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On the value distribution of meromorphic mappings of covering spaces over C^m into algebraic varieties

Dedicated to Professor M. Ozawa on his 60th birthday

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Introduction.

The purpose of this paper is to study the meromorphic mappings $f: X \to V$ of a finite analytic (ramified) covering space X over the m-dimensional complex vector space $\boldsymbol{C}^m$ with projection $\pi: X \to \boldsymbol{C}^m$ into a complex projective manifold V of dimension n from the view point of the Nevanlinna theory; especially, we are interested in inequalities of the second main theorem type. In the case where $m=1$ and V is the 1-dimensional complex projective space $\boldsymbol{P}^1(\boldsymbol{C})$, Selberg [**17**] proved the first and the second main theorems for $f: X \to \boldsymbol{P}^1(\boldsymbol{C})$. In the case where $\dim X \geqq \dim V = \operatorname{rank} f (=\sup\{\dim X - \dim_x f^{-1}(f(x)); x \in X\}) \geqq 1$, we proved the second main theorem and defect relations for f and divisors on V, generalizing the above results of Selberg and the Carlson-Griffiths-King theory [**2**] and [**5**] (see [**10**], [**11**] and [**19**]).

Here we deal with the case where $\operatorname{rank} f$ does not necessarily equal $\dim V$. Stoll [**20**] obtained the Ahlfors-Weyl theory for linearly non-degenerate meromorphic mappings from a parabolic manifold into $\boldsymbol{P}^n(\boldsymbol{C})$ which applies to the case of $f: X \to \boldsymbol{P}^n(\boldsymbol{C})$ (see [**20**, Theorem 11.8]). When X is an affine algebraic curve (or the domain of f may be the punctured disc Δ^* in $\boldsymbol{C}$), we proved an inequality of the second main theorem type for $f: X \to V$ in terms of logarithmic 1-forms along the given divisors on V (see [**12**], [**13**] and [**14**]), and applied it to obtain a generalization of big Picard's theorem (see [**14**]). In the present paper we extend this inequality to the case where X is a finite analytic covering space over $\boldsymbol{C}^m$, and give an application.

Let D be an effective reduced divisor on V such that the closed image of the quasi-Albanese mapping $\alpha: V-D \to A_{V-D}$ is of dimension n and of general type (cf. [**6**] and [**7**]). Here we identify D with its support. Let $f: X \to V$ be a meromorphic mapping. We say that f is degenerate with respect to the complete linear system $|L|$ of a holomorphic line bundle L over V if $f(X) \subset \operatorname{Supp} E$ for some $E \in |L|$, where $\operatorname{Supp} E$ denotes the support of the divisor E; otherwise, f is said to be non-degenerate with respect to $|L|$. Assume that $f(X) \not\subset D$ and

f is non-degenerate with respect to $|K_V \otimes [D]|$, where K_V denotes the canonical line bundle over V and $[D]$ the line bundle determined by D. Then we prove the following inequality of the second main theorem type in section 3 (see section 1 for notation):

MAIN THEOREM (3.2). *Let the notation be as above. Then there is a positive constant K independent of f such that*

$$KT_f(r) \leqq N(r, \operatorname{Supp} f^*D) + N(r, \operatorname{Supp} R)$$
$$+O(\delta \log r) + O(\log^+ T_f(r)) \quad \|_{E(\delta)}$$

for arbitrary $\delta \in (0, 1)$, where R denotes the ramification divisor of $\pi: X \to \boldsymbol{C}^m$.

For the proof of the Main Theorem (3.2) we show a lemma on logarithmic derivatives for $f: X \to V$ and logarithmic 1-forms on V in section 1 (Lemma (1.6)). By making use of this lemma and the higher jet bundles over V investigated in section 2, we prove the Main Theorem (3.2).

At the end of section 3, applying the arguments used there, we remark an explicit second main theorem and defect relations for linearly non-degenerate $f: X \to \boldsymbol{P}^n(\boldsymbol{C})$ and hyperplanes of $\boldsymbol{P}^n(\boldsymbol{C})$ in general position (see (3.16), (3.17) and [**20**, Theorem 11.8]). Let X be the domain of existence of $f: X \to V$ (see section 1) and D as in the Main Theorem (3.2). We see by the Main Theorem (3.2) that if $N(r, \operatorname{Supp} f^*D) = o(T_f(r))$ and $N(r, \operatorname{Supp} R) = o(T_f(r))$ or $O(\log r)$, then f is degenerate with respect to $|K_V \otimes [D]|$, or $\pi: X \to \boldsymbol{C}^m$ is an algebraic covering space and f is rational (Theorem (3.12)).

In section 4 we apply the Main Theorem (3.2) to obtain the following multiplicity inequalities. Let $L \to V$ be a holomorphic line bundle over V and $|L|$ the complete linear system. Put $|L|' = \{D \in |L|; D \text{ is reduced and irreducible}\}$. Let $\{D_i\}_{i=1}^q$ be a finite family of $|L|'$ satisfying some conditions (cf. section 4). Let $f: X \to V$ be a meromorphic mapping which is non-degenerate with respect to $|L|'$ and $|L^\lambda \otimes K_V|$ with certain positive integer λ (cf. section 4), such that $N(r, \operatorname{Supp} R) = o(T_f(r))$. Let ν_i, $1 \leqq i \leqq q$, be positive integers such that

$$f^*D_i \geqq \nu_i \operatorname{Supp} f^*D_i \qquad \text{for all } i.$$

Then there is a positive constant K' independent of f and $\{D_i\}_{i=1}^q$ such that

$$\sum_{i=1}^q \frac{1}{\nu_i} \geqq \frac{K'}{\lambda} q \qquad \text{(Theorem (4.4)).}$$

Here we set $\nu_i = +\infty$ if $f(D) \cap D_i = \emptyset$.

For meromorphic functions on $\boldsymbol{C}$, R. Nevanlinna proved that $\sum_{i=1}^q 1/\nu_i \geqq q-2$ (cf. [**9**]), and for linearly non-degenerate holomorphic curves from $\boldsymbol{C}$ into the n-dimensional complex projective space $\boldsymbol{P}^n(\boldsymbol{C})$ and for hyperplanes in general position, H. Cartan [**3**] proved that $\sum_{i=1}^q 1/\nu_i \geqq (q-n-1)/n$. The above inequality extends these inequalities to the case of meromorphic mappings from the finite analytic covering space X into the general V, while the constant K' is not effectively computed. For linearly non-degenerate meromorphic mappings $f: X\rightarrow \boldsymbol{P}^n(\boldsymbol{C})$ and hyperplanes of $\boldsymbol{P}^n(\boldsymbol{C})$ in general position, one gets the following inequality:

$$\sum_{i=1}^{q}\frac{1}{\nu_i}\geqq\frac{q-2kn+n-1}{n} \qquad \text{(see (4.6))}.$$

See [16] and [10] for the equidimensional case.

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1. Lemma on logarithmic derivatives.

Let $\pi: X\rightarrow \boldsymbol{C}^m$ be a finite analytic covering space; that is, X is a normal complex space with finite proper surjective holomorphic mapping π. We denote by k its sheet number and by R the ramification divisor of $\pi: X\rightarrow \boldsymbol{C}^m$. For $z=(z^1, \cdots, z^m)\in \boldsymbol{C}^m$ we set

$$\|z\|=\Big(\sum_{i=1}^{m}|z^i|^2\Big)^{1/2},$$

$$B(r)=\{z\in \boldsymbol{C}^m;\ \|z\|<r\} \qquad (r>0),$$

$$X(r)=\pi^{-1}(B(r)),$$

$$\varphi=\pi^*dd^c\|z\|^2=\frac{i}{2\pi}\pi^*\partial\bar\partial\|z\|^2,$$

where $d=\partial+\bar\partial$ and $d^c=(i/4\pi)(\bar\partial-\partial)$. Let Z be a Weil divisor on X and define the counting function of Z by

$$N(r, Z)=\frac{1}{k}\int_1^r\frac{dt}{t}\int_{X(t)\cap Z}\varphi^{m-1}.$$

Let $S(X)$ denote the set of singular points of X and F be a multiplicative meromorphic function on $X-S(X)$; i.e., F is a multivalued meromorphic function on $X-S(X)$ such that the absolute value $|F|$ is one-valued. Then we have the zero divisor $(F)_0$ and the polar divisor $(F)_\infty$ on $X-S(X)$ determined by F. Since $\operatorname{codim} S(X)\geqq 2$, the divisors $(F)_0$ and $(F)_\infty$ uniquely extend to divisors on X, which are respectively denoted by the same $(F)_0$ and $(F)_\infty$. Take a holomorphic function ζ_1 on X such that $(\zeta_1)\geqq(F)_\infty$, where (ζ_1) denotes the divisor on X determined by ζ_1. Put

$$\xi_2 = \log|F\zeta_1| \qquad \text{on } X - S(X).$$

Then ξ_2 is plurisubharmonic on $X-S(X)$. By [4] ξ_2 uniquely extends to a plurisubharmonic function on X. Hence we have

$$\log|F| = \xi_2 - \xi_1 \qquad \text{on } X - S(X), \tag{1.1}$$

where $\xi_1 = \log|\zeta_1|$ and ξ_2 are plurisubharmonic functions on X. It follows that $\log|F|$ is locally integrable on X. Then we have the following Poincaré-Lelong current equation for F:

$$dd^c[\log|F|^2] = (F)_0 - (F)_\infty \qquad \text{on } X, \tag{1.2}$$

where $dd^c[\log|F|]$ stands for that it is taken in the sense of current. To see (1.2), we first note that (1.2) holds on $X-S(X)$ by the ordinary Poincaré-Lelong equation. In the same way as in [**5**, section 1 (a)], we see that

$$\left\{\begin{array}{l}\text{if } \xi \text{ is a plurisubharmonic function on } X, \text{ then} \\ \text{analytic subsets of codimension } \geqq 2 \text{ of } X \text{ are sets of} \\ \text{measure zero for the positive current } dd^c[\xi] \text{ on } X.\end{array}\right. \tag{1.3}$$

It is obvious that analytic subsets of codimension $\geqq 2$ of X are sets of measure zero for the positive currents $(F)_0$ and $(F)_\infty$. By (1.1), (1.3) and the fact that $\operatorname{codim} S(X) \geqq 2$, (1.2) holds on X. We have by (1.1), (1.2) and [**10**, Lemma 3.1] (cf. also [**18**])

$$\frac{1}{k}\int_{\partial X(r)} \log|F|\eta - \frac{1}{k}\int_{\partial X(1)} \log|F|\eta = N(r.\ (F)_0) - N(r,\ (F)_\infty), \tag{1.4}$$

where $\eta = \pi^*(d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1})$. We set

$$m(r,\ F) = \frac{1}{k}\int_{\partial X(r)} \log^+|F|\eta,$$

where $\log^+|F| = \max\{\log|F|,\ 0\}$, and

$$T(r,\ F) = m(r,\ F) + N(r,\ (F)_\infty).$$

By (1.4) we have the first main theorem for F:

$$T(r,\ F) = T\left(r,\ \frac{1}{F}\right) + \frac{1}{k}\int_{\partial X(1)} \log|F|\eta. \tag{1.5}$$

The function $T(r,\ F)$ is called the Nevanlinna characteristic function of F. Here one notes that

$$\frac{1}{k}\int_{\partial X(r)} \eta = 1.$$

Let V be a complex projective manifold and $f: X \to V$ be a meromorphic mapping. We say that X is *the domain of existence* of f if f separates the fibers of $\pi: X\rightarrow \boldsymbol{C}^m$ in the sense of [10]; that is, there is a point $z\in \boldsymbol{C}^m$ such that π is unramified at all points of $\pi^{-1}(z)$, f is holomorphic at all points of $\pi^{-1}(z)$ and $f(x)\neq f(y)$ for all distinct points x and y of $\pi^{-1}(z)$.

Let Ω be a differential form of type (1, 1) on V and define the characteristic function of f with respect to Ω by

$$T_f(r;\Omega)=\frac{1}{k}\int_1^r\frac{dt}{t}\int_{X(t)} f^*\Omega\wedge\varphi^{m-1}.$$

In the rest of this section, we assume that Ω is the positive form associated with a hermitian metric h on X, and denote $T_f(r;\Omega)$ by $T_f(r)$.

Let D be an effective reduced divisor on V and $\Omega_V^1(\log D)$ the sheaf of germs of logarithmic 1-forms along D (cf., e.g., [6]). Assume that $f(X)\not\subset D$. For a global logarithmic 1-form $\omega\in H^0(M, \Omega_V^1(\log D))$ we put

$$f^*\omega=\sum_{j=1}^m \xi_j\pi^*dz^j$$

on X, where ξ_j are meromorphic functions on X.

LEMMA (1.6). *Let ξ_j be as above. Then we have*

$$m(r, \xi_j)=O(\delta\log r)+O(\log^+T_f(r)) \quad \|_{E(\delta)}{}^{1)}$$

for arbitrary $\delta\in(0, 1)$.

PROOF. By [6] (cf. also [12]), there is a basis $\{\omega^i\}_{i=1}^p$ of $H^0(V, \Omega_V^1(\log D))$ such that ω^i, $1\leqq i\leqq q$ $(q\leqq p)$, are holomorphic 1-forms and

$$\omega^i=d\log\theta^i, \qquad q+1\leqq i\leqq p,$$

where θ^i are multiplicative meromorphic functions on V such that the supports $\mathrm{Supp}(\theta^i)$ of the divisors (θ^i) are contained in D. Put

$$f^*\omega^i=\sum_{j=1}^m \xi_j^i\pi^*dz^j \qquad \text{and} \qquad \omega=\sum_{i=1}^p c_i\omega^i,$$

where $c_i\in\boldsymbol{C}$. We have

$$m(r, \xi_j)\leqq\sum_{i=1}^p m(r, \xi_j^i)+\sum_{i=1}^p \log^+|c_i|+\log p.$$

Hence we may assume that ω is one of $\{\omega^i\}_{i=1}^p$.

Let ω be holomorphic. Then there is a constant $C_1>0$ such that

$$|\omega(v)|^2\leqq C_1h(v, \bar{v})$$

for all holomorphic tangent vector $v\in T(V)$ on V. Put $S=\pi(\mathrm{Supp}\,R)$. Then the

1) The symbol "$\|_{E(\delta)}$" stands for that the stated (in)equality holds as r tends to infinity outside an exceptional subset $E(\delta)$ of $(0, \infty)$ which is a union of intervals with finite total measure and depends on δ in this case.

restriction $\pi|(X-\pi^{-1}(S)): X-\pi^{-1}(S)\to \boldsymbol{C}^m-S$ is locally biholomorphic. We regard the vector fields $\partial/\partial z^i$ as meromorphic vector fields on X which is holomorphic on $X-\pi^{-1}(S)$. Then $\xi_j=\omega(f_*(\partial/\partial z^j))$ on $X-\pi^{-1}(S)$. Put

$$f^*h=\sum_{j,l}\frac{i}{\pi}H_{j\bar{l}}\pi^*(dz^j\cdot d\bar{z}^l),$$

$$f^*\Omega=\sum_{j,l}\frac{i}{2\pi}H_{j\bar{l}}\pi^*(dz^j\wedge d\bar{z}^l).$$

Then we have

$$f^*h\Big(\frac{\partial}{\partial z^j},\frac{\partial}{\partial \bar{z}^l}\Big)=\frac{1}{\pi}H_{j\bar{l}},$$

so that

$$\begin{aligned}|\xi_j|^2\varphi^m&\leqq C_1\frac{1}{\pi}H_{j\bar{j}}\varphi^m\leqq\frac{C_1}{\pi}\sum_{l=1}^m H_{l\bar{l}}\varphi^m\\&=\frac{mC_1}{\pi}f^*\Omega\wedge\varphi^{m-1}\qquad\text{on}\quad X-\pi^{-1}(S).\end{aligned}$$

Hence, as in [**14**, section 2], we have

$$\begin{aligned}m(r,\xi_j)&=\frac{1}{k}\int_{\partial X(r)}\log^+|\xi_j|\eta\\&=\frac{1}{2k}\int_{\partial X(r)}\log^+|\xi_j|^2\eta\leqq\frac{1}{2k}\int_{\partial X(r)}\log(1+|\xi_j|^2)\eta\\&\leqq\frac{1}{2}\log\Big(1+\frac{1}{k}\int_{\partial X(r)}|\xi_j|^2\eta\Big)\\&=\frac{1}{2}\log\Big(1+\frac{1}{k}\cdot\frac{1}{r^{2m-1}}\cdot\frac{d}{dr}\int_{X(r)}|\xi_j|^2\varphi^m\Big)\\&\leqq\frac{1}{2}\log\Big(1+\frac{1}{k}\cdot\frac{1}{r^{2m-1}}\Big(\int_{X(r)}|\xi_j|^2\varphi^m\Big)^{1+\delta}\Big)\quad\|_{E_1(\delta)}\\&=\frac{1}{2}\log\Big(1+\frac{1}{k}\cdot\frac{1}{r^{2m-1}}\Big(r^{2m-1}\frac{d}{dr}\int_1^r\frac{dt}{t^{2m-1}}\int_{X(t)}|\xi_j|^2\varphi^m\Big)^{1+\delta}\Big)\quad\|_{E_1(\delta)}\\&\leqq\frac{1}{2}\log\Big(1+\frac{1}{k}r^{(2m-1)\delta}\Big(\int_1^r\frac{dt}{t^{2m-1}}\int_{X(t)}|\xi_j|^2\varphi^m\Big)^{(1+\delta)^2}\Big)\quad\|_{E_2(\delta)}\\&\leqq\frac{1}{2}\log\Big(1+\frac{1}{k}r^{(2m-1)\delta}\Big(\frac{mC_1}{\pi}\int_1^r\frac{dt}{t^{2m-1}}\int_{X(t)}f^*\Omega\wedge\varphi^m\Big)^{(1+\delta)^2}\Big)\quad\|_{E_2(\delta)}\\&\leqq\frac{(1+\delta)^2}{2}\log^+T_f(r)+\frac{2m-1}{2}\delta\log r+O(1)\quad\|_{E_2(\delta)}.\end{aligned}$$

Let ω be of the type $d\log\theta$, where θ is a multiplicative meromorphic function on V. Let $I(f)$ be the indeterminacy locus of f and put $G=\theta\circ f$ on $X-I(f)$.

Since $\operatorname{codim} I(f) \geqq 2$, G extends to a multiplicative meromorphic function on $X-(I(f) \cap S(X))$ which is denoted by the same G. Put

$$dG = \sum_{j=1}^{m} G_j \pi^* dz^j .$$

Then we have that $\xi_j = G_j/G$. To prove our lemma, we use the curvature method due to [**5**] in the same way as in [**22**] (cf. [**14**]). For constants $\varepsilon>0$, $a_0>0$ and $b_0>1$ we put

$$\Psi = \frac{a_0(|w|+|w|^{-1})^{2+2\varepsilon}}{(\log b_0(1+|w|^2))^2(\log b_0(1+|w|^{-2}))^2} \psi_0 ,$$

where $\psi_0 = (1+|w|^2)^{-2}(i/2\pi) dw \wedge d\bar{w}$ on $\boldsymbol{P}^1(\boldsymbol{C})$ with inhomogeneous coordinate w is the standard Fubini-Study-Kähler form. By [**5**, Proposition (6.9)] there are constants $a_0>0$, $b_0>1$ and $\varepsilon>0$ such that

$$\operatorname{Ric} \Psi \geqq (|w|+|w|^{-1})^{-2\varepsilon} \Psi . \tag{1.7}$$

Since $|G|$ is one-valued, we have

$$\begin{aligned} G^*\Psi &= \frac{a_0(|G|+|G|^{-1})^{2+2\varepsilon}}{(\log b_0(1+|G|^2))^2(\log b_0(1+|G|^{-2}))^2} G^*\psi_0 \\ &= \frac{a_0(|G|+|G|^{-1})^{2\varepsilon}}{(\log b_0(1+|G|^2))^2(\log b_0(1+|G|^{-2}))^2} \\ &\quad \times \frac{1}{|G|^2} \sum_{j,l} G_j \bar{G}_l \pi^* \left(\frac{i}{2\pi} dz^j \wedge d\bar{z}^l \right) . \end{aligned}$$

Therefore we have

$$G^*\Psi \wedge \varphi^{m-1} = \sum_{j=1}^{m} A_j \frac{1}{m} \varphi^m ,$$

where

$$A_j = \frac{a_0(|G|+|G|^{-1})^{2\varepsilon}}{(\log b_0(1+|G|^2))^2(\log b_0(1+|G|^{-2}))^2} |\xi_j|^2 .$$

Putting $\Phi_j = \bigwedge_{l \neq j} (i/2\pi) dz^l \wedge d\bar{z}^l$, we have by (1.7)

$$\begin{aligned} (G^* \operatorname{Ric} \Psi) \wedge \Phi_j &\geqq (|G|+|G|^{-1})^{2\varepsilon} G^*\Psi \wedge \Phi_j \\ &= (|G|+|G|^{-1})^{-2\varepsilon} A_j \frac{1}{m!} \varphi^m \\ &= \frac{a_0}{(\log b_0(1+|G|^2))^2(\log b_0(1+|G|^{-2}))^2} |\xi_j|^2 \frac{1}{m!} \varphi^m . \end{aligned}$$

Taking dd^c as currents, we have

$$dd^c \left[\log \frac{a_0(|G|+|G|^{-1})^{2\varepsilon}}{(\log b_0(1+|G|^2))^2(\log b_0(1+|G|^{-2}))^2} |\xi_j|^2 \right]$$

$$=G^* \operatorname{Ric} \Psi+(\xi_j)_0-(\xi_j)_\infty-\varepsilon((G)_0+(G)_\infty)$$

$$\geqq G^* \operatorname{Ric} \Psi-(1+\varepsilon)((G)_0+(G)_\infty)-R$$

on $X-S(X)$. Using the same method to show (1.2) by (1.1) and (1.3), one sees that the above current equation holds on X. Then the similar computation to [**22**, section 4] and [**14**, section 2] yields

$$\text{(1.8)} \qquad m(r, \xi_j) \leqq 8 \log^+(T(r, G)+N(r, R))+12 \log^+ T(r, G) \\ +(2m-1)\delta \log r+O(1) \quad \|_{E(\delta)}\,.$$

In the same way as in [**14**, (2.10)], it follows that

$$\text{(1.9)} \qquad T(r, G) \leqq C_2 T_f(r)+O(1)\,,$$

where C_2 is a positive constant independent of f. Moreover, by [**10**, Proposition 1] there are a finite analytic covering space $\pi' : X' \to \boldsymbol{C}^m$, a finite holomorphic mapping $\lambda : X \to X'$ and a meromorphic mapping $f' : X' \to V$ such that $\pi=\pi' \circ \lambda$, $f=f' \circ \lambda$ and X' is the domain of existence of f'. Put $f'^* d \log \theta=\sum \xi'_j \pi'^* dz^j$. Then we have

$$\text{(1.10)} \qquad m(r, \xi_j)=m(r, \xi'_j)\,, \qquad T_f(r\,; \Omega)=T_{f'}(r\,; \Omega)\,.$$

Let R' be the ramification divisor of $\pi' : X' \to \boldsymbol{C}^m$. Then we see by (1.8)$\sim$(1.10) that

$$m(r, \xi_j)=m(r, \xi'_j) \\ \leqq 8 \log^+(T_{f'}(r)+N(r, R')) \\ +12 \log^+ T_{f'}(r)+(2m-1)\delta \log r+O(1) \quad \|_{E(\delta)}\,.$$

By [**10**, Lemma 4.1] there is a positive constant C_3 independent of f' such that

$$\text{(1.11)} \qquad N(r, R') \leqq C_3 T_{f'}(r)+O(1)\,.$$

Therefore we obtain

$$m(r, \xi_j) \leqq 20 \log^+ T_{f'}(r)+(2m-1)\delta \log r+O(1) \quad \|_{E(\delta)} \\ =20 \log^+ T_f(r)+(2m-1)\delta \log r+O(1) \quad \|_{E(\delta)}\,.$$

Q.E.D.

The polar divisors of ξ_j are at most $\operatorname{Supp}(f^*D+R)$. Therefore we have by Lemma (1.6)

$$T(r, \xi_j) \leqq N(r, \operatorname{Supp} f^*D)+N(r, \operatorname{Supp} R) \\ +O(\delta \log r)+O(\log^+ T_f(r)) \quad \|_{E(\delta)}\,.$$

Let $\nu=(\nu_1, \cdots, \nu_m)$, $\nu_j \in \boldsymbol{Z}$, $\nu_j \geqq 0$, be a multi-index and put $D_\nu=\partial^{|\nu|}/(\partial z^1)^{\nu_1} \cdots (\partial z^m)^{\nu_m}$ with $|\nu|=\sum \nu_j$. Let $D_\nu \xi_j$ be the partial derivative of ξ_j with respect to D_ν.

Then the polar divisors of $D_\nu \xi_j$ are at most $|\nu| \operatorname{Supp}(f^*D+R)$. Using Lemma (1.6) successively, we have the following.

LEMMA (1.12). *Let the notation be as above. Then we have*

$$m(r, D_\nu \xi_j) = O(\delta \log r) + O(\log^+ T_f(r)) \quad \|_{E(\delta)},$$

$$T(r, D_\nu \xi_j) \leqq |\nu| (N(r, \operatorname{Supp} f^*D) + N(r, \operatorname{Supp} R)) + O(\delta \log r) + O(\log^+ T_f(r)) \quad \|_{E(\delta)}$$

for arbitrary $\delta \in (0, 1)$.

2. Quasi-Albanese mapping and subvarieties of quasi-Abelian varieties.

Let V be a complex projective manifold and D an effective reduced divisor on V. Let $\Omega_V^1(\log D)$ be the sheaf of germs of logarithmic 1-forms along D (cf., e.g., [6] and [7]). Let $\{\omega^1, \cdots, \omega^q, \omega^{q+1}, \cdots, \omega^p\}$ be a basis of $H^0(V, \Omega_V^1(\log D))$ such that $\omega^1, \cdots, \omega^q$ are holomorphic 1-forms on V and

$$\omega^i = d \log \theta^i, \qquad q+1 \leqq i \leqq p,$$

where θ^i are multiplicative meromorphic functions over V such that $\operatorname{Supp}(\theta^i) \subset D$. Taking a fixed point $O \in X$, we consider the p-ple multivalued holomorphic function

$$\left(\int_O^x \omega^1, \cdots, \int_O^x \omega^p\right) \in \boldsymbol{C}^p \qquad (x \in V-D).$$

Then the subgroup $\Gamma = \left\{\left(\int_\gamma \omega^1, \cdots, \int_\gamma \omega^p\right); \gamma \in H_1(V-D, \boldsymbol{Z})\right\}$ of $\boldsymbol{C}^p$ is discrete and we have a holomorphic mapping

$$\alpha : \ x \in V-D \longrightarrow \left(\int_O^x \omega^1, \cdots, \int_O^x \omega^p\right) \in \boldsymbol{C}^p/\Gamma = A_{V-D},$$

which is called the quasi-Albanese mapping (cf. [6]). The group A_{V-D} is an algebraic group such that the sequence

$$0 \longrightarrow (\boldsymbol{C}^*)^{p-q} \longrightarrow A_{V-D} \longrightarrow A_V \longrightarrow 0$$

is exact, where $\boldsymbol{C}^*$ denotes the multiplicative group of non-zero complex numbers and A_V the Albanese variety of V ([6]). We call A_{V-D} the quasi-Albanese variety of $V-D$, which has the similar universal property to the Albanese variety ([6]). In general, a complex algebraic group A is called a quasi-Abelian variety if A carries an exact sequence

$$0 \longrightarrow (\boldsymbol{C}^*)^t \longrightarrow A \longrightarrow A_0 \longrightarrow 0,$$

where A_0 is an Abelian variety ([6]).

Let $W \subset A$ be a subvariety of dimension n of a quasi-Abelian variety A of dimension N. Then by [**6**] and [**7**] we have

$$(2.1) \qquad \begin{cases} W \text{ is of general type if and only if the} \\ \text{subgroup } \{a \in A\,; W+a=W\} \text{ is finite.} \end{cases}$$

Let W_{reg} denote the set of regular points of W and $i_{W_{\mathrm{reg}}}: W_{\mathrm{reg}} \to A$ the natural inclusion mapping. Let $J_k(W_{\mathrm{reg}})$ (resp. $J_k(A)$) be the k-th jet bundle over W_{reg} (resp. A) of holomorphic mappings from neighborhoods of the origin of $\boldsymbol{C}$ into W_{reg} (resp. A) (cf., e.g., [**15**, section 2]). Then $i_{W_{\mathrm{reg}}}$ naturally induces the injective bundle morphism

$$(i_{W_{\mathrm{reg}}})_* : \ J_k(W_{\mathrm{reg}}) \longrightarrow J_k(A).$$

Let $\omega^1, \cdots, \omega^N$ be the basis of invariant 1-forms on A (logarithmic 1-forms on A). Then $\omega^1 \wedge \cdots \wedge \omega^N$ does not vanish anywhere on A. By making use of $\{\omega^1, \cdots, \omega^N\}$, $J_k(A)$ has the triviality, $J_k(A) \cong A \times (\boldsymbol{C}^N)^k$. Let $\mu: J_k(A) \to (\boldsymbol{C}^N)^k$ be the projection and set

$$I_{W_{\mathrm{reg}}k} = \mu \circ (i_{W_{\mathrm{reg}}})_* : \ J_k(W_{\mathrm{reg}}) \longrightarrow (\boldsymbol{C}^N)^k.$$

Assume that W is of general type. Let $\varDelta = \{z \in \boldsymbol{C}\,; |z| < 1\}$ be the unit disc, $g: \varDelta \to W_{\mathrm{reg}}$ a holomorphic mapping and $J_k(g): \varDelta \to J_k(W_{\mathrm{reg}})$ its k-th prolongation. As a corollary of [**14**, Lemma (4.4)] we have the following:

Lemma (2.2). *Let $g: \varDelta \to W_{\mathrm{reg}}$ be algebraically non-degenerate (i.e., $g(\varDelta)$ is Zariski dense in W_{reg}). Then the differential of $I_{W_{\mathrm{reg}}k}$*

$$d\,I_{W_{\mathrm{reg}}k} : \ T(J_k(W_{\mathrm{reg}})) \longrightarrow T((\boldsymbol{C}^N)^k)$$

is injective at $J_k(g)(0)$ for all large k, where $T(\cdot)$ denotes the holomorphic tangent bundle.

Remark (2.3). There always exists an algebraically non-degenerate holomorphic mapping from $\varDelta$ into any algebraic variety.

For the above Remark (2.3), it is sufficient to see that there is an analytically non-degenerate holomorphic mapping $g: \varDelta \to \varDelta^n$; that is, $g(\varDelta)$ is not contained in any proper analytic subset of $\varDelta^n$. For instance, let $a_1, \cdots, a_n$ be real positive numbers linearly independent over the rational number field $\boldsymbol{Q}$ and $\varphi: \varDelta \to H$ a biholomorphic mapping from $\varDelta$ into the upper half plane H of $\boldsymbol{C}$. Put

$$G: \ z \in H \longrightarrow (e^{ia_1 z}, \cdots, e^{ia_n z}) \in \varDelta^n.$$

Then it is easy to see that $g \circ \varphi: \varDelta \to \varDelta^n$ is analytically non-degenerate.

In the following we improve Lemma (2.2) and show that we can generically take $k=n$. Let $\{\varPhi_1, \cdots, \varPhi_s\}$ be the maximal linearly independent system of

$$\{(i_{W_{\mathrm{reg}}})^*(\omega^{j_1}\wedge\cdots\wedge\omega^{j_n})\ ;\ 1\leqq j_1<\cdots<j_n\leqq N\}\,.$$

We say that $g:\Delta\to W_{\mathrm{reg}}$ is non-degenerate with respect to $\{\Phi_j\}_{j=1}^s$ if $g(\Delta)\not\subset\{\sum c_j\Phi_j=0\}$ for any $(c_1,\cdots,c_s)\in\boldsymbol{C}^s-\{O\}$.

LEMMA (2.4). *Let $g:\Delta\to W_{\mathrm{reg}}$ be non-degenerate with respect to $\{\Phi_j\}_{j=1}^s$. Then the differential*

$$dI_{W_{\mathrm{reg}}n}\ :\ T(J_n(W_{\mathrm{reg}}))\longrightarrow T((\boldsymbol{C}^N)^n)$$

is injective at some point $J_n(g)(z)$ with $z\in\Delta$.

PROOF. We put $\omega^{j\prime}=(i_{W_{\mathrm{reg}}})^*\omega^j$ for $1\leqq j\leqq N$. We may assume that $\omega^{1\prime}\wedge\cdots\wedge\omega^{n\prime}\not\equiv0$. Put

$$\omega^{j\prime}=\sum_{i=1}^n Q_i^j\omega^{i\prime}\,,\qquad n+1\leqq j\leqq N\,,$$

where Q_i^j are rational functions on W_{reg} and written as

$$Q_i^j=\pm\frac{\omega^{1\prime}\wedge\cdots\wedge\widehat{\omega^{i\prime}}\wedge\cdots\wedge\omega^{n\prime}\wedge\omega^{j\prime}}{\omega^{1\prime}\wedge\cdots\wedge\omega^{n\prime}}\ {}^{2)}\,.$$

Put $W'_{\mathrm{reg}}=\{\omega^{1\prime}\wedge\cdots\wedge\omega^{n\prime}\neq0\}$. Take $z\in\Delta$ so that $g(z)\in W'_{\mathrm{reg}}$. Let $(x^1,\cdots,x^n)$ be a holomorphic local coordinate system in a neighborhood U of $g(z)$ such that $dx^j=\omega^{j\prime}$. Let $T(J_k(W'_{\mathrm{reg}}))|U\cong U\times(\boldsymbol{C}^n)^k$ be the local trivialization with respect to $(x^1,\cdots,x^n)$. Let $(Z_l^i)_{1\leqq l\leqq k}^{1\leqq i\leqq n}$ be the standard coordinate system of $(\boldsymbol{C}^n)^k$, where l correspond to the orders of jets. Then $I_{W_{\mathrm{reg}}k}$ is written as

$$I_{W_{\mathrm{reg}}k}(x,(Z_l^i))=((Z_l^i)_{1\leqq l\leqq k}^{1\leqq i\leqq n},(Y_l^i)_{1\leqq l\leqq k}^{n+1\leqq i\leqq N})\in(\boldsymbol{C}^N)^k\,,$$

where Y_l^i are polynomials in $Z_{l'}^i$, $1\leqq l'\leqq l$, $1\leqq i\leqq n$ and the partial derivatives of Q_i^j of order $<l$. Therefore we infer from Lemmas (2.2) and (2.4) that there are n rational functions $Q_1,\cdots,Q_n$ among $\{Q_i^j\}$ which form a transcendental basis of the rational function field of W_{reg} over $\boldsymbol{C}$. Put $k=n$. By the detailed expression of Y_l^i in the proof of Lemma 2.3 of [**15**, p. 8] we see that some $n+n^2$ $(=\dim J_n(W_{\mathrm{reg}}))$ minor of $dI_{W_{\mathrm{reg}}n}$ at $J_n(g)(z)$ is

$$d(z)=\begin{vmatrix}\dfrac{d}{dz}Q_1(g(z))\cdots\dfrac{d}{dz}Q_n(g(z))\\ \cdot\qquad\cdots\qquad\cdot\\ \dfrac{d^n}{dz^n}Q_1(g(z))\cdots\dfrac{d^n}{dz^n}Q_n(g(z))\end{vmatrix}\,.$$

Since the functions 1, $Q_1(g(z)),\cdots,Q_n(g(z))$ are linearly independent over $\boldsymbol{C}$, $d(z)\not\equiv0$ on Δ. Hence $dI_{W_{\mathrm{reg}}n}$ is injective at $J_n(g)(z)$ with $d(z)\neq0$. Q.E.D.

---

2) The symbol "$\widehat{\omega^{i\prime}}$" stands for that it is deleted.

3. An inequality of the second main theorem type.

Let V be a complex projective manifold of dimension n and D an effective reduced divisor on V. Let $\alpha: V-D\to A_{V-D}$ be the quasi-Albanese mapping and W the closed image of α. Let $\mu: J_n(A_{V-D})\cong A_{V-D}\times(\boldsymbol{C}^N)^n\to(\boldsymbol{C}^N)^n$ ($N=\dim A_{V-D}$) be the projection and $I_{W_{\mathrm{reg}}n}: J_n(W_{\mathrm{reg}})\to(\boldsymbol{C}^N)^n$ the rational mapping defined in section 2. We set

$$\Psi_n(V-D)=\mu\circ\alpha_*: \ J_n(V-D)\longrightarrow(\boldsymbol{C}^N)^n.$$

Let α' be the mapping α regarded as a mapping into W. Then we have

$$\Psi_n(V-D)=I_{W_{\mathrm{reg}}n}\circ\alpha'_* \qquad \text{on } J_n((V-D)\cap\alpha'^{-1}(W_{\mathrm{reg}})). \tag{3.1}$$

Let $\pi: X\to\boldsymbol{C}^m$ be a finite analytic covering space with ramification divisor R and $f: X\to V$ a meromorphic mapping such that $f(X)\not\subset D$. We fix a hermitian metric form Ω on V and set $T_f(r)=T_f(r;\Omega)$.

MAIN THEOREM (3.2). *Assume that $f: X\to V$ is non-degenerate with respect to $|K_V\otimes[D]|$, $\dim W=n$, and W is of general type. Then there is a positive constant K independent of f such that*

$$KT_f(r)\leqq N(r, \operatorname{Supp} f^*D)+N(r, \operatorname{Supp} R)$$
$$+O(\delta\log r)+O(\log^+T_f(r)) \quad \|_{E(\delta)}$$

for arbitrary $\delta\in(0, 1)$.

PROOF. Take a point $z_0\in X-\operatorname{Supp}R$ so that f is holomorphic at z_0 and $f(z_0)\notin D$. Then there is a neighborhood of z_0 which is biholomorphic to the polydisc $\Delta^n(\rho;\pi(z_0))\subset\boldsymbol{C}^m$ of radius $\rho>0$ with center $\pi(z_0)$. We identify the neighborhood with $\Delta^n(\rho;\pi(z_0))$ and write $z_0=\pi(z_0)$. Let $(z^1,\cdots,z^m)$ be the standard coordinate system of $\boldsymbol{C}^m$. For the sake of simplicity, we assume that $z_0=O$, and put $\Delta^n(\rho;z_0)=\Delta^n(\rho)$. Let $\gamma:\Delta\to\Delta^n(\rho)$ be an analytically non-degenerate holomorphic mapping such that $\gamma(0)=O$ (see Remark (2.3)). Then the holomorphic mapping $f\circ\gamma:\Delta\to V$ is non-degenerate with respect to $|K_V\otimes[D]|$. Take small $\sigma>0$ so that $f\circ\gamma(\Delta(\sigma))\subset V-D$, where $\Delta(\sigma)=\{t\in\boldsymbol{C}: |t|<\sigma\}$. Let $\{\omega^1,\cdots,\omega^N\}$ be a basis of $H^0(V, \Omega_V^1(\log D))$. We may assume that $\omega^1\wedge\cdots\wedge\omega^n\neq 0$ at $f(z_0)$. Take a local coordinate neighborhood $U(x^1,\cdots,x^n)$ of $f(z_0)$ in $V-D$ such that $f(z_0)=(0,\cdots,0)$ and $\omega^i=dx^i$, $1\leqq i\leqq n$. Then the Jacobian matrix of the quasi-Albanese mapping $\alpha: V-D\to A_{V-D}$ is everywhere of maximal rank on U, so that $\alpha(f(z_0))\in W_{\mathrm{reg}}$ and

$$\alpha_*: \ J_n(U)\longrightarrow J_n(W_{\mathrm{reg}}) \tag{3.3}$$

is a bundle into-isomorphism. Let σ be smaller if necessary, so that the composed mapping $\alpha\circ f\circ\gamma:\Delta(\sigma)\to W_{\mathrm{reg}}$ is non-degenerate in the sense of Lemm (2.4).

Then $dI_{W_{\mathrm{reg}}n}: T(J_n(W_{\mathrm{reg}}))\to T((\boldsymbol{C}^N)^n)$ is injective at $J_n(\alpha\circ f\circ\gamma)(t_0)$ for some $t_0\in\Delta(\sigma)$. By making use of holomorphic automorphisms of $\Delta(\sigma)$, we may assume that $t_0=0$. Letting σ and ρ be smaller if necessary, we have a holomorphic mapping

$$f(z\,;t)=f(z+\gamma(t))\ :\ \Delta^n(\rho)\times\Delta(\sigma)\longrightarrow U\subset V-D\,.$$

Taking the n-th jet of $f(z\,;t)$ in the valuable t at $t=0$ for each $z\in\Delta^n(\rho)$, we obtain a holomorphic mapping

$$\tilde{f}\ :\ \Delta^n(\rho)\longrightarrow J_n(U)\subset J_n(V-D)$$

such that $p\circ\tilde{f}(z)=f(z)$ and $\tilde{f}(0)=J_n(\alpha\circ f\circ\gamma)(0)$, where $p: J_n(V-D)\to V-D$ is the projection. By (3.1), $d\Psi_n(V-D): T(J_n(V-D))\to T((\boldsymbol{C}^N)^n)$ is injective at $\tilde{f}(0)$. Let $(Z_l^i)_{1\le l\le n}^{1\le i\le N}$ be the coordinate system in $(\boldsymbol{C}^N)^n$ as in the proof of Lemma (2.4). We fix an embedding $V\subset\boldsymbol{P}^{k_0}(\boldsymbol{C})$ and let $\varphi_1,\cdots,\varphi_{k_0}$ be the generators of the rational function field of $\boldsymbol{P}^{k_0}$ restricted over V. We may assume that $f(X)$ is not contained in any polar locus of φ_k $(1\leqq k\leqq k_0)$. We regard the rational functions φ_k as rational functions on $J_n(V-D)$ through the projection p. Then φ_k are algebraic over the rational function field generated by $(\Psi_n(V-D))^*Z_l^i$, $1\leqq l\leqq n$, $1\leqq i\leqq N$. Moreover, applying the same proof as in [13, Lemma (3)] to $\tilde{f}:\Delta^n(\rho)\to J_n(V-D)$, we have only finitely many polynomials in T_k, $1\leqq k\leqq k_0$, and Z_l^i, $1\leqq l\leqq n$, $1\leqq i\leqq N$

$$P_{ks}(T_k,\ Z_l^i)=P_{ks0}(Z_l^i)(T_k)^{d_{ks}}+\cdots+P_{ksd_{ks}}(Z_l^i)\,,$$

$1\leqq s\leqq s_k$, which are independent of f, such that

$$\text{(3.4)}\qquad\begin{cases} P_{ks}(\varphi_k(f(z)),\ (\Psi_n(V-D))^*Z_l^i(\tilde{f}(z)))\equiv 0\,, & 1\leqq s\leqq s_k\,,\\ P_{ks0}((\Psi_n(V-D))^*Z_l^i(\tilde{f}(z))\not\equiv 0 & \text{with some } s\end{cases}$$

for all k and $z\in\Delta^n(\rho)$. We put

$$f^*\omega^i=\sum_{j=1}^m \xi_j^i dz^j\,,$$

$$z^j\circ\gamma(t)=c_1^jt+\cdots+c_n^jt^n+\text{higher term.}$$

Then the meromorphic functions $(\Psi_n(V-D))^*Z_l^i(f(z))$ are polynomials in ξ_j^i, partial derivatives $D_\nu\xi_j^i$ of ξ_j^i of order $|\nu|\leqq l-1$ and c_h^j, $1\leqq h\leqq l$, $1\leqq j\leqq m$, where $\nu=(\nu_1,\cdots,\nu_m)$ are multi-indices and $D_\nu=\partial^{|\nu|}/(\partial z^1)^{\nu_1}\cdots(\partial z^m)^{\nu_m}$. Substituting these polynomials in (3.4), we have only finitely many polynomials

$$\tilde{P}_{ks}(T_k,\ \Xi_{\nu j}^i)=\tilde{P}_{ks0}(\Xi_{\nu j}^i)T_k^{d_{ks}}+\cdots+\tilde{P}_{ksd_{ks}}(\Xi_{\nu j}^i)$$

in T_k and $\Xi_{\nu j}^i$ $(|\nu|\leqq n-1)$ such that

$$
(3.5)\qquad \begin{cases} \tilde{P}_{ks}(\varphi_k(f(z)),\, D_\nu \xi^i_j(z)) \equiv 0\,, & 1 \leqq s \leqq s_k\,, \\ \tilde{P}_{ks0}(D_\nu \xi^i_j(z)) \not\equiv 0 & \text{with some } s \end{cases}
$$

for $z \in \Delta^n(\rho)$ and $k=1, 2, \cdots, k_0$. Since $\Delta^n(\rho)$ is an open subset of X, the assertion (3.5) holds for all $z \in X$. Applying the proof of Valiron's theorem [**21**, section 1], we have

$$
(3.6)\qquad K_1 T(r,\, f^*\varphi_k) \leqq \sum_{\nu, i, j} T(r,\, D_\nu \xi^i_j) + O(1)\,, \qquad 1 \leqq k \leqq k_0\,,
$$

where K_1 is a positive constant independent of f. Here one remarks that

(3.7) K_1 depends only on the degrees of the polynomials P_{ks}.

We infer from (3.6) and Lemma (1.12) that

$$
(3.8)\qquad \begin{aligned} K_2 T(r,\, f^*\varphi_k) \leqq{} & N(r,\, \mathrm{Supp}\, f^*D) + N(r,\, \mathrm{Supp}\, R) \\ & + O(\delta \log r) + O(\log^+ T_f(r)) \quad \|_{E(\delta)}\,, \qquad 1 \leqq k \leqq k_0\,. \end{aligned}
$$

On the other hand, there is a positive constant K_3 independent of f such that

$$
(3.9)\qquad K_3 T_f(r) \leqq \max_{1 \leqq k \leqq k_0} T(r,\, f^*\varphi_k) + O(1)\,.
$$

We have by (3.8) and (3.9)

$$
K T_f(r) \leqq N(r,\, \mathrm{Supp}\, f^*D) + N(r,\, \mathrm{Supp}\, R) + O(\delta \log r) + O(\log^+ T_f(r)) \quad \|_{E(\delta)}\,,
$$

where K is a positive constant independent of f. Q.E.D.

We say that the finite analytic covering space $\pi : X \to \boldsymbol{C}^m$ is algebraic if X is an algebraic variety and π is rational: In this case, X is necessarily affine algebraic.

COROLLARY (3.10). *Let D and $f : X \to V$ be as in the Main Theorem* (3.2). *Assume that $\pi : X \to \boldsymbol{C}^m$ is algebraic. Then there is a positive constant K independent of f such that*

$$
K T_f(r) \leqq N(r,\, \mathrm{Supp}\, f^*D) + O(\log^+(r T_f(r))) \quad \|_E\,.
$$

The proof is clear by the Main Theorem (3.2) and the following lemma:

LEMMA (3.11) ([**10**, p. 274]). *The finite analytic covering space $\pi : X \to \boldsymbol{C}^m$ is algebraic if and only if $N(r, R) = O(\log r)$. Moreover, in the case where $\pi : X \to \boldsymbol{C}^m$ is algebraic, f is rational if and only if $T_f(r) = O(\log r)$.*

THEOREM (3.12). *Let D be a divisor on V as in the Main Theorem* (3.2) *and $f : X \to V$ a meromorphic mapping such that $f(X) \not\subset D$. Assume that X is the domain of existence of f, and that*

$$
N(r,\, \mathrm{Supp}\, R) = o(T_f(r)) \ \ \textit{or} \ \ O(\log r)\,, \qquad N(r,\, \mathrm{Supp}\, f^*D) = o(T_f(r))\,.
$$

Then we have that

i) *f is degenerate with respect to* $|K_V \otimes [D]|$,

or

ii) $\pi : X \to \boldsymbol{C}^m$ *is algebraic and* f *is rational.*

PROOF. Suppose that f is non-degenerate with respect to $|K_V \otimes [D]|$. Then by the Main Theorem (3.2) and the assumptions we see that

$$T_f(r) = O(\log r) \quad \|_E . \tag{3.13}$$

Since $T_f(r)$ is a convex increasing function in $\log r$, the estimate (3.13) implies that

$$T_f(r) = O(\log r).$$

Hence by Lemma (3.11) we see that $\pi : X \to \boldsymbol{C}^m$ is algebraic and f is rational.

Q.E.D.

REMARKS TO THE CASE OF $V = \boldsymbol{P}^n(\boldsymbol{C})$.

Let $\pi : X \to \boldsymbol{C}^m$ and R be as above. Let $V = \boldsymbol{P}^n(\boldsymbol{C})$ and $f : X \to \boldsymbol{P}^n(\boldsymbol{C})$ be a linearly non-degenerate meromorphic mapping. Stoll [**20**, Theorem 11.8] proved the second main theorem for f and hyperplanes of $\boldsymbol{P}^n(\boldsymbol{C})$ in general position. Here, applying the arguments of the present section, we give a little more explicit inequality.

As seen in (1.10), we may assume without loss of generality that X is the domain of existence of f. Let k be the sheet number of $\pi : X \to \boldsymbol{C}^m$ and $T_f(r)$ the characteristic function of f with respect to the standard Fubini-Study-Kähler form on $\boldsymbol{P}^n(\boldsymbol{C})$. Then by [**10**, Lemma 4.1], the estimate (1.11) takes the form of

$$N(r, R) \leqq (2k-2) T_f(r) + O(1). \tag{3.14}$$

Let $\{D_i\}_{i=1}^q$ be a finite family of hyperplanes of $\boldsymbol{P}^n(\boldsymbol{C})$ in general position. By making use of "Lemma on logarithmic derivatives" due to Vitter [**22**] (cf. also [**1**]) and Cartan's method [**3**] combined with the arguments of the present section and the computation of the ramification divisor R (cf. [**17**] and [**16**, Proposition 3]), we have

$$\begin{aligned}(q-n-1) T_f(r) \leqq & \sum_{i=1}^q N_n(r, f^* D_i) + n N(r, \operatorname{Supp} R) \\ & + O(\delta \log r) + O(\log^+ T_f(r)) \quad \|_{E(\delta)} ,\end{aligned} \tag{3.15}$$

where $N_n(r, f^* D_i)$ are counting functions defined as follows: If $\operatorname{Supp} Z = \bigcup_{j=1}^{\infty} Z_j$ is the irreducible decomposition of the support of a divisor Z on X and $Z = \sum_{j=1}^{\infty} \nu_j Z_j$, then we set

$$N_n(r, Z) = \sum_{j=1}^{\infty} \min\{\nu_j, n\} N(r, Z_j).$$

It follows from (3.14) and (3.15) that

$$(3.16)\qquad (q-(2kn-n+1))T_f(r)\leqq\sum_{i=1}^{q}N_n(r,\ f^*D_i)+O(\delta\log r)$$
$$+O(\log^+T_f(r))\quad \|_{E(\delta)}\,.$$

Setting $\delta_n(D_i)=1-\overline{\lim}_{r\to\infty}N_n(r,\ f^*D_i)/T_f(r)$, we have a defect relation:

$$(3.17)\qquad \sum_{i=1}^{q}\delta_n(D_i)\leqq 2kn-n+1\,.$$

4. Application to multiplicity inequalities.

Let $L\to V$ be a holomorphic line bundle over an n-dimensional complex projective manifold V. Let $|L|$ be the complete linear system of L and $|L|'=\{D\in|L|\,;D$ is irreducible and reduced$\}$. Then $|L|'$ is a Zariski open subset of $|L|$. Assume that there are l divisors $D_1,\cdots,D_l\in|L|'$ such that $D=\sum_{i=1}^{l}D_i$ satisfies the conditions of the Main Theorem (3.2). For example, if L is ample, then there are such divisors in $|L^j|'$ for large j. Let λ be the smallest one among such positive integers l. Let Ξ be the set of $(D_1,\cdots,D_\lambda)\in\prod_1^\lambda|L|'$ such that $D=\sum_{i=1}^{\lambda}D_i$ satisfies the conditions of the Main Theorem (3.2). Then Ξ is a Zariski open subset of $\prod_1^\lambda|L|'$ and symmetric with respect to the permutations.

Let $\pi:X\to \boldsymbol{C}^m$ be a finite analytic covering space and R its ramification divisor. Let $f:X\to V$ be a meromorphic mapping which is non-degenerate with respect to $|L|'$ and $|K_V\otimes L^\lambda|$, and $T_f(r)$ the characteristic function of f with respect to a fixed hermitian metric form on V. Let $(D_1,\cdots,D_\lambda)\in\Xi$ and $\Psi_n(V-\sum_{i=1}^{\lambda}D_i)$ be the mapping defined by (3.1). Then the mapping

$$(y,\ (D_1,\cdots,D_\lambda))\in J_n(V)\times\Xi\ \longrightarrow\ \Psi_n(V-\textstyle\sum D_i)(y)\in(\boldsymbol{C}^N)^n$$

is a rational mapping which is regular on $\{(y,\ (D_1,\cdots,D_\lambda))\in J_n(V)\times\Xi\,;\,y\notin\sum D_i\}$. By the construction of the polynomials P_{ks} in (3.4) (see the proof of [**13**, Lemma (3)]) and by (3.7), we see that there is a positive constant K independent of f such that

$$(4.1)\qquad KT_f(r)\leqq\sum_{i=1}^{\lambda}N(r,\ \mathrm{Supp}\,f^*D_i)+N(r,\ \mathrm{Supp}\,R)$$
$$+O(\delta\log r)+O(\log^+T_f(r))\quad\|_{E(\delta)}$$

for all $(D_1,\cdots,D_\lambda)\in\Xi$. Let $T_f(r\,;L)$ be the characteristic function of f with respect to the (1, 1) form representing the first Chern class of L. Then we have

$$(4.2)\qquad \begin{cases} N(r,\ f^*D)\leqq T_f(r\,;L)+O(1) & \text{for}\quad D\in|L|\,,\\ T_f(r\,;L)\leqq CT_f(r)\,, \end{cases}$$

where C is a positive constant independent of f (cf. [**2**], [**5**], [**10**], [**18**] and [**19**]).

Let $\{D_i\}_{i=1}^q$ be an arbitrary finite family of divisors in $|L|'$ such that $(D_{i_1}, \cdots, D_{i_\lambda}) \in \Xi$ for all λ divisors D_{i_j} of $\{D_i\}_{i=1}^q$. Let ν_i be positive integers such that

$$\nu_i \operatorname{Supp} f^*D_i \leqq f^*D_i, \qquad 1 \leqq i \leqq q.$$

In case $f(X) \cap D_i = \emptyset$, we set $\nu_i = +\infty$. Then for all λ divisors $D_{i_1}, \cdots, D_{i_\lambda}$ of $\{D_i\}_{i=1}^q$, we have by (4.1) and (4.2)

$$\begin{aligned} K &\leqq \sum_{j=1}^{\lambda} \frac{N(r, \operatorname{Supp} f^*D_{i_j})}{T_f(r)} + \frac{N(r, \operatorname{Supp} R)}{T_f(r)} \\ &\quad + O\Big(\frac{\delta \log r}{T_f(r)}\Big) + O\Big(\frac{\log^+ T_f(r)}{T_f(r)}\Big) \quad \|_{E(\delta)} \\ &\leqq C \sum_{j=1}^{\lambda} \frac{N(r, f^*D_{i_j})}{T_f(r\,;L)} \cdot \frac{1}{\nu_{i_j}} + O\Big(\frac{N(r, \operatorname{Supp} R)}{T_f(r)}\Big) \\ &\quad + O\Big(\frac{\delta \log r}{T_f(r)}\Big) + O\Big(\frac{\log^+ T_f(r)}{T_f(r)}\Big) \quad \|_{E(\delta)}. \end{aligned}$$

Thus we have

$$K \leqq C \sum_{j=1}^{\lambda} \frac{1}{\nu_{i_j}} + O(\delta) + O\Big(\varlimsup_{r\to\infty} \frac{N(r, \operatorname{Supp} R)}{T_f(r)}\Big),$$

so that

$$K' \leqq \sum_{j=1}^{\lambda} \frac{1}{\nu_{i_j}} + O\Big(\varlimsup_{r\to\infty} \frac{N(r, \operatorname{Supp} R)}{T_f(r)}\Big), \tag{4.3}$$

where $K' = K/C$. Now assume that $N(r, \operatorname{Supp} R) = o(T_f(r))$. Then it follows from (4.3) that

$$K' \leqq \sum_{j=1}^{\lambda} \frac{1}{\nu_{i_j}}$$

for all λ integers $\nu_{i_1}, \cdots, \nu_{i_\lambda}$. Hence we get

$$\sum_{i=1}^{q} \frac{1}{\nu_i} \geqq \frac{K'}{\lambda} q.$$

Therefore we have proved the following theorem.

THEOREM (4.4). *Let $f: X \to V$ be a meromorphic mapping which is non-degenerate with respect to $|L|'$ and $|K_V \otimes L^\lambda|$, $\{D_i\}_{i=1}^q$ a finite family of divisors in $|L|'$ and ν_i, $1 \leqq i \leqq q$, positive integers such that $\nu_i \operatorname{Supp} f^*D_i \leqq f^*D_i$ for all i. Assume the following conditions:*

i) *$N(r, \operatorname{Supp} R) = o(T_f(r))$,*

ii) *$(D_{i_1}, \cdots, D_{i_\lambda}) \in \Xi$ for all λ divisors $D_{i_1}, \cdots, D_{i_\lambda}$ of $\{D_i\}_{i=1}^q$.*

Then there is a positive constant K' independent of f and $\{D_i\}_{i=1}^q$ such that

$$\sum_{i=1}^{q}\frac{1}{\nu_i}\geqq\frac{K'}{\lambda}q\,.$$

EXAMPLE. Let $X=\boldsymbol{C}^m$, $V=\boldsymbol{P}^n(\boldsymbol{C})$ and L be the hyperplane bundle over $\boldsymbol{P}^n(\boldsymbol{C})$. Then we see that

$$\lambda=n+2 \qquad \text{and} \qquad K=K'=\frac{1}{n} \qquad \text{(cf. [12] and (3.15))}$$

and that $K_{\boldsymbol{P}^n(\boldsymbol{C})}\otimes L^{n+2}=L$. Therefore $f:\boldsymbol{C}^m\to\boldsymbol{P}^n(\boldsymbol{C})$ is non-degenerate with respect to $|L|'$ and $|K_{\boldsymbol{P}^n(\boldsymbol{C})}\otimes L^{\lambda}|$ if and only if f is linearly non-degenerate. In this case, Theorem (4.4) implies

$$\sum_{i=1}^{q}\frac{1}{\nu_i}\geqq\frac{q}{(n+2)n} \tag{4.5}$$

Let $\pi:X\to\boldsymbol{C}^m$ be a k-sheeted finite analytic covering space and $f:X\to\boldsymbol{P}^n(\boldsymbol{C})$ a linearly non-degenerate meromorphic mapping. Then the second main theorem (3.16) implies

$$\sum_{i=1}^{q}\frac{1}{\nu_i}\geqq\frac{q-2kn+n-1}{n}\,. \tag{4.6}$$

Therefore, putting $k=1$, we see that (4.5) is not sharp: The reason may lie in the fact that the inequality of the second main theorem type proved in the Main Theorem (3.2) is not good enough to derive defect relations.

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