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The ordinary Z_2-homology theory and singular bordism theories

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Introduction.

The first part (§ 1~§ 5) of this paper is devoted to describe the ordinary Z_2-homology theory from the viewpoint of the singular bordism theory developed in [**2**].

Let $\mathcal{K}$ be the category whose objects are homogeneous finite simplicial complexes which have mod 2 boundaries and whose morphisms are *pl*-imbeddings which take boundary into boundary.

Then we have our main result (Theorem 2 in § 4) which shows that the ordinary Z_2-homology theory is reconstructed with $\mathcal{K}$ by the analogous method of [**2**].

The latter part (§ 6 and § 7) is devoted to explain the relation of singular complex theory and singular bordism theory.

For the purpose, we introduce notions of bordism category and (generalized) singular bordism theory. Our bordism category is a generalization of both chain complex and cobordism category defined by R. E. Stong [**4**] from the categorical point of view. A bordism category defines a singular bordism theory if it is regular, that is, it has finite sums and an initial object. Every chain complex becomes a bordism category as discrete category, but unfortunately it is not regular and induces no singular bordism theory. Therefore the classical singular complex theory is not singular bordism theory.

On the other hand, our category $\mathcal{K}$ is a regular bordism category and induces a singular bordism theory which defines the ordinary Z_2-homology theory.

§ 1. Z_2-complexes.

(1.1) *Simplicial complexes* which we consider in this paper are finite simplicial complexes in some Euclidean spaces. The underlying space of a simplicial complex K is denoted by $|K|$ as usual if it is necessary to distinguish them, but we will often write K for $|K|$ for convenience. We regard the empty set

$\emptyset$ as a simplex so that $\emptyset \in K$ for any simplicial complex K. Also by the same symbol $\emptyset$ we denote the complex which consists of only one simplex $\emptyset$. For two simplexes s and s' of K, we write $s' \prec s$, if s' is a face of s, so that $\emptyset \prec s$ and $s \prec s$ for any simplex s.

(1.2) A *map* from a simplicial complex K to a simplicial complex L is a continuous map $f: |K| \rightarrow |L|$, and is denoted by $f: K \rightarrow L$.

A *subdivision* K' of K is a complex such that $|K'| = |K|$ and each simplex of K' is contained in a simplex of K. We write $K' < K$ if K' is a subdivision of K.

A map $f: K \rightarrow L$ between finite simplicial complexes is *piecewise linear* if ther exists a subdivision K' of K so that f maps each simplex of K' linearly into a simplex of L.

(1.3) A simplicial complex K is a n-dimensional *Z_2-complex* if K is made up of only finite n-simplexes and their faces. In particular empty complex $\emptyset$ is a Z_2-complex. If K is a n-dimensional Z_2-complex, then we may define the *boundary* ∂K of K as follows. ∂K is the subcomplex of K which is made up of $(n-1)$-simplexes which are faces of an odd number of n-simplexes of K, and their faces. In particular we notice that $\partial K^0 = \emptyset$ for any 0-dimensional Z_2-complex K^0, and $\partial \emptyset = \emptyset$. Then ∂K is again a Z_2-complex.

An *incidence number* $[s^m : s^{m-1}]$ is an element of Z_2 such that

$$\begin{aligned} [s^m : s^{m-1}] &= 1\,, \quad \text{if} \quad s^{m-1} \text{ is a face of } s^m\,, \\ &= 0\,, \quad \text{if} \quad s^{m-1} \text{ is not a face of } s^m\,. \end{aligned}$$

In particular, $[s^0 : \emptyset] = 1$.

The following properties are checked by the standard methods from our definitions:

PROPOSITION 1.

(1) $s^{m-1} \prec s^m$ *if and only if* $[s^m : s^{m-1}] = 1$.

(2) *If K is a n-dimensional Z_2-complex, then*

$$s^{n-1} \in \partial K \textit{ if and only if } \sum_{s^n \in K} [s^n : s^{n-1}] = 1 \ (n \geqq 1)\,.$$

(3) *For any pair of s^n and s^{n-2} of K,*

$$\sum_{s^{n-1} \in K} [s^n : s^{n-1}][s^{n-1} : s^{n-2}] = 0 \ (n \geqq 1)\,.$$

(4) $\partial\partial K = \emptyset$ *for any Z_2-complex K.*

(1.4) Since the composition of *pl*-imbeddings is a *pl*-imbedding, there exists a category $\mathcal{K}$ whose objects are Z_2-complexes and whose morphisms are *pl*-imbeddings which take boundary into boundary (cf. § 7). This category $\mathcal{K}$ has *finite sums* by the disjoint union and has an *initial object* given by the empty complex $\emptyset$. By $K+L$ we dente an object which is a sum for K and L.

(1.5) Let K and L be Z_2-complexes.
If K is simplicial isomorphic to L, we write $K\cong L$, and if K is piecewise linear isomorphic to L, we write $K\simeq L$.

Then we have easily:

PROPOSITION 2.

(1) *$K\cong L$ implies $K\simeq L$.*

(2) *$K\simeq L$ implies that there are K' and L' such that $K'<K$, $L'<L$ and $K'\cong L'$.*

(3) *$K'<K$ implies $\partial K'\simeq\partial K$.* (4) *$K\simeq L$ implies $\partial K\simeq\partial L$.*

PROOF. (1), (2) and (3) are well-known.

Proof of (4). Let $K\simeq L$. From (2), there exist $K'<K$ and $L'<L$ such that $K'\cong L'$, therefore $\partial K'\cong\partial L'$. From (3), $\partial K\simeq\partial K'\cong\partial L'\simeq\partial L$.

§ 2. Bordism groups of the category $\mathcal{K}$.

(2.1) A Z_2-complex K is *closed* if $\partial K=\emptyset$, K *bounds* if there is a Z_2-complex W with ∂W piecewise linear isomorphic to K. Generally we say that two closed Z_2-complexes K and L are *bordant* if there exists a Z_2-complex W such that $\partial W\simeq K+L$. This will be written $K\sim L$; we refer to this relation as the *bordism relation.*

We have easily:

PROPOSITION 3.

(1) *K closed and $K\simeq L$ implies L closed.*

(2) *K bounds and $K\simeq L$ implies L bounds.*

(3) *$\sim$ is an equivalence relation on the closed Z_2-complexes.*

PROOF. (1) and (2) are trivially verified.

Proof of (3). (i) Since K is closed, we have $\partial(I\times K)\simeq K+K$. This implies $K\sim K$. (ii) $K\sim L$ implies $\partial W\simeq K+L$ for some W. $K+L\simeq L+K$ implies $\partial W\simeq L+K$. This means $L\sim K$. (iii) Let $\partial U\simeq K+L$ and $\partial V\simeq L+N$. We may suppose that L is a full subcomplex in U and V. Let $W=U\underset{L}{\oplus}V$ be a complex obtained from $U+V$ by identifying two subcomplexes of U and V which are *pl*-isomorphic to L. Then we have easily $W\in\mathcal{K}$ and $\partial W\simeq K+N$. This completes the proof.

(2.2) The bordism relation $\sim$ is an equivalence relation on the class of closed n-dimensional Z_2-complexes. The resulting set $\Omega_n(\mathcal{K})$ of equivalence classes is an abelian group with addition induced by disjoint union. Since $\partial(I\times K)\simeq K+K$ for each closed Z_2-complex K, every element in $\Omega_n(\mathcal{K})$ has order two.

(2.3) THEOREM 1.

$$\Omega_n(\mathcal{K})=\begin{cases} 0 & if\ n\neq 0, \\ Z_2 & if\ n=0. \end{cases}$$

PROOF. Let K be any closed n-dimensional Z_2-complex for $n>0$, and $v*K$ be the corn over K (where $*$ denotes the join of a point v and K), then we have $\partial(v*K)=K$. This shows $\Omega_n(\mathcal{K})=0$ for $n>0$. For $n=0$, any 0-dimensional Z_2-complex K^0 is closed by the definition. Now let r be numbers of vertices of K^0, then $\partial(v*K^0)=K^0$ if r is even, and $\partial(v*K^0)=K^0+v$ if r is odd, which shows

$$K^0\sim 0 \quad \text{if } r \text{ is even, and}$$
$$K^0\sim v \quad \text{if } r \text{ is odd.}$$

Since a point v is closed, but not bounds, we have $\Omega_0(\mathcal{K})=Z_2$.

§ 3. Singular Z_2-complexes of pairs of spaces.

(3.1) Let (X, A) be a pair of a space X and its subspace A. A *n-singular Z_2-complex in* (X, A) is a pair (K, f) consisting of a n-dimensional Z_2-complex K and a continuous map $f: (K, \partial K)\to(X, A)$. If $A=\emptyset$ then of course $\partial K=\emptyset$ also.

Two singular Z_2-complexes (K, f) and (L, g) are *equivalent*, written $(K, f)\simeq(L, g)$, if there is a pl-homeomorphism $\psi: K\to L$ such that $f=g\circ\psi$.

A n-singular Z_2-complex (K, f) in (X, A) is said to *bord* if and only if there is $(n+1)$-dimensional Z_2-complex W and a map $F: W\to X$ for which

(1) there is a n-dimensional Z_2-complex K' such that

$$|K'|\subset|\partial W| \quad\text{and}\quad F(|\partial W|\setminus|K'|)\subset A,$$

(2) $(K', F|K')\simeq(K, f)$.

From two singular Z_2-complexes (K, f) and (L, g) in (X, A) a disjoint union $(K+L, f+g)$ is defined where $K+L\simeq K\cup L$, $K\cap L=\emptyset$, $f+g|K=f$ and $f+g|L=g$.

A pair (K, f) and (L, g) of singular Z_2-complexes in (X, A) are *bordant* if and only if the disjoint union $(K+L, f+g)$ bords in (X, A). This will be written $(K, f)\equiv(L, g)$.

By the definition we have easily:

LEMMA 1. *$(K, f)\simeq(L, g)$ implies $(K, f)\equiv(L, g)$.*
In particular, $K'<K$ and $f'=f$ implies $(K', f')\equiv(K, f)$.

Combining Lemma 1 with the fact that if $|L|\subset|K|$, then there are L' and K' such that $L'<L$, $K'<K$ and $L'\subset K'$, we can say as follows:

(K, f) bords in (X, A) if and only if there exists (K', f') with $(K', f')\simeq(K, f)$ such that there is a $(n+1)$-dimensional Z_2-complex W' and a map $F': W'\to X$ for which

(1)′ $K'\subset\partial W'$ and $F'(\partial W'\backslash K')\subset A$,

(2)′ $F'|K'=f'$.

PROPOSITION 4. *$\equiv$ is an equivalence relation on the singular Z_2-complexes in (X, A).*

PROOF. In order to see that (K, f) in (X, A) is bordant to itself, form the Z_2-complex $I\times K$ where I is the unit interval, and define a map $F: I\times K\rightarrow X$ by $F(t, x)=f(x)$ for $t\in I$ and $x\in K$. Then $0\times K\cup 1\times K$ is a subcomplex of $\partial(I\times K)$ such that $F|0\times K\cup 1\times K=f+f$ and $F(\partial(I\times K)\backslash 0\times K\cup 1\times K)\subset A$. Hence $(K+K, f+f)$ bords.

The bordism relation $\equiv$ is clearly symmetric. Finally for the proof of transitivity, suppose $(K, f)\equiv(L, g)$ and $(L, g)\equiv(N, h)$, where K, L and N are n-dimensional Z_2-complexes. Then there are $(n+1)$-dimensional Z_2-complexes W and Z with maps $F: W\rightarrow X$ and $G: Z\rightarrow X$. We may suppose without loss of generality that L is full in W and Z respectively. Let C be the complex obtained from $W+Z$ by identifying two subcomplexes in W and Z which are isomorphic to L; we write $C=W\underset{L}{\oplus}Z$. Then C is a $(n+1)$-dimensional Z_2-complex, and we can construct a map $H: C\rightarrow X$ so that (C, H) gives $(K, f)\equiv(N, h)$.

(3.2) LEMMA 2. *Let K be any n-dimensional Z_2-complex. Suppose P and Q are closed disjoint subsets of the space $|K|$. Then there exists a Z_2-complex L such that*

(1) *L is a n-dimensional subcomplex of an iterated barycentric subdivision Sd^kK of K.*

(2) *$P\subset|L|$ and $|L|\cap Q=\emptyset$.*

(3) *$|\partial L|\cap P=|\partial K|\cap P$.*

PROOF. Let ρ be the distance of P and Q, then $\rho>0$ and $\mathrm{mesh}(Sd^kK)<\rho$ for some large integer k ($\geqq 2$), where Sd^kK is the k^{th} barycentric subdivision of K.

For any point x of $|Sd^kK|$, $s(x)$ denotes the smallest simplex of Sd^kK which contain x. Define $St(s(x))$ by

$$St(s(x))=\{\sigma\,|\,\sigma\prec\tau \quad\text{and}\quad s(x)\prec\tau\in Sd^kK\}.$$

Clearly it is a n-dimensional Z_2-complex. Finally we define $St(P)$ as follows:

$$St(P)=\bigcup_{x\in p}St(s(x)).$$

$St(P)$ is also a n-dimensional Z_2-complex.

From the above constructions we can see easily that $P\subset|St(P)|$ and $Q\cap|St(P)|=\emptyset$. To complete the proof of Lemma 2 is only necessary to verify that $St(P)$ has the property (3). Let $L=St(P)$. Then it is sufficient to show that $|\partial L|\cap P=|\partial Sd^kK|\cap P$. Suppose given any point x of $|\partial L|\cap P$. Then $x\in|\partial L|$ implies $s(x)\in\partial L$. Since ∂L is a Z_2-complex, there is a $(n-1)$-simplex s^{n-1} such that $s(x)\prec s^{n-1}\in\partial L$. It follows from $s^{n-1}\in\partial L$ that the number of n-

simlexes s^n in L with $s^{n-1} \prec s^n$ is odd. And write them $s_1^n, \cdots, s_q^n$. Let τ^n be any n-simplex of $Sd^k K$ with $s^{n-1} \prec \tau$. Then $s(x) \prec s^{n-1} \prec \tau \in Sd^k K$. By the definition of $St(P)$, we have $\tau \in L = St(P)$. Therefore there is no other n-simplex τ with $s^{n-1} \prec \tau^n$ and $\tau^n \neq s_1^n, \cdots, s_q^n$. Thus we have $s^{n-1} \in \partial Sd^k K$. It follows that $s(x) \in \partial Sd^k K$ which shows $x \in |\partial Sd^k K|$. Since $x \in P$, we have $x \in |\partial Sd^k K| \cap P$. This proves that $|\partial L| \cap P \subset |\partial Sd^k K| \cap P$. Finally we must prove that $|\partial L| \cap P \supset |\partial Sd^k K| \cap P$. Suppose given x of $|\partial Sd^k K| \cap P$. Then $x \in |\partial Sd^k K|$ implies $s(x) \in \partial Sd^k K$. Since $\partial Sd^k K$ is a Z_2-complex, there is a $(n-1)$-simplex s^{n-1} with $s(x) \prec s^{n-1}$. $s^{n-1} \in \partial Sd^k K$ implies that the number of n-simplexes s^n with $s^{n-1} \prec s^n$ is odd. We denote them by $s_1^n, \cdots, s_q^n$. Then $x \in P$, $s(x) \prec s_1^n, \cdots, s_q^n \in Sd^k K$. From the definition of $L = St(P)$, it follows that $s_1^n, \cdots, s_q^n \in L$. Thus we have $s^{n-1} \in \partial L$, so that $s(x) \in \partial L$. Hence $x \in |\partial L|$. This completes the proof of Lemma 2.

§ 4. The singular bordism groups of pairs of spaces.

(4.1) Denote the bordism class of (K^n, f) by $[K^n, f]$, and the collection of all such bordism classes by $\mathcal{H}_n(X, A)$. An abelian group structure is imposed on $\mathcal{H}_n(X, A)$ by disjoint union; that is, $[K_1^n, f_1] + [K_2^n, f_2] = [K_1^n + K_2^n, f_1 + f_2]$. It is seen that the class of all (K^n, f) which bord forms the zero element. Let $\mathcal{H}_*(X, A)$ be the weak direct sum $\sum \mathcal{H}_n(X, A)$.

Given a map $\psi : (X, A) \to (Y, B)$ there is associated a natural homomorphism $\psi_* : \mathcal{H}_n(X, A) \to \mathcal{H}_n(Y, B)$ given by $\psi_*[K^n, f] = [K^n, \psi \circ f]$. There is also a homomorphism $\tilde{\partial}_n : \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A, \emptyset)$ given by $\tilde{\partial}_n[K^n, f] = [\partial K^n, f | \partial K^n]$. It is easy to see that $\tilde{\partial}_n$ is well defined and associative.

The following proposition is trivially verified:

PROPOSITION 5.

(1) *If* $i : (X, A) \to (X, A)$ *is the identity map then* $i_* : \mathcal{H}_n(X, A) \to \mathcal{H}_n(X, A)$ *is the identity automorphism.*

(2) *If* $\psi : (X_1, A_1) \to (X_2, A_2)$ *and* $\theta : (X_2, A_2) \to (X_3, A_3)$, *then* $(\theta \circ \psi)_* = \theta_* \circ \psi_*$.

(3) *For any map* $\psi : (X, A) \to (Y, B)$ *the diagram*

$$\begin{array}{ccc} \mathcal{H}_n(X, A) & \xrightarrow{\tilde{\partial}_n} & \mathcal{H}_{n-1}(A, \emptyset) \\ \downarrow \psi_* & & \downarrow (\psi | A)_* \\ \mathcal{H}_n(Y, B) & \xrightarrow{\tilde{\partial}_n} & \mathcal{H}_{n-1}(B, \emptyset) \end{array}$$

is commutative.

Proposition 5 shows $\{\mathcal{H}_n(\ ,\ ), \tilde{\partial}_n\}$ is a covariant functor on the category of pairs of spaces, and $\tilde{\partial}_n$ is a natural transformation.

(4.2) Our main theorem is the following, which will be proved in the next section.

THEOREM 2. *On the category of pairs of spaces and maps the singular bordism functors*

$$\{\mathcal{H}_*(X, A), \phi_*, \tilde{\partial}_*\}$$

satisfies the Eilenberg-Steenrod axioms for a homology theory. For a single point pt., we have

$$\mathcal{H}_n(pt., \emptyset) \approx \Omega_n(\mathcal{K}) \approx \begin{cases} 0 & n \neq 0, \\ Z_2 & n = 0. \end{cases}$$

§ 5. The Eilenberg-Steenrod axioms.

(5.1) $$\mathcal{H}_n(pt., \emptyset) \approx \Omega_n(\mathcal{K}).$$

PROOF. For any closed Z_2-complex K^n, a map $f: K^n \to pt.$ is unique. Inversely given any singular Z_2-complex (K^n, f) in (pt., $\emptyset$), K^n is closed. Therefore there is a one to one correspondence between closed Z_2-complexes and singular Z_2-complexes in (pt., $\emptyset$), which shows $\mathcal{H}_n$(pt., $\emptyset$)$\approx \Omega_n(\mathcal{K})$.

(5.2) *If $\phi_0, \phi_1: (X, A) \to (Y, B)$ are homotopic, then $\phi_{0*} = \phi_{1*}$.*

PROOF. Let $h: (I \times X, I \times A) \to (Y, B)$ be a homotopy joining ϕ_0 and ϕ_1. For (K^n, f) a singular Z_2-complex in (X, A) define $\theta: I \times K \to Y$ by $\theta(t, x) = h(t, f(x))$; then $\theta(0, x) = \phi_0 \circ f(x)$ and $\theta(1, x) = \phi_1 \circ f(x)$. Now $I \times K^n$ is a $(n+1)$-dimensional Z_2-complex and $\partial(I \times K^n) = (\partial I \times K^n) \cup (I \times \partial K^n)$. Thus $1 \times K^n \cup 0 \times K^n$ is a subcomplex of $\partial(I \times K^n)$, and $\theta(I \times \partial K^n) \subset B$. Hence $[K^n, \phi_0 \circ f] = [K^n, \phi_1 \circ f]$.

(5.3) We need a lemma before proving exactness.

LEMMA 3. *Let L be a n-dimensional Z_2-complex and a subcomplex of a closed n-dimensional Z_2-complex K. If $f: K \to X$ is a map with $f(K \backslash L) \subset A$, then $[K, f] = [L, f|L]$ in $\mathcal{H}_n(X, A)$.*

PROOF. Define $F: I \times K^n \to X$ by $F(t, x) = f(x)$. Now $\partial(I \times K^n) = \partial I \times K^n$ and $1 \times L^n \cup 0 \times K^n$ is a subcomplex of $\partial(I \times K^n)$, for K^n is closed. Since $F(1 \times (K^n \backslash L^n)) \subset A$ then $[K^n, f] = [L^n, f|L^n]$.

(5.4) PROPOSITION 6. *For every pair (X, A) the sequence*

$$\cdots \xrightarrow{\tilde{\partial}} \mathcal{H}_n(A, \emptyset) \xrightarrow{i_*} \mathcal{H}_n(X, \emptyset) \xrightarrow{j_*} \mathcal{H}_n(X, A) \xrightarrow{\tilde{\partial}} \mathcal{H}_{n-1}(A, \emptyset) \longrightarrow \cdots$$

is exact.

PROOF. (5.4.1) $$\tilde{\partial} j_* = 0.$$

If $[K^n, f] \in \mathcal{H}_n(X, \emptyset)$, then $\partial K = \emptyset$ and $\tilde{\partial} j_*[K^n, f] = [\partial K^n, jf|\partial K^n] = [\emptyset, \phi] = 0$.

(5.4.2) $$\mathrm{Im}\, j_* \supset \mathrm{Ker}\, \tilde{\partial}.$$

Consider $[K^n, f] \in \mathcal{H}_n(X, A)$ in the kernel of $\tilde{\partial}$; then $\tilde{\partial}[K^n, f] = [\partial K^n, f|\partial K^n] = 0$ in $\mathcal{H}_{n-1}(A, \emptyset)$.

By Lemma 1 we may suppose that ∂K^n is full in K^n. Since $[\partial K^n, f|\partial K] = 0$ in $\mathcal{H}_{n-1}(A, \emptyset)$, there is a Z_2-complex W^n and a map $F: W^n \to A$ with $\partial K^n = \partial W^n$ and $F|\partial K^n = f|\partial K^n$. It is also no loss of generality to suppose that ∂W^n is full in W^n. Construct a complex Z^n identifying the boundary ∂W^n of W^n with the boundary ∂K^n of K^n from the disjoint sum $W^n + K^n$; we write $Z^n = W \underset{\partial W \equiv \partial K}{\oplus} K$. Then Z^n is a closed Z_2-complex. Define $g: Z^n \to X$ by $g|W = F$ and $g|K = f$, then $[Z^n, g] \in \mathcal{H}_n(X, \emptyset)$ and $j_*[Z^n, g] = [Z^n, jg]$ coincides with $[K^n, f]$ by Lemma 3.

(5.4.3) $$j_* i_* = 0.$$

If $[K^n, f] \in \mathcal{H}_n(A, \emptyset)$, then $\partial K^n = \emptyset$, $f: K^n \to A$ and $j_* i_*[K^n, f] = [K^n, jif]$. Define $F: I \times K^n \to X$ by $F(t, x) = f(x)$, then $0 \times K^n$ is a subcomplex of $\partial(I \times K^n)$, $F|0 \times K^n = f$ and $F(\partial(I \times K^n) \backslash 0 \times K^n) \subset A$. Hence $[K^n, f] = 0$ in $\mathcal{H}_n(X, A)$.

(5.4.4) $$\operatorname{Im} i_* \supset \operatorname{Ker} j_*.$$

Let $[K^n, f]$ be an element of $\mathcal{H}_n(X, \emptyset)$ with $j_*[K^n, f] = 0$ in $\mathcal{H}_n(X, A)$. Then we may find a representative (K, f) for which there is a Z_2-complex W^{n+1} and a map $F: W^{n+1} \to X$ such that

(1) $$K^n \subset \partial W^{n+1},$$

(2) $$F|K^n = f \quad \text{and} \quad F(\partial W^{n+1} \backslash K^n) \subset A.$$

Let Z^{n+1} be the complex obtained from the disjoint sum $W^{n+1} + (I \times K^n)$ by identifying the subcomplex K^n of W^{n+1} with the subcomplex $1 \times K^n$ of $I \times K^n$, i.e. $Z^{n+1} = W^{n+1} \underset{K \equiv 1 \times K}{\oplus} I \times K^n$.

Define $G: Z^{n+1} \to X$ by $G|W^{n+1} = F$ and $G(t, x) = f(x)$ for $t \in I$ and $x \in K$. Denote $0 \times K^n$ by K^n, then

(3) $$K^n \subset \partial Z^{n+1},$$

(4) $$G|K^n = f \quad \text{and} \quad G(\partial Z^{n+1} \backslash K^n) \subset A.$$

Let $L^n = \partial Z^{n+1} \backslash K^n$, then by the construction $\partial L^n = \emptyset$, $L^n \cap K^n = \emptyset$ and $\partial Z^{n+1} = L^n + K^n$. Hence $G|K^n \cup L^n = f \cup G|L^n$. Since $[L^n, G|L^n] \in \mathcal{H}_n(A, \emptyset)$, $i_*[L^n, G|L^n]$ coincides with $[K^n, f]$ in $\mathcal{H}_n(X, \emptyset)$.

(5.4.5) $$i_* \tilde{\partial} = 0.$$

Let $[K^n, f]$ be an element of $\mathcal{H}_n(X, A)$ such that $i_* \tilde{\partial}[K^n, f] = [\partial K^n, if|\partial K^n] = 0$ in $\mathcal{H}_{n-1}(X, \emptyset)$. Since ∂K is the boundary of K^n and $f|\partial K^n$ is a restriction of $f: K^n \to X$, the singular Z_2-complex $(\partial K^n, f|\partial K^n)$ bords in $(X, \emptyset)$.

(5.4.6) $$\operatorname{Im}\tilde{\partial}\supset\operatorname{Ker} i_*.$$

Let $[K^{n-1}, f]$ be an element of $\mathcal{H}_{n-1}(A, \emptyset)$ such that $i_*[K^{n-1}, f]=[K^{n-1}, if]=0$ in $\mathcal{H}_{n-1}(X, \emptyset)$. Since $f: K^{n-1}\to A\subset X$, $\partial K^{n-1}=\emptyset$ and $[K^{n-1}, f]=0$ in $\mathcal{H}_{n-1}(X, \emptyset)$, we may find a representative (K, f) for which there is a Z_2-complex W^n and a map $F: W^n\to X$ with $K^{n-1}=\partial W^n$ and $F|K^{n-1}=f$. Then $F: (W^n, \partial W^n)\to (X, A)$, $[W^n, F]\in\mathcal{H}_n(X, A)$ and $\tilde{\partial}[W^n, F]=[\partial W^n, F|\partial W^n]=[K^{n-1}, f]$ in $\mathcal{H}_{n-1}(A, \emptyset)$.

(5.5) PROPOSITION 7.

If U is an open set with $\bar{U}\subset\operatorname{Int} A$, then the inclusion $i: (X\setminus U, A\setminus U)\to(X, A)$ induces an isomorphism $i_*: \mathcal{H}_n(X\setminus U, A\setminus U)\approx\mathcal{H}_n(X, A)$.

PROOF. (5.5.1) *i^* is an epimorphism.*
Let (K^n, f) be a singular Z_2-complex in (X, A); $f: (K^n, \partial K^n)\to(X, A)$. Let $P=f^{-1}(X\setminus\operatorname{Int} A)$, $Q=f^{-1}(\bar{U})$. There exists, by Lemma 2, a Z_2-complex L^n with $P\subset|L^n|$ and $Q\cap|L^n|=\emptyset$. Furthermore L^n is a subcomplex of a sufficiently fine barycentric subdivision K'^n of K^n. Now we show that $f(\partial L)\subset A$. Let x be any point of $|\partial L|$. In the case $x\notin P$, it follows from the definition of P that $f(x)\in A$. In the case when $x\in P$, we know from Lemma 2 that $x\in|\partial L|\cap P=|\partial K|\cap P$. Therefor $x\in|\partial K|$. Since $f(\partial K)\subset A$, we know that $f(x)\in A$. Thus $[L^n, f|L^n]\in\mathcal{H}_n(X\setminus U, A\setminus U)$.

Define $F: I\times K'^n\to X$ by $F(t, x)=f(x)$. Then $1\times L^n\cup 0\times K'^n$ is a subcomplex of $\partial(I\times K'^n)=\partial I\times K'^n\cup I\times\partial K'^n$ and $F(\partial(I\times K'^n)\setminus 1\times L^n\cup 0\times K'^n)\subset A$. Hence $[L^n, f|L^n]=[K'^n, f]=[K^n, f]$ in $\mathcal{H}_n(X, A)$.

(5.5.2) *i_* is a monomorphism.*
Let $[K^n, f]$ be any element of the kernel of i_*;

$$\begin{tikzcd}
(K^n, \partial K) \arrow[r, "f"] \arrow[dr, "if"'] & (X\setminus U, A\setminus U) \arrow[d, "i"] \\
 & (X, A)
\end{tikzcd}$$

and $i_*[K^n, f]=[K^n, if]=0$ in $\mathcal{H}_n(X, A)$.

Let $P_1=f^{-1}(X\setminus\operatorname{Int} A)$ and $Q_1=f^{-1}(\bar{U})$, then there exists, by Lemma 2, a Z_2-complex K_1^n such that it is a subcomplex of a sufficiently fine barycentric subdivision K'^n of K^n with $P_1\subset|K_1^n|$ and $Q_1\cap|K_1^n|=\emptyset$.

Consider $F_1: (I\times K'^n)\to X\setminus U$ by $F_1(t, x)=f(x)$, then

$$F_1|K_1\cup K'=f|K_1\cup f \quad\text{and}\quad F_1(\partial(I\times K'^n)\setminus K_1\cup K')\subset A\setminus U.$$

Hence $[K_1, f|K_1]=[K, f]$ in $\mathcal{H}_n(X\setminus U, A\setminus U)$. Let $f|K_1=f_1$, then $f_1(K_1)\cap\bar{U}=\emptyset$. Since $[K_1, f_1]=[K, f]=0$ in $\mathcal{H}_n(X, A)$, we may find a representative (K_1, f_1) for which there is a Z_2-complex W^{n+1} and a map $F: W^{n+1}\to X$ such that

(1) $$K_1 \subset \partial W^{n+1},$$

(2) $$F|K_1 = f_1 \quad \text{and} \quad F(\partial W \backslash K_1) \subset A.$$

Let $P' = F^{-1}(X \backslash \mathrm{Int}\, A)$ and $Q = F^{-1}(\bar{U})$, then P' and Q are closed sets with $P' \cap Q = \emptyset$. Now let $P = P' \cup K_1$, then P is also a closed set. Since $K_1 \cap Q = \emptyset$ by the construction of K_1, P and Q are closed sets with $P \cap Q = \emptyset$. By Lemma 2, there exists a Z_2-complex $W_1 \subset W$ with $P \subset |W_1|$ and $Q \cap |W_1| = \emptyset$. Since $K_1 \subset W_1 \subset W$, $K_1 \subset P$ and $|\partial W| \cap P = |\partial W_1| \cap P$, we know that $K_1 \subset \partial W_1$.

Next we show that $F(\partial W_1 \backslash K_1) \subset A$. Let $x \in |\partial W_1 \backslash K_1|$. In the case when $x \notin P$, we know that $x \notin F^{-1}(X \backslash \mathrm{Int}\, A)$. It follows that $F(x) \notin (X \backslash \mathrm{Int}\, A)$ which shows $F(x) \in A$. In the case when $x \in P$, we see that $x \in |\partial W_1| \cap P = |\partial W| \cap P$. $x \in |\partial W|$ and $x \notin K_1$ implies $x \in |\partial W \backslash K_1|$. Since $F(\partial W \backslash K_1) \subset A$, we know $F(x) \in A$. This completes the proof of $F(\partial W_1 \backslash K_1) \subset A$. Since $K_1 \subset \partial W_1$, $F(\partial W_1 \backslash K_1) \subset A \backslash U$ and $F|W_1 \colon W_1 \to X \backslash U$, then $[K_1, f_1] = 0$ in $\mathcal{H}_n(X \backslash U, A \backslash U)$.

§ 6. Bordism categories.

(6.1) A *category with sum* $\mathcal{C} = \langle \mathcal{C}, +, \emptyset \rangle$ is a category $\mathcal{C}$ with a bifunctor $+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $\emptyset$ of $\mathcal{C}$ and following three natural isomorphisms;

(1) $$A + B \simeq B + A,$$

(2) $$(A+B)+C \simeq A+(B+C),$$

(3) $$\emptyset + A \simeq A.$$

Notice that every category with sum is a monoidal category in the sense of S. MacLane [**3**].

(6.2) A *bordism category* $\mathcal{C} = \langle \mathcal{C}, \partial, +, \emptyset \rangle$ is a category with sum $\langle \mathcal{C}, +, \emptyset \rangle$, a functor $\partial \colon \mathcal{C} \to \mathcal{C}$ and following three natural isomorphisms;

(1) $$\partial \circ \partial A \simeq \emptyset \text{ for any object } A \text{ of } \mathcal{C},$$

(2) $$\partial(A+B) \simeq \partial A + \partial B,$$

(3) $$\partial \emptyset \simeq \emptyset.$$

(6.3) If $\mathcal{C} = \langle \mathcal{C}, \partial, +, \emptyset \rangle$ is a bordism category, we say that the objects A and B are *bordant* if there exist objects U and V of $\mathcal{C}$ such that $A + \partial U \simeq B + \partial V$. This will be written $A \equiv B$. Then we have easily that $\equiv$ is an equivalence relation on the objects of $\mathcal{C}$. Denote the equivalence class of A by $[A]$ and define the *sum* $[A]+[B]$ by $[A+B]$. An object A is *closed* if $\partial A \simeq \emptyset$ and A is *bounds* if $A \equiv \emptyset$.

(6.4) The bordism semi-group $\Omega(\mathcal{C})$ of the bordism category $\mathcal{C}$ is the set of equivalence classes of closed objects of $\mathcal{C}$ with the operation induced by the sum in $\mathcal{C}$. Indeed $\Omega(\mathcal{C})$ is a commutative semi-group with a unit.

(6.5) A *graded bordism category* $\mathcal{C}=\{\langle\mathcal{C}_n, +_n, \emptyset_n\rangle, \partial_n\}_{n\in Z}$ is a sequence $\{\langle\mathcal{C}_n, +_n, \emptyset_n\rangle\}_{n\in Z}$ of categories with sum, a sequence $\{\partial_n\}$ of functors $\partial_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ and following natural isomorphisms;

(1) $$\partial_n(A+B)\simeq\partial_n A+\partial_n B\,,$$

(2) $$\partial_{n-1}\circ\partial_n A\simeq\emptyset_{n-2}\,,$$

(3) $$\partial_n\emptyset_n\simeq\emptyset_{n-1}\,.$$

Then $\langle\mathcal{C}, \partial, +, \emptyset\rangle=\langle\{\mathcal{C}_n\}, \{\partial_n\}, \{+_n\}, \{\emptyset_n\}\rangle$ is a bordism category. Objects of $\mathcal{C}_n$ and morphisms in $\mathcal{C}_n$ will be called respectively n-objects of $\mathcal{C}$ and n-morphisms in $\mathcal{C}$. n-objects A and B of $\mathcal{C}$ are *bordant* if there exist $(n+1)$-objects U and V of $\mathcal{C}$ such that $A+\partial U\simeq B+\partial V$. This will be written $A\equiv_n B$. Then we may define a *n-dimensional bordism semi-group* $\Omega_n(\mathcal{C})$ for each $n\in Z$ as the set of equivalence classes of closed n-objects of $\mathcal{C}$, and we have $\Omega(\mathcal{C})\approx\{\Omega_n(\mathcal{C})\}_{n\in Z}$.

(6.6) A category is *discrete* when every morphism is an identity. Every set C is the set of objects of a discrete category (just add one identity morphism $c\to c$ for each $c\in C$), and every discrete category is so determined by its set of objects. For every abelian group C with an operation $+$ and a unit 0, $\langle C, +, 0\rangle$ is regarded as a discrete category with sum.
For every chain complex (C, d), $\langle C, d, +, 0\rangle$ is a discrete bordism category. We have easily:

THEOREM 3. *Every chain complex is a discrete graded bordism category, and its homology groups coincide with its bordism groups.*

§ 7. Singular bordism theories.

(7.1) A category with sum $\mathcal{C}=\langle\mathcal{C}, +, \emptyset\rangle$ is *regular*, when $+$ is the finite sums in $\mathcal{C}$ and $\emptyset$ is an initial object of $\mathcal{C}$. A bordism category $\langle\mathcal{C}, \partial, +, \emptyset\rangle$ is *regular*, if $\langle\mathcal{C}, +, \emptyset\rangle$ is regular.
(7.2) Let $\mathcal{X}=\langle\mathcal{X}, +, \emptyset\rangle$ be a regular category with sum. A *singular bordism theory* $\langle\mathcal{C}, F, l\rangle$ on $\mathcal{X}$ is a triple in which;

(1) $\mathcal{C}=\langle\mathcal{C}, \partial, +, \emptyset\rangle$ is a regular bordism category,

(2) $F : \mathcal{C}\to\mathcal{X}$ is a functor such that

i) $$F(A+B)=F(A)+F(B)\,,$$

ii) $$F(\emptyset)=\emptyset\,,$$

(3) $l : F \circ \partial \rightarrow F$ is a natural transformation.

(7.3) Let $\langle C, F, l\rangle$ is a singular bordism theory on $\mathscr{X}$. For any object X of $\mathscr{X}$, form a category C/X whose objects are pairs (A, f) with A an object of C and $f \in \mathrm{Mor}_{\mathscr{X}}(F(A), X)$ and whose morphisms are given by letting $\mathrm{Mor}_{C/X}((A, f), (B, g))$ be the set of morphisms $\phi \in \mathrm{Mor}_C(A, B)$ such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\phi)} & F(B) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

commutes.

If (A, f) and (B, g) are objects of C/X and $A+B$ is a finite sum for A and B in C, then $F(A+B)$ is a finite sum for $F(A)$ and $F(B)$ in $\mathscr{X}$. The morphisms f and g give a unique morphism $f+g : F(A)+F(B) \rightarrow X$, and $(A, f)\tilde{+}(B, g)=(A+B, f+g)$ is the sum of (A, f) and (B, g) in C/X. If $\emptyset$ is an initial object of C and $\phi : F(\emptyset) \rightarrow X$ is the unique morphism, then $\tilde{\emptyset}=(\emptyset, \phi)$ is an initial object of C/X. From our definitions and regularity of $\mathscr{X}$ and C, we have:

PROPOSITION 8. *$\langle C/X, \tilde{+}, \tilde{\emptyset}\rangle$ is a regular category with sum.*

(7.4) Let $\tilde{\partial}(A, f)=(\partial A, f\circ l_A)$ and $\tilde{\partial}(\emptyset, \phi)=(\emptyset, \phi)$, then $\tilde{\partial}\tilde{\partial}(A, f)=(\partial\partial A, f\circ l_A\circ l_{\partial A}) \simeq(\emptyset, \phi)$.

If $\tilde{\phi} : (A, f) \rightarrow (B, g)$ is a morphism in C/X, then there exists a morphism $\phi : A \rightarrow B$ in C such that the diagram

$$\begin{array}{ccccc} F(\partial A) & \xrightarrow{\iota_A} & F(A) & \xrightarrow{f} & X \\ \downarrow{\scriptstyle F(\partial\phi)} & & \downarrow{\scriptstyle F(\phi)} & \nearrow{\scriptstyle g} & \\ F(\partial B) & \xrightarrow{\iota_B} & F(B) & & \end{array}$$

commutes.

Let $\tilde{\partial}(\tilde{\phi})=\partial\phi$ to define the functor $\tilde{\partial} : C/X \rightarrow C/X$. It is easily checked that the functor $\tilde{\partial} : C/X \rightarrow C/X$ has properties (1), (2) and (3) of (6.2).

PROPOSITION 9. *$\langle C/X, \tilde{\partial}, \tilde{+}, \tilde{\emptyset}\rangle$ is a regular bordism category.*

(7.5) Define $\tilde{F} : C/X \rightarrow \mathscr{X}$ by $\tilde{F}(A, f)=F(A)$ for any object (A, f) of C/X and $\tilde{F}(\tilde{\phi})=F(\phi)$ for any morphism $\tilde{\phi} : (A, f) \rightarrow (B, g)$ such that $\tilde{\phi}=\phi : A \rightarrow B$. Then $\tilde{F}\{(A, f)\tilde{+}(B, g)\}=\tilde{F}\{(A+B, f+g)\}=F(A+B)=\tilde{F}(A, f)+\tilde{F}(B, g)$ and $\tilde{F}(\emptyset, \phi)=F(\emptyset)=\emptyset$.

Define the natural transformation $\tilde{l} : \tilde{F}\tilde{\partial} \rightarrow F$ by $\tilde{l}_{(A,f)}=l_A : F(\partial A) \rightarrow F(A)$ for an object (A, f) of C/X. From the above definitions and Proposition 9, we have following theorem;

THEOREM 4. *If* $\langle \mathcal{C}, F, l\rangle$ *is a singular bordism theory on* $\mathcal{X}$ *then* $\langle \mathcal{C}/X, \tilde{F}, \tilde{l}\rangle$ *is also a singular bordism theory on* $\mathcal{X}$ *for any object* X *of* $\mathcal{X}$.

(7.6) Let $\mathcal{K}$ be the category whose n-objects are n-dimensional Z_2-complexes and whose n-morphisms are pl-imbeddings which take boundary into boundary. This category $\mathcal{K}$ has finite sums $+$ given by the disjoint union and has an initial object $\emptyset$ given by the empty Z_2-complex, so that $\langle \mathcal{K}, \partial, +, \emptyset\rangle$ is a regular graded bordism category.

Let $\mathcal{X}$ be the category of topological spaces and continuous maps, then $\mathcal{X}$ has finite sums $+$ given by the disjoint union and an initial object $\emptyset$ given by the empty space, so that $\langle \mathcal{X}, +, \emptyset\rangle$ is a regular category with sum. Let F: $\mathcal{K}\rightarrow\mathcal{X}$ be the forgetful functor, then the natural transformation $l: F\partial\rightarrow F$ are given by the inclusion $l_K: |\partial K|\rightarrow|K|$ in $\mathcal{X}$ for each object K of $\mathcal{K}$. $\langle \mathcal{K}, F, l\rangle$ is a singular bordism theory on $\mathcal{X}$ and this singular bordism theory defines the ordinary Z_2-homology theory on $\mathcal{X}$.

For any space X, $\langle \mathcal{K}/X, \tilde{F}, \tilde{l}\rangle$ is also a singular bordism theory on $\mathcal{X}$, and $\Omega_n(\mathcal{K}/X)\approx H_n(X; Z_2)$, $\Omega_n(\mathcal{K}/X/Y)\approx H_n(X\times Y; Z_2)$.

REMARK 1) We may consider the oriented case by a suitable generalization of the notion of orientation.

REMARK 2) Given two singular bordism theories on $\mathcal{X}$, there exists their product singular bordism theory on $\mathcal{X}$.

(7.7) Other examples are classical bordism theories (cf. [2] and [4]). Let $\mathcal{D}_0$ be the graded category whose n-objects are oriented n-dimensional compact differential manifolds and whose n-morphisms are orientation preserving differential imbeddings which take boundary into boundary. Let F be the forgetful functor $F: \mathcal{D}_0\rightarrow\mathcal{X}$, then there exists a natural transformation $l: F\partial\rightarrow F$ and $\langle \mathcal{D}_0, F, l\rangle$ is the singular bordism theory on $\mathcal{X}$ which define a generalized homology theory on $\mathcal{X}$.

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