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Pinching theorem for the real projective space

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§ 1. Introduction.

Let M be an n-dimensional connected and complete Riemannian manifold whose sectional curvature K satisfies

(1.1) $$1/4<\delta \leqq K \leqq 1 \quad \text{for any plane section}.$$

If M is simply connected and $\delta \doteqdot 0.85$, then M is diffeomorphic to the standard sphere (see [5]). In the present paper we shall establish a differentiable pinching theorem for the real projective space. Our pinching number is independent of the dimension.

MAIN THEOREM. *Let M be a connected and complete Riemannian manifold with* (1.1). *Assume that the fundamental group $\pi_1(M)$ of M is*

(1.2) $$\pi_1(M)=Z_2.$$

Then there exists a constant $\delta_0 \in (1/4, 1)$ such that

(1.3) $$\delta>\delta_0$$

implies M to be diffeomorphic to the real projective space.

§ 2. Preliminaries.

Throughout this paper, let M satisfy both (1.1) and (1.2). We denote by d the distance function on M with respect to the Riemannian metric. The diameter $d(M)$ of M is defined by $d(M):=\mathrm{Max}\,\{d(x, y)\,;\, x, y \in M\}$ and we set $d(p, q):=d(M)$. Let $\tilde{M}$ be the universal Riemannian covering manifold of M and π the covering projection. For any point $x \in M$, we denote by $\tilde{x}_1$, $\tilde{x}_2$ $\in \tilde{M}$ the elements of the inverse image $\pi^{-1}(x)$, and by $C(x)$ the cut locus of x. Under the assumptions (1.1) and (1.2), we see in [4] that

$$\pi/2 \leqq d(x, C(x)) \leqq \pi/2\sqrt{\delta}, \qquad \pi/2 \leqq d(M) \leqq \pi/2\sqrt{\delta}$$

hold for any $x \in M$. Since for any $x \in M$ and any $y \in C(x)$, each minimizing geodesic from x to y has no conjugate pair, they are joined by two and just two distinct minimizing geodesics. Let E be defined by

$$E := \{\tilde{y} \in \tilde{M};\ d(\tilde{p}_1, \tilde{y}) = d(\tilde{p}_2, \tilde{y})\}\ .$$

Then we observe

$$\pi(E) = C(p)\,.$$

Especially E is a hypersurface diffeomorphic to S^{n-1}. Let $f: \tilde{M} \to \tilde{M}$ be the deck transformation. Then f is a fixed point free involution and it leaves E invariant.

Next we observe that for any $\tilde{y} \in \tilde{M}$, $f(\tilde{y}) \in C(\tilde{y})$. Hence we have

$$d(\tilde{y}, f(\tilde{y})) \geqq \pi$$

from the cut locus theorem due to Klingenberg [**2**]. For any $y \in C(p)$, let γ_1, γ_2 be the shortest connections between p and y (each emanating from p) and $\tilde{\gamma}_1, \tilde{\gamma}_2$ the lifted geodesics joining $\tilde{p}_1$ to $\tilde{y}_1, \tilde{y}_2$ respectively. By construction they have the same length l, and $\pi/2 \leqq l \leqq \pi/2\sqrt{\delta}$. Therefore we have the geodesic quadrangle $(\tilde{\gamma}_1, f\cdot\tilde{\gamma}_2^{-1}, f\cdot\tilde{\gamma}_1, \tilde{\gamma}_2^{-1})$ with the same edge length and the vertices $\tilde{p}_1, \tilde{y}_1, \tilde{p}_2$ and $\tilde{y}_2$. Moreover from Toponogov's comparison theorem (see [**2**]) all of the edge angles are bounded from below by $\pi\sqrt{\delta} > \pi/2$, which is proved in Lemma 3.1. Therefore as is shown in [**5**], all of the shortest geodesics emanating from $\tilde{p}_i$ to points on E can be deformed simultaneously in a thin neighborhood of E so that they hit orthogonally to E. In fact, let $\lambda: \tilde{M} \to \boldsymbol{R}$ be the function defined by $\lambda(\tilde{x}) := d(\tilde{p}_1, \tilde{x}) - d(\tilde{p}_2, \tilde{x})$, $\tilde{x} \in \tilde{M}$. Then 0 is a regular value of λ, and hence there exists an open interval I of 0 contained in the set of regular values of λ such that $\lambda^{-1}(I) \subset B_\pi(\tilde{p}_1) \cap B_\pi(\tilde{p}_2)$ and all of shortest connections joinning $\tilde{p}_i$ to points on E are transversal to each of the hypersurfaces $\lambda^{-1}(\{a\})$, $a \in I$, where $B_\pi(\tilde{p}_i)$ is by definition the open metric ball in $\tilde{M}$ with the radius π and the center at $\tilde{p}_i$. Thus we get the family of loops at p covering simply M, so that any loop has the same tangent vectors at p as one of the original biangles. *If all of these loops can be deformed simultaneously to simply closed smooth curves, then M is diffeomorphic to the real projective space.* For this purpose we consider the involutive diffeomorphism $\varphi: S_p(1) \to S_p(1)$ ($S_p(1) \subset M_p$ is by definition the unit hypersphere in the tangent space M_p centered at the origin) caused by the deck transformation as follows: For each $u \in S_p(1)$, $\varphi(u)$ is the unit tangent vector such that

$$\exp_p l\cdot u = \exp_p l\cdot\varphi(u) \in C(p)\,, \qquad u \neq \varphi(u)\,, \qquad l \in [\pi/2,\ \pi/2\sqrt{\delta}\,]\,.$$

Clearly φ is a fixed point free involutive diffeomorphism.

Now the problem is how to construct a homotopy $\{\Phi_t\}$ $(0 \leqq t \leqq 1)$ of diffeomorphisms on $S_p(1)$ satisfying

$$\text{(2.1)} \qquad \begin{cases} \Phi_t^2 = \text{identity} & \text{for each } t \in [0, 1]\,, \\ \Phi_0 = \varphi\,, & \Phi_1 = \text{antipodal map}\,. \end{cases}$$

The essential tool for proving (2.1) is the following (see [**5**])

DIFFEOTOPY THEOREM. *Let h be a diffeomorphism on the standard k-sphere $S^k \subset R^{k+1}$. Assume that*

$$\beta := \operatorname{Max}\{\sphericalangle(u, h(u))\,;\ u \in S^k\} \leqq \pi/2\,, \tag{2.2}$$

$$\varepsilon := \operatorname{Max}\{\sphericalangle(A, dhA)\,;\ A \in TS^k\} < \cos^{-1}\Big\{-\cos\beta\sqrt{\frac{\sin\beta}{\beta}}\Big\}. \tag{2.3}$$

Then h is diffeotopic to the identity via the following homotopy of diffeomorphisms: For each $u \in S^k$, let $\gamma_u : [0, 1] \to S^k$ be the shortest great circle arc parametrized proportionally to arc length such that $\gamma_u(0) := u$, $\gamma_u(1) := h(u)$. Let $H_t(u) := \gamma_u(t)$. Then H_t is a diffeomorphism for all $t \in [0, 1]$.

Let us consider the following $\phi : S_p(1) \to S_p(1)$

$$\phi(u) := -\varphi(u)\,. \tag{2.4}$$

Then φ is diffeotopic to the antipodal map if and only if ϕ is diffeotopic to the identity.

The final step of the proof is to find out δ_0 such that (1.3) yields the diffeotopy conditions (2.2) and (2.3) for ϕ. In fact if ϕ satisfies the conditions then there exists the homotopy $\{\Psi_t\}$ $(0 \leqq t \leqq 1)$ of diffeomorphisms obtained in the diffeotopy theorem. From construction we see

$$\Psi_{1/2}(u) = -\Psi_{1/2}(\varphi(u))\,, \qquad \text{for any } u \in S_p(1)\,.$$

Setting

$$\Phi_t := \Psi_{t/2} \circ \varphi \circ \Psi_{t/2}^{-1}\,, \tag{2.5}$$

we see that Φ_t satisfies (2.1) and hence M is diffeomorphic to the real projective space.

§ 3. Construction of the involutive diffeotopy.

LEMMA 3.1. *$\sphericalangle(u, \phi(u)) \leqq \pi(1-\sqrt{\delta}) < \pi/2$ holds for any $u \in S_p(1)$.*

PROOF. For any $u \in S_p(1)$, we have the geodesic quadrangle with edges $(\tilde{\gamma}_1, f\cdot\tilde{\gamma}_2^{-1}, f\cdot\tilde{\gamma}_1, \tilde{\gamma}_2^{-1})$ in $\tilde{M}$, where $d\pi(\tilde{\gamma}_1'(0)) = u$, $d\pi(\tilde{\gamma}_2'(0)) = -\phi(u)$. Apply Toponogov's comparison theorem to the isosceles triangle with vertices $\tilde{p}_1$, $\tilde{y}_1$ and $\tilde{y}_2$, where $\tilde{y}_i := \tilde{\gamma}_i(l) \in E$, $l = 1, 2$. The conclusion is obvious from $l \in [\pi/2, \pi/2\sqrt{\delta}]$ and $d(\tilde{y}_1, \tilde{y}_2) \geqq \pi$.

LEMMA 3.2. *There exists $\alpha(\delta)$ such that*

$$\lim_{\delta \to 1} \alpha(\delta) = 0\,, \tag{3.1}$$

$$d\Big(\exp_p \frac{\pi}{2} A, \exp_p \frac{\pi}{2} \frac{d\varphi A}{\|d\varphi A\|}\Big) \leqq \alpha(\delta) \quad \textit{for any } A \in TS_p(1),\ \|A\| = 1\,, \tag{3.2}$$

where A and $d\varphi A$ on the left hand side of (3.2) *are identified with those translated parallely in M_p to the origin.*

PROOF. For any $u \in S_p(1)$ and any $A \in T_u S_p(1)$, let $a: I \to S_p(1)$ be a curve fitting A (i.e., $a(0)=u$, $a'(0)=A$ and I is an open interval containing 0). Let $\gamma_1, \gamma_2: [0, l_0] \to M$ be the shortest geodesics such that $\gamma_1'(0)=u$, $\gamma_2'(0)=\varphi(u)$, $\gamma_1(l_0)=\gamma_2(l_0) \in C(p)$. We define the smooth function $s \to l(s)$, $s \in I$ by $l_0=l(0)$, $\exp_p l(s) \cdot a(s) \in C(p)$. We denote by $V^i: [0, l_0] \times I \to M$ the 1-parameter geodesic variation along γ_i

$$
\begin{aligned}
V^1(t, s) &:= \exp_p t\frac{l(s)}{l(0)} \cdot a(s)\,, \\
V^2(t, s) &:= \exp_p t\frac{l(s)}{l(0)} \cdot \varphi(a(s))\,.
\end{aligned} \tag{3.3}
$$

Obviously we see $V_s^1(l_0)=V_s^2(l_0) \in C(p)$ for any $s \in I$, where $V_s^i(t):=V^i(t, s)$. Let Y_i be the Jacobi field associated with V^i and Z_i its normal component. From $L(V_s^1)=L(V_s^2)$ for any $s \in I$ ($L(\ )$ denotes the length of curve) and $Y_i(0)=0$,

$$
Y_i(t)=Z_i(t)+c \cdot t \cdot \gamma_i'(t)\,, \tag{3.4}
$$

where c is a constant such that $|c| \leqq \frac{2}{\pi\sqrt{\delta}} \cot \frac{\pi\sqrt{\delta}}{2}$. This follows immediately from $\sphericalangle(Y_i(l_0), Z_i(l_0)) \leqq \frac{\pi}{2}(1-\sqrt{\delta})$ and $\|Y_i(l_0)\| \leqq 1/\left(\sqrt{\delta}\sin\frac{\pi\sqrt{\delta}}{2}\right)$. From construction, follows

$$
Z_1'(0)=A\,, \qquad Z_2'(0)=d\varphi A\,, \tag{3.5}
$$

and

$$
Y_1(l_0)=Y_2(l_0)\,, \qquad \|Z_1(l_0)\|=\|Z_2(l_0)\| \tag{3.6}
$$

where $Z_i'=\nabla_{\gamma_i'} Z_i$. Let P_i be the parallel field along γ_i such that $P_i(0)=Z_i'(0)/\|Z_i'(0)\|$, and $b_i: [0, l_0] \to M$ be defined by

$$
b_i(t) := \exp_{\gamma_i(t)} \frac{\pi}{2} P_i(t)\,.
$$

We shall apply Berger's comparison theorem (see [1] and the "equator estimate" in [5]) to b_i to get

$$
L(b_i) \leqq \frac{\pi}{2\sqrt{\delta}} \cos \frac{\pi\sqrt{\delta}}{2}\,. \tag{3.7}
$$

Making use of the approximation theorem for Jacobi fields (see 8 and 9 in [5]), we can find a function $\bar{\Theta}(\delta)$ such that

$$
\lim_{\delta \to 1} \bar{\Theta}(\delta)=0\,, \qquad \sphericalangle(P_i(l_0), Z_i(l_0)) \leqq \bar{\Theta}(\delta)\,.
$$

From (3.6), we have a bound for $\sphericalangle(P_1(l_0), P_2(l_0)) \leqq 2\cos^{-1}\left\{\sin\frac{\pi\sqrt{\delta}}{2}\cos\bar{\Theta}(\delta)\right\}$ $=:\Theta(\delta)$. In fact, applying the cosine rule for the spherical trigonometry to the triangle $(Y_1(l_0)/\|Y_1(l_0)\|, Z_1(l_0)/\|Z_1(l_0)\|, P_1(l_0))$, we get $\sphericalangle(Y_1(l_0), P_1(l_0)) \leqq \cos^{-1}\left\{\sin\frac{\pi\sqrt{\delta}}{2}\cdot\cos\bar{\Theta}(\delta)\right\}$. Thus we get

$$d\left(\exp_p\frac{\pi}{2}A, \exp_p\frac{\pi}{2}\frac{d\varphi A}{\|d\varphi A\|}\right)=d(b_1(0), b_2(0)) \leqq L(b_1)+d(b_1(l_0), b_2(l_0))+L(b_2)$$

$$\leqq \frac{\pi}{\sqrt{\delta}}\cos\frac{\pi\sqrt{\delta}}{2}+\frac{1}{\sqrt{\delta}}\cos^{-1}\left\{\cos^2\frac{\pi\sqrt{\delta}}{2}+\sin^2\frac{\pi\sqrt{\delta}}{2}\cos\Theta(\delta)\right\}=:\alpha(\delta).$$

PROPOSITION 3.3. *Let δ be taken so as to satisfy*

$$\alpha(\delta) \leqq \left(2-\frac{1}{\sqrt{\delta}}\right)\pi. \tag{3.8}$$

Then for any $A \in TS_p(1)$, we have either

$$0 \leqq \sphericalangle(A, d\varphi A) \leqq \alpha(\delta), \tag{3.9}$$

or else

$$\pi\sqrt{\delta}-\left\{\alpha(\delta)+\pi\left(\frac{1}{\sqrt{\delta}}-1\right)\right\} \leqq \sphericalangle(A, d\varphi A) \leqq \pi. \tag{3.10}$$

PROOF. Let $\tilde{\sigma}_i\colon [0, m] \to \tilde{M}$ be the shortest geodesic such that $\tilde{\sigma}_i(0)=\tilde{p}_1$, $d\pi\cdot\tilde{\sigma}_1'(0)=A(\in M_n)$, $\pi(\tilde{\sigma}_1(m))=\pi(\tilde{\sigma}_2(m)) \in C(p)$ and $\tilde{\tau}\colon [0, \pi/2] \to \tilde{M}$ be such that $\tilde{\tau}(0)=\tilde{p}_1$, $d\pi(\tilde{\tau}'(0))=d\varphi A/\|d\varphi A\|$ $(\in M_p)$. Because of $\pi(\tilde{\tau}(\pi/2))=b_2(0)$, we have from (3.2), either $d(\tilde{\tau}(\pi/2), \tilde{\sigma}_1(\pi/2)) \leqq \alpha(\delta)$ or else $d(\tilde{\tau}(\pi/2), f\circ\tilde{\sigma}_1(\pi/2)) \leqq \alpha(\delta)$. Thus we get either

$$d(\tilde{\tau}(\pi/2), \tilde{\sigma}_1(m)) \leqq \alpha(\delta)+\frac{\pi}{2}\left(\frac{1}{\sqrt{\delta}}-1\right),$$

or else

$$d(\tilde{\tau}(\pi/2), \tilde{\sigma}_2(m)) \leqq \alpha(\delta)+\frac{\pi}{2}\left(\frac{1}{\sqrt{\delta}}-1\right).$$

(3.8) ensures that one of the circumferences of the triangles $(\tilde{p}_1, \tilde{\sigma}_i(\pi/2), \tilde{\tau}(\pi/2))$ is less than 2π. Hence we can apply Rauch theorem to the "smaller" triangle to get an upper bound for the angle $\sphericalangle(A, d\pi\cdot\tilde{\tau}'(0))$. From the former case we get (3.9) and from the latter (3.10).

PROOF OF THE MAIN THEOREM. From now on *let δ be taken so as to satisfy*

$$\alpha(\delta) < \frac{\pi}{2}\left(1+\sqrt{\delta}-\frac{1}{\sqrt{\delta}}\right). \tag{3.11}$$

It follows from the continuity of $A \to \sphericalangle(A, d\varphi A)$, that (3.11) yields one of the inequalities (3.9) or (3.10). We want to find out $\delta_0' \in (1/4, 1)$ in such a way that $\delta > \delta_0'$ implies (3.10) for all $A \in TS_p(1)$. For this purpose *we suppose that there exists* $A \in TS_p(1)$ *for which* (3.9) *holds*. Then $\sphericalangle(X, d\varphi X) \leqq \alpha(\delta)$ holds for all $X \in TS_p(1)$. We shall make use of the special closed geodesic to derive a contradiction. Let $\gamma : [0, d(M)] \to M$ be a shortest connection joining p to q. Then γ can be extended to the simply closed geodesic $\gamma : [0, 2d(M)] \to M$ (see [4]). Set $\gamma_1(t) := \gamma(t)$, $\gamma_2(t) := \gamma(2d(M)-t)$, $t \in [0, d(M)]$. We consider the lifted map $\tilde{\varphi} : S_{\tilde{p}_1} \to S_{\tilde{p}_1}$, $d\pi \cdot \tilde{\varphi} = \varphi$. Obviously we have $\tilde{\varphi}(\tilde{\gamma}_1'(0)) = \tilde{\gamma}_2'(0)$, where we use the same notations as in Lemma 3.2. The quadrangle $(\tilde{\gamma}_1, f \cdot \tilde{\gamma}_2^{-1}, f \cdot \tilde{\gamma}_1, \tilde{\gamma}_2^{-1})$ forms the simply closed geodesic with vertices $\tilde{p}_i$ and $\tilde{y}_i := \tilde{\gamma}_i(d(M))$. Because $\tilde{\gamma}_i'(d(M))$ is normal to $T_{\tilde{y}}E$, the Jacobi field $\tilde{Y}_i$ along $\tilde{\gamma}_i$ with the initial conditions $\tilde{Y}_i(0) := 0$, $d\pi \tilde{Y}_1'(0) := A$, $d\pi \tilde{Y}_2'(0) := d\varphi A$ is normal to $\tilde{\gamma}_i$ for any $A \in T_{d\pi(\tilde{\gamma}_1'(0))}S_p(1)$. We denote by $\tilde{P}_i$ the parallel field along $\tilde{\gamma}_i$ such that $\tilde{P}_i(0) = \tilde{Y}_i'(0)/\|\tilde{Y}_i'(0)\|$. Let $\tilde{\alpha}_i : [0, \pi/2] \to \tilde{M}$ be the geodesic such that $\tilde{\alpha}_i(0) = \tilde{y}_i$, $\tilde{\alpha}_i'(0) = \tilde{Y}_i(d(M))/\|\tilde{Y}_i(d(M))\|$. From $\tilde{\alpha}_2(0) = f(\tilde{\alpha}_1(0))$, $\tilde{\alpha}_2'(0) = df\tilde{\alpha}_0'(0)$, follows $f(\tilde{\alpha}_1(s)) = \tilde{\alpha}_2(s)$ for any $s \in [0, \pi/2]$. Because of $f(\tilde{y}) \in C(\tilde{y})$, we have a lower bound for the distance

$$d(\tilde{\alpha}_1(s), \tilde{\alpha}_2(s)) \geqq \pi \qquad \text{for any } s \in [0, \pi/2]. \tag{3.12}$$

On the other hand, we have an upper bound for the distance

$$\begin{aligned} d(\tilde{\alpha}_1(\pi/2), \tilde{\alpha}_2(\pi/2)) &\leqq d(\tilde{\alpha}_1(\pi/2), \exp_{\tilde{\gamma}_1(d(M))}\frac{\pi}{2}\tilde{P}_1(d(M)) \\ &\quad + \frac{\pi}{2\sqrt{\delta}}\cos\frac{\pi\sqrt{\delta}}{2} + \alpha(\delta) + \frac{\pi}{2\sqrt{\delta}}\cos\frac{\pi\sqrt{\delta}}{2} \\ &\quad + d\Big(\exp_{\tilde{\gamma}_2(d(M))}\frac{\pi}{2}\tilde{P}_2(d(M)),\ \tilde{\alpha}_2(\pi/2)\Big) \\ &\leqq \frac{\pi}{\sqrt{\delta}}\cos\frac{\pi\sqrt{\delta}}{2} + \alpha(\delta) + \frac{2}{\sqrt{\delta}}\cos^{-1}\Big\{\cos^2\frac{\pi\sqrt{\delta}}{2} + \sin^2\frac{\pi\sqrt{\delta}}{2}\cos\bar{\Theta}(\delta)\Big\}. \end{aligned} \tag{3.13}$$

Hence we can find δ_0' in such a way that $\delta > \delta_0'$ implies the right hand side of (3.13) is smaller than π.

Finally we shall check the second diffeotopy condition for ψ. From Lemma 3.1, $\beta := \mathrm{Max}\,\{\sphericalangle(u, \psi(u))\,;\, u \in S_p(1)\} \leqq \pi(1-\sqrt{\delta}) < \pi/2$. From $\psi = -\varphi$ and (3.10) (assuming $\delta > \delta_0'$), follows $\varepsilon := \mathrm{Max}\,\{\sphericalangle(A, d\varphi A)\,;\, A \in TS_p(1)\} \leqq \pi(1-\sqrt{\delta}) + \alpha(\delta) + \pi\Big(\frac{1}{\sqrt{\delta}} - 1\Big)$. Hence we can find δ_0 such that $\delta > \delta_0$ implies (2.3) for ψ. Thus the proof of the main theorem is completed.

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