists R_x \text{ and } \exists z_x \in \boldsymbol{R}^n \text{ such that } |Q_{R_x}(z_x) \setminus \Omega_{x,R_x,\tau}| \geq \sigma |Q_{R_x}(z_x)|.$$

Here $\Omega_{x,R_x,\tau}$ is the connected component of $\Omega \cap Q_{R_x/\tau}(z_x)$ such that $x \in \Omega \cap Q_{R_x}(z_x)$.

We will impose the condition

$$M := \sup_{x\in\Omega} M_x < \infty, \tag{8.1}$$

where $M_x = R_x^{1-\frac{n}{q}} \|\mu\|_{L^q(\Omega_{x,R_x,\tau})}$. In [**8**], instead of (8.1), it was assumed that

$$\sup_{x\in\Omega} R_x \|\mu\|_{L^\infty(\Omega_{x,R_x,\tau})} < \infty.$$

We also refer to a recent paper [**1**] and references therein for results on ABP type maximum principles, strong maximum principle, Liouville type theorems, etc. for viscosity solutions of fully nonlinear PDE having superlinear, at most quadratic, growth in Du with bounded and continuous coefficients in unbounded domains.

THEOREM 8.1. *Let $\Omega \subset \boldsymbol{R}^n$ satisfy* (*wG*) *and let* (2.3) *hold. Suppose that $f \in L^p_+(\Omega)$ and $\mu \in L^q_+(\Omega)$ satisfies* (8.1). *There exists $C_{11} = C_{11}(n, \lambda, \Lambda, p, q, M, \sigma, \tau) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution* (*resp., supersolution*) *of*

$$\mathscr{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega$$

$$(\textit{resp., } \mathscr{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \text{in } \Omega)$$

such that $\sup_\Omega u < \infty$ (*resp.,* $\inf_\Omega u > -\infty$), *then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + C_{11} \sup_{x \in \Omega} R_x^{2-\frac{n}{p}} \|f\|_{L^p(\Omega_{x,R_x,\tau})}$$

$$\left(\textit{resp., } \sup_\Omega(-u) \leq \sup_{\partial\Omega}(-u) + C_{11} \sup_{x \in \Omega} R_x^{2-\frac{n}{p}} \|f\|_{L^p(\Omega_{x,R_x,\tau})}\right).$$

PROOF. Fix any $x \in \boldsymbol{R}^n$. By (wG), we choose $R = R_x > 0$ and $z = z_x \in \boldsymbol{R}^n$ such that $x \in Q_R(z)$ and

$$|Q_R(z) \setminus \Omega_{x,R,\tau}| \geq \sigma |Q_R(z)|. \tag{8.2}$$

We may suppose $z = 0$ by translation. Moreover, by scaling, we may suppose that u is an L^p-viscosity subsolution of

$$\mathscr{P}^-(D^2u) - \hat{\mu}(y)|Du| = \hat{f}(y) \quad \text{in } \frac{1}{R}\Omega_{x,R,\tau},$$

where $\hat{\mu}(y) = R\mu(Ry)$ and $\hat{f}(y) = R^2 f(Ry)$ (and where f and μ are equal to 0 outside of Ω). We notice that (8.1) implies that $\|\hat{\mu}\|_{L^q(Q_{1/\tau})} \leq M$.

Setting $v = K - u$, where $K := \sup_\Omega u$, we denote by v_m^- the function

$$v_m^-(y) = \begin{cases} \min\{K - u(y), m\} & \text{for } y \in \frac{1}{R}\Omega_{x,R,\tau}, \\ m & \text{for } y \in Q_{1/\tau} \setminus \frac{1}{R}\Omega_{x,R,\tau}, \end{cases}$$

where $m = K - \sup_{\partial\Omega} u$. It now follows from Theorem 6.1 and (8.2) that

$$\sigma\Big(K - \sup_{\partial\Omega} u\Big) \le C_9'\Big(K - u(x) + \|\hat{f}\|_{L^n(Q_{1/\tau})}\Big)$$

for some $C_9' = C_9'(n, \lambda, \Lambda, p, q, M, \tau) \ge 1$. Therefore we have

$$u(x) \le \left(1 - \frac{\sigma}{C_9'}\right)K + \frac{\sigma}{C_9'}\sup_{\partial\Omega} u + \|\hat{f}\|_{L^p(Q_{1/\tau})}.$$

Therefore, taking the supremum over $x \in \Omega$, we find

$$K \le \sup_{\partial\Omega} u + \frac{C_9'}{\sigma}\sup_{x\in\Omega} R_x^{2-\frac{n}{p}}\|f\|_{L^p(\Omega_{x,R_x,\tau})}. \qquad \square$$

9. Appendix.

We prove several technical results about L^p-viscosity solutions. We refer to [**9**] for the definition of the semi-jets $J^{2,\pm}$.

PROPOSITION 9.1 (cf. Proposition 3.4 in [**7**]). *Let F satisfy* (3.1)–(3.3), *let $f \in L^p(\Omega)$ and let one of the following conditions be satisfied*:

$$\begin{cases} (1) \quad q > n,\ q \ge p \ge n, \\ (2) \quad q > n > p > p_0,\ p(mq - n) > nq(m-1). \end{cases} \tag{9.1}$$

If $u \in C(\Omega)$ is an L^p-viscosity subsolution (*resp., supersolution*) *of* (2.1), *then there exists a null set $N \subset \Omega$ such that for $x \in \Omega \setminus N$*

$$\begin{aligned} &F(x, u(x), p, X) \le f(x) \quad (\textit{resp.},\ \ge f(x)) \\ &\textit{provided}\ \ (p, X) \in J^{2,+}u(x) \quad (\textit{resp.},\ J^{2,-}u(x)). \end{aligned}$$

In particular, if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ is an L^p-viscosity solution of (2.1), *then it is an L^p-strong solution of* (2.1).

REMARK 9.2. When $m = 1$, the second property of (9.1) automatically holds.

PROOF. We will only present proof in the case (2) and $m > 1$ as the proofs in the other cases are similar but much easier. Set $G(x, r, p, X) = F(x, r,$

$p, X) - f(x)$ for $(x, r, p, X) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^n$. For $(p, X) \in \boldsymbol{R}^n \times S^n$ we denote by $L(p, X)$ the set of points $x \in \Omega$ such that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |G(x, u(x), p, X) - G(y, u(y), p, X)|^p dy = 0. \tag{9.2}$$

Let $\mathscr{C} \subset \boldsymbol{R}^n \times S^n$ be a countable dense set. For $\mu, c \in L^p_+(\Omega)$ in (3.3), we then define

$$E = \bigcap_{(p,X) \in \mathscr{C}} L(p, X) \bigcap \left\{ x \in \Omega \;\middle|\; \lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |\mu(x) - \mu(y)|^p dy = 0 \right\}.$$

This set has full measure, *i.e.* $|E| = |\Omega|$ and it is easy to see that (9.2) holds for every $(x, p, X) \in E \times \boldsymbol{R}^n \times S^n$.

We will show that if $(p, X) \in J^{2,+}u(x)$ for $x \in E$, then

$$G(x, u(x), p, X) \leq 0.$$

Similar statement holds for the supersolution case.

By translation we may assume that $x = 0$. Suppose that the above inequality fails, i.e. there is $\delta > 0$ such that

$$2\delta \leq G(0, u(0), p, X).$$

Then for small $\eta > 0$, setting $Y = X + 2\eta I$, we find $r_0 > 0$ such that

$$u(x) + \frac{\eta}{2}|x|^2 \leq \phi(x) := u(0) + \langle p, x \rangle + \frac{1}{2}\langle Yx, x \rangle \quad \text{for } |x| \leq r_0, \tag{9.3}$$

and

$$\delta \leq G(0, u(0), p, Y). \tag{9.4}$$

Conditions (3.2) and (3.3) imply that $\psi = u - \phi$ is an L^p-viscosity subsolution of

$$\begin{cases} \mathscr{P}^-(D^2\psi) - C\mu(x)|D\psi|^m - C\mu(x)|D\psi| = g(x) & \text{in } B_r, \\ \psi \leq -\dfrac{\eta}{2}r^2 & \text{on } \partial B_r, \end{cases} \tag{9.5}$$

where

$$g(x) = -\delta + C\mu(x)|Yx| + G(0, u(0), p, Y) - G(x, u(x), p, Y).$$

Since $0 \in E$ we have

$$\|g\|_{L^p(B_r)} \leq o(r^{\frac{n}{p}}). \tag{9.6}$$

The idea now is to show that this cannot happen using maximum principle. Unfortunately the estimate of Theorem 2.6 is nonlinear and it is not obvious how it scales when the diameter of the domain goes to 0. However after careful analysis one can obtain a scaled version of (2.15) presented below in Lemma 9.3. Using (9.6) and (9.7), we easily conclude the proof since $\psi(0) = 0$ and this cannot happen for small r. $\square$

Lemma 9.3. *If $\Omega = B_r$ in Theorem 2.6, then in case* (ii) *we have*

$$\sup_{B_r} u \leq \sup_{\partial B_r} u + Cr^{2-\frac{n}{p}} \sum_{k=0}^{N+1} \|\mu_m\|_{L^q(B_r)}^{a_k} \|f\|_{L^p(B_r)}^{m^k}. \tag{9.7}$$

Proof. We will rescale equation (2.12). Set $w(x) = u(rx)$. Then w is an L^p-viscosity subsolution of

$$\mathscr{P}^-(D^2 w) - \frac{\mu_m(rx)}{r^{m-2}} |Dw|^m - r\mu_1(rx)|Dw| = r^2 f(rx) \quad \text{in } B_1. \tag{9.8}$$

Denote

$$\tilde{\mu}_m(x) = \frac{\mu_m(rx)}{r^{m-2}}, \quad \tilde{\mu}_1(x) = r\mu_1(rx), \quad \tilde{f}(x) = r^2 f(rx).$$

Then for $r \leq 1$

$$\|\tilde{\mu}_k\|_{L^q(B_1)} = r^{2-k-\frac{n}{q}} \|\mu_k\|_{L^q(B_r)} \quad (k = 1, m), \quad \|\tilde{f}\|_{L^p(B_1)} = r^{2-\frac{n}{p}} \|f\|_{L^p(B_r)}. \tag{9.9}$$

First we need to convince ourselves that condition (2.14) is preserved under scaling. We have

$$\|\tilde f\|^{m^N(m-1)}_{L^p(B_1)}\|\tilde\mu_m\|^{m^N}_{L^q(B_1)}=r^{m^N(m-\frac{n(m-1)}{p}-\frac{n}{q})}\|f\|^{m^N(m-1)}_{L^p(B_r)}\|\mu_m\|^{m^N}_{L^q(B_r)}<r^{m^N(m-\frac{n(m-1)}{p}-\frac{n}{q})}\delta.$$

However $m-\frac{n(m-1)}{p}-\frac{n}{q}>0$ by (2.13) so (2.14) is satisfied. In fact the above shows that $\delta\to+\infty$ as $r\to 0$.

To show (9.7) we proceed by induction. We define

$$M_0=\|\tilde f\|_{L^p(B_1)},\quad M_{k+1}=\|\tilde\mu_m\|_{L^q(B_1)}M_k^m\quad\text{for } k\ge 0.$$

By (9.9) we can assume that

$$M_k\le r^{2-\frac{n}{p}}\|\mu_m\|^{a_k}_{L^q(B_r)}\|f\|^{m^k}_{L^p(B_r)}$$

since it is satisfied for $k=0$. Then for $r\le 1$

$$\begin{aligned} M_{k+1} &\le r^{2-m-\frac{n}{q}}r^{(2-\frac{n}{p})m}\|\mu_m\|^{a_{k+1}}_{L^q(B_r)}\|f\|^{m^{k+1}}_{L^p(B_r)}\\ &= r^{2-\frac{n}{p}}r^{m-\frac{n(m-1)}{p}-\frac{n}{q}}\|\mu_m\|^{a_{k+1}}_{L^q(B_r)}\|f\|^{m^{k+1}}_{L^p(B_r)}\le r^{2-\frac{n}{p}}\|\mu_m\|^{a_{k+1}}_{L^q(B_r)}\|f\|^{m^{k+1}}_{L^p(B_r)}.\end{aligned}$$

where we again have used by (2.13). Therefore, by (2.15), we obtain

$$\sup_{B_r}u=\sup_{B_1}w\le\sup_{\partial B_1}w+C\sum_{k=0}^{N+1}M_k\le\sup_{\partial B_r}u+Cr^{2-\frac{n}{p}}\sum_{k=0}^{N+1}\|\mu_m\|^{a_k}_{L^q(B_r)}\|f\|^{m^k}_{L^p(B_r)}.\quad\square$$

We can now show the stability result for (2.1), which is needed to establish the strong solvability of extremal PDE in section 7.

PROPOSITION 9.4. *Let* $F,F_k:\Omega\times\boldsymbol{R}\times\boldsymbol{R}^n\times S^n\to\boldsymbol{R},k=1,2,\dots$ *satisfy* (3.1)–(3.3) *with* $m\ge 1,\lambda,\Lambda>0,\mu\in L^q_+(\Omega),c\in L^p_+(\Omega)$ *and modulus* ω, *let* $f,f_k\in L^p(\Omega),k=1,2,\dots,$ *and let one of* (9.1) *hold.*

Let u_k *be* L^p*-viscosity subsolutions* (*resp., supersolutions*) *of*

$$F_k(x,u_k,Du_k,D^2u_k)=f_k(x)\quad\text{in }\Omega.$$

Assume also that for every $B_r(x)\subset\Omega$, $u_k\to u$ *uniformly in* $B_r(x)$ *as* $k\to\infty$, *and for* $\phi\in W^{2,p}(B_r(x))$

$$\lim_{k\to+\infty}\|(G[\phi]-G_k[\phi])^+\|_{L^p(B_r(x))}=0\tag{9.10}$$

$$\left(\textit{resp.,}\quad \lim_{k\to+\infty}\|(G[\phi]-G_k[\phi])^-\|_{L^p(B_r(x))}=0\right),$$

where

$$G_k[\phi](y)=F_k(x,u_k(x),D\phi(x),D^2\phi(x))-f_k(x),\ \textit{and}$$
$$G[\phi](y)=F(x,u(x),D\phi(x),D^2\phi(x))-f(x).$$

Then u is an L^p-viscosity subsolution (*resp., supersolution*) *of*

$$F(x,u,Du,D^2u)=f(x)\quad \textit{in}\ \Omega.$$

PROOF. Again we will only show the result for case (2) when $m>1$. Suppose, contrary to the claim, that there exist $y\in\Omega$, $r>0$, $\varphi\in W^{2,p}(B_r(y))$ such that $u-\varphi$ has a maximum at y over $B_r(y)\Subset\Omega$ but

$$F(x,u(x),D\varphi(x),D^2\varphi(x))-f(x)\geq\delta\quad \textit{a.e.}\ \text{in}\ B_r(y).$$

Without loss of generality we may assume that $y=0\in\Omega$ and $(u-\varphi)(0)=0$. Define $w_k(x)=u_k(x)-\varphi(x)-\eta|x|^2$, where $\eta=\delta/(2n\Lambda)$, and set $\varepsilon_k=\sup_{B_r}|u_k(x)-u(x)|$.

Using (3.2) and (3.3) we easily obtain that w_k is an L^p-viscosity subsolution of

$$\begin{aligned}&\mathscr{P}^-(D^2w_k)-C_1\mu(x)|Dw_k|^m-C_1\mu(x)(|D\varphi(x)|^{m-1}+|2\eta x|^{m-1}+1)|Dw_k|\\ =\ & c(x)\omega(\varepsilon_k)+G[\varphi+\eta|x|^2](x)-G_k[\varphi+\eta|x|^2](x)\\ &\qquad\qquad\qquad +C_1\mu(x)(|D\varphi(x)|^{m-1}+|2\eta x|^{m-1}+1)|2\eta x|\end{aligned}$$

in B_r for some constant $C_1=C_1(m)$. Because of condition (2), the function $\mu(x)(|D\varphi(x)|^{m-1}+|2\eta x|^{m-1}+1)\in L^{\hat p}(B_r)$ for some $\hat p>n$. Moreover, denoting $h(x):=C_1\mu(x)(|D\varphi(x)|^{m-1}+|2\eta x|^{m-1}+1)|2\eta x|$ and using that $\mu\in L^q(\Omega)$, $D\varphi\in L^{p^*}(B_r)$, we have

$$\|h\|_{L^p(B_r)}\leq C_2r^{\frac{n}{p}+\alpha}$$

for some C_2 and $\alpha>0$. Therefore it follows from Lemma 9.3 and (9.10) that

$$\sup_{B_r}w_k\leq\sup_{\partial B_r}w_k+C_3r^{2-\frac{n}{p}}r^{\frac{n}{p}+\alpha}+\sigma(k),$$

where $\sigma(k) \to 0$ as $k \to +\infty$. This leads to a contradiction by first choosing small r and then letting $k \to +\infty$ since

$$\lim_{k\to+\infty} \sup_{B_r} w_k = 0 \quad \text{and} \quad \lim_{k\to+\infty} \sup_{\partial B_r} w_k \leq -\eta r^2. \qquad \square$$

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