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On (G, Γ, n, q)-translation planes

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1. Introduction.

Several authors have studied the translation plane π which satisfies the following conditions:

(i) Each component of a spread set Γ of π is a subspace of $V(2n, q)$.

(ii) A collineation group G of π leaves a set Δ of $q+1$ components of Γ invariant and acts transitively on $\Gamma-\Delta$.

Any translation plane satisfying (i) and (ii) is called a (G, Γ, n, q)-plane and Δ as in (ii) is denoted by $\Delta(\pi)$.

The known classes of (G, Γ, n, q)-planes are (i) the desarguesian planes of square order, (ii) the Hall planes, (iii) the planes of order 5^{2n} with n odd constructed by Narayana Rao and Satyanarayana [**9**], (iv) the desarguesian planes of cubic order and (v) the LR-16 and JW-16 [**7**]. Recently we presented a generalization of (ii) and (iii), which are also $(G, \Gamma, 2, q)$-planes [**3**]. We note that the "n" are rather small for these examples. In his paper [**6**] V. Jha has shown $n=2$ under the additional assumptions that (a) q is a prime, (b) G fixes at least two components of Δ and (c) $O_q(G)$ is a Sylow q-subgroup of G. Moreover the author has proved in [**2**] that $n=2$ or 3 if (a) q is a prime and (b) $O_q(G)$ has a nontrivial element which leaves at least two components of Δ fixed.

In this paper we generalize these results. The following theorem is a generalization of Theorem 1 of [**2**].

THEOREM 1. *Let π be a (G, Γ, n, q)-plane with characteristic p. Set $\Delta=\Delta(\pi)$ and $q=p^m$. Then one of the following holds.*

(i) *$O_p(G)$ is semi-regular on $\Delta-\{A\}$ for some $A\in\Delta$.*

(ii) *$n=2$.*

(iii) *$n=3$ and $q\equiv 1 \pmod 2$. Moreover the length of each G-orbit on Δ is divisible by $\theta(n, q)$. Here $\theta(n, q)=\prod_{t\in\Phi}(q+1)_t$ if $q\equiv 1 \pmod 4$ or $\theta(n, q)=\prod_{t\in\Phi\cup\{2\}}(q+1)_t$ if $q\equiv -1 \pmod 4$, where Φ is the set of prime p-primitive divisors of $p^{2m}-1$.*

In the case $q^n\equiv -1 \pmod 4$ we prove the following theorem.

THEOREM 2. *Let π be a (G, Γ, n, q)-plane with characteristic p. If $q^n\equiv -1 \pmod 4$ and $O_p(G)\neq 1$, then $n=3$.*

The following theorem is a generalization of Theorem A of [**6**].

THEOREM 3. *Let π be a (G, Γ, n, q)-plane with characteristic p. If $O_p(G)\neq 1$ and G has at least two fixed components of $\Delta(\pi)$, then $n=2$.*

The desarguesian plane of order 27 satisfies the assumption of Theorem 2 and the classes of the planes (ii), (iii) as above satisfy the assumption of Theorem 3.

2. The (G, Γ, n, q)-planes.

In this section we assume that π is a (G, Γ, n, q)-plane with characteristic p. Set $q=p^m$. We use the following notations.

T : the group of translations of π.

$T(A)$: the group of translations with center A.

n_p : the highest power of a prime p dividing a positive integer n.

$F(H)$: the fixed structure consisting of points and lines of π fixed by a nonempty subset H of G.

$\Delta=\Delta(\pi)$, $\bar{\Gamma}=\Gamma-\Delta$, $M=O_p(G)$.

Other notations are standard and taken largely from [**1**], [**4**] and [**8**].

Let $\bar{\pi}$ be the projective plane associated with π. Throughout the paper we identify Γ with the set of points on the line at infinity of $\bar{\pi}$. All sets and groups in this section are finite.

LEMMA 1. *Let p^r be a power of a prime p with $r>1$ and let t be a prime p-primitive divisor of p^r-1. If $p^i-1\equiv 0 \pmod t$ for some $i>1$, then $i\equiv 0 \pmod r$.*

PROOF. Set $i=kr+s$ with $0\leqq s<r$. Since $p^{kr+s}-1=p^s(p^{kr}-1)+(p^s-1)$ and $p^{kr}-1\equiv 0 \pmod t$, we have $0\equiv p^i-1\equiv p^s-1 \pmod t$. Hence $s=0$ and so $i\equiv 0 \pmod r$.

LEMMA 2. *Let t be a prime p-primitive divisor of $p^{m(n-1)}-1$ and X a nontrivial t-subgroup of G. If X centralizes a subgroup Y of M and XY fixes a point $A\in\Gamma$, then either* (i) *$C_{T(A)}(X)\ngeqq C_{T(A)}(Y)$ and $|C_{T(A)}(Y)|\geqq q^{n-1}$ or* (ii) *$n=2$ and X is a group of homologies with axis OA.*

PROOF. Set $U=T(A)$. Then U is an elementary abelian p-group of order q^n and XY normalizes U. By Theorem 5.2.3 of [**1**], $U=C_U(X)\times[U, X]$. If $[U, X]=1$, then $U=C_U(X)$ and so X is a group of homologies with axis OA. Hence $q^n\equiv 1 \pmod t$. By Lemma 1, $mn\equiv 0 \pmod{m(n-1)}$ and (ii) follows. If $[U, X]\neq 1$, then $1\neq C_{[U,X]}(Y)$ as Y normalizes $[U, X]$. Hence $C_U(X)\ngeqq C_U(Y)$ $\geqq C_{[U,X]}(Y)$ and so X acts nontrivially on $C_U(Y)$. Thus $|C_U(Y)|\geqq q^{n-1}$ and (i) follows.

LEMMA 3. *Let t be a prime p-primitive divisor of $p^{m(n-1)}-1$ and let R be a Sylow t-subgroup of G. Then the following hold.*

(i) *Assume $n \neq 3$. Then R fixes a point of Δ.*

(ii) *Assume $n \neq 2$. Let S be a nontrivial subgroup of R which fixes a point of Γ. Then $F(S)$ is a subplane of order q. Moreover, $F(S) \cap \Gamma = \Delta$ and S is semi-regular on Γ when $M \neq 1$.*

(iii) *Assume $n \neq 2$. If R fixes a point of Δ, then $F(R) \cap \Gamma = \Delta$ and R is semi-regular on Γ.*

PROOF. Assume $F(R) \cap \Delta = \emptyset$. Then $t \mid q+1$. Since $q+1 \mid q^2-1$, $n \leqq 3$. If $n=2$, then $t \mid (q+1, q-1)=1$ or 2, a contradiction. Thus (i) holds.

Let $A \in F(S) \cap \Gamma$. Set $U=T(A)$ and $p^k = |C_U(S)|$. If $p^k = q^n$, then S is a group of homologies with axis OA. Hence $t \mid (q^n-1, q^{n-1}-1)=q-1$ and so $n=2$, contrary to the assumption. Therefore $0 \leqq k < mn$. By Theorem 5.3.2 of [**1**], S centralizes no nontrivial element of $U/C_U(S)$. Hence $t \mid |U/C_U(S)|-1=p^{mn-k}-1$. Applying Lemma 1, $mn-k \equiv 0 \pmod{m(n-1)}$ and so $k=m$. Therefore $|C_U(S)|=q$.

Since $t \nmid |\Gamma - \{A\}| = q^n$, S fixes another point $B \in \Gamma - \{A\}$. By the similar argument as above, $|C_{T(B)}(S)|=q$. Hence $F(S)$ is a subplane of order q. Thus we obtain the former half of (ii). Hence (iii) holds as $F(R) \cap \Gamma = \emptyset$.

To prove the latter half of (ii) we may assume $F(R) \cap \Gamma = \emptyset$ by (iii). Hence $n=3$ by (i). Since $|F(M) \cap \Delta| \geqq 3$ and $F(M) \cap \Gamma = \emptyset$, $F(M)$ is a subplane of order at most q and therefore $1 \neq |C_T(M)| \leqq q^2$. As $F(R) \cap \Gamma = \emptyset$, R does not centralize $C_T(M)$, so that $|C_T(M)|=q^2$ $(=q^{n-1})$ and $F(M)$ is a subplane of order q with $F(M) \cap \Gamma = \Delta$. By Bruck's theorem (Theorem 3.7 of [4]), M is semi-regular on Γ. From this $|M| \mid |\Gamma|_p = q$ and so R centralizes M. Let $A, B \in F(S) \cap \Delta$, $A \neq B$. Then S centralizes $C_{T(A)}(M)$ and $C_{T(B)}(M)$. Therefore $C_T(S) \geqq \langle C_{T(A)}(M), C_{T(B)}(M) \rangle = C_T(M)$ and so $F(S)=F(M)$. Hence S is semi-regular on Γ by Bruck's theorem. Thus we obtain the latter half of (ii).

LEMMA 4. *There exists a prime p-primitive divisor of $p^{m(n-1)}-1$ except in the following cases.*

(i) *$(m, n)=(1, 3)$ or $(2, 2)$ and p is a Mersenne prime.*

(ii) *$(m, n)=(1, 2)$ or $(6, 2)$.*

(iii) *$p=2$ and $(m, n)=(1, 7)$, $(2, 4)$ or $(3, 3)$. Moreover M is semi-regular on $\Delta - \{A\}$ for some $A \in \Delta$.*

PROOF. Suppose false. By Zsigmondy's theorem (Theorem 6.2 of [8]), $p=2$, $m(n-1)=6$ and M contains a Baer involution w with $|F(w) \cap \Delta| \geqq 2$. Hence $m=2$ and $n=4$.

Let R be a Sylow 7-subgroup of G. Since $|\Gamma| = 2^2 \cdot 3^2 \cdot 7 \mid |G|$, $R \neq 1$. Clearly R fixes Δ pointwise. Let $x \in R - \{1\}$ and assume $F(x)$ is a subplane of order 2^i. Since $F(x) \supset \Delta$ and $2 \leqq i \leqq 4$, we have $i=2$ by Bruck's theorem. Hence $F(x) \cap \Gamma = \Delta$. Thus R is semi-regular on Γ and so $R \cong Z_7$.

Assume $M_\Delta = \{u \in M \mid F(u) \supset \Delta\} = 1$. Then M is isomorphic to a subgroup of D_8, the dihedral group of order 8. Since $w \in M$ and every nontrivial characteristic subgroup of M is 1/2-transitive and faithful on $\bar{\Gamma}$, one of the following occurs: (a) $M\cong D_8$ and each M-orbit on $\bar{\Gamma}$ is of length 4. (b) $M\cong Z_2\times Z_2$ and each M-orbit on $\bar{\Gamma}$ is of length 2. Let J be the set of involutions in M. Then $\bar{\Gamma}=\bigcup_{u\in J}(F(u)\cap\bar{\Gamma})$ and $F(u)\cap F(u')\cap\bar{\Gamma}=\emptyset$ for u, $u'\in J$, $u\neq u'$. Hence $|J|\times\sqrt{2^8}\geqq|\bar{\Gamma}|=4^4-4$ and so $|J|>4^2-1$, a contradiction.

Next assume $M_\Delta\neq 1$ and set $N=\Omega_1(Z(M_\Delta))$. Since $G\rhd N\neq 1$, $F(N)$ is a subplane of order 4 and $F(N)\cap\Gamma=\Delta$. Hence $|C_U(N)|=4$, where $U=T(A)$, $A\in\Delta$. If $|N|\leqq 4$, then R centralizes N and so $C_U(R)\geqq C_U(N)\geqq C_{[U,R]}(N)\neq 1$. This is a contradiction by Theorem 5.2.3 of [**1**]. Thus $|N|\geqq 8$.

Let $B\in\bar{\Gamma}$. Then $F(N_B)$ is a Baer subplane of order 2^4 as $F(N_B)\supset F(N)$. Hence N_B is semi-regular on $\bar{\Gamma}-\bar{\Gamma}\cap F(N_B)$. Thus each N-orbit on $\bar{\Gamma}$ is of length 4 and $|N|=8$ or 16. If $|N|=16$, then R centralizes an involution $x\in N$. Therefore R acts on $\bar{\Gamma}-\bar{\Gamma}\cap F(x)$, contrary to the semi-regularity of R on $\bar{\Gamma}$. If $|N|=8$, then $\bar{\Gamma}=\bigcup_{1\neq x\in N}F(x)\cap\bar{\Gamma}$ and so $4^4-4=7(4^2-4)$, a contradiction.

LEMMA 5. *Assume $n\neq 2$. Let t be a p-primitive divisor of $p^{m(n-1)}-1$ and let R be a Sylow t-subgroup of G. Then*

(i) *If a subgroup S $(\neq 1)$ of R fixes a point of Δ, then $F(S)$ is a desarguesian subplane of π of order q.*

(ii) *Either M is faithful on Δ or R is semi-regular on Δ.*

(iii) *Assume $n>3$. Then $|F(M)\cap OA|\geqq q^{n-1}+1$ for some $A\in\Delta$ and*

$$M\leqq \mathrm{Aut}(GF(q))\left\{\begin{pmatrix}1&0&x\\0&1&0\\0&0&1\end{pmatrix}\middle|\ x\in GF(q)\right\}\ (\leqq\Gamma L(3,q)).$$

PROOF. By Lemma 3 (ii), $F(S)$ is a subplane of π of order q. Let A, $B\in F(S)\cap\Delta$, $A\neq B$ and let I be an affine fixed point of S with $I\notin OA\cup OB$. Set $A=(\infty)$, $B=(0)$, $O=(0,0)$, $I=(1,1)$ and let Q be a coordinatizing quasifield relative to O, A, B, I. Set $K=\mathrm{Kern}(Q)$. Since Γ is a set of $GF(q)$-subspaces, $K\geqq GF(q)$. To prove (i) we may assume that π is not desarguesian. In particular $|K|<q^{n-1}$. Since $S\leqq \mathrm{Aut}(Q)$, we have $S\leqq\mathrm{Aut}_K(Q)$ and $K=GF(q)$. Therefore $F(S)$ is a subplane of order q coordinatized by K, hence (i) holds.

Deny (ii). Then $N=M_\Delta\neq 1$ and there exists S which satisfies the assumption of (i). Let A, B, I, Q and K be as above. By Lemma 2, $S\nleqq C_G(N)$ and so $|N|\geqq q^{n-1}$. In particular π is not desarguesian. By a similar argument as above $S\leqq\mathrm{Aut}_K(Q)$ and $F(S)$ is a desarguesian plane of order q. $C_{T(A)}(N)$ and $C_{T(B)}(N)$ are S-invariant subgroups of order q as $F(N)$ is a subplane of order q. Hence $C_T(S)\geqq\langle C_{T(A)}(N), C_{T(B)}(N)\rangle=C_T(N)$ and so $F(S)=F(N)$. Therefore $N\leqq\mathrm{Aut}_K(Q)$. By Proposition 6.12 of [**5**], $(p,m,n)=(2,1,4)$ and $|N|\geqq q^3$. As $|\bar{\Gamma}|=q(q^2+q+1)$, every N-orbit on $\bar{\Gamma}$ is of length at most q. Let $C\in\bar{\Gamma}$. Then $|N_C|\geqq q^2$ and $F(N_C)$ is a Baer subplane of order q^2. Hence N_C is semi-regular on $\bar{\Gamma}-\bar{\Gamma}\cap F(N_C)$. Since N is 1/2-transitive on $\bar{\Gamma}$, this is a contradiction. Thus (ii) holds.

To prove (iii) we may assume $M\neq 1$. If $F(M)\cap\Delta=\{A\}$ for some $A\in\Delta$, then $A\in F(R)\cap\Delta$. By (i) and (ii), $F(R)$ is a desarguesian subplane of order q and M is faithful on Δ. By Lemma 3 (iii), $[M, R]\leqq(MR)_{\Delta}\cap M\leqq M_{\Delta}=1$. Applying Lemma 2, $|C_{T(A)}(M)|\geqq q^{n-1}$. Since M acts on the desarguesian plane $F(R)$ of order q and $|F(M)\cap F(R)\cap OA|\geqq p+1\geqq 3$, we have (iii) in this case.

Assume $|F(M)\cap\Delta|\geqq 2$. If $F(M)\cap\Gamma\neq\Delta$, then $|C_T(M)|<q^2$ and so R centralizes $C_T(M)$. Hence $F(R)\cap\Gamma=\Delta$ and $M_{\Delta}=1$ by (ii) and Lemma 3 (iii). Therefore $[M, R]\leqq(MR)_{\Delta}\cap M=M_{\Delta}=1$ and so $|C_T(M)|\geqq q^{n-1}\geqq q^2$ by Lemma 2, a contradiction. Hence $F(M)\cap\Gamma=\Delta$. By (ii) and Lemma 3 (i), $n=3$, contrary to the assumption.

LEMMA 6. *Assume $n\neq 2$. If there exists a prime p-primitive divisor of $p^{m(n-1)}-1$, then $|F(M)\cap\Delta|\geqq 2$ or M has at least q^{n-1} affine fixed points on OA for some $A\in\Delta$.*

PROOF. Let t be a prime p-primitive divisor of $p^{m(n-1)}-1$ and R a Sylow t-subgroup of G. Assume $F(M)\cap\Delta=\{A\}$ for some $A\in\Delta$. Then $A\in F(R)\cap\Delta$ and by Lemma 3 (iii) R fixes Δ pointwise. Hence, by Lemma 5 (ii), M is faithful on Δ. Therefore $[M, R]\leqq(MR)_{\Delta}\cap M\leqq M_{\Delta}=1$. By Lemma 2, $|C_{T(A)}(M)|\geqq q^{n-1}$. Thus the lemma holds.

LEMMA 7. *Assume $n\neq 2$. If there exists a p-primitive divisor t of $p^{m(n-1)}-1$, then one of the following holds.*

(i) *M is semi-regular on $\Delta-\{A\}$ for some $A\in\Delta$.*

(ii) *$n=3$ and a Sylow t-subgroup of G is semi-regular on Δ.*

PROOF. Let R be a Sylow t-subgroup of G. Assume M contains a nontrivial element v such that $|F(v)\cap\Delta|\geqq 2$ and assume $F(S)\cap\Delta\neq\emptyset$ for a nontrivial subgroup S of R. Then S fixes Δ pointwise by Lemma 3 (ii). Applying Lemma 5 (ii), $M_{\Delta}=1$ and so $[M, S]\leqq(MS)_{\Delta}\cap M=M_{\Delta}=1$. By Lemma 2, M has at least q^{n-1} affine fixed points on OA for some $A\in\Delta$. Since $v\in M$ and $|F(v)\cap\Delta|\geqq 2$, $F(v)$ is a subplane whose order is at least q^{n-1}. This contradicts Bruck's theorem. Therefore we have the lemma.

LEMMA 8. *Assume $n\neq 2$ and $q\equiv 0 \pmod 2$. If there exists a prime 2-primitive divisor t of $2^{m(n-1)}-1$, then the following hold.*

(i) *M is semi-regular on $\Delta-\{A\}$ for some $A\in\Delta$.*

(ii) *$C_G(M)$ contains a Sylow t-subgroup of G.*

(iii) *If $M\neq 1$, then every nontrivial t-subgroup fixes Δ pointwise and has no fixed point on Γ.*

PROOF. Let R be a Sylow t-subgroup of G. We may assume $M\neq 1$. If $M_{\Delta}\neq 1$, then $F(M_{\Delta})$ is a subplane of order q. By Lemma 7, $n=3$. Let w be an involution in M_{Δ}. Then $F(w)$ is a subplane of order $\sqrt{q^3}$ by Baer's theorem and $F(M_{\Delta})$ is a subplane of $F(w)$. It follows from Bruck's theorem that $q^2\leqq\sqrt{q^3}$, a contradiction. Hence $M_\Delta=1$. If $|F(M)\cap\Delta|\geqq 2$, then $F(M)$ is a subplane whose order is less than q as $M_\Delta=1$. Therefore $|C_T(M)|<q^2$ and R centralizes $C_T(M)$, so that $F(R)\cap\Delta\neq\emptyset$. This contradicts Lemma 7. Thus $|F(M)\cap\Delta|=1$.

Set $F(M)\cap\Delta=\{A\}$. Then R fixes A and therefore M is semi-regular on $\Delta-\{A\}$ by Lemma 7. Moreover, $F(R)\cap\Gamma=\Delta$ and R is semi-regular on $\bar{\Gamma}$ by Lemma 3 (iii). Therefore (i) and (iii) hold. Since $[M, R]\leqq(MR)_\Delta\cap M=1$, (ii) holds.

LEMMA 9. *Let V be an elementary abelian p-group of order p^r with $p^r\equiv-1$* (mod 4) *and S a 2-subgroup of the automorphism group of V. If an involution x in S inverts V, then $\langle x\rangle$ is a direct factor of S.*

PROOF. Since $p^r\equiv-1$ (mod 4), r is odd and $p\equiv-1$ (mod 4). We may assume that $V=V(r, p)$ and $S\leqq GL(r, p)$. Then $x=-E$, where E is the unit matrix of degree r. Since r is odd, the determinant of $-E$ is equal to -1. Hence $\langle x\rangle\times SL(r, p)$ is a normal subgroup of $GL(r, p)$ of index $(p-1)/2$. Since $(p-1)/2$ is odd, $S\leqq\langle x\rangle\times SL(r, p)$ and hence $S=\langle x\rangle\times(S\cap SL(r, p))$. Thus $\langle x\rangle$ is a direct factor of S.

LEMMA 10. *Let S be a Sylow 2-subgroup of G.I f $q^n\equiv-1$* (mod 4), *then the following hold.*

(i) *S is dihedral or semi-dihedral and $Z(S)=S_\Gamma\cong Z_2$.*

(ii) *$|S|\geqq 4(q+1)_2$ and $S_A\cong Z_2\times Z_2$ for each $A\in\Delta$.*

PROOF. Set $W=S_\Gamma$. Since $|\bar{\Gamma}|=q(q^{n-1}-1)\mid|G|$ and $q^2-1\mid q^{n-1}-1$, $2(q+1)_2\mid|S/W|$. Hence $|S_A|\geqq 2|W|$ for some point $A\in\Delta$ as $|\Delta|=q+1$. Let $B\in F(S_A)\cap(\Delta-\{A\})$.

First we show that $W\neq 1$. Assume $W=1$ and let x be an involution in $Z(S_A)$. Since $q^n\equiv-1$ (mod 4), every involution in S is a homology by Baer's theorem. Since $S_B\geqq S_A$, we may assume that x is an (A, OB)-homology. Hence $S_A=C_S(x)$. In particular $|S_A|\geqq 4$. By Lemma 4.22 of [4], $S_{(B,OA)}=1$ as $W=1$. Hence S_A has a unique involution and it inverts $T(A)$. However, by Lemma 9, S_A contains a subgroup isomorphic to $Z_2\times Z_2$ as $|S_A|\geqq 4$, a contradiction. Thus $W=1$. Since $4\nmid q^n-1$, $S_\Gamma=W\cong Z_2$. In particular $|S_A|\geqq 2|W|=4$.

Set $\langle z\rangle=W$ and $V=S_A$. If $V_{(A,OB)}=1$, then V acts fixed point freely on $T(A)$ and z inverts $T(A)$. By Lemma 9, V contains a subgroup isomorphic to $Z_2\times Z_2$. By Lemma 4.22 of [4], $V_{(A,OB)}\neq 1$, a contradiction. Hence $V_{(A,OB)}\neq 1$ and similarly $V_{(B,OA)}\neq 1$.

Set $\langle u\rangle=V_{(A,OB)}$. Then $\langle u\rangle\cong Z_2$ and $C_S(u)=V$ as $u\in Z(V)$. Assume $|V|>4$ and set $V=V/\langle u\rangle$. Then V acts on $T(B)$ and z inverts $T(B)$. Hence $V=\langle z\rangle\times U$ for a subgroup U of V with $u\in U$ by Lemma 8. Since $U_\Gamma=1$ and $u\in U$, U acts fixed point freely on $T(A)$ and u inverts $T(A)$. Therefore U contains a subgroup isomorphic to $Z_2\times Z_2$. This implies that $U_\Gamma\neq 1$, a contradiction. Thus $|V|=4$. In particular $V\cong Z_2\times Z_2$.

As $V \leqq S_B$ and $F(V) \cap \Gamma = \{A, B\}$, we have $V = S_B$. Since $C_S(u) = V$, S is dihedral or semi-dihedral by a lemma of [**10**]. Therefore any involution in S is S-conjugate to an involution in V. Hence if $S_C \neq 1$ for some $C \in \Delta$, then $C = A^s$ or B^s for a suitable element $s \in S$. Thus $|S_C| = |V| = 4$ and the lemma holds.

3. Proof of the theorems.

PROOF OF THEOREM 1. Assume $n \neq 2$ and $|F(v) \cap \Delta| \geqq 2$ for some $v \in M - \{1\}$. Let Φ be the set of prime p-primitive divisors of $p^{m(n-1)} - 1$. Suppose $\Phi = \emptyset$. Then $(m, n) = (1, 3)$ and $q^n \equiv -1 \pmod 4$ by Lemma 4. Applying Lemma 10, the length of each G-orbit on Δ is divisible by $(q+1)_2$. Hence (iii) holds. Suppose $\Phi \neq \emptyset$. By Lemma 8, q is odd. Then (iii) follows immediately from Lemmas 7 and 10.

PROOF OF THEOREM 2. Assume $n \neq 3$. By Lemma 10, $|F(M) \cap \Delta| \geqq 2$. Hence there exists a prime p-primitive divisor t of $p^{m(n-1)} - 1$ by Lemma 4. As $q^n \equiv -1 \pmod 4$, $n \neq 2$. It follows from Lemma 7 that $n = 3$.

PROOF OF THEOREM 3. Let Φ be the set of prime p-primitive divisors of $p^{m(n-1)} - 1$. Let $A, B \in F(G) \cap \Delta$, $A \neq B$. If Φ is not empty, then $n = 2$ applying Theorem 1. Hence we may assume Φ is empty.

Suppose $n \neq 2$. Then, by Lemma 4, $(m, n) = (1, 3)$ and p is a Mersenne prime. The order of a Sylow 2-subgroup S of G is divisible by 8 as $|\bar{\Gamma}| \mid |G|$. Since $|\Delta - \{A, B\}| = p - 1 \equiv 2 \pmod 4$, there is a subgroup T of S of index 2 such that $|F(T) \cap \Delta| = 5$. As $mn = 3$, any involution of T is a homology by Baer's theorem. Therefore $|T| \mid |OA - \{O, A\}| = p^3 - 1 \equiv 2 \pmod 4$ and so $|T| \leqq 2$. This implies $|S| \leqq 4$, a contradiction. Thus $n = 2$.

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