J. Math. Soc. Japan
Vol. 34, No. 4, 1982

Invariant subspaces of unitary operators

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(Received Aug. 25, 1980)
(Revised April 13, 1981)

§ 1. Introduction.

Let $\mathcal{H}$ be a separable, complex Hilbert space and U a unitary operator on $\mathcal{H}$. If

$$\mathcal{H}=M(C)=\cdots\oplus U^{*}C\oplus C\oplus UC\oplus U^{2}C\oplus\cdots$$

for some closed subspace C of $\mathcal{H}$, then U is said to be a bilateral shift of multiplicity n, where U^{*} is the adjoint operator of U and n is the dimension of C. When U is an isometry with $\mathcal{H}=M_{+}(C)=C\oplus UC\oplus U^{2}C\oplus\cdots$, then U is said to be a unilateral shift of multiplicity n. The study of invariant subspaces of shifts was begun by Beurling [1]. He characterized all invariant subspaces of unilateral shifts of multiplicity one. Lax [8] extended Beurling's result to unilateral shifts of finite multiplicity. Helson-Lowdenslager [6] and Halmos [5] characterized all invariant subspaces of bilateral shifts of arbitrary multiplicity. We are going to study invariant subspaces of unitary operators in order to generalize previous results concerning bilateral shifts.

A unitary operator U on $\mathcal{H}$ is said to be pure if every invariant subspace for U is reducing. Every unitary operator U can be written as the direct sum of a bilateral shift and a pure unitary operator [**2**; pp. 62-63]. Namely, there exists a closed subspace C of $\mathcal{H}$ such that $U^{n}C$, $n=0, \pm 1, \pm 2, \cdots$ are mutually orthogonal and the restriction of U to $\mathcal{H}\ominus M(C)$ is pure. Setting $\mathcal{K}=\mathcal{H}\ominus M(C)$, we get a decomposition

$$(*)\qquad \mathcal{H}=M(C)\oplus\mathcal{K}\,,\qquad U=S\oplus V\,,$$

so that $S=U|M(C)$ is a bilateral shift and $V|\mathcal{K}$ is pure. It is known that such decomposition is not necessarily unique. In fact, any bilateral shift of infinite multiplicity allows infinitely many non-isomorphic decompositions of the form (*). We call a decomposition $\mathcal{H}=M(C_{0})\oplus\mathcal{K}_{0}$ of the form (*) maximal if we have $M(C')=M(C_{0})$ for any decomposition $\mathcal{H}=M(C')\oplus\mathcal{K}'$ of the form (*) with

This research was partially supported by Grant-in-Aid for Scientific Research (No. 56540052), Ministry of Education.

$M(C')\supseteqq M(C_0)$. In Section 3, we shall show that, for any unitary operators, maximal decompositions exist and are mutually isomorphic.

In Section 4, we shall generalize the characterization of invariant subspaces of bilateral shifts of arbitrary multiplicity. Fix a maximal decomposition $\mathcal{H}=M(C)\oplus\mathcal{K}$ for U. We set

$$\mathcal{H}_+^\alpha=M_+(C_\alpha)\oplus\mathcal{K}_\alpha\,,$$

where C_α is a closed subspace of C and $\mathcal{K}_\alpha$ is an invariant subspace of $\mathcal{K}$.

A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is called invariant if $Uf\in\mathcal{M}$ for every $f\in\mathcal{M}$. We say that an invariant subspace $\mathcal{M}$ is a Beurling-Wiener subspace if it has the form

$$\mathcal{M}=V\mathcal{H}_+^\alpha\oplus V_0PM(C)\,,$$

where P is a projection in the commutant $\{U\}'$ of U with $PM(C)\subseteqq M(C)$, and V and V_0 are partial isometries in $\{U\}'$ with initial spaces $\mathcal{H}_+^\alpha$ and $PM(C)$, respectively. We shall show that every invariant subspace $\mathcal{M}$ for U is a Beurling-Wiener subspace.

Let U be a unitary operator on $\mathcal{H}$ with spectral measure $E(\delta)$. U is said to be absolutely continuous (singular) if the measure $\mu(\delta)=(E(\delta)f, f)$ for each $f\in\mathcal{H}$ is absolutely continuous (singular). In general, we have $\mathcal{H}=\mathcal{H}_{\mathrm{ab}}\oplus\mathcal{H}_{\mathrm{sing}}$, $\mathcal{H}_{\mathrm{ab}}$ and $\mathcal{H}_{\mathrm{sing}}$ are reducing subspaces for U so that $U|\mathcal{H}_{\mathrm{ab}}$ is absolutely continuous while $U|\mathcal{H}_{\mathrm{sing}}$ is singular (cf. [5]).

Let C_0 be a separable, complex Hilbert space, and $d\theta$ the Lebesgue measure on the unit circle. Then $L^2_{C_0}$ is the Hilbert space of all weakly measurable functions f from the unit circle to C_0 for which $\int_0^{2\pi}\|f(e^{i\theta})\|^2_{C_0}d\theta<\infty$. Suppose $e^{i\theta}$ is the identity function on the unit circle so that the bilateral shift on $L^2_{C_0}$ is given by $S_0: f\to e^{i\theta}f$. A measurable range function $J=J(e^{i\theta})$ in C_0 is a function on the circle taking values in the family of closed subspaces of C_0 such that the orthogonal projection $P(e^{i\theta})$ on $J(e^{i\theta})$ is weakly measurable in the operator sense. For each measurable range function J, let $\mathcal{L}_J$ be the set of all functions f in $L^2_{C_0}$ such that $f(e^{i\theta})$ lies in $J(e^{i\theta})$ almost everywhere. It is known (cf. [7, p. 59]) that every reducing subspace $\mathcal{M}$ of S_0 has the form $\mathcal{L}_J$ for some measurable range function J. It is known [3, pp. 55-56] that a unitary operator is absolutely continuous if and only if it is a restriction to a reducing subspace of a bilateral shift. Hence for a unitary operator U, its absolutely continuous part U_{ab} is unitary equivalent to an $S_0|\mathcal{L}_J$ for some measurable range function J.

§ 2. Measurable range functions.

Let $J=J(e^{i\theta})$ be a measurable range function taking values in the family of closed subspaces of a separable Hilbert space C_0. For each measurable set E on the unit circle, χ_E is the characteristic function of E. For each measurable range function J, let $\dim J(e^{i\theta})$ be the dimension of closed subspace $J(e^{i\theta})$. When J_1 and J_2 are measurable range functions in C_0, we shall write $J_1 \supseteqq J_2$ if $J_1(e^{i\theta}) \supseteqq J_2(e^{i\theta})$ a.e. If $J_1 \supseteqq J_2$ and $J_1 \subseteqq J_2$, we shall write $J_1=J_2$. When J is a measurable range function with $E=\{\theta ; \dim J(e^{i\theta}) \geqq 1\}$, we call Φ a rigid function for J if Φ is a measurable function on the unit circle with values in C_0, $\Phi(e^{i\theta}) \in J(e^{i\theta})$ a.e. and $\|\Phi(e^{i\theta})\|_{C_0}=1$ a.e. on E. We say that J is generated by rigid functions, if there exists a family of rigid functions $\{\Phi_\alpha\}$ for J such that, for almost every $e^{i\theta} \in E$, $\{\Phi_\alpha(e^{i\theta})\}$ forms an orthonormal basis of $J(e^{i\theta})$. In general, J is not generated by rigid functions in J. In this section, we shall show that there exists a measurable range function $J_0 \subseteqq J$ which is maximal among those range functions generated by rigid functions for J, i.e. if J_{10} is a measurable range function $\subseteqq J$ which is generated by rigid functions and $J_{10} \supseteqq J_0$, then $J_{10}=J_0$.

THEOREM 1. *Let J be a measurable range function in C_0 and set $F=\{\theta ; \dim J(e^{i\theta}) \geqq 1\}$. Then*

(1) $J(e^{i\theta})=\sum_{k=0}^{\infty} \oplus J_k(e^{i\theta})$ *a.e., where $J_0, J_1, \cdots$ are measurable range functions having the following properties: J_0 is a maximal measurable range function $\subseteqq J$ which is generated by rigid functions, and J_j, for each $j \geqq 1$, is a maximal measurable range function in $J_{j-1,1}$ which is generated by rigid functions, where $J_{j-1,1}(e^{i\theta}) = J(e^{i\theta}) \ominus \sum_{k=0}^{j-1} \oplus J_k(e^{i\theta})$ a.e. In this case, $\dim J_k(e^{i\theta})$ is a constant n_k a.e. on the set $F_k=\{e^{i\theta} : \dim J_k(e^{i\theta}) \geqq 1\}$.*

(2) $\dim J(e^{i\theta})=\sum_{k=0}^{\infty} n_k \chi_{F_k}(e^{i\theta})$ *a.e. If $n_j=\infty$ for some j, then $d\theta(F_{j+1})=0$ and so* $\dim J(e^{i\theta})=\sum_{k=0}^{j} n_k \chi_{F_k}(e^{i\theta})$ *a.e.*

(3) *If $J=\sum_{k=0}^{\infty} \oplus J_{1k}$ is another decomposition satisfying the property stated in (1), then* $\dim J_{1k}(e^{i\theta})=\dim J_k(e^{i\theta})$ *a.e.* $(k=0, 1, 2, \cdots)$.

The theorem above is an immediate consequence of the following lemmas.

LEMMA 1. *Let J be a measurable range function with $J \subseteqq C_0$ and set $E=\{\theta ; \dim J(e^{i\theta}) \geqq 1\}$. Then, there exists Φ in $L^2_{C_0}$ such that $\|\Phi(e^{i\theta})\|_{C_0}=1$ a.e. on E and $\Phi(e^{i\theta}) \in J(e^{i\theta})$ a.e.*

PROOF. Let $e_0, e_1, \cdots$ be an orthonormal basis for C_0 and set $k_n(e^{i\theta})=P(e^{i\theta})e_n$, where $P(e^{i\theta})$ is the othogonal projection on $J(e^{i\theta})$; then $J(e^{i\theta})$ is the closed linear span of $\{k_n(e^{i\theta})\}_{n=0}^{\infty}$. If $\theta \in \{\theta ; \|k_0(e^{i\theta})\|_{C_0} \neq 0\}$, set

$$\phi_0(e^{i\theta})=k_0(e^{i\theta})/\|k_0(e^{i\theta})\|_{C_0}$$

and, otherwise, set $\phi_0(e^{i\theta})=0$. If $\theta\in\{\theta\,;\|k_1(e^{i\theta})-(k_1(e^{i\theta}),\ \phi_0(e^{i\theta}))_{C_0}\phi_0(e^{i\theta})\|_{C_0}\neq 0\}$, set

$$\phi_1(e^{i\theta})=\frac{k_1(e^{i\theta})-(k_1(e^{i\theta}),\ \phi_0(e^{i\theta}))_{C_0}\phi_0(e^{i\theta})}{\|k_1(e^{i\theta})-(k_1(e^{i\theta}),\ \phi_0(e^{i\theta}))_{C_0}\phi_0(e^{i\theta})\|_{C_0}}$$

and, otherwise, set $\phi_1(e^{i\theta})=0$. In general, we can use the orthogonalization procedure due to Schmidt. Hence $J(e^{i\theta})$ is the closed linear span of $\{\phi_n(e^{i\theta})\}_{n=0}^{\infty}$ such that $\|\phi_n(e^{i\theta})\|_{C_0}=\chi_{E_n}(e^{i\theta})$ a.e. for some measurable set E_n as $n=0, 1, 2, \cdots$ and $(\phi_n(e^{i\theta}),\ \phi_m(e^{i\theta}))_{C_0}=0$ a.e. as $n\neq m$. Moreover $\bigcup_{n=0}^{\infty}E_n\subseteqq E$ and $d\theta\left(\bigcup_{n=0}^{\infty}E_n\right)=d\theta(E)$. A desired function Φ can now be obtained by setting

$$\Phi(e^{i\theta})=\phi_0(e^{i\theta})+\sum_{n=0}^{\infty}(1-\chi_{\bigcup_{j=0}^{n}E_j}(e^{i\theta}))\phi_{n+1}(e^{i\theta})\,.$$

LEMMA 2. *Let J be any measurable range function in C_0 and set $E=\{\theta:\dim J(e^{i\theta})\geqq 1\}$. Suppose $d\theta(E)>0$ and define $d=d_J=\text{ess.}\inf\{\dim J(e^{i\theta}):\theta\in E\}$. Then there exists a measurable range function $J_0\subseteqq J$ such that J_0 is generated by d rigid functions for J and such that $J(e^{i\theta})=J_0(e^{i\theta})$ for almost every θ with $\dim J(e^{i\theta})=d$.*

PROOF. We take a rigid function Φ_1 for J by using Lemma 1 and set $J^{(1)}(e^{i\theta})=J(e^{i\theta})\ominus[\Phi_1(e^{i\theta})]$, where [ ] denotes the closed subspace generated by the vectors in the bracket. Then $J^{(1)}$ is a measurable range function and ess.inf$\{\dim J^{(1)}(e^{i\theta}):\theta\in E\}=d-1$. If $d-1\geqq 1$, then we can take another rigid function Φ_2 for $J^{(1)}$. We then set $J^{(2)}(e^{i\theta})=J^{(1)}(e^{i\theta})\ominus[\Phi_2(e^{i\theta})]=J(e^{i\theta})\ominus[\Phi_1(e^{i\theta}),\ \Phi_2(e^{i\theta})]$. A finite or countable induction will provide a sequence of d rigid functions $\Phi_1, \Phi_2, \cdots$. We denote by $J_0(e^{i\theta})$ the closed subspace generated by those Φ_n's. Our construction shows that J_0 has the desired property.

The proof of Theorem 1 is now obvious; for we have only to repeat the same process for $J\ominus J_0$ and use the induction, if necessary.

§ 3. Maximal decompositions of unitary operators.

Let U be a unitary operator on a separable Hilbert space $\mathscr{H}$. In this section, we show, by using Theorem 1, that maximal decompositions for U exist and are mutually isometrically isomorphic.

THEOREM 2. *Let U be a unitary operator on $\mathscr{H}$.*

(1) *There are closed subspaces C and $\mathscr{K}$ of $\mathscr{H}$ such that*

$$\mathscr{H}=M(C)\oplus\mathscr{K}\quad(U=S\oplus V)$$

is a maximal decomposition for U and $\mathscr{K}\supset\mathscr{H}_{\text{sing}}$.

(2) *If* $\dim C=\infty$ *in* (1), *then* $\mathcal{K}=\mathcal{H}_{\mathrm{sing}}$ *and so* $V=U_{\mathrm{sing}}$.

(3) *If* $\mathcal{H}=M(C_1)\oplus\mathcal{K}_1$ $(U=S_1\oplus V_1)$ *is another maximal decomposition of* U, *then we have*

$$M(C_1)=WM(C) \quad and \quad \mathcal{K}_1=W_0\mathcal{K}\,,$$

where W *and* W_0 *are partial isometries in* $\{U\}'$ *with initial spaces* $M(C)$ *and* $\mathcal{K}$, *respectively. Hence* S_1 *and* V_1 *are unitary equivalent to* S *and* V, *respectively.*

PROOF. If $\mathcal{H}=\mathcal{H}_{\mathrm{ab}}\oplus\mathcal{H}_{\mathrm{sing}}$, where $U|\mathcal{H}_{\mathrm{ab}}$ is absolutely continuous and $U|\mathcal{H}_{\mathrm{sing}}$ is singular, then we can assume $\mathcal{H}_{\mathrm{ab}}=\mathcal{L}_J$ for some measurable range function J in some Hilbert space C_0. Set $F=\{\theta\,;\dim J(e^{i\theta})\geqq 1\}$. When $d\theta(F)\neq 2\pi$, $U|\mathcal{H}_{\mathrm{ab}}$ is pure and so U is pure. For if $U|\mathcal{H}_{\mathrm{ab}}$ is not pure, then there exists Φ in $\mathcal{H}_{\mathrm{ab}}$ such that $U^n\Phi$ is orthogonal to Φ for all integers n. This implies that $\|\Phi(e^{i\theta})\|_{C_0}=1$ a.e. and $\Phi(e^{i\theta})\in J(e^{i\theta})$ a.e. and so $d\theta(F)=2\pi$. This contradiction implies that $U|\mathcal{H}_{\mathrm{ab}}$ is pure and hence $\mathcal{H}=\mathcal{K}$ is the desired maximal decomposition. When $d\theta(F)=2\pi$, there exists by (1) of Theorem 1 a maximal measurable range function $J_0\subseteqq J$ which is generated by rigid functions $\{\Phi_j\}_{j=1}^n$ such that $\|\Phi_j(e^{i\theta})\|_{C_0}=1$ a.e. for $1\leqq j\leqq n$. Suppose C is a closed subspace of $\mathcal{H}_{\mathrm{ab}}$ generated by $\{\Phi_j\}_{j=1}^n$ and $\mathcal{K}=\mathcal{L}_{J\ominus J_0}\oplus\mathcal{H}_{\mathrm{sing}}$. Then $\mathcal{H}=M(C)\oplus\mathcal{K}$ is the desired maximal decomposition and this implies (1).

If $\dim C=\infty$, (2) of Theorem 1 implies $\mathcal{L}_{J\ominus J_0}=\{0\}$ and so $\mathcal{K}=\mathcal{H}_{\mathrm{sing}}$ and this implies (2).

By (3) of Theorem 1, if $\mathcal{H}=M(C_1)\oplus\mathcal{K}_1$ $(U=S_1\oplus V_1)$ is another maximal decomposition for U, then $\dim C_1=\dim C$. It is well known that there exists a partial isometry W in $\{U\}'$ with initial space $M(C)$ such that $M(C_1)=WM(C)$. We shall show that there exists a partial isometry W_0 in $\{U\}'$ with initial space $\mathcal{K}$ such that $\mathcal{K}_1=W_0\mathcal{K}$. Since $\mathcal{K}\cap\mathcal{K}_1\supseteqq\mathcal{H}_{\mathrm{sing}}$, we can assume $\mathcal{H}=\mathcal{H}_{\mathrm{ab}}=\mathcal{L}_J$ for some measurable range function J in some Hilbert space C_0. These two decompositions give two decompositions for the measurable range function J such that

$$J=J_0\oplus J_1=J_{10}\oplus J_{11}\,,$$

where $M(C)=\mathcal{L}_{J_0}$, $\mathcal{K}=\mathcal{L}_{J_1}$, $M(C_1)=\mathcal{L}_{J_{10}}$ and $\mathcal{K}_1=\mathcal{L}_{J_{11}}$. Suppose $J_1=\sum_{j=0}^{\infty}\oplus J_{1,j}$ and $J_{11}=\sum_{j=0}^{\infty}\oplus J_{11,j}$ are the decompositions in Theorem 1, then $J=J_0\oplus\sum_{j=0}^{\infty}\oplus J_{1,j}$ and $J=J_{10}\oplus\sum_{j=0}^{\infty}\oplus J_{11,j}$ are the decompositions of J in Theorem 1, too. For each j, set $F_{1,j}=\{\theta\,;\dim J_{1,j}(e^{i\theta})\geqq 1\}$ and $F_{11,j}=\{\theta\,;\dim J_{11,j}(e^{i\theta})\geqq 1\}$, then $\dim J_{1,j}(e^{i\theta})=n_{1,j}\chi_{F_{1,j}}(e^{i\theta})$ and $\dim J_{11,j}(e^{i\theta})=n_{11,j}\chi_{F_{11,j}}(e^{i\theta})$ a.e. Then,

$$\mathcal{K}=\mathcal{L}_{J_1}=\sum_{j=0}^{\infty}\oplus\chi_{F_{1,j}}M(C_{1,j})$$

and

$$\mathcal{K}_1=\mathcal{L}_{J_{11}}=\sum_{j=0}^{\infty}\oplus\chi_{F_{11,j}}M(C_{11,j}),$$

where $C_{1,j}$ and $C_{11,j}$ are closed subspaces of $\mathcal{K}$ and $\mathcal{K}_1$, respectively and $\dim C_{1,j}=n_{1,j}$ and $\dim C_{11,j}=n_{11,j}$. Since $\dim J_{1,j}(e^{i\theta})=\dim J_{11,j}(e^{i\theta})$ a.e. $(j=0, 1, 2, \cdots)$, $\dim C_{1,j}=\dim C_{11,j}$ and $F_{1,j}=F_{11,j}$. This implies that there exists a partial isometry W_0 in $\{U\}'$ with initial space $\mathcal{K}$ such that $\mathcal{K}_1=W_0\mathcal{K}$.

COROLLARY 1. *Let C' be a closed subspace of $\mathcal{H}$ such that $M(C')\subseteqq\mathcal{H}$ and suppose $\mathcal{H}=M(C)\oplus\mathcal{K}$ is a maximal decomposition. Then* $\dim C'\leqq\dim C$.

4. Invariant subspaces of unitary operators.

In this section, we shall generalize the characterization of invariant subspaces of bilateral shifts of arbitrary multiplicity. Let U be a unitary operator on a separable Hilbert space $\mathcal{H}$. Fix a maximal decomposition $\mathcal{H}=M(C)\oplus\mathcal{K}$ $(U=S\oplus V)$ for U. As in Section 1, set

$$\mathcal{H}_+^{\alpha}=M_+(C_\alpha)\oplus\mathcal{K}_\alpha,$$

where C_α is a closed subspace of C and $\mathcal{K}_\alpha$ is an invariant subspace of $\mathcal{K}$.

THEOREM 3. *Every invariant subspace $\mathcal{M}$ for U is a Beurling-Wiener subspace, that is, $\mathcal{M}$ has the form*

$$\mathcal{M}=V\mathcal{H}_+^{\alpha}\oplus V_0PM(C),$$

where P is a projection in $\{U\}'$ with $PM(C)\subseteqq M(C)$, and V and V_0 are partial isometries in $\{U\}'$ with initial space $M(C_\alpha)\oplus\mathcal{K}_\alpha$ and $PM(C)$, respectively.

PROOF. We can write $\mathcal{M}=M_+(C_{\mathcal{M}})\oplus\mathcal{M}_{-\infty,\mathrm{ab}}\oplus\mathcal{M}_{-\infty,\mathrm{sing}}$ for some closed subspace $C_{\mathcal{M}}$ of $\mathcal{M}$, where $\mathcal{M}_{-\infty}=\bigcap_n U^n\mathcal{M}$. By Corollary 1, $\dim C_{\mathcal{M}}\leqq\dim C$ and then it is well known that there exists a partial isometry V_1 in $\{U\}'$ with initial space $M(C_\alpha)$, where C_α is a closed subspace of C with $\dim C_\alpha=\dim C_{\mathcal{M}}$ and $M_+(C_{\mathcal{M}})=V_1M_+(C_\alpha)$.

Case I. Suppose $\dim C=\infty$. Then, by (2) of Theorem 2, we have $\mathcal{H}=M(C)\oplus\mathcal{H}_{\mathrm{sing}}$. Since $\mathcal{M}_{-\infty,\mathrm{sing}}\subseteqq\mathcal{H}_{\mathrm{sing}}=\mathcal{K}$ and $\mathcal{M}_{-\infty,\mathrm{ab}}\subseteqq\mathcal{H}_{\mathrm{ab}}=M(C)$, there exist projections P_α and P in $\{U\}'$ with $\mathcal{M}_{-\infty,\mathrm{sing}}=P_\alpha\mathcal{K}$ and $\mathcal{M}_{-\infty,\mathrm{ab}}=PM(C)$. Set $V=V_1+P_\alpha$, $V_0=P$ and $\mathcal{K}_\alpha=P_\alpha\mathcal{K}$, then $\mathcal{M}=V\mathcal{H}_+^{\alpha}\oplus V_0PM(C)$.

Case II. Suppose $\dim C<\infty$. Let $\mathcal{M}_{-\infty}=M(B_1)\oplus\mathcal{N}_1$ is a maximal decomposition for $U|\mathcal{M}_{-\infty}$, then we have

$$\mathcal{M}=M_+(C_{\mathcal{M}})\oplus M(B_1)\oplus\mathcal{N}_1.$$

We may assume $\mathcal{H}_{\mathrm{ab}}=\mathcal{L}_J$ for some measurable range function J and so there exists a measurable range function J_1 with $J_1\subseteqq J$ such that $\mathcal{N}_1=\mathcal{L}_{J_1}\oplus\mathcal{N}_{1,\mathrm{sing}}$ and $\mathcal{M}_{-\infty,\mathrm{sing}}=\mathcal{N}_{1,\mathrm{sing}}$.

Set $M(B_2)=M(C_{\mathcal{M}})\oplus M(B_1)$ and $\tilde{\mathcal{M}}=M(C_{\mathcal{M}})\oplus\mathcal{M}_{-\infty}$, then $\tilde{\mathcal{M}}=M(B_2)\oplus\mathcal{N}_1$ is a maximal decomposition for $U|\tilde{\mathcal{M}}$. For if $\tilde{\mathcal{M}}=M(B_2')\oplus\mathcal{N}_1'$ is a decomposition of the form (*) for $U|\tilde{\mathcal{M}}$ with $M(B_2')\supseteqq M(B_2)$, then there exists a closed subspace B_2'' in $\tilde{\mathcal{M}}$ such that $M(B_2'')=M(B_2')\ominus M(B_2)$ because $\dim B_2<\infty$ and $\dim B_2'<\infty$. Then $\mathcal{N}_1\supseteqq\mathcal{M}(B_2'')$ and so $M(B_2')=M(B_2)$ since $\mathcal{M}_{-\infty}=M(B_1)\oplus\mathcal{N}_1$ is a maximal decomposition. Suppose $\tilde{\mathcal{M}}^{\perp}=\mathcal{H}\ominus\tilde{\mathcal{M}}=M(B_3)\oplus\mathcal{N}_2$ is a maximal decomposition for $U|\tilde{\mathcal{M}}^{\perp}$. Then $\mathcal{H}=M(B_2)\oplus M(B_3)\oplus\mathcal{N}_1\oplus\mathcal{N}_2$. Suppose

$$\mathcal{N}_1\oplus\mathcal{N}_2=M(B_4)\oplus\mathcal{N}_4$$

is a maximal decomposition for $U|\mathcal{N}_1\oplus\mathcal{N}_2$. Set $M(C_1)=M(B_2)\oplus M(B_3)\oplus M(B_4)$ and $\mathcal{K}_1=\mathcal{N}_4$, then

$$\mathcal{H}=M(C_1)\oplus\mathcal{K}_1$$

is a maximal decomposition for U. There exists a measurable range function $J_2\subseteqq J$ such that $\mathcal{N}_2=\mathcal{L}_{J_2}\oplus\mathcal{N}_{2,\mathrm{sing}}$. If there exist measurable range functions Q_1 and Q_2 with $Q_i\subseteqq J_i$ $(i=1, 2)$ such that $M(B_4)=\mathcal{L}_{Q_1}\oplus\mathcal{L}_{Q_2}$, then

$$\mathcal{M}=M_+(C_{\mathcal{M}})\oplus M(B_1)\oplus\mathcal{L}_{Q_1}\oplus\mathcal{L}_{J_1\ominus Q_1}\oplus\mathcal{N}_{1,\mathrm{sing}}\,.$$

Moreover $M_+(C_{\mathcal{M}})\oplus M(B_1)\oplus\mathcal{L}_{Q_1}\subseteqq M(C_1)$ and $\mathcal{L}_{J_1\ominus Q_1}\oplus\mathcal{N}_{1,\mathrm{sing}}\subseteqq\mathcal{K}_1$. Since $\mathcal{H}=M(C_1)\oplus\mathcal{K}_1$ is a maximal decomposition for U, (3) in Theorem 2 implies that $\mathcal{M}$ is a Beurling-Wiener subspace. We shall show that there exist measurable range functions Q_1 and Q_2 with $Q_i\subseteqq J_i$ $(i=1, 2)$ such that $M(B_4)=\mathcal{L}_{Q_1}\oplus\mathcal{L}_{Q_2}$. Set $F_j=\{\theta\,;\dim J_j(e^{i\theta})\geqq 1\}$ $(j=1, 2)$. If $d\theta(F_1\cup F_2)\neq 2\pi$, then $M(B_4)=\{0\}$ and so $Q_1(e^{i\theta})=Q_2(e^{i\theta})=\{0\}$ a.e. If $d\theta(F_1\cup F_2)=2\pi$, suppose $J_j=\sum_{k=0}^{\infty}\oplus J_{jk}$ is the decomposition in (1) of Theorem 1. Set $F_{jk}=\{\theta\,;\dim J_{jk}(e^{i\theta})\geqq 1\}$ $(j=1, 2$, and $k=0, 1, 2, \cdots)$, then $F_{jk}\supsetneqq F_{jk+1}$. Let m be the largest number such that $d\theta(F_{10}\cup F_{2m})=2\pi$, then the m is a finite number because $\dim C<\infty$. By (1) of Theorem 1, $\dim J_{jk}(e^{i\theta})=n_{jk}\chi_{F_{jk}}(e^{i\theta})$ a.e. $(j=1, 2$ and $k=0, 1, 2, \cdots)$. If $n_{10}=\sum_{k=0}^{m}n_{2k}$, set $Q_{10}=J_{10}$ and $Q_{20}=\left(\sum_{k=0}^{m}\oplus J_{2k}\right)(1-\chi_{F_{10}})$, then $\dim Q_{10}(e^{i\theta})=n_{10}\chi_{F_{10}}(e^{i\theta})$ a.e., $\dim Q_{20}(e^{i\theta})=n_{10}(1-\chi_{F_{10}}(e^{i\theta}))$ a.e. and so $\dim(Q_{10}(e^{i\theta})+Q_{20}(e^{i\theta}))=n_{10}$ a.e. If $n_{10}<\sum_{k=0}^{m}n_{2k}$, then there exists a measurable range function Q_{20} such that $Q_{20}\subset\left(\sum_{k=0}^{m}\oplus J_{2k}\right)(1-\chi_{F_{10}})$ and $\dim Q_{20}(e^{i\theta})=n_{10}(1-\chi_{F_{10}}(e^{i\theta}))$ a.e. Set $Q_{10}=J_{10}$, then $\dim(Q_{10}(e^{i\theta})+Q_{20}(e^{i\theta}))=n_{10}$ a.e. If $n_{10}>\sum_{k=0}^{m}n_{2k}$, there exist measurable range functions Q_{10} and Q_{20} with $Q_{10}\subset J_{10}$ and $Q_{20}\subset\left(\sum_{k=0}^{m}\oplus J_{2k}\right)(1-\chi_{F_{10}})$ such that $\dim Q_{10}(e^{i\theta})=\left(\sum_{k=0}^{m}n_{2k}\right)\chi_{F_{10}}(e^{i\theta})$ a.e. and $\dim Q_{20}(e^{i\theta})=\left(\sum_{k=0}^{m}n_{2k}\right)(1-\chi_{F_{10}}(e^{i\theta}))$ a.e. Then $\dim(Q_{10}(e^{i\theta})+Q_{20}(e^{i\theta}))=$

$\sum_{k=0}^{m} n_{2k}$ a.e. Suppose $J_1^{(2)}$ and $J_2^{(2)}$ are measurable range functions such that $J_1^{(2)}(e^{i\theta})=J_1(e^{i\theta})\ominus Q_{10}(e^{i\theta})$ and $J_2^{(2)}(e^{i\theta})=J_2(e^{i\theta})\ominus Q_{20}(e^{i\theta})$ a.e. Applying the same reasoning to measurable range functions $J_1^{(2)}$ and $J_2^{(2)}$ in place of J_1 and J_2, we obtain measurable range functions Q_{11} and Q_{21} such that $Q_{11}\subset J_1^{(2)}$, $Q_{21}\subset J_2^{(2)}$ and $\dim(Q_{11}(e^{i\theta})+Q_{21}(e^{i\theta}))=n_1$ a.e. Iterating, we obtain measurable range functions $\{Q_{jk}\}_{k=0}^{s}$ ($j=1, 2$) such that $Q_{jk}\subseteqq J_j$ ($j=1, 2$ and $0\leqq k\leqq s$) and $\dim(Q_{1k}(e^{i\theta})+Q_{2k}(e^{i\theta}))=n_k$ a.e. The s is a finite number because $\dim C<\infty$. Set $Q_j=\sum_{k=0}^{s}\oplus Q_{jk}$, then $Q_j\subseteqq J_j$ ($j=1, 2$) and so $M(B_4)=\mathcal{L}_{Q_1}\oplus\mathcal{L}_{Q_2}$.

If $\mathcal{N}=V\mathcal{K}'$ for some partial isometry V in $\{U\}'$ with initial space $\mathcal{K}'$, where $\mathcal{K}'$ is an invariant subspace $\subset\mathcal{K}$, then we say $\mathcal{N}$ is a pure set. If $\mathcal{N}$ is a pure set, then $U|\mathcal{N}$ is a pure unitary operator. The converse is not valid. If $\mathcal{K}_1$ gives a maximal decomposition $\mathcal{H}=M(C_1)\oplus\mathcal{K}_1$ for U, then $\mathcal{K}_1$ is a pure set by (3) of Theorem 2.

COROLLARY 2. *An invariant subspace for U has the form $\mathcal{M}=V\mathcal{H}_+^{\alpha}$ for some partial isometry V in $\{U\}'$ with an initial space $\mathcal{H}_+^{\alpha}=M(C_\alpha)\oplus\mathcal{K}_\alpha$ if and only if $\bigcap_n U^n(\mathcal{M}\ominus\mathcal{N})=\{0\}$ for some pure set $\mathcal{N}$ with $\mathcal{N}\subseteqq\mathcal{M}$.*

COROLLARY 3. *If $\mathcal{M}$ is an invariant subspace in $\mathcal{H}_+$ for U, then $\mathcal{M}=V\mathcal{H}_+^{\alpha}$ for some partial isometry V in $\{U\}'$ with an initial space $\mathcal{H}_+^{\alpha}=M(C_\alpha)\oplus\mathcal{K}_\alpha$.*

COROLLARY 4. *If $\mathcal{M}$ is a reducing subspace for U, then $\mathcal{M}=V\mathcal{K}_\alpha\oplus V_0PM(C)$, where P is a projection in $\{U\}'$ with $PM(C)\subseteqq M(C)$, and V and V_0 are partial isometries in $\{U\}'$ with initial space $\mathcal{K}_\alpha$ and $PM(C)$, respectively.*

The author wishes to express his thanks to Mr. Katutoshi Takahashi for very helpful conversations. He is very grateful to the referee who gave a simplification of the proof in Theorem 1.

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