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Remarks on conditional expectations in von Neumann algebra

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1. Introduction. The conditional expectation has been studied by several authors, e.g. [1] F. Combes, [5] I. Kovács and J. Szüces, [6] M. Nakamura and T. Turumaru and [9] H. Umegaki. Here in this note, we shall make a detailed study on the conditional expectation T_ϕ from M to $(M^{\Sigma\phi})e_\phi$ (See [1]). We then apply it to the strict semi-finiteness of weight.

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2. Conditional expectation. Given a weight ϕ on a von Neumann algebra M, we denote by m_ϕ the *-subalgebra spanned by $n_\phi^* n_\phi$ where $n_\phi=\{x\in M;\ \phi(x^*x)<+\infty\}$. The linear extension on m_ϕ of $\phi|_{(m_\phi)+}$ will be denoted by $\dot{\phi}$.

The following theorem is a slight modification of [8] Theorem 3, which plays a crucial role in our study. The σ_t-invariance of T follows from the uniqueness of T.

THEOREM 1. *Let M be a von Neumann algebra, ϕ a faithful normal semi-finite weight on M, N a von Neumann subalgebra of M on which $\phi|_{N+}$ is semi-finite.*

Then the following two statements are equivalent;

(i) *N is invariant under the modular automorphism group σ_t associated with ϕ.*

(ii) *There exists a unique σ-weakly continuous conditional expectation T from M on N such that $\phi(x)=\phi\circ T(x)$ for all $x\in M_+$.*

By excluding the condition "$\phi|_{N+}$ is semi-finite" in the above Theorem 1, we get the following proposition.

PROPOSITION 2. *Let M be a von Neumann algebra, ϕ a faithful normal semi-finite weight on M, N a von Neumann subalgebra, e_0 the greatest projection in the σ-weak closure of $m_\phi|_{N+}$.*

Then the following two statements are equivalent;

(i) *e_0Ne_0 is invariant under the modular automorphism group $\Sigma=\{\sigma_t\}$ associated with ϕ.*

(ii) *e_0 is a projection of the subalgebra M^Σ of fixed points of M for Σ and* *there exists a unique σ-weakly continuous conditional expectation T from M on e_0Ne_0 such that $\phi(e_0xe_0)=\phi\circ T(x)$ for all $x\in M_+$.*

PROOF. (i)$\rightarrow$(ii).

Since e_0Ne_0 is invariant under $\Sigma, \sigma_t(e_0)\leqq e_0$ for all t and hence $e_0\in M^{\Sigma}$. We define a weight ψ by $\psi=\phi|_{(M_{e_0})_+}$. Then it follows from [7] Theorem 3.6 that ψ is a faithful normal semi-finite weight on M_{e_0}. Moreover by the construction of $e_0, \psi|_{(N_{e_0})_+}$ is also semi-finite on N_{e_0}.

The modular automorphism group σ_t^{ψ} associated with ψ is the restriction $\sigma_t|_{M_{e_0}}$ on M_{e_0}, therefore N_{e_0} is invariant under σ_t^{ψ}. By Theorem 1 there exists a σ-weakly continuous conditional expectation T_1 from M_{e_0} on N_{e_0} such that $\psi(x)=\psi\circ T_1(x)$ for all $x\in(M_{e_0})_+$. Then putting a σ-weakly continuous conditional expectation T by $T(x)=T_1(e_0xe_0)$ for all $x\in M$, we get;

$$\phi(e_0xe_0)=\phi\circ T(x)$$

for all x in M_+.

Let T' be another conditional expectation of the same properties.

For each $x\in(m_\phi)_+$, we get;

$$\begin{aligned}&\phi((T(x)-T'(x))^*(T(x)-T'(x)))\\&=\phi[(T\{T(x)-T'(x))^*x\}-T'\{(T(x)-T'(x))^*x\}]\\&=\dot{\phi}[e_0\{(T(x)-T'(x))^*\}e_0]-\dot{\phi}[e_0\{(T(x)-T'(x))^*x\}e_0]\\&=0.\end{aligned}$$

Since ϕ is faithful, we get;

$$T(x)-T'(x)=0 \quad \text{for all} \quad x\in(m_\phi)_+$$

Since m_ϕ is σ-weakly dense in M, T_1 and T_2 are σ-weakly continuous $T_1(x)=T_2(x)$ for all $x\in M$.

(ii)$\rightarrow$(i) The first part of statement (ii) implies that ψ is semi-finite as before. By applying Theorem 1 to M_{e_0}, N_{e_0}, $\psi, \psi|_{(N_{e_0})+}$ and $T|_{(M_{e_0})}$ instead of $M, N, \phi, \phi|_{N+}$ and T, it follows from Theorem 1 that N_{e_0} is invariant under σ_t^{ψ} for all $t\in\boldsymbol{R}$. On the other hand, e_0Ne_0 is invariant under Σ since $\sigma_t^{\psi}=\sigma_t|_{M_{e_0}}$.

We shall recall some definitions from [1]. Let ϕ be a faithful normal semi-finite weight on M_+. Put

$$A_\phi=\{x\in n_\phi^*\cap n_\phi\,;\ \dot{\phi}(xy)=\dot{\phi}(yx) \quad \text{for all} \quad y\in n_\phi^*\cap n_\phi\}$$

and let M_ϕ denote the σ-weak closure of A_ϕ.

COROLLARY 3. *Let M be a von Neumann algebra, ϕ a faithful normal semi-finite weight on M_+ with the modular automorphism group $\Sigma=\{\sigma_t\}$, e_0 the* *greatest projection of* $m_\phi|_{M^\Sigma}$.

Then there exists a unique σ*-weakly continuous conditional expectational* T *from* M *onto* $(M^\Sigma)_{e_0}$ *such that;*

$$\phi(e_0 x e_0)=\phi(Tx)$$

for all $x\in M_+$. *Moreover* $M_\phi=e_0 M^\Sigma e_0$.

PROOF. The first part of statement can be proved by replacing M^Σ in exchange for N in Proposition 2. Therefore we may have only to show that M_ϕ is the σ-weak closure of $m_\phi|_{(M^\Sigma)_+}$.

For each $x\in m_\phi|_{(M^\Sigma)_+}$ we see by [7] Theorem 3.6

$$\dot{\phi}(xz)=\dot{\phi}(zx) \quad \text{for all} \quad z\in m_\phi,$$

which implies $\phi(x(y^*z))=\phi((y^*z)x)$ for all $z, y\in n_\phi^*\cap n_\phi$

$$\begin{aligned}&<\pi_\phi(y^*)\eta_\phi(z)|\eta(x^*)>\\ &=<\eta_\phi(x)|\pi_\phi(z^*)\eta_\phi(y)>\\ &=<\eta_\phi(x)|S\pi_\phi(y^*)\eta_\phi(z)>\\ &=<\Delta^{1/2}\phi\pi_\phi(y^*)\eta_\phi(z)|J_\phi\eta_\phi(x)>.\end{aligned}$$

Since m_ϕ is a σ-weakly dense *-subalgebra of M, there exists a net $\{u_\lambda\}$ in $(m_\phi)_+$ such that $\{u_\lambda\}$ converges σ-strongly to 1 with $\|u_\lambda\|\leqq 1$ for all λ. Put $y_\lambda=\pi^{-1/2}\int_{-\infty}^{\infty}(\exp -t^2)\sigma_t(u_\lambda)dt$, then y_λ is an element of $(m_\phi)_+$ which is analytic for σ_t, moreover $\sigma_\alpha(y_\lambda)$ converges strongly to 1 and $\sigma_\alpha(y_\lambda)$ is bounded for all $\alpha\in\boldsymbol{C}$. [See [7] Lemma 5.2.]

Replacing y_λ by y, we get;

$$\begin{aligned}&<\pi_\phi(y_\lambda^*)\eta_\phi(z)|\eta_\phi(x^*)>\\ &=<\pi_\phi(\sigma_{-i/2}(y_\lambda))\Delta^{1/2}{}_\phi\eta_\phi(z)|J_\phi\eta_\phi(x)>.\end{aligned}$$

Therefore
$$\begin{aligned}&<\eta_\phi(z)|\eta_\phi(x^*)>=\lim_\lambda<\pi_\phi(y_\lambda^*)\eta_\phi(z)|\eta_\phi(x^*)>\\ &=\lim_\lambda<\pi_\phi(\sigma_{-i/2}(y_\lambda))\Delta^{1/2}{}_\phi\eta_\phi(z)|J_\phi\eta_\phi(x)>\\ &=<\Delta^{1/2}{}_\phi\eta_\phi(z)|J_\phi\eta_\phi(x)>,\end{aligned}$$

which implies $\dot{\phi}(xz)=\dot{\phi}(zx)$ for all $z\in n_\phi^*\cap n_\phi$. By the definition of M_ϕ we get $m_\phi|_{(M^\Sigma)_+}\subset M_\phi$.

Conversely for $x\in A_\phi$, it follows from [7] Theorem 3.6 that $x\in M^\Sigma$, then by [1] Lemma 2.2 the σ-weak closure of $m_\phi|_{(M^\Sigma)_+}$ contains A_ϕ.

DEFINITION 4. T and e_0 in Corollary 3 are written by T_ϕ and e_ϕ respectively and T_ϕ is called the conditional expectation associated with ϕ.

THEOREM 5. (*The characterization of* e_ϕ.)

The projection e_ϕ is the greatest projection in $\{e\in(M^\Sigma)_p\,;\ M_e$ *is* Σ*-finite*$\}$.

PROOF. We shall show that M_{e_ϕ} is Σ-finite. It follows from the uniqueness of T_ϕ that $T_\phi\sigma_t(x)=T_\phi(x)$ for all $x\in M$ and $t\in\boldsymbol{R}$. If $x\in(m_\phi)_+$, $y\mapsto\phi(T_\phi(x)yT_\phi(x))$ is Σ-invariant normal positive linear functional on M since $T_\phi(x)$ is in $M^\Sigma\cap(m_\phi)_+$ for $x\in(m_\phi)_+$ [See [7] Theorem 3.6].

We suppose; $y\in(e_\phi Me_\phi)_+$ and $\phi(T_\phi(x)yT_\phi(x))=0$ for all $x\in(m_\phi)_+$. $T_\phi(x)yT_\phi(x)=0$ for all $x\in(m_\phi)_+$ since ϕ is faithful. Since m_ϕ is a σ-weakly dense *-subalgebra of M, T_ϕ is σ-weakly continuous and $T_\phi(1)=e_\phi$, we get $y=e_\phi ye_\phi=0$, which implies M_{e_ϕ} is Σ-finite.

Conversely we suppose that M_e is Σ-finite with $e\in M^\Sigma$. By the definition of Σ-finiteness, there exists a family of Σ-invariant normal positive linear functional $\{\omega_i\}_{i\in I}$ on M_e such that the support $s(\omega_i)$ of ω_i is mutually orthogonal with $\sum_{i\in I}s(\omega_i)=e$ [See [5]].

Put

$$\psi=\sum_{i\in I}\omega_i\,.$$

Then ψ is a $\{\sigma_t|_{M_e}\}$-invariant faithful normal semi-finite weight on $(M_e)_+$. On the other hand $\phi|_{(M_e)_+}$ is semi-finite on $(M_e)_+$ and its modular automor phism group $\{\sigma_t^{\phi|_M}\}$ proves to be $\{\sigma_t^\phi|_{M_e}\}$.

By Radon-Nikodym Theorem in [7], there exists a unique non-singular positive self-adjoint operator h is affiliated with $(M^\Sigma)_e$ such that $\psi(\cdot)=\phi|_{M_e}(h\cdot)$, then $\phi|_{M_e}(\cdot)=\psi(h^{-1}\cdot)$ and h is affiliated with $(M_e)^{\Sigma\psi}$. It follows from [7] Theorem 3.6 and $s(\omega_i)\in(m_\psi)_+$ that

$$\left(e-e\left(\frac{1}{n}\right)\right)s(\omega_i)\left(e-e\left(\frac{1}{n}\right)\right)\in(m_\phi)_+\ \text{ where } h=\int_0^\infty\lambda de(\lambda)\,.$$

Since ω_i is Σ-invariant, $s(\omega_i)$ is a projection of M^Σ, and then $\left(e-e\left(\frac{1}{n}\right)\right)s(\omega_i)\left(e-e\left(\frac{1}{n}\right)\right)$ is in $(M^\Sigma)_e$.

By the definition of e_ϕ, we get;

$$\left(e-e\left(\frac{1}{n}\right)\right)s(\omega_i)\left(e-e\left(\frac{1}{n}\right)\right)\leqq e_\phi\quad\text{for all}\quad n\in\boldsymbol{N}.$$

Since h is non-singular, $e_\phi\geqq w-\lim_{n\to\infty}\left(e-e\left(\frac{1}{n}\right)\right)s(\omega_i)\left(e-e\left(\frac{1}{n}\right)\right)=es(\omega_i)e$ then $e\leqq e_\phi$ because $\sum_{i\in I}s(\omega_i)=e$.

Therefore e_ϕ is the greatest projection.

In the following Corollary the equivalence of condition (i), (iv) and (v) was proved by Combes [2] 3.4 Théorème.

COROLLARY 6. *Let ϕ be a faithful normal semi-finite weight on M_+. The* *following statements are equivalent;*

(i) ϕ *is strictly semi-finite.*

(ii) $M_\phi = M^\Sigma$.

(iii) $e_\phi = 1$.

(iv) M *is* Σ*-finite.*

(v) *There exists a* σ*-weakly continuous conditional expectation* T *from* M *onto* M^Σ *such that* $\phi(x) = \phi\circ T(x)$ *for all* $x \in M_+$. *Moreover* T *in* (v) *is* T_ϕ.

PROOF. (i)$\leftrightarrow$(ii)$\leftrightarrow$(iii)$\leftrightarrow$(iv)$\rightarrow$(v) follow from Corollary 3 and Theorem 5.

(v)$\rightarrow$(iii). For $x \in (m_\phi)_+$, we see that $T(x) \in (m_\phi|_{M^\Sigma})_+$.

In the proof of Corollary 3 we have already shown $m_\phi|_{(M^\Sigma)_+} \subset M_\phi$, which implies, $T((m_\phi)_+) \subset M_\phi$.

Since T is σ-weakly continuous and m_ϕ is σ-weakly dense in M, $M^\Sigma = T(\overline{m_\phi}^{\sigma-W}) \subset \overline{T(m_\phi)}^{\sigma-W} \subset M_\phi$. Since $M^\Sigma \supset M_\phi$, we get $M_\phi = M^\Sigma$.

The last statement $T = T_\phi$ follows the uniqueness of T_ϕ.

THEOREM 7. *Let* M *(resp.* N*) be a von Neumann algebra,* ϕ *(resp.* ψ*) a faithful normal semi-finite weight on* M *(resp.* N*).*

Then $$e_\phi \otimes e_\psi = e_{\phi\otimes\psi}\,.$$

PROOF. *Let* $\Sigma = \{\sigma_t\}$ (resp. $\Sigma^\psi = \{\rho_t\}$) be the modular automorphism group associated with ϕ (resp. ψ), M^Σ (resp. $M^{\Sigma\psi}$) the subalgebra of fixed points of M (resp. N) for Σ (resp. Σ^ψ).

We shall prove that $$e_\phi \otimes e_\psi \geqq e_{\phi\otimes\psi}.$$

Since $(M\otimes N)_{e_{\phi\otimes\psi}}$ is $\Sigma\otimes\Sigma^\psi$-finite by Theorem 5, there exists a family of $\Sigma\otimes\Sigma^\psi$-invariant positive linear functional $\{\omega_i\}_{i\in I}$ on $M\otimes N$ such that $\sum_{i\in I} s(\omega_i) = e_{\phi\otimes\psi}$.

Put $$\tilde{\omega}_i(x) = \omega_i(x\otimes 1) \quad \text{for all} \quad x \in M_+\,.$$

Then we get $$s(\omega_i) \leqq s(\tilde{\omega}_i)\otimes 1\,.$$

On the other hand, since $\tilde{\omega}_i$ is Σ-invariant normal positive linear functional on M, we get $s(\tilde{\omega}_i) \leqq e_\phi$ by Theorem 5, which implies $s(\omega_i) \leqq s(\tilde{\omega}_i)\otimes 1 \leqq e_\phi \otimes 1$.

Similarly we get; $s(\omega_i) \leqq 1\otimes e_\psi$ so that $s(\omega_i) \leqq e_\phi\otimes e_\psi$ for all $i \in I$, therefore $e_{\phi\otimes\psi} = \sum_{i\in I} s(\omega_i) \leqq e_\phi \otimes e_\psi$.

By the definitions of e_ϕ, e_ψ, $e_{\phi\otimes\psi}$ and of tensor product of weights [See **3** or **4**], we get;

$$m_\phi|_{M^\Sigma} \odot m_\psi|_{M^{\Sigma\psi}} \subset m_{\phi\otimes\psi}|_{(M\otimes N)^{\Sigma\otimes\Sigma^\psi}}$$

and hence $$e_\phi \otimes e_\psi \leqq e_{\phi\otimes\psi}\,.$$

Then we finally get $$e_\phi \otimes e_\psi = e_{\phi\otimes\psi}.$$

PROPOSITION 8. *Let ϕ (resp. ψ) be a faithful normal semi-finite weight on M_+ (resp. N_+).*

ϕ and ψ are strictly semi-finite if and only if $\phi\otimes\psi$ is strictly semi-finite.

PROOF. It follows from Theorem 7 and Corollary 6.

REMARK 9. The result in Proposition 8 has already mentioned without its proof in [4].

References

[1] F. Combes, Poids et espérences conditionelles dans algèbres de von Neumann, Bull. Soc. Math. France, **99** (1971), 73-112.

[2] F. Combes, Poids associé à une algèbre hilbertienne á gauche, Compositio Math., **23** (1971), 49-77.

[3] A. Connes, Une classification des facteurs de type III, Ann. Sci. École Norm. Sup., **6** (1973), 133-252.

[4] Y. Katayama, The tensor product of weights, Proc. Japan Acad., **50** (1974), 430-432.

[5] I. Kovács and Szüces, Ergodic type theorems in von Neumann algebras, Acta sci. Math. Szeged, **27** (1966), 233-246.

[6] M. Nakamura and T. Tsurumaru, Expectations in an operator algebra, Tôhoku Math. J., **6** (1954), 182-188.

[7] G.K. Pedersen and M. Takesaki, The Radon-Nikodym theorem for von Neumann algebras, Acta Math., **130** (1973), 53-87.

[8] M. Takesaki, Conditional expectations in von Neumann algebra, J. Funct. Anal., **9** (1971), 306-321.

[9] H. Umegaki, Conditional expectation in an operator algebra, Tôhoku Math. J. **6** (1954), 177-181.

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