J. Math. Soc. Japan
Vol. 28, No. 2, 1976

On characteristic classes of riemannian, conformal and projective foliations

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(Received Oct. 5, 1974)

Introduction.

In this paper we shall study certain families of foliations with structures defined below. Our purpose is to prove a vanishing theorem for their characteristic classes.

Let M be a smooth n-manifold and TM its tangent bundle. Let E be an integrable smooth $(n-q)$-subbundle of TM. A foliated structure is then given on M by a system of local integrals $\mathcal{F}=\{f_\lambda\}$ of E, which satisfies the atlas condition: for each pair of local submersions $f_\lambda: U_\lambda \to \boldsymbol{R}^q$ and $f_\mu: U_\mu \to \boldsymbol{R}^q$, and for each $x \in U_\lambda \cap U_\mu$, there exists a local diffeomorphism $\gamma_{\mu\lambda}^x$ with $f_\mu = \gamma_{\mu\lambda}^x \circ f_\lambda$ in some neighborhood of x. $\mathcal{F}$ is called a G-foliation if we can take the $\{\gamma_{\mu\lambda}^x\}$ as local automorphisms of some G-structure. The principal object of this paper is a study of G-foliations associated with second order G-structures. Among those structures the conformal or projective ones have been known to be the most significant (cf. Ochiai [19]).

Our main theorem is stated as follows:

MAIN THEOREM. *Let $\mathcal{F}$ be a conformal (resp. projective) foliation of codimension q on a smooth manifold M (see* § 1 *for the precise definition). Suppose $q \geqq 3$ (resp. $q \geqq 2$). Then for the normal bundle $\nu = TM/E$ of $\mathcal{F}$, we have*

$$(*) \qquad \mathrm{Pont}^k(\nu; \boldsymbol{R}) = 0 \qquad \text{for} \quad k > q,$$

where $\mathrm{Pont}^k(\nu; \boldsymbol{R})$ *contained in* $H^k(M; \boldsymbol{R})$ *is the k-th homogeneous part of the Pontrjagin ring generated by the real Pontrjagin classes of ν.*

Note that each riemannian foliation (see § 1) may be regarded as a conformal as well as a projective one. In the course of the proof of the Main Theorem, it can be seen that (*) holds for every riemannian foliation (cf. § 4). This is a theorem of Pasternack [21]. A riemannian foliation is a G-foliation associated with the riemannian structure, a first order G-structure, and is nothing but a foliation with bundle-like metric in the sense of Reinhart [23].

Our theorem may be illustrated as follows. As is well-known, smooth fibre bundles serve as trivial examples of foliations. It is not difficult to verify that this foliation is riemannian and (*) holds trivially. Then our result says that as far as the vanishing of real Pontrjagin classes is concerned, riemannian, conformal and projective foliations are similar to those induced by smooth fibre bundles. However, the phenomena change considerably in the exotic characteristic classes defined by Bott-Haefliger [7] (see the Introduction of Bott [4]). That is, every exotic characteristic class for riemannian foliations is trivial (cf. § 4), while conformal or projective foliations have examples with non-trivial ones. Indeed consider a horocyclic foliation of the geodesic flow on the unit tangent bundle of a compact riemannian manifold of constant negative curvature (see § 1, Example 2). It is a typical example which is conformal and has non-trivial exotic characteristic classes (Godbillon-Vey [11], Yamato [28]). This was also noticed by Roussarie and Thurston.

With the aid of our vanishing theorem, we can investigate the relation between the Pontrjagin numbers of the virtual normal bundle and the behavior near singularities of conformal or projective foliations. Similar problems can be seen in Baum-Bott [1] and Obata [18]. We will take up this problem on another occasion.

This paper is divided into four sections with the following titles:

1. Definitions and examples
2. Cartan connections on the normal bundle
3. Pontrjagin ring of the normal bundle
4. Final miscellany

The manifolds under consideration are always smooth of class C^∞ and paracompact. For the basic knowledge of characteristic classes and of the Chern-Weil theory, see Borel-Hirzebruch [3], Bott [6], Kobayashi-Nomizu [16, vol. II]. For general background material on foliations, see Haefliger [12]. Lawson [17], Reeb [22].

§ 1. Definitions and examples.

We will begin with the precise definition of folitions considered in this paper in the way relevant to our purpose.

DEFINITION 1.1. Let M be a smooth n-manifold. On M a *riemannian* (resp. *conformal, projective*) *foliation* of codimension q $\mathcal{F}=\{B, U_\lambda, f_\lambda, \gamma^x_{\mu\lambda}\}$ is given by the following data:

(1) An auxiliary (not necessarily connected) riemannian q-manifold B*).

(2) An open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of M for some index set Λ and for each λ a smooth submersion $f_\lambda\colon U_\lambda \to B$ called a distinguished mapping.

*) In the case of projective foliation, we may only assume that B is a smooth manifold with a projective structure.

(3) For $x\in U_{\lambda}\cap U_{\mu}$ there is a local isometry (resp. conformal transformation, projective transformation) $\gamma_{\mu\lambda}^{x}$ from a neighborhood of $f_{\lambda}(x)$ onto a neighborhood of $f_{\mu}(x)$ satisfying $f_{\mu}=\gamma_{\mu\lambda}^{x}\circ f_{\lambda}$ on a neighborhood of x.

Notice that if we drop the condition that B is riemannian and require the $\{\gamma_{\mu\lambda}^{x}\}$ only to be local diffeomorphisms of B, then we recover a definition of ordinary foliations (see, for example, [**12**]).

Given a riemannian, conformal or projective foliation $\mathcal{F}$ on M, let E be the subbundle of the tangent bundle TM of M satisfying $E_{x}=\operatorname{Ker}(df_{\lambda}|_{x})$ for $x\in U_{\lambda}$. Then E is tangent to the leaves of $\mathcal{F}$ and the quotient bundle $\nu=TM/E$ is referred to as the normal bundle of $\mathcal{F}$. E is integrable, that is, its space of (local) smooth sections is closed under the bracket operation.

Before proceeding let us examine some examples of these foliations.

EXAMPLE 1 (Riemannian foliations). It should be noted that riemannian foliations are nothing but foliations with bundle-like metrics in the sense of Reinhart [**23**]. Smooth fibre bundles are primary examples of riemannian foliations. One of the distinctive properties of riemannian foliations is that these occur frequently from compact Lie group actions. In fact a Lie group acting by isometries on a riemannian n-manifold M having all orbits of dimension $n-q$ generates a riemannian foliation of codimension q on M (cf. Pasternack [**21**], Reinhart [**23**]). A locally free action of a compact Lie group is of this sort.

There are many other examples of riemannian foliations [**23**], although riemannian ones place severe restrictions on themselves. For instance a riemannian foliation of codimension one has a strong stability [**23**], which the Reeb foliation of S^{3} fails to satisfy.

EXAMPLE 2 (Conformal foliations). Let

$$L=O(q+1,1)/\{\pm\text{identity}\}$$
$$\cong\{X\in GL(q+2;\boldsymbol{R})\,;\,{}^{t}XSX=S\}/\{\pm\text{identity}\},\ \text{where}$$
$$S=\begin{pmatrix}0&0&-1\\0&I_{n}&0\\-1&0&0\end{pmatrix},$$

$$L_{0}=\left\{\begin{pmatrix}a&0&0\\ *&A&0\\ *&*&a^{-1}\end{pmatrix}\in O(q+1,1)\,;\ A\in O(q),\ a\in\boldsymbol{R}^{*}\right\}/\{\pm\text{identity}\}.$$

Define subgroups $K\subset H\subset G$ of the Lie group L respectively by

$$G=\text{the identity component of } L,$$

$$H=L_0\cap G,$$

$$K=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & A & 0\\ 0 & 0 & 1\end{pmatrix};\ A\in SO(q)\right\}.$$

Then a conformal foliation of codimension q is constructed as follows (see Yamato [28], also Tomter [27]). G/K is an open orientable $(2q+1)$-manifold, and G/H is a q-manifold. Note that the manifold $B=G/H$ is diffeomorphic to the unit q-sphere S^q in $\boldsymbol{R}^{q+1}$, and each element of L acts on B as a conformal transformation with respect to the riemannian structure naturally induced from S^q (see, for example, Kobayashi [14], Obata [18]). Set $\tilde{M}=G/K$. $\tilde{M}$ is foliated by the fibres of the fibre bundle $\pi:\tilde{M}=G/K\to B=G/H$. We denote this foliation by $\tilde{\mathcal{F}}$. It is clear that $\tilde{\mathcal{F}}$ is a G-invariant foliation of codimension q on $\tilde{M}$. By a theorem of Borel [2], G admits a discrete subgroup D such that the quotient space $M=D\backslash\tilde{M}$ is a closed orientable $(2q+1)$-manifold and the natural projection $p:\tilde{M}\to M$ is a covering mapping of M with D as the group of covering transformations. In fact, since G is connected and semi-simple, G has a discrete uniform subgroup D' and D' has a proper, normal torsionfree subgroup of finite index D. Since the foliation $\tilde{\mathcal{F}}$ is G-invariant, M has a codimension q foliation $\mathcal{F}$ induced naturally from $\tilde{\mathcal{F}}$. $\mathcal{F}$ is a desired conformal foliation of codimension q on M. To prove this we have only to recall that each element of D is a conformal transformation on B. In fact a system of local submersions $f_\lambda: U_\lambda\to B$ can be defined in a way that each U_λ is an open neighborhood of M evenly covered by $p:\tilde{M}\to M$ and each f_λ is a composition mapping $\pi\circ p^{-1}$ which is determined up to D. Then each $\gamma^x_{\mu\lambda}$ with $f_\mu=\gamma^x_{\mu\lambda}\circ f_\lambda$ is an element of D.

The importance of this example is two-fold. First, Yamato [28] proved that for each integer $q\geqq 1$ all the exotic characteristic classes for this foliation of codimension q which correspond to the canonical generators of $H^{2q+1}(WO_q)$ are non-zero in $H^{2q+1}(M;\boldsymbol{C})$ (see also Fuks [10], Haefliger [13]). Second, this foliation may be regarded as a horocyclic foliation of the geodesic flow on the unit tangent bundle of a compact riemannian $(q+1)$-manifold of constant negative curvature [27, 28].*)

EXAMPLE 3 (Projective foliations). By a slight modification of the procedure of Example 2 we can give a family of projective foliations. Let

*) The horocyclic foliation is the integration of the distribution given by the geodesic flow vector and the stable vectors.

$$L = PGL(q+1\,;\,\boldsymbol{R})$$
$$= GL(q+1\,;\,\boldsymbol{R})/\text{center},$$
$$L_0 = \left\{\begin{pmatrix} A & 0 \\ * & a \end{pmatrix} \in GL(q+1\,;\,\boldsymbol{R})\,;\ A \in GL(q\,;\,\boldsymbol{R}),\ a \in \boldsymbol{R}^*\right\}/\text{center}\,.$$

Define subgroups $K \subset H \subset G$ of L by

$$G = \text{the identity component of } L,$$
$$H = L_0 \cap G,$$
$$K = \left\{\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix};\ A \in SO(q)\right\}.$$

Then the manifold $\tilde{M} = G/K$ has a G-invariant foliation $\tilde{\mathcal{F}}$ whose leaves are the fibres of the fibre bundle $\tilde{M} = G/K \to G/H$. Notice that the manifold $B = G/H$ is diffeomorphic to a real projective q-space $\boldsymbol{R}P^q$ and each element of L operates on B as a projective transformation with respect to the riemannian structure induced naturally from $\boldsymbol{R}P^q$ [**14**]. Let D be a discrete subgroup of G such that $M = D \backslash \tilde{M}$ is a closed manifold [**2**]. Then a projective foliation $\mathcal{F}$ of codimension q on M is induced naturally from $\tilde{\mathcal{F}}$.

Another simple construction of a family of projective foliations can be obtained as a generalization of Roussarie's example [**6**, **11**]. These are foliations on $\Gamma \backslash SL(q+1\,;\,\boldsymbol{R})$, where Γ is a discrete uniform subgroup [**2**]. The foliation is naturally induced from the fibering $G \to G/H$.

The above two families of foliations also provide examples of foliations with non-trivial exotic characteristic classes (see [**10**, **13**]).

REMARK 1.2. It is obvious that every riemannian foliation is a conformal one as well as a projective one. However, as will be seen in § 4, the converse is generally not true. In fact those foliations described in Example 2 and 3 do not admit any structure of a riemannian foliation.

Finally note a rather trivial fact that every smooth foliation of codimension one may be regarded as a conformal as well as a projective foliation. But we do not know how this helps the study of codimension one foliations.

§ 2. Cartan connections on the normal bundle.

The goal of this section is roughly speaking to introduce Cartan connections on the normal bundles of the foliations under consideration. Since the theory of Cartan connections seems not so familiar as that of linear connections, the exposition of this section is more self-contained than is usually expected.

For the complete development, see Kobayashi [14], Kobayashi-Nagano [15], Ogiue [20]. A slightly different approach can be found in Tanaka [24, 25].

For our purpose we construct bundles of higher order contact, in particular, of second order contact. First some general concepts are recalled. Let N be a smooth n-manifold. Let U and V be neighborhoods of the origin 0 in $\boldsymbol{R}^m$. Two mappings $f: U \to N$ and $g: V \to N$ such that $x=f(0)=g(0)$ give rise to the same r-jet at x if they have the same partial derivatives up to order r at 0. The r-jet at x given by f is denoted by $j_x^r(f)$. If f is a diffeomorphism of a neighborhood of 0 in $\boldsymbol{R}^n$ onto an open subset of N, then the r-jet $j_x^r(f)$ at x is called an *r-frame* at $x=f(0)$. Clearly, a 1-frame is an ordinary linear frame.

Let $\mathcal{F}=\{B, U_\lambda, f_\lambda, \gamma_{\mu\lambda}^x\}$ be a conformal or projective foliation of codimension q on a smooth n-manifold M. We first construct the bundle of r-frames on B. Define $G^r(q)$ by the set of r-frames $j_0^r(g)$ of $\boldsymbol{R}^q$ at 0, where g is a diffeomorphism from a neighborhood of $0 \in \boldsymbol{R}^q$ onto a neighborhood of $0 \in \boldsymbol{R}^q$. Then $G^r(q)$ is a group with multiplication defined by

$$j_0^r(g) \cdot j_0^r(g') = j_0^r(g \circ g') .$$

The set of r-frames of B, denoted by $P^r(B)$, is a principal bundle over B with projection π, $\pi(j_x^r(f)) = x = f(0)$, and with group $G^r(q)$, which acts on $P^r(B)$ on the right by

$$j_x^r(f) \cdot j_0^r(g) = j_x^r(f \circ g) \qquad \text{for } \ j_x^r(f) \in P^r(B),\ j_0^r(g) \in G^r(g) .$$

$P^r(B)$ is called the bundle of r-frames of B. Note that $P^1(B)$ is nothing but the bundle of linear frames of B with group $G^1(q) = GL(q; \boldsymbol{R})$. It is sufficient for our purpose to consider only $P^2(B)$ and $P^1(B)$.

Now we consider the foliated manifold M. We will define the bundle of transversal r-frames of M. Let U be a neighborhood of the origin 0 in $\boldsymbol{R}^q$ and $(x^1, \cdots, x^q)$ the natural coordinate system in $\boldsymbol{R}^q$. Let $f: U \to M$ be a mapping transversal to the foliation $\mathcal{F}$ in the sense that for each $x \in U$ and for each distinguished mapping $f_\lambda: U_\lambda \to B$ with $y=f(x) \in U_\lambda$, $f_\lambda \circ f$ is a submersion. Take a local coordinate system $(y^1, \cdots, y^{n-q}, y^{n-q+1}, \cdots, y^n)$ of M distinguished by the f_λ, that is, with respect to this coordinate system f_λ is given by the mapping

$$(y^1, \cdots, y^{n-q}, y^{n-q+1}, \cdots, y^n) \longrightarrow (y^{n-q+1}, \cdots, y^n) .$$

Let $g: V \to M$ be a mapping transversal to $\mathcal{F}$ defined on a neighborhood V of 0 in $\boldsymbol{R}^q$ such that $g(0)=y=f(0)$. Then the r-jets $u=j_y^r(f)$ and $v=j_y^r(g)$ at y defined by f and g respectively are expressed as follows:

$$\begin{aligned} u &: y^A \circ f(x) = u^A + \sum u_{j_1}^A x^{j_1} + \cdots + \sum u_{j_1 \cdots j_r}^A x^{j_1} \cdots x^{j_r} , \\ v &: y^A \circ g(x) = v^A + \sum v_{j_1}^A x^{j_1} + \cdots + \sum v_{j_1 \cdots j_r}^A x^{j_1} \cdots x^{j_r} , \end{aligned} \tag{2.1}$$

where $u^A_{j_1\cdots j_k}$ and $v^A_{j_1\cdots j_k}$ are symmetric with respect to $j_1, \cdots, j_k$ and $1\leqq A\leqq n$, $1\leqq j_1, \cdots, j_k, \cdots, j_r\leqq q$.

DEFINITION 2.1. Two mappings $f: U\to M$ and $g: V\to M$ such that $y=f(0)=g(0)$ which are transversal to a codimension q foliation $\mathcal{F}$ is said to define the same *transversal r-frame* at x if in the above expression (2.1) we have

$$u^A=v^A,\quad u^\alpha_{j_1j_2}=v^\alpha_{j_1j_2},\quad u^\alpha_{j_1\cdots j_r}=v^\alpha_{j_1\cdots j_r},$$

where $1\leqq A\leqq n$, $n-q+1\leqq\alpha\leqq n$ and $1\leqq j_1, \cdots, j_r\leqq q$. The transversal r-frame of M defined by f at y is written as $\tilde{j}^r_y(f)$.

Notice that if $(y^1, \cdots, y^{n-q}, y^{n-q+1}, \cdots, y^n)$ and $(z^1, \cdots, z^{n-q}, z^{n-q+1}, \cdots, z^n)$ are two distinguished local coordinate systems defined by local submersions $f_\lambda: U_\lambda\to B$ and $f_\mu: U_\mu\to B$ respectively, then the functions giving the change of coordinates

$$z^A=z^A(y^1, \cdots, y^n)\qquad \text{on}\quad U_\lambda\cap U_\mu$$

must satisfy the equations

$$\partial z^\alpha/\partial y^a=0\qquad \text{for}\quad 1\leqq a\leqq n-q<\alpha\leqq n$$

(see, for example, [17]). Thus the definition of a transversal r-frame is independent of the choice of distinguished coordinate systems and hence is well-defined.*)

The set of transversal r-frames on M, denoted by $P^r(\nu)$, is a principal bundle over M with projection π, $\pi(\tilde{j}^r_y(f))=y=f(0)$, and with group $G^r(q)$. Here $G^r(q)$ is the set of r-frames $j^r_0(g)$ of $\boldsymbol{R}^q$ at 0, which acts on $P^r(\nu)$ on the right by

$$\tilde{j}^r_y(f)\cdot j^r_0(g)=\tilde{j}^r_y(f\circ g)\qquad \text{for}\quad \tilde{j}^r_y(f)\in P^r(\nu),\ j^r_0(g)\in G^r(q).$$

$P^r(\nu)$ is called the bundle of transversal r-frames on M. In particular $P^1(\nu)$ is regarded as the bundle of linear frames of the normal bundle ν of the foliation $\mathcal{F}$ with group $G^1(q)=GL(q\,;\boldsymbol{R})$.

From the construction it is almost obvious that we have the following lemma, whose proof is routine.

LEMMA 2.2. *Let $\mathcal{F}$ be a foliation as above. Then for each distinguished mapping $f_\lambda: U_\lambda\to B$*

$$P^r(\nu)\,|\,U_\lambda=f_\lambda^{-1}(P^r(B)\,|_{f_\lambda(U_\lambda)})$$

holds, where $f_\lambda^{-1}(\ )$ denotes the bundle induced by f_λ.

REMARK 2.3. It should be noticed that in the above construction the assumption that a given foliation $\mathcal{F}$ is conformal or projective is irrelevant.

*) By a coordinate free expression, $\tilde{j}^r_y(f)$ can be written as $j^r_y(f_\lambda\circ f)$. The well-definedness follows directly from this expression.

Indeed the construction of the above bundles can be done for any foliation, and Lemma 2.2 is also true in that case.

From now on we are mainly interested in $P^2(\nu)$ and $P^1(\nu)$.

Let L and L_0 be as in Example 2 or 3 in § 1. *L_0 can be considered as a subgroup of the group $G^2(q)$ defined above.* The correspondence is given as follows. Let o denote the origin of the homogeneous space L/L_0. We consider each element g of L_0 as a transformation of L/L_0 leaving o fixed, and thus as a diffeomorphism from a neighborhood of $0\in \boldsymbol{R}^q$ onto a neighborhood of $0\in \boldsymbol{R}^q$. It can be easily verified that $j_0^2(g)=$identity if and only if $g=$identity, that is, every element of L_0 is determined by its partial derivatives of order 1 or 2 at o (cf. [**15**, **20**]). Hence L_0 is isomorphic to the group of 2-jets $\{j_0^2(g)\,;\, g\in L_0\}$. Identify $\boldsymbol{R}^q$ with the subspace V of the Lie algebra $\mathfrak{l}$ of L defined respectively by

$$V=\left\{\begin{pmatrix} 0 & {}^t v & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}\right\} \text{ in the case of Example 2},$$

$$V=\left\{\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\right\} \text{ in the case of Example 3}.$$

Then the mapping

$$\boldsymbol{R}^q = V \xrightarrow{\exp} L \xrightarrow{\text{proj}} L/L_0$$

gives a diffeomorphism from a neighborhood of $0\in \boldsymbol{R}^q$ onto a neighborhood of $o\in L/L_0$, a local coordinate system around $o\in L/L_0$. With respect to this coordinate system, each 2-jet $j_0^2(g)$ is an element of the group $G^2(q)$. Therefore L_0 can be considered as a subgroup of $G^2(q)$.

Then a conformal or projective structure is defined as follows.

DEFINITION 2.4. Let N be a smooth q-manifold and $P^2(N)$ the bundle of 2-frames on N with group $G^2(q)$. A principal subbundle P of $P^2(N)$ with group $L_0\subset G^2(q)$ is called a *conformal* or *projective structure* on N according as whether L_0 is as in Example 2 or Example 3 in § 1.

Let P be a conformal or projective structure. We will introduce a conformal or projective connection on P respectively. This can be done in a unified manner from the viewpoint of Cartan connections.

Since L_0 acts on P on the right, every element A of the Lie algebra $\mathfrak{l}_0$ of L_0 defines a vertical vector field on P, called the fundamental vector field corresponding to A (see [**16**; vol. I]). This vector field will be denoted by A^*. For each element $a\in L_0$, the right translation by a acting on P will be denoted by R_a.

DEFINITION 2.5. A *Cartan connection* in the bundle P is a 1-form ω on P with values in the Lie algebra $\mathfrak{l}$ of L satisfying the following conditions:

(a) $\omega(A^*)=A$ for every $A\in\mathfrak{l}_0$,

(b) $(R_a)^*\omega=ad(a^{-1})\omega$ for every $a\in L_0$, where $ad(a^{-1})$ is the adjoint action of a^{-1} on $\mathfrak{l}$,

(c) $\omega(X)\neq 0$ for every non-zero vector X of P.

Note that the condition (c) implies an absolute parallelizability of P. Moreover

REMARK 2.6. A Cartan connection in P is not a connection in P in the usual sense, for ω is not $\mathfrak{l}_0$-valued (cf. [**16**; vol. I]). It can be however considered as a connection in a larger bundle P^L obtained by enlarging the structure group of P to L, namely,

$$P^L=P\times_{L_0}L\,.$$

Then P is a subbundle of P^L and *a Cartan connection in P can be uniquely extended to a usual connection form on P^L*, also denoted by ω. In fact, if Y is a vector of P^L, then $Y=R_{a*}(X)+Z$ where X is a vector tangent to P and $a\in L$ and, consequently, Z is a vector tangent to a fibre of P^L so that Z can be extended to a unique fundamental vector field A^* of P^L with $A\in\mathfrak{l}$. We then set

$$\omega(Y)=ad(a^{-1})(\omega(X))+A\,.$$

It is an easy matter to verify that $\omega(Y)$ is well-defined and is a connection form on P^L.

Then the following have been well-known [**14**, **15**, **20**].

PROPOSITION 2.7. *Let P be a conformal (resp. projective) structure on a smooth q-manifold. Suppose $q\geqq 3$ (resp. $q\geqq 2$). Then there is a unique Cartan connection on P called the normal conformal (resp. normal projective) connection.*

Since the precise definition of the normal conformal or normal projective connection is not used in the subsequent discussion, we omit the description but emphasize that their uniqueness is essential and implies the property (3) of the following Proposition 2.8.

PROPOSITION 2.8. *Let N be a riemannian q-manifold and f a (local) conformal (resp. projective) transformation on N. Then*

(1) *N has a conformal structure as well as a projective structure, each of which is induced naturally from a riemannian structure of N.*

(2) *The isomorphism f_* of the bundle $P^2(N)$ of 2-frames induced from f preserves a conformal (resp. projective) structure P on N in* (1).

(3) *Assume that $q\geqq 3$ (resp. $q\geqq 2$), then f^* (restricted to P) preserves the normal conformal (resp. normal projective) connection ω on P.*

We are now in a position to prove the theorem fundamental to our proof of the Main Theorem.

THEOREM 2.9. *Let* $\mathcal{F}=\{B, f_\lambda, U_\lambda, \gamma^x_{\mu\lambda}\}$ *be a conformal (resp. projective) foliation of codimension* q *on a smooth* n*-manifold* M. *Suppose* $q\geqq 3$ *(resp.* $q\geqq 2$*). Then the following hold.*

(1) *There is the normal conformal (resp. normal projective) connection* ω *on a natural conformal (resp. projective) structure* $P(B)$ *of* B.

(2) *There is induced a subbundle* $P(\nu)$ *of* $P^2(\nu)$ *of the transversal 2-frames with group* L_0, *which is lifted in the sense that for each distinguished mapping* $f_\lambda: U_\lambda\to B$ *we have*

$$P(\nu)|_{U_\lambda}=f_\lambda^{-1}(P(B)|_{f_\lambda(U_\lambda)}),$$

where $f_\lambda^{-1}(\ )$ *denotes the bundle induced by* f_λ.

(3) *There is induced a projectable connection form* $\tilde{\omega}$ *on the extended bundle* $P(\nu)^L=P(\nu)\times_{L_0}L$, *that is, for each distinguished mapping* $f_\lambda: U_\lambda\to B$ *we have*

$$\tilde{\omega}=f_\lambda{}^*\omega,$$

where ω *is a connection form on the extended bundle* $P(B)^L=P(B)\times_{L_0}L$ *and is obtained naturally from the normal conformal (resp. normal projective) connection on* $P(B)$ *in a way as in Remark* 2.6, *and* $f_\lambda{}^*$ *is the mapping from the* 1*-forms on* $P(B)^L$ *to that of* $P(\nu)^L$ *induced naturally from* f_λ.

PROOF. (1) Since B is a riemannian q-manifold with q sufficiently large, this is a consequence of Proposition 2.8 (1) combined with Proposition 2.7.

(2) It is known in Lemma 2.2 that for each distinguished mapping f_λ we have

$$P^2(\nu)|_{U_\lambda}=f_\lambda^{-1}(P^2(B)|_{f_\lambda(U_\lambda)}).$$

Since $\mathcal{F}$ is a conformal (resp. projective) foliation, each $\gamma^x_{\mu\lambda}$ is a conformal (resp. projective) transformation on B. Thus the isomorphism $\gamma^x_{\mu\lambda *}$ induced from $\gamma^x_{\mu\lambda}$ on $P^2(B)$ preserves a natural conformal (resp. projective) structure $P(B)$. Hence on $U_\lambda\cap U_\mu$

$$\begin{aligned} f_\mu^{-1}(P(B)|_{f_\mu(U_\lambda\cap U_\mu)}) &= f_\lambda^{-1}(\gamma^{x\,-1}_{\mu\lambda}(P(B)|_{f_\mu(U_\lambda\cap U_\mu)})) \\ &= f_\lambda^{-1}(P(B)|_{f_\lambda(U_\lambda\cap U_\mu)}) \end{aligned}$$

holds. From this it is not difficult to see that the pull back data $\{f_\lambda^{-1}(P(B)|_{f_\lambda(U_\lambda)})\}$ define a subbundle $P(\nu)$ of $P^2(\nu)$ with group L_0.

(3) First we prove that the normal conformal (resp. normal projective) connection ω on $P(B)$ can be lifted to a 1-form $\tilde{\omega}$ on $P(\nu)$. By virtue of Proposition 2.8 (3), each $\gamma^x_{\mu\lambda}{}^*$ preserves ω, since $\gamma^x_{\mu\lambda}$ is a local conformal (resp. projective) transformation on B. Then on $U_\lambda\cap U_\mu$ we see that

$$f_\mu{}^*(\omega)=f_\lambda{}^*(\gamma^x_{\mu\lambda}{}^*\omega)=f_\lambda{}^*(\omega).$$

Hence ω is pulled back to a 1-form $\tilde{\omega}$ on $P(\nu)$, and $\tilde{\omega}=f_\lambda^*\omega$ holds for each f_λ. Then we extend ω and $\tilde{\omega}$ to 1-forms on $P(B)^L$ and $P(\nu)^L$ respectively in the same way as in Remark 2.6, which are also denoted by ω and $\tilde{\omega}$. We extend the induced mappings $f_{\lambda *}$ and f_λ^* naturally outside the subbundles $P(\nu)$ and $P(B)$ of the corresponding bundles $P(\nu)^L$ and $P(B)^L$. It is now obvious that for each thus extended mapping f_λ^* we have

$$\tilde{\omega}=f_\lambda^*\omega\,.$$

This completes the proof. Q. E. D.

§ 3. Pontrjagin ring of the normal bundle.

The aim of this section is to prove the Main Theorem stated in the introduction. Our proof is based on the Chern-Weil construction of characteristic classes from the curvature of a connection.

Let G be a Lie group with finitely many components. We denote by B_G the classifying space for G. In the following only principal bundles with group G over a smooth manifold M of fixed dimension n are considered, so it can be assumed that B_G is the N-classifying space by taking N large. For later convenience we will always assume this for every B_G under consideration.

Our first goal is to compare the real cohomology of B_L with that of $B_{O(q+1)}$ or $B_{GL(q+1)}$, where L is as in Example 2 or 3 in § 1. We treat the conformal case and the projective case separately, for there are some differences between them in the details.

(I) Conformal case

Let $\mathcal{F}$ be a conformal foliation of codimension q on M with normal bundle ν. Let L and L_0 be as in Example 2 in § 1. Our first assertion is that there are isomorphisms

$$H^*(B_{O(q)}\,;\boldsymbol{R})\cong H^*(B_{L_0}\,;\boldsymbol{R})\,,$$

$$H^*(B_{O(q+1)}\,;\boldsymbol{R})=H^*(B_L\,;\boldsymbol{R})\,.$$

To see this, let $i: O(q)\to L_0$ and $j: O(q+1)\to O(q+1,1)$ be the natural injections. Then we have fibrations and a bundle map such that the following diagram is commutative.

$$\begin{array}{ccccc} O(q) & \xrightarrow{\ i\ } & L_0 & \longrightarrow & L_0/O(q) \\ \downarrow & & \downarrow & & \downarrow \\ O(q+1) & \xrightarrow{\ j\ } & L & \longrightarrow & L/O(q+1)\,. \end{array}$$

Note that there are diffeomorphisms

$$L_0/O(q) \cong \boldsymbol{R}^{q+1}, \qquad L/O(q+1) \cong \boldsymbol{R}^{q+1},$$

which imply that i and j are homotopy equivalences, for $\boldsymbol{R}^{q+1}$ is contractible. Hence we have the homotopy equivalences

$$i_* : B_{O(q)} \cong B_{L_0}, \qquad j_* : B_{O(q+1)} \cong B_L,$$

from which our first assertion follows.

Let $P(\nu)$ be the principal bundle defined in Theorem 2.9 (2) and $P(\nu)^L$ its extended bundle $P(\nu)\times_{L_0}L$. Let $f_\nu : M \to B_{GL(q)}$, $f_{\nu\oplus 1} : M \to B_{GL(q+1)}$, $f_{L_0} : M \to B_{L_0}$ and $f_L : M \to B_L$ be the classifying maps of the principal bundles ν, $\nu\oplus 1$, $P(\nu)$ and $P(\nu)^L$ respectively. Then it follows from the construction of $P(\nu)^L$ (cf. Tanaka [25]) that the following diagram is homotopy commutative

$$\begin{tikzcd}
 & B_{L_0} \arrow[r] & B_L \\
M \arrow[ur, "f_{L_0}"] \arrow[r] \arrow[dr, "f_\nu"'] \arrow[urr, bend left=50, "f_L"] \arrow[drr, bend right=50, "f_{\nu\oplus 1}"'] & B_{O(q)} \arrow[u, "i_*", "\simeq"'] \arrow[d, "j_*"', "\simeq"] \arrow[r] & B_{O(q+1)} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
 & B_{GL(q)} \arrow[r] & B_{GL(q+1)}
\end{tikzcd}$$

where the vertical arrows are homotopy equivalences and three horizontal arrows on the right are mappings induced by the respective natural inclusions.*)

Our second assertion is as follows. For each Pontrjagin class $p_i(\nu)$ of ν in $H^i(M : \boldsymbol{R})$, there exists an element $\tilde{p}_i$ in $H^i(B_L ; \boldsymbol{R})$ such that $f_L{}^*\tilde{p}_i$ coincides with $p_i(\nu)$. To prove this, recall that

$$H^*(B_{GL(q+1)} ; \boldsymbol{R}) \cong H^*(B_{O(q+1)} ; \boldsymbol{R}) \cong H^*(B_L ; \boldsymbol{R}).$$

Then it is easy to find an element $\tilde{p}_i \in H^*(B_L ; \boldsymbol{R})$ with the property

*) Without reference to the construction, the commutativity of the diagram can be seen by the fact that $P(\nu)$ is a reduction of the bundle $P^2(\nu)$ and that $O(q)$ is a deformation retract of $G^2(q)$. There exists a unique reduction of $P^2(\nu)$ to an $O(q)$-principal bundle.

$$p_i(\nu)=p_i(\nu\oplus 1)=f_{\nu\oplus 1}{}^*\tilde{p}_i=f_L{}^*\tilde{p}_i\,,$$

by virtue of the above diagram.

(II) Projective case

Let $\mathcal{F}$ be a projective foliation of codimension q on M with normal bundle ν. Let L and L_0 be as in Example 3 in § 1. Then we have the following commutative diagram

$$\begin{tikzcd} GL(q) \arrow[r, "i"] \arrow[d] & L_0 \arrow[r] \arrow[d] & L_0/GL(q) \\ PGL(q+1) \arrow[r, equal] & L & \end{tikzcd}$$

where $i: GL(q)\to L_0$ is the natural injection and the horizontal sequence is a fibration. The space $L_0/GL(q)$ is diffeomorphic to $\boldsymbol{R}^{q+1}$ and hence is contractible. Therefore the homomorphism i is a homotopy equivalence

$$i: GL(q)\cong L_0\,,$$

which then induces the homotopy equivalence

$$i_*: B_{GL(q)}\cong B_{L_0}\,.$$

Note that the center of $GL(q+1;\boldsymbol{R})$ is the subgroup defined by

$$\{aI_{q+1}\,;\, a\in\boldsymbol{R}^*\},$$

where $\boldsymbol{R}^*$ denotes the multiplicative group of non-zero reals. Since $\boldsymbol{R}^*$ is homotopy equivalent to $\boldsymbol{Z}_2$, the natural map

$$B_{GL(q+1)}\longrightarrow B_{PGL(q+1)}$$

induces the following isomorphism in real cohomology

$$H^*(B_L\,;\boldsymbol{R})\cong H^*(B_{GL(q+1)}\,;\boldsymbol{R})\,.$$

Consider the following diagram

$$\begin{tikzcd}
& B_{L_0} \arrow[r] & B_L \arrow[d, equal] \\
M \arrow[ur, "f_{L_0}"] \arrow[r] \arrow[dr, "f_\nu"'] \arrow[urr, bend left=50, "f_L"] \arrow[drr, bend right=50, "f_{\nu\oplus 1}"'] & B_{GL(q)} \arrow[u, "\simeq"'] \arrow[r] \arrow[d, equal] & B_{PGL(q+1)} \\
& B_{GL(q)} \arrow[r] & B_{GL(q+1)} \arrow[u]
\end{tikzcd}$$

where f_ν, $f_{\nu\oplus 1}$, f_{L_0} and f_L are classifying maps of the principal bundles ν, $\nu\oplus 1$, $P(\nu)$ and $P(\nu)^L$ respectively as in Case (I). On account of the construction of the principal bundle $P(\nu)^L$ defined in Theorem 2.9 (2) (cf. Tanaka [**24**]), the diagram is homotopy commutative. Hence, for each Pontrjagin class $p_i(\nu)$ of ν in $H^i(M; \boldsymbol{R})$, there exists an element $\tilde{p}_i$ in $H^i(B_L; \boldsymbol{R})$ with the property

$$p_i(\nu) = p_i(\nu\oplus 1) = f_{\nu\oplus 1}{}^* \tilde{p}_i = f_L{}^* \tilde{p}_i .$$

Thus we have shown the following

PROPOSITION 3.1. *Let $\mathcal{F}$ be a conformal or projective foliation on M with normal bundle ν. Let $P(\nu)^L$ be the principal bundle defined in Theorem* 2.9 (2) *and $f_L: M \to B_L$ its classifying map. Then, for each Pontrjagin class $p_i(\nu)$ of ν in $H^i(M; \boldsymbol{R})$, there exists an element $\tilde{p}_i$ in $H^i(B_L; \boldsymbol{R})$ such that $p_i(\nu)$ coincides with $f_L{}^* \tilde{p}_i$.*

Let $\mathrm{Pont}^i(P(\nu)^L; \boldsymbol{R})$ denote the i-dimensional homogeneous part of the Pontrjagin ring of $P(\nu)^L$ generated by the elements $p_j(P(\nu)^L) = f_L{}^* \tilde{p}_j$, $\tilde{p}_j \in H^j(B_L; \boldsymbol{R})$, for $j \leqq i$. Then Proposition 3.1 claims that

$$\mathrm{Pont}^i(\nu; \boldsymbol{R}) \cong \mathrm{Pont}^i(P(\nu)^L; \boldsymbol{R}) . \tag{3.1}$$

This is our second goal.

Before going into the proof of the Main Theorem, recall some basic facts about the Chern-Weil theory. Let $I(L)$ be the set of $ad(L)$-invariant symmetric multilinear mappings on the Lie algebra $\mathfrak{l}$ of L. $I(L)$ is a commutative graded algebra over $\boldsymbol{R}$, whose elements are called characteristic maps. $I(L)$ gives information about the real cohomology of B_L. This is done by the (N-universal) Chern-Weil homomorphism

$$w: I(L) \longrightarrow H^*(B_L; \boldsymbol{R}).$$

Consider the following sequence

$$I(L) \xrightarrow{w} H^*(B_L; \boldsymbol{R}) \xrightarrow{i^*} H^*(B_{O(q)}; \boldsymbol{R})$$

where i^* is the induced mapping from the natural inclusion $i: O(q) \rightarrow L$. Our third assertion is that $i^* \circ w$ is a surjection. In fact, we have the following diagram

$$\begin{tikzcd} & H^*(B_L; \boldsymbol{R}) \arrow[ld, "f_L^*"'] \arrow[r, "w"] \arrow[d, "i^*"] & I(L) \arrow[d, "r"] \\ H^*(M; \boldsymbol{R}) & H^*(B_{GL(q)}; \boldsymbol{R}) \cong H^*(B_{O(q)}; \boldsymbol{R}) \arrow[l, "f_\nu^*"] \arrow[r, "w"] & I(O(q)) \end{tikzcd}$$

where r denotes the restriction mapping. First, notice that the square on the right is commutative. This follows from standard properties of N-universal connections (cf. Narasimhan and Ramanan [**29**]). It is well-known that the Chern-Weil homomorphism

$$w: I(O(q)) \longrightarrow H^*(B_{O(q)}; \boldsymbol{R})$$

is an isomorphism. More precisely, the i-th real Pontrjagin class $p_i \in H^{4i}(B_{O(q)}; \boldsymbol{R})$ is given by w as

$$p_i = w(P_i),$$

where $P_i \in I(O(q))$ is the characteristic map defined by the formula

$$\det(\lambda I - (1/2\pi)A) = \sum_{k=0}^{[n/2]} P_k(\underbrace{A, \cdots, A}_{2k})\lambda^{n-2k} + Q,$$

Q denoting the terms involving the n-odd powers of λ. It is now easy to see that $P_i \in I(O(q))$ lies in the image $r(I(L))$ and $i^* \circ w$ is surjective.

Recall that from our previous discussion the triangle on the left in the above diagram is commutative. The fundamental consequence of the Chern-Weil theory is then summerized as follows:

PROPOSITION 3.2. *For each element $p_i(\nu) \in \mathrm{Pont}^i(\nu; \boldsymbol{R})$, there exists a characteristic map $\phi \in I(L)$ of degree $i/2$ such that*

$$p_i(\nu) = [\phi((1/2\pi)\tilde{\Omega})],$$

where $\tilde{\Omega}$ is the curvature form of a connection on $P(\nu)^L$ and $[\cdot]$ denotes cohomology class in $H^*(M; \boldsymbol{R})$.

As the final goal, we will now prove our Main Theorem.

PROOF OF MAIN THEOREM.

First recall the data of the theorem. $\mathcal{F}=\{B, U_\lambda, f_\lambda, \gamma^x_{\mu\lambda}\}$ is a conformal (resp. projective) foliation of codimension $q\geqq 3$ (resp. $\geqq 2$) on M with normal bundle ν.

Let ω and $\tilde{\omega}$ be the connection forms on $P(B)^L$ and $P(\nu)^L$ respectively as in (3) of Theorem 2.9. Recall that $\tilde{\omega}=f_\lambda{}^*\omega$ holds for each distinguished mapping f_λ. Then we have by naturality of the exterior derivative

$$\tilde{\Omega}=f_\lambda{}^*\Omega\,,$$

where $f_\lambda{}^*$ is the natural induced mapping on 2-forms. Thus, for any characteristic map $\phi\in I(L)$ and each f_λ

$$\phi((1/2\pi)\tilde{\Omega})\,|_{U_\lambda}=f_\lambda{}^*(\phi((1/2\pi)\Omega))\,. \tag{3.2}$$

Note here that $\phi((1/2\pi)\Omega)$, and hence $\phi((1/2\pi)\tilde{\Omega})$ vanishes by (3.2), if the degree of $\phi>q/2$, since $\dim B=q$. Consequently, by virtue of Proposition 3.2,

$$\mathrm{Pont}^k(\nu\,;\boldsymbol{R})=0 \qquad \text{for} \quad k>q\,.$$

This completes the proof. Q. E. D.

§ 4. Final miscellany.

1. Let $\mathcal{F}$ be a smooth codimension q foliation with normal bundle ν. Then the Bott vanishing theorem or integrability criterion [**4**, **5**, **6**] asserts that

$$\mathrm{Pont}^k(\nu\,;\boldsymbol{R})=0 \qquad \text{for} \quad k>2q\,.$$

It is recently known that this gives a sharp bound on dimensions. For details, see Thurston [**26**].

2. Bott and Haefliger [**7**] constructed new computable characteristic classes, called secondary or exotic, of foliations (see also Bott [**6**], Haefliger [**12**]). We gave in § 1 some examples of foliations with non-trivial such characteristic classes. On the other hand, we have a theorem, which may be well-known;

THEOREM 4.1. *Let $\mathcal{F}$ be a riemannian foliation on a smooth manifold M. Then every exotic characteristic class for $\mathcal{F}$ is trivial.*

Before the proof, we will recall the Bott-Haefliger construction. For details, see [**6**]. Consider the graded differential algebra over $\boldsymbol{R}$

$$WO_q=E(u_1, u_3, \cdots, u_{2[q/2]-1})\otimes\boldsymbol{R}_q[c_1, \cdots, c_q]$$

with $d(u_i\otimes 1)=c_i$ for odd i and $d(1\otimes c_i)=0$ for all i (degree $u_i=2i-1$, degree $c_i=2i$), where E denotes an exterior algebra and $\boldsymbol{R}_q[c_1, \cdots, c_q]$ is the polynomial algebra in the c_i's modulo elements of total degree $>2q$. Then, given a smooth codimension q foliation $\mathcal{F}$ on M, there is a graded algebra homomorphism

$$\lambda_{\mathcal{F}}: WO_q \longrightarrow \Lambda^*(M)$$

into the de Rham algebra on M, defined in terms of two connections, called metric*) and basic**) respectively, on the normal bundle ν of $\mathcal{F}$ and unique up to chain homotopy. Its key feature is that $\lambda_{\mathcal{F}}$ restricted to $E(u_1, u_3, \cdots, u_{2[q/2]-1})$ is a trivial mapping if two such connections are identical. The elements of $\lambda_{\mathcal{F}}(H^*(WO_q))$ in $H^*(M; \boldsymbol{C})$ except the Pontrjagin classes of ν are called the exotic characteristic classes for $\mathcal{F}$.

It is now clear that to prove Theorem 4.1 it suffices to show the following

LEMMA 4.2. *Let $\mathcal{F}=\{B, U_\lambda, f_\lambda, \gamma^x_{\mu\lambda}\}$ be a riemannian foliation of codimension q on M. Then on the normal bundle ν of $\mathcal{F}$ we can choose a metric connection ω_0 and a basic connection ω_1 in such a way that ω_0 and ω_1 are identical.*

PROOF. Let g and ω be a riemannian metric and a riemannian connection on B respectively. Since g and ω are both invariant under local isometries on B, in particular under the $\gamma^x_{\mu\lambda}$, it is easily seen that they are pulled back to a metric on ν and a metric connection on ν respectively. We take this metric connection as ω_0. Now, remark that by definition a basic connection on ν is the connection given by glueing together by a partition of unity the connections induced on each restricted bundle $\nu|U_\lambda$ from a connection on B by the submersion $f_\lambda: U_\lambda \to B$ (see Bott [4]; it should be noted that Bott's definition in [6] is equivalent to that in [4]). Hence the above ω_0 is also a basic connection on ν. This completes the proof.

3. Let $\mathcal{F}=\{B, U_\lambda, f_\lambda, \gamma^x_{\mu\lambda}\}$ be a riemannian foliation of codimension q on M. Although $\mathcal{F}$ is considered as a conformal as well as a projective foliation, the construction of the bundle of 2-frames is extra baggage in this case, at least to obtain the vanishing theorem. Indeed, in the course of the proof of Lemma 4.2, we have seen that a riemannian connection ω on B can be pulled back to a metric connection ω_0 of the normal bundle ν of $\mathcal{F}$. With this projectable connection ω_0 it is immediately verified by the Chern-Weil construction that

$$\mathrm{Pont}^k(\nu; \boldsymbol{R})=0 \qquad \text{for} \quad k>q\,.$$

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*) A metric connection ∇^0 on the normal bundle $\nu=T(M)/E$ with a metric g is one such that

$$dg(Z_1, Z_2)(X)=g(\nabla^0_X(Z_1), Z_2)+g(Z_1, \nabla^0_X(Z_2)),$$

for any $X\in\Gamma(T(M))$, $Z_1, Z_2\in\Gamma(\nu)$.

**) A basic connection ∇^1 on ν is one such that

$$\nabla^1_X(Z)=\pi[X, \tilde{Z}],$$

for any $X\in\Gamma(E)$, $Z\in\Gamma(\nu)$, where $\tilde{Z}\in\Gamma(T(M))$ is such that $\pi(\tilde{Z})=Z$, $\pi: T(M)\to\nu$ denoting the natural projection.

This is nothing but an idea of Pasternack [**21**].

In the case of a conformal or projective foliation, however, we need the bundle of 2-frames to introduce a certain projectable connection on the normal bundle, since a riemannian connection on B is in general not preserved under local conformal or projective transformations on B.

The existence and uniqness of a normal Cartan connection are proved in Ochiai [**19**] in a general setting. Indeed they are related to the vanishing of certain Spencer cohomology groups. One can expect to generalize our method to G-foliations associated with such second order G-structures.

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