J. Math. Soc. Japan
Vol. 34, No. 4, 1982

Complex crystallographic groups I

By Syoshi TOKUNAGA and Masaaki YOSHIDA

(Received Nov. 20, 1980)
(Revised March 13, 1981)

§ 0. Introduction.

Let $\boldsymbol{E}(n)$ be the complex motion group acting on the n-dimensional complex euclidean space $X=\boldsymbol{C}^n$. We define a crystallographic group Γ on X to be a discrete subgroup of $\boldsymbol{E}(n)$ with compact quotient. The problem is to find crystallographic groups and to investigate the structure of the quotient space $M=X/\Gamma$ and the ramification of the natural map $X \to M$.

For $n=1$, solution of the problem is well known. For $n \geqq 2$, several authors studied fixed point free crystallographic groups and the quotient manifolds by them ([**14**], [**15**], [**17**]). We are interested in the groups which admit fixed points. For two dimensional crystallographic reflection groups, Shvartsman ([**13**]) proved the rationality of the quotient spaces and determined all splittable groups and the quotient spaces.

In this paper, we study geometric and topological properties of the quotient varieties and prove the vanishing of the plurigenera under certain conditions and obtain the characterization of crystallographic reflection groups. Two dimensional groups whose point groups are generated by reflections are investigated in detail: We prove that the quotient varieties are rational except for the group $\Gamma_{NR}(\tau)$ and the desingularization of the quotient variety by $\Gamma_{NR}(\tau)$ is an Enriques surface. All the two dimensional crystallographic reflection groups and the quotient varieties by them are determined.

The authors would like to express their hearty thanks to Mr. J. Kaneko for valuable discussions during the preparation of the present paper. The authors would also like to thank the referee for helpful suggestions, which led to the improvement of the original manuscript.

§ 1. Preliminaries and notations.

1.1. We shall use the following notations:

$X \cong \boldsymbol{C}^n$: n-dimensional complex euclidean space
$\boldsymbol{A}(n)=\{(A, a) \mid A \in GL(n, \boldsymbol{C}),\ a \in \boldsymbol{C}^n\}$: affine transformation group on X

$U(n)$: unitary group of size n
$\boldsymbol{E}(n)=\{(A, a)\in \boldsymbol{A}(n) \mid A\in U(n)\}$: complex motion group on X
$T\cong \boldsymbol{C}^n$: tangent space of X.

We have the exact sequence

$$0 \longrightarrow T \xrightarrow{\alpha} \boldsymbol{E}(n) \xrightarrow{\beta} U(n) \longrightarrow 1$$

where $\alpha(a)=(I, a)$ and $\beta(A, a)=A$; the linear part of (A, a).

1.2. Let Γ be an n-dimensional crystallographic group, that is, Γ is a discrete subgroup of $\boldsymbol{E}(n)$ with compact quotient. Then we have the following exact sequence (cf. [**16**])

$$0 \longrightarrow L \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} G \longrightarrow 1 .$$

We call L the lattice of Γ and G the point group of Γ. Remark that the lattice L is invariant under G. For the details of crystallography, see, for example, [**2**].

1.3. DEFINITION. $A\in U(n)$ is called a (unitary) reflection if A is of finite order, $A\neq I$, and has exactly $n-1$ eigenvalues equal to 1. The unique nontrivial eigenvalue of A is denoted by $\mu=\mu(A)$.

Set

$H=H(A)$: hyperplane of T fixed pointwise by A
$\boldsymbol{r}=\boldsymbol{r}(A)$: a normalized base of the orthogonal complement of H
$P(\boldsymbol{r}, A)$, $P(H, A)$: projection to $\boldsymbol{C}\boldsymbol{r}$, H, respectively.

Then we have

$$A=I+(\mu-1)P(\boldsymbol{r}, A) . \tag{1.3.1}$$

DEFINITION. A reflection group $G\subset U(n)$ is a finite group generated by reflections.

For the naming and the matrix representations of the imprimitive group $G(m, p, n)$ and the primitive group, we follow Shephard-Todd ([**12**]).

1.4. DEFINITION. An element $g\in \boldsymbol{E}(n)$ is called a reflection if g is of finite order, $g\neq$identity, and keeps a hyperplane $H(g)\subset X$ pointwise fixed.

REMARK. $g=(A, a)\in \boldsymbol{E}(n)$ is a reflection if and only if $A\in U(n)$ is a reflection and a is parallel to $\boldsymbol{r}(A)$.

DEFINITION. $\Gamma\subset \boldsymbol{E}(n)$ is called a (crystallographic) reflection group if Γ is a crystallographic group generated by finitely many reflections in $\boldsymbol{E}(n)$. In particular, Γ is called a splittable group if the point group G of Γ is generated by n reflections.

§2. G-invariant lattices.

2.1. PROPOSITION 2.1.1. *Let L be a lattice in T and $A\in U(n)$ a reflection of order l. If L is A-invariant, then we have $lP(\boldsymbol{r}, A)L\subset L\cap \boldsymbol{Cr}$ and $lP(H, A)L\subset L\cap H$.*

PROOF. Immediate from the representation (1.3.1).

COROLLARY 2.1.2. *Let L be a lattice in T and G an irreducible reflection group. If L is G-invariant, then there exists a sublattice of L which is a direct product of one dimensional lattices.*

2.2. LEMMA 2.2. *Let $G=G(m, p, n)$ and $m>1$. Then there exists a G-invariant lattice L if and only if $m=2$, 3, 4 or 6.*

PROOF. We can restrict ourselves to consider the group $G(m, m, 2)$, which is generated by $A=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}$ and $B=\begin{pmatrix} & \theta\\ \theta^{-1} & \end{pmatrix}$, $\theta=e^{2\pi i/m}$. Let L be a $G(m, m, 2)$-invariant lattice. Multiplying a suitable constant to L, we have that $L\cap \boldsymbol{Cr}(A)=(\boldsymbol{Z}+\tau\boldsymbol{Z})\begin{pmatrix} 1\\ -1\end{pmatrix}$ for some τ. Since L is invariant under B and A, we have

$$2P(\boldsymbol{r}, A)\left\{B\begin{pmatrix} 1\\ -1\end{pmatrix}\right\}=2P(\boldsymbol{r}, A)\begin{pmatrix} -\theta\\ \theta^{-1}\end{pmatrix}\in L\cap \boldsymbol{Cr}(A).$$

This is equivalent to $\theta+\theta^{-1}\in \boldsymbol{Z}$, which is possible only when $m=2$, 3, 4 and 6. Conversely, if $m=2$, 3, 4 or 6, it is easy to show that the direct product of the one dimensional lattices with the suitable moduli is invariant under the operation of $G(m, p, n)$.

2.3. We shall find every irreducible reflection group $G\subset U(2)$ and a G-invariant lattice L. Let (G, L) be a pair of an irreducible reflection group $G\subset U(2)$ and a G-invariant lattice L. Two pairs (G, L) and (G', L') are said to be equivalent if and only if there exists a matrix A such that $G'=AGA^{-1}$ and $L'=AL$. On account of the results in 2.1 and 2.2, we find a complete system of representatives:

Table I

G	G-invariant lattices L
$G(2, 1, 2)$	$L^2(\tau)$, $L^2(\tau)+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix} 1\\ 1\end{pmatrix}$, $L^2(\tau)+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix} 1\\ \tau\end{pmatrix}+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix} \tau\\ 1\end{pmatrix}$
$G(4, 2, 2)$	$L^2(i)$, $L^2(i)+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix} 1\\ 1\end{pmatrix}$
$G(4, 1, 2)$	$L^2(i)$, $L^2(i)+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix} 1\\ 1\end{pmatrix}$

$G(3,1,2)$	$L^2(\zeta),\ L^2(\zeta)+\boldsymbol{Z}\frac{1+\zeta}{3}\binom{1}{1},\ L^2(\zeta)+\boldsymbol{Z}\frac{1+\zeta}{3}\binom{1}{2}$
$G(6,2,2)$	$L^2(\zeta),\ L^2(\zeta)+\boldsymbol{Z}\frac{1+\zeta}{3}\binom{1}{1}$
$G(6,3,2)$	$L^2(\zeta),\ L^2(\zeta)+\boldsymbol{Z}\frac{1}{2}\binom{1}{\zeta}+\boldsymbol{Z}\frac{1}{2}\binom{\zeta}{1}$
$G(6,1,2)$	$L^2(\zeta)$
$G(3,3,2)$	$L(\tau)\binom{1}{-1}+L(\tau)\binom{\zeta^2}{\zeta}$ $L(\tau)\binom{1}{-1}+L(\tau)\binom{\zeta^2}{\zeta}+\boldsymbol{Z}\frac{1}{3}\left(\binom{1}{-1}-\binom{\zeta^2}{\zeta}\right)$ $L(\tau)\binom{1}{-1}+L(\tau)\binom{\zeta^2}{\zeta}+L(\tau)\frac{1}{3}\left(\binom{1}{-1}-\binom{\zeta^2}{\zeta}\right)$
$G(6,6,2)$	$L(\tau)\binom{1}{-1}+L(\tau)\binom{\zeta^2}{\zeta}$ $L(\tau)\binom{1}{-1}+L(\tau)\binom{\zeta^2}{\zeta}+\boldsymbol{Z}\frac{1}{3}\left(\binom{1}{-1}-\binom{\zeta^2}{\zeta}\right)$
[4]	$L(\zeta)u+L(\zeta)v,\ \ L(\zeta)u+L(\zeta)v+\boldsymbol{Z}\frac{1}{2}(u+v)$, $L(\zeta)u+L(\zeta)v+L(\zeta)\frac{1}{2}(u+v)$ where $u=\binom{1}{\varepsilon^2+\sqrt{2}\,\varepsilon^3\zeta^2},\ v=\binom{\varepsilon^2}{\varepsilon^6+\sqrt{2}\,\varepsilon^5\zeta^4}$
[5]	$L(\zeta)\binom{1}{\varepsilon^2+\sqrt{2}\,\varepsilon^3\zeta^4}+L(\zeta)\binom{\varepsilon^2}{\varepsilon^6+\sqrt{2}\,\varepsilon^5\zeta^2}$
[8]	$L(i)\binom{1}{0}+L(i)\frac{1+i}{2}\binom{1}{1}$

Here we used the following notations:

$$L(\tau)=\boldsymbol{Z}+\tau\boldsymbol{Z} \qquad (\operatorname{Im}\tau>0),$$

$$L^2(\tau)=L(\tau)\binom{1}{0}+L(\tau)\binom{0}{1},$$

$i=\sqrt{-1}$, $\zeta=\exp(2\pi i/6)$, $\varepsilon=\exp(2\pi i/8)$, and [4], [5], [8] are the Shephard-Todd's number of primitive groups (cf. Table I, II in [12]).

§ 3. Algebro-geometric and topological properties of quotient varieties.

3.1. Consider a crystallographic group Γ with the lattice L and the point group G. Put $\Gamma^*=\alpha(L)$, and let A denote the quotient manifold of X by Γ^*. Let M be the quotient variety of X by Γ and $\tilde{M}$ a desingularization of M. For the algebro-geometric terminology in this and the next section, we refer to [**9**].

LEMMA 3.1.1. *Let K_X and $K_{\tilde{M}}$ be the canonical line bundles of X and $\tilde{M}$, respectively. Let $H^0(X, \mathcal{O}(mK_X))^\Gamma$ be the linear space of Γ-invariant holomorphic sections of the line bundle $K_X^{\otimes m}$.*

Then (i) *we have the natural inclusion:*

$$\iota: H^0(\tilde{M}, \mathcal{O}(mK_{\tilde{M}})) \hookrightarrow H^0(X, \mathcal{O}(mK_X))^\Gamma ,$$

(ii) *every element of $H^0(X, \mathcal{O}(mK_X))^\Gamma$ is a constant multiple of $(dx_1\wedge \cdots \wedge dx_n)^m$, where $x_1, \cdots, x_n$ are the coordinates of X, and* (iii) $\dim_C H^0(X, \mathcal{O}(mK_X))^\Gamma \leqq 1$.

PROOF. Obvious.

PROPOSITION 3.1.2. (i) *If there exists a reflection in Γ, then the plurigenera $p_m(\tilde{M})$ of M vanishes for $m=1, 2, \cdots$.* (ii) *If the point group G of Γ is irreducible, then the irregularity $q(\tilde{M})$ of $\tilde{M}$ vanishes.*

PROOF. (i) Let ω be a holomorphic section of $(K_{\tilde{M}})^{\otimes m}$. We shall denote by φ and π the natural mappings $X\to M$ and $\tilde{M}\to M$, respectively. Suppose there exists a reflection $g\in\Gamma$, of order l. Let the hyperplane $H(g)$ be represented by $x_1=0$, and $P\in H(g)$ a point such that the variety consisting of every fixed point of Γ is nonsingular at P. Then we have the local representations at $\varphi(P)\in M$ and $P\in X$:

$$(\pi^{-1})^*\omega=f(y_1, x_2, \cdots, x_n)(dy_1\wedge dx_2\wedge \cdots \wedge dx_n)^m ,$$

$$(\pi^{-1}\circ\varphi)^*\omega=f(x_1^l, x_2, \cdots, x_n)(lx_1^{l-1})^m(dx_1\wedge dx_2\wedge \cdots \wedge dx_n)^m$$

where $f(y_1, x_2, \cdots, x_n)$ is holomorphic at $\varphi(P)$. By the lemma above, $f(x_1^l, x_2, \cdots, x_n)x_1^{m(l-1)}$ must be constant. This implies $f=0$ and so $\omega=0$.

(ii) Let v be a holomorphic 1-form on $\tilde{M}$. Then the form $(\pi^{-1}\circ\varphi)^*v$ is a linear combination of $dx_1, \cdots, dx_n$ with constant coefficients and is G-invariant. Since an irreducible group has no invariant of degree 1, we have $v=0$. Q. E. D.

REMARK. J. Noguchi showed a similar result by making use of the Nevanlinna theory at a seminar in Kyoto, June 1980.

COROLLARY 3.1.3. *In addition to the assumption in Proposition* 3.1.2, (i), (ii), *suppose $n=2$ and G is a reflection group. Then M is rational.*

PROOF. Corollary 2.1.2 asserts that A is an abelian variety and so $\tilde{M}$ is algebraic. Thus the rationality is deduced from the Castelnuovo criterion (cf. [**9**]). Q. E. D.

COROLLARY 3.1.4 ([**13**]). *If Γ is a two dimensional crystallographic reflection* *group, then M is rational.*

3.2. We shall give a characterization of crystallographic reflection groups. Let S denote the set of singular points of M.

THEOREM 3.2.1. *Crystallographic group Γ is generated by reflections if and only if $M-S$ is simply connected.*

PROOF. Put $\tilde{S}=\{Q\in X \mid \Gamma_Q\neq R_Q\}$, where Γ_Q is the isotropy subgroup of Γ at Q, and R_Q is a subgroup of Γ generated by every reflection of which fixed hyperplane passes through Q. It is easy to show that $\tilde{S}$ is a complex subvariety of X and $\operatorname{codim}_C\tilde{S}\geqq 2$. In particular, $X-\tilde{S}$ is simply connected. On the other hand, by Chevalley-Shephard-Todd theorem ([**3**], [**12**]), we conclude that $\varphi\tilde{S}=S$. It is known that the quotient of a simply connected manifold by a properly discontinuous group Γ is also simply connected if and only if Γ is generated by transformations with fixed points ([**1**]). Note that Γ operates on $X-\tilde{S}$ and $M-S=(X-\tilde{S})/\Gamma$. Thus we conclude that if Γ is generated by reflections then the quotient space $M-S$ is simply connected.

On the contrary, assume that $M-S$ is simply connected. Then Γ is generated by transformations with fixed points in $X-\tilde{S}$. By the definition of $\tilde{S}$, Γ is generated by reflections. Q.E.D.

COROLLARY 3.2.2. *If the quotient space M is biholomorphic to the n-dimensional projective space, then Γ is generated by reflections.*

3.3. PROPOSITION 3.3.1. *Let Γ be a two dimensional crystallographic group, and C a curve in M such that $C\cap S=\emptyset$. Then the self-intersection number $C\cdot C$ is non-negative.*

PROOF. Let $\tilde{C}\subset A$ be the preimage of C by the natural map $A\to M$. The curve $\tilde{C}$ has a small displacement $\tilde{C}_\varepsilon=\tilde{C}+\varepsilon$ $(\varepsilon\in \boldsymbol{C}^2)$ in A. Q.E.D.

PROPOSITION 3.3.2. *Assumption as above. If the point group G of Γ is irreducible, then $C\cdot C>0$.*

PROOF. There exists an element $g\in G$ such that the curve $g\tilde{C}_\varepsilon$ intersects $\tilde{C}$. Q.E.D.

COROLLARY 3.3.3. *Assumption as in Proposition 3.3.2. If M is a non-singular rational surface, then M is biholomorphic to the two dimensional projective space $\boldsymbol{P}^2$.*

§4. Non-rational quotients.

4.1. We define the two dimensional crystallographic group $\Gamma_{NR}(\tau)$ to be the subgroup of $\boldsymbol{E}(2)$ generated by

$$\begin{pmatrix} & 1 & 0 \\ 1 & & 1/2 \end{pmatrix}, \quad \begin{pmatrix} -1 & & \tau/2 \\ & 1 & \tau/2 \end{pmatrix} \qquad \operatorname{Im}\tau>0$$

and (I, ω), $\omega\in L=L^2(\tau)+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$. The point group of $\Gamma_{NR}(\tau)$ is equal to $G(2, 1, 2)$.

THEOREM 4.1.1. *Let Γ be a two dimensional crystallographic group whose point group is an irreducible reflection group. If Γ is not conjugate, in $\boldsymbol{A}(2)$, to $\Gamma_{NR}(\tau)$ for any τ ($\mathrm{Im}\,\tau>0$), then the quotient variety M of X by Γ is rational.*

Let $M_{NR}(\tau)$ be the quotient variety of X by the group $\Gamma_{NR}(\tau)$.

THEOREM 4.1.2. (i) *The desingularization $\tilde{M}_{NR}(\tau)$ of $M_{NR}(\tau)$ is an Enriques surface, i.e. $q=p_g=0$ and $p_2=1$.*

(ii) *Set* $\Gamma^*=\left\{(I, \omega)\,\middle|\,\omega\in L^2(\tau)+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix}1\\ \tau\end{pmatrix}\right\}$, $\Gamma''=\langle(-I, 0), \Gamma^*\rangle$, *and*

$$\Gamma'=\left\langle\left(\begin{matrix} & -1 \\ 1 & \end{matrix}\quad\begin{matrix}\tau/2\\(1+\tau)/2\end{matrix}\right), \Gamma''\right\rangle,$$

where $\langle\alpha, \beta, \cdots\rangle$ denotes the group generated by $\alpha, \beta, \cdots$. Then we have the inclusion relations all of which are of index 2:

$$\Gamma^*\lhd\Gamma''\lhd\Gamma'\lhd\Gamma_{NR}(\tau)\,.$$

(iii) *The desingularization $\tilde{M}'$ of $M'=X/\Gamma'$ is a $K3$-surface and the natural map $M'\to M_{NR}(\tau)$ can be lifted to the two sheeted unramified covering: $\tilde{M}'\to\tilde{M}_{NR}(\tau)$.*

4.2. The proof of Theorems 4.1.1 and 4.1.2 is divided into several steps. In this subsection, we prepare some lemmas. Let Γ be a crystallographic group with the point group G.

LEMMA 4.2.1. *If there exists an element $A\in G$ such that* $(\det A)^m\neq1$, *then* $p_m(\tilde{M})=0$.

PROOF. Direct consequence of Lemma 3.1.1, (i) and (ii). Q.E.D.

COROLLARY 4.2.2. *Let $n=2$. If the point group G is an irreducible reflection group containing a reflection of order >2. Then M is rational.*

Let L be the lattice of Γ. By 1.2 and 1.3 we obtain two lemmas:

LEMMA 4.2.3. *Let $A\in G$ be a reflection of order l and $(A, a)\in\Gamma$. Then we have $P(H, A)a\in(1/l)L\cap H$.*

LEMMA 4.2.4. *Let $A\in G$ be a reflection of order l and $(A, a)\in\Gamma$. There exists a reflection $g\in\Gamma$ with the linear part $\beta(g)=A^k$ ($k\,|\,l$, $k\neq l$) if and only if $P(H, A)a\in(1/k)P(H, A)L$.*

REMARK. For a reflection $A\in G$ of order l, we have

$$L\cap H\subset P(H, A)L\subset\frac{1}{k}P(H, A)L\subset\frac{1}{l}L\cap H\,.$$

COROLLARY 4.2.5. *If there exists a reflection $A\in G$ of order l such that $P(H, a)L=(1/k')L\cap H(A)$ for some k' ($k'\,|\,l$ and $k'\neq1$), then M is rational.*

PROOF. By Lemma 4.2.3, Lemma 4.2.4 and Proposition 3.1.2. Q.E.D.

4.3. For each pair (G, L) in Table I, we shall examine every extension Γ of L by G. If the point group G of Γ is conjugate to $G(4, 1, 2)$, $G(3, 1, 2)$, $G(6, 2, 2)$, $G(6, 1, 2)$, [4], [5] or [8], then M is rational. Indeed, by definition, every group above contains a reflection of order >2. Thus we can apply Corollary 4.2.2.

Let G be conjugate to $G(3, 3, 2)$, $G(6, 6, 2)$, $G(6, 3, 2)$, $G(2, 1, 2)$ or $G(4, 2, 2)$, and L be any lattice invariant under G. By making use of Corollary 4.2.5, we can check that for any extension Γ of L by G, the quotient M is rational, except possibly for the two groups $\Gamma_{NR}(\tau)$ and Γ_R defined below:

$$\Gamma_R=\left\langle\begin{pmatrix} & 1 & a\\ 1 & & \end{pmatrix},\ \begin{pmatrix} & i & b\\ -i & & \end{pmatrix},\ \begin{pmatrix} 1 & & c\\ & -1 & \end{pmatrix},\ (I, \omega);\ \omega\in L\right\rangle$$

where

$$a=\frac{1}{4}\begin{pmatrix}1+i\\1+i\end{pmatrix},\qquad b=\frac{1}{4}\begin{pmatrix}1+i\\1-i\end{pmatrix},\qquad c=\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix},$$

$$L=L^2(i)+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}.$$

The point group of the group Γ_R is equal to $G(4, 2, 2)$.

4.4. Let $\tilde{M}_R$ be the desingularization of $M_R=X/\Gamma_R$. We shall compute the plurigenera $p_2(\tilde{M}_{NR}(\tau))$ and $p_2(\tilde{M}_R)$.

LEMMA 4.4.1. *The groups $\Gamma_{NR}(\tau)$ and Γ_R have only discrete fixed points.* (i) *For every fixed point P of $\Gamma_{NR}(\tau)$, the isotropy subgroup $I(P)$ is equal to* $\left\langle\begin{pmatrix}-1 & \\ & -1\end{pmatrix}\right\rangle$ *or* $\left\langle\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\right\rangle$. (ii) *There exists a fixed point P of Γ_R such that the isotropy subgroup $I(P)$ is equal to* $\left\langle\begin{pmatrix}i & \\ & i\end{pmatrix}\right\rangle$.

PROOF. Recall that the group $\Gamma_{NR}(\tau)$ and Γ_R contain no reflections. Thus (i) is obvious. For (ii), we have only to note that the element $\begin{pmatrix} & i & b\\ -i & & \end{pmatrix}\begin{pmatrix} & 1 & a\\ 1 & & \end{pmatrix}$ $\times\begin{pmatrix}1 & & c\\ & -1 & \end{pmatrix}$ has the linear part $\begin{pmatrix}i & \\ & i\end{pmatrix}$. Q.E.D.

Let Q be an isolated fixed point of Γ in X, γ a generator of the isotropy subgroup of Γ at Q, and (U, x, y) a sufficiently small coordinate neighbourhood of Q in X. Then $\varphi(U)\subset M$ is isomorphic to $U/\langle\gamma\rangle$. Let $\rho: V\to U/\langle\gamma\rangle$ be the desingularization of the singularity of $U/\langle\gamma\rangle$.

LEMMA 4.4.2. (i) *If* $\gamma=\begin{pmatrix}-1 & \\ & -1\end{pmatrix}$ *or* $\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, *then the form* $(\rho^{-1}\circ\varphi)^*(dx\wedge dy)^{\otimes m}$ *on V is holomorphic.* (ii) *If* $\gamma=\begin{pmatrix}i & \\ & i\end{pmatrix}$, *then the form*

$(\rho\circ\varphi)^*(dx\wedge dy)^{\otimes m}$ *has poles.*

PROOF. Easy (cf. [8]).

COROLLARY 4.4.3. (i) $\iota: H^0(M, \mathcal{O}(mK_{\tilde{M}_{NR}}))\rightleftarrows H^0(X, \mathcal{O}(mK_X))^{\Gamma_{NR}(\tau)}$ *is isomorphic,* (ii) $\iota: H^0(\tilde{M}, \mathcal{O}(mK_{\tilde{M}_R}))\hookrightarrow H^0(X, \mathcal{O}(mK_X))^{\Gamma_R}$ *is not surjective.*

4.5. PROOF OF THEOREM 4.1.1. By Corollary 4.4.3, (ii) and Lemma 3.1.1, (iii), we conclude that $p_2(\tilde{M}_R)=0$. Since $q=0$, this proves the rationality of M_R.

4.6. PROOF OF THEOREM 4.1.2. The determinant of every element of the point group of $\Gamma_{NR}(\tau)$ is ± 1. Thus we have

$$\dim H^0(X, \mathcal{O}(K_X))^{\Gamma_{NR}(\tau)}=0$$

and

$$\dim H^0(X, \mathcal{O}(2K_X))^{\Gamma_{NR}(\tau)}=1\,.$$

These equalities with Corollary 4.4.3, (i) proves (i). (ii) is easy. (iii) We have only to repeat the similar arguments as above to show $p_g(\tilde{M}')=p_2(\tilde{M}')=1$. One can show that the fixed points and the linear parts of the isotropy subgroup of Γ' coincide with those of $\Gamma_{NR}(\tau)$. Thus the group $\Gamma_{NR}(\tau)/\Gamma'$ acts freely on M'. This implies that the map $M'\to M_{NR}(\tau)$ is unramified.

§ 5. Two dimensional crystallographic reflection groups.

In this section, we shall find every two dimensional crystallographic group generated by reflections and study the quotient varieties by them.

5.1. The two dimensional weighted projective space of type (k, l, m) is, by definition, the quotient variety of $\boldsymbol{C}^3-\{0\}$ by the equivalence relation $\sim: (x, y, z)\sim(\lambda^k x, \lambda^l y, \lambda^m z)$, $\lambda\in\boldsymbol{C}-\{0\}$.

THEOREM 5.1. *Every two dimensional crystallographic reflection group is conjugate in* $\boldsymbol{A}(2)$ *to one of the twenty groups in Table II. The quotient spaces by them are isomorphic to the weighted projective spaces, of which types are also in the table. Each group, except* $(4, 2)_2$, *is the semidirect product* $G\ltimes L$ *of the point group* G *and the lattice* L.

Table II

Name	Γ	Type of M
$(2, 1)_0$	$G(2, 1, 2)\ltimes L^2(\tau)$	(1, 1, 1)
$(2, 1)_1$	$G(2, 1, 2)\ltimes\left\{L^2(\tau)+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\}$	(1, 1, 2)
$(4, 2)_0$	$G(4, 2, 2)\ltimes L^2(i)$	(1, 1, 2)
$(4, 2)_1$	$G(4, 2, 2)\ltimes\left\{L^2(i)+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\}$	(1, 1, 1)

$(4, 2)_2$	$\left\langle \begin{pmatrix} & i & 0\\ -i & & 0\end{pmatrix}, \begin{pmatrix} & 1 & 0\\ 1 & & 0\end{pmatrix}, \begin{pmatrix} 1 & & 0\\ & -1 & (1+i)/2\end{pmatrix},\right.$ $\left.(I, \omega),\ \omega\in L^2(i)+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\rangle$	$(1, 1, 4)$
$(4, 1)_0$	$G(4, 1, 2)\ltimes L^2(i)$	$(1, 1, 1)$
$(4, 1)_1$	$G(4, 1, 2)\ltimes\left\{L^2(i)+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\}$	$(1, 1, 2)$
$(3, 1)_0$	$G(3, 1, 2)\ltimes L^2(\zeta)$	$(1, 1, 1)$
$(3, 1)_1$	$G(3, 1, 2)\ltimes\left\{L^2(\zeta)+\boldsymbol{Z}\frac{1+\zeta}{3}\begin{pmatrix}1\\2\end{pmatrix}\right\}$	$(1, 1, 3)$
$(6, 2)_0$	$G(6, 2, 2)\ltimes L^2(\zeta)$	$(1, 1, 2)$
$(6, 2)_1$	$G(6, 2, 2)\ltimes\left\{L^2(\zeta)+\boldsymbol{Z}\frac{1+\zeta}{3}\begin{pmatrix}1\\2\end{pmatrix}\right\}$	$(1, 2, 3)$
$(6, 3)_0$	$G(6, 3, 2)\ltimes L^2(\zeta)$	$(1, 1, 3)$
$(6, 3)_1$	$G(6, 3, 2)\ltimes\left\{L^2(\zeta)+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix}1\\\zeta\end{pmatrix}+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix}\zeta\\1\end{pmatrix}\right\}$	$(1, 3, 4)$
$(6, 1)_0$	$G(6, 1, 2)\ltimes L^2(\zeta)$	$(1, 1, 1)$
$(3, 3)_0$	$G(3, 3, 2)\ltimes\left\{L(\tau)\begin{pmatrix}1\\-1\end{pmatrix}+L(\tau)\begin{pmatrix}\zeta^2\\\zeta\end{pmatrix}\right\}$	$(1, 1, 1)$
$(6, 6)_0$	$G(6, 6, 2)\ltimes\left\{L(\tau)\begin{pmatrix}1\\-1\end{pmatrix}+L(\tau)\begin{pmatrix}\zeta^2\\\zeta\end{pmatrix}\right\}$	$(1, 1, 2)$
$(6, 6)_1$	$G(6, 6, 2)\ltimes\left\{L(\tau)\begin{pmatrix}1\\-1\end{pmatrix}+L(\tau)\begin{pmatrix}\zeta^2\\\zeta\end{pmatrix}\right.$ $\left.+\boldsymbol{Z}\frac{1}{3}\left(\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}\zeta^2\\\zeta\end{pmatrix}\right)\right\}$	$(1, 2, 3)$
$[4]_0$	$[4]\ltimes\left\{L(\zeta)\begin{pmatrix}1\\\varepsilon^2+\sqrt{2}\,\varepsilon^3\zeta^2\end{pmatrix}+L(\zeta)\begin{pmatrix}\varepsilon^2\\\varepsilon^6+\sqrt{2}\,\varepsilon^5\zeta^4\end{pmatrix}\right\}$	$(1, 1, 2)$
$[5]_0$	$[5]\ltimes\left\{L(\zeta)\begin{pmatrix}1\\\varepsilon^2+\sqrt{2}\,\varepsilon^3\zeta^4\end{pmatrix}+L(\zeta)\begin{pmatrix}\varepsilon^2\\\varepsilon^6+\sqrt{2}\,\varepsilon^5\zeta^2\end{pmatrix}\right\}$	$(1, 2, 3)$
$[8]_0$	$[8]\ltimes\left\{L(i)\begin{pmatrix}1\\0\end{pmatrix}+L(i)\frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\}$	$(1, 2, 3)$

REMARK. The point group and the lattice of the group $(4, 2)_2$ are the same to those of the groups $(4, 2)_1$.

REMARK. Shvartsman ([13]) found the splittable groups $(2, 1)_0$, $(2, 1)_1$, $(4, 1)_0$, $(4, 1)_1$, $(3, 1)_0$, $(3, 1)_1$, $(6, 1)_0$, $(3, 3)_0$, $(6, 6)_0$, $(6, 1)_1$, $[4]_0$, $[5]_0$, $[8]_0$ and determined the quotient spaces by them.

Proof of Theorem 5.1 shall be given in 5.3.

5.2. In this subsection, we shall give every black and white groups (inclusion relations of index 2) in the two dimensional crystallographic reflection groups. For two crystallographic groups Γ and Γ', we shall use the following notations:

$\Gamma' \lhd \Gamma$: Γ' is a subgroup of Γ of index 2

$\Gamma' \lesssim \Gamma$: There exists an element $g \in \boldsymbol{A}(2)$ such that $\Gamma' \lhd g\Gamma g^{-1}$.

PROPOSITION 5.2. *The following is the complete list of black and white groups in the two dimensional crystallographic reflection groups:*

1) $(2, 1)_0 \lhd (2, 1)_1$	2) $(2, 1)_1 \lesssim (2, 1)_0$	3) $(2, 1)_0 \lhd (4, 2)_0$
4) $(2, 1)_1 \lhd (4, 2)_1$	5) $(2, 1)_1 \lesssim (4, 2)_2$	6) $(4, 2)_0 \lhd (4, 1)_0$
7) $(4, 2)_0 \lhd (4, 2)_1$	8) $(4, 1)_0 \lhd (4, 1)_1$	9) $(4, 2)_1 \lhd (4, 1)_1$
10) $(4, 2)_2 \lesssim (4, 2)_0$	11) $(4, 2)_1 \lesssim (4, 2)_0$	12) $(3, 1)_0 \lhd (6, 2)_0$
13) $(6, 2)_0 \lhd (6, 1)_0$	14) $(3, 1)_1 \lhd (6, 2)_1$	15) $(3, 3)_0 \lhd (6, 6)_0$
16) $(6, 6)_1 \lesssim (6, 3)_0$	17) $(6, 6)_1 \lesssim (6, 3)_1$.	

In the relations 3), 4) *and* 5), *the lattices of the groups* $(2, 1)_0$ *and* $(2, 1)_1$ *are* $L^2(i)$ *and* $L^2(i)+\boldsymbol{Z}\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$ *respectively.*

Since the proof of the proposition is easy but tiresome, we omit it.

5.3. PROOF OF THEOREM 5.1. For each pair (G, L) of Table I, we consider every extension of L by G, and pick up every group, up to conjugacy in $\boldsymbol{A}(2)$, generated by reflections. If the point group G of Γ is generated by two reflections, then Γ is the semidirect product $G \ltimes L$ of G and L. These are the splittable groups, determined by Shvartsman.

Let G be one of the following three groups: $G(4, 2, 2)$, $G(6, 2, 2)$ and $G(6, 3, 2)$. Recall that these groups are generated by three reflections. Set $\theta=\exp(2\pi i/m)$, $q=m/p$ and

$$P=\begin{pmatrix} & 1\\ 1 & \end{pmatrix}, \quad Q=\begin{pmatrix} & \theta\\ \theta^{-1} & \end{pmatrix}, \quad T=\begin{pmatrix} 1 & \\ & \theta^{p}\end{pmatrix}.$$

Then the group $G(m, p, 2)$ is generated by P, Q and T with the generating relations (cf. [11]):

$$P^2=Q^2=T^q=1, \quad TPQ=PQT, \quad PT^{-1}PT=TPT^{-1}P=(PQ)^p. \tag{5.3.1}$$

Suppose the group Γ with the point group $G=G(m, p, 2)$ is generated by reflections. Taking conjugate by a parallel displacement if necessary, we conclude that Γ contains three reflections $P_1=(P, 0)$, $Q_1=(Q, 0)$ and $T_1=\left(T, \begin{pmatrix}0\\c\end{pmatrix}\right)$ for some $c\in \boldsymbol{C}$. We shall seek for possible values of c. Let L be the lattice of Γ.

LEMMA 5.3.1. *We have* (i) $\begin{pmatrix}0\\(\theta-1)c\end{pmatrix}\in L$ *and* (ii) $\begin{pmatrix}-c\\c\end{pmatrix}\in L$.

PROOF. By (5.3.1), the linear parts of $T_1P_1Q_1$ and $P_1Q_1T_1$ are the same. The respective vector parts are $\begin{pmatrix}0\\c\end{pmatrix}$ and $\begin{pmatrix}c\\\theta c\end{pmatrix}$. Thus 1.2 gives $\begin{pmatrix}0\\(\theta-1)c\end{pmatrix}\in L$. Calculate the vector parts of $P_1T_1^{-1}P_1T_1$, $T_1P_1T_1^{-1}P_1$ and $(P_1Q_1)^p$ and note that $\begin{pmatrix}\theta^p & \\ & 1\end{pmatrix}\in G(m, p, 2)$. Then we have $\begin{pmatrix}-c\\c\end{pmatrix}\in L$. Q.E.D.

By making use of (i) and (ii), we shall determine c. (I) For $L=L(\theta)\begin{pmatrix}1\\0\end{pmatrix}+L(\theta)\begin{pmatrix}0\\1\end{pmatrix}$, we have, by (ii) above, $c\in L(\theta)$. (II) If $m=6$, i.e. $\theta=\zeta$, then $c\in L(\zeta)$. Indeed (i) implies $\begin{pmatrix}0\\\zeta^2 c\end{pmatrix}\in L(\zeta)\begin{pmatrix}0\\1\end{pmatrix}\subset L$. (III) Let $G=G(4, 2, 2)$ and $L=L(i)\begin{pmatrix}0\\1\end{pmatrix}+L(i)\begin{pmatrix}1\\0\end{pmatrix}+\boldsymbol{Z}\frac{1+i}{2}\begin{pmatrix}1\\1\end{pmatrix}$. (i) and (ii) imply $\begin{pmatrix}0\\(i-1)c\end{pmatrix}\in L(i)\begin{pmatrix}0\\1\end{pmatrix}$ and $2c\in L(i)$. Thus we have $c\equiv 0$ or $(1+i)/2 \bmod L(i)$.

$c \bmod L(\theta)$ determines the group Γ, because we have $\Gamma=\langle P_1, Q_1, T_1, (I|\omega); \omega\in L\rangle$ and $\begin{pmatrix}0\\1\end{pmatrix}L(\theta)\subset L$. Remark that if $c\in L(\theta)$, then Γ is the semidirect product $G\ltimes L$. We omit the proof of the fact that the groups obtained above are generated by reflections.

For the quotient spaces, we utilize the results of Shvartsman ([**13**]) and the inclusion relations in Proposition 5.2. Let Γ_1 and Γ_2 be crystallographic reflection groups such that Γ_1 is a normal subgroup of Γ_2 of index 2. Suppose that X/Γ_1 is a weighted projective space of type (k, l, m). Since Γ_2 is generated by reflections, there exists a reflection γ of order 2 such that $\Gamma_2=\langle\Gamma_1, \gamma\rangle$. Thus the set of fixed points of $\gamma \bmod \Gamma_1$ on X/Γ_1 contains a divisor. This implies that X/Γ_2 is isomorphic to the weighted projective space of type $(2k, l, m)$, $(k, 2l, m)$ or $(k, l, 2m)$. Therefore, we have only to study the fixed points of Γ_1 and γ to determine the type of the weighted projective space X/Γ_1.

References

[1] M.A. Armstrong, On the fundamental group of an orbit space, Proc. Cambridge Philos. Soc., **61** (1965), 639-646.

[2] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, Crystallographic groups of four-dimensional space, John Wiley & Sons, 1978.

[3] C. Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math., **77** (1955), 778-782.

[4] A.M. Cohen, Finite complex reflection groups, Ann. Sci. École Norm. Sup., **9** (1976), 379-436.

[5] H.S.M. Coxeter, Regular complex polytopes, Cambridge Univ. Press, 1974.
[6] D.W. Crowe, Some two dimensional unitary groups generated by three reflections, Canad. J. Math., **13** (1961), 418-426.
[7] L. Flatto, Invariants of finite reflection groups, L'enseignement mathématique II^e serie, **24** (1978), 237-292.
[8] S. Hattori and M. Yoshida, Local theory of Fuchsian systems with certain discrete monodromy groups III, Funkcial. Ekvac., **22** (1979), 1-40.
[9] S. Iitaka, Algebraic geometry I, II, III, Iwanami, 1977.
[10] J. Kaneko, S. Tokunaga and M. Yoshida, Complex crystallographic groups II, J. Math. Soc. Japan, **34** (1982), 595-605.
[11] G.C. Shephard, Abstract definitions for reflection groups, Canad. J. Math., **9** (1957), 273-276.
[12] G.C. Shephard and I.A. Todd, Finite unitary reflection groups, Canad. J. Math., **6** (1954), 274-304.
[13] O.V. Shvartsman, Chevalley's theorem for complex crystallographic groups generated by reflections in the affine space $\boldsymbol{C}^2$, Uspehi Mat. Nauk, **34** (1979), 249-250. Translation: Russian Math. Surveys, **34** (1979), 239-240.
[14] T. Suwa, Compact quotient spaces of $\boldsymbol{C}^2$ by affine transformation groups, J. Differential Geometry, **19** (1975), 239-252.
[15] K. Uchida and H. Yoshihara, Discontinuous groups of affine transformations of $\boldsymbol{C}^3$, Tôhoku Math. J., **28** (1976), 89-94.
[16] J.A. Wolf, Spaces of constant curvature, McGraw-Hill, 1967.
[17] H. Yoshihara, On deformations of hyperelliptic manifolds, Proc. Internat. Symp. on Algebraic Geometry, Kyoto, 1977, 735-737.

Syoshi TOKUNAGA
Department of Mathematics
Kyusyu University
Fukuoka 812
Japan

Masaaki YOSHIDA
Department of Mathematics
Kyusyu University
Fukuoka 812
Japan