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Hyperbolic links with Brunnian properties

To the memory of Professor Shichirô Oka

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1. Introduction.

In his paper of 1892 [**3**], Hermann Brunn constructed a link with what is called the Brunnian property, that is, the link itself is nontrivial, yet every proper sublink is trivial, see [**21**, p. 67]. His link is the origin of the links in this paper.

A k-link L in S^{k+2} is *splittable* if there exists a $(k+2)$-disk B^{k+2} in S^{k+2} satisfying $L\cap B^{k+2}\neq\emptyset$, $L\cap\partial B^{k+2}=\emptyset$, and $L\cap(S^{k+2}-B^{k+2})\neq\emptyset$. Generalizing the Brunnian property, H. Debrunner considered the splitting problem of a link. Given a k-link $L=L_1\cup L_2\cup\cdots\cup L_n$ in S^{k+2}, let $\mathfrak{A}$ be a family of those subsets S of $I_n=\{1, 2, \cdots, n\}$ for which the sublink $L_S=\bigcup_{i\in S}L_i$ of L does not split. Then we call L has the *Brunnian property of type* $\mathfrak{A}$. For convenience, we assume that $\emptyset$, $\{i\}\notin\mathfrak{A}$ for all $i\in I_n$. For this family of subsets $\mathfrak{A}$, the following condition must be satisfied:

(*) If $S, T\in\mathfrak{A}$ and $S\cap T\neq\emptyset$, then $S\cup T\in\mathfrak{A}$.

Conversely, for any family $\mathfrak{A}$ of subsets of I_n with (*), H. Debrunner [**4**] constructed a k-link with n components for each $k\geqq 2$ and $n\geqq 2$ having the Brunnian property of type $\mathfrak{A}$. Furthermore, the author [**8**] constructed such an example for each $k\geqq 1$ and $n\geqq 2$, where the link is a satellite link, that is, a link with nontrivial companions. See [**8**] for other references of this problem.

In this paper we restrict ourselves to 1-links in S^3. In § 4, we show that for any family $\mathfrak{A}$ of subsets of I_n with (*), there exists a link L with Brunnian property of type $\mathfrak{A}$ such that the exterior of each sublink L_S for $S\in\mathfrak{A}$ is atoroidal (Theorem 4), that is, irreducible and boundary-irreducible and contains no non-boundary-parallel incompressible annuli and tori. Such a link is hyperbolic by Thurston's theorem, see [**13**]. Note that a link is nonsplittable if and only if its exterior is irreducible.

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The main tools for proving that the exterior of a link is atoroidal are Myers' results [**14, 15**] (see § 2) and Soma's simple tangle [**22**]. They may originate in the tangle theory of Kirby and Lickorish [**9**]. A link is prime if it is nonsplittable and any 2-sphere in S^3, which meets the link transversely in two points, bounds a ball meeting the link in an unknotted spanning arc. In [**10**] Lickorish showed that summing together two prime tangles always produces a prime link with 1 or 2 components. As the analogue for a simple link, that is, a nonsplittable link whose exterior contains no non-boundary-parallel incompressible tori, Soma introduced the notion of a simple tangle. In § 3, we show that a prime tangle and a simple tangle have some properties of Myers (Theorems 1-3). Using this, we see that summing together two simple tangles produces a link with (atoroidal) hyperbolic exterior (Corollary 3.1), which has been already observed by Y. Nakanishi and A. Kawauchi (unpublished). Compare Nakanishi [**17, 18**] and Quách [**19**].

Two oriented links L_0 and L_1 in a 3-manifold M are *concordant* if they can be joined by locally flat disjoint annuli in $M\times[0, 1]$, where $L_i\subset M\times\{i\}$. Kirby and Lickorish [**9**] and Livingston [**11**] have shown that any knot K_0 in S^3 is concordant to a prime knot K_1, and Bleiler [**2**] has shown that K_1 can be chosen to have the same Alexander polynomial as K_0, see also Quách [**20**]. Nakanishi [**16**] has shown that any link L_0 in S^3 is concordant to a prime link L_1 with the same Alexander invariant, and Soma [**22**] has shown that L_1 can be chosen to be simple. Here the Alexander invariant of a link L in S^3 is the homology of the universal abelian cover of S^3-L, which has a natural module structure, see [**21**, Section 7 I]. On the other hand, Myers [**15**] has shown that any link in a 3-manifold whose boundary contains no 2-spheres is concordant to a link with hyperbolic exterior, where the definition of hyperbolic structure is extended. As an application of the construction in § 4, we show in § 5 that any link in S^3 is concordant to a link with the same Alexander invariant whose exterior is (atoroidal) hyperbolic (Theorem 5).

The author is grateful to Y. Nakanishi and M. Sakuma for their helpful suggestions and conversations.

2. Preliminaries.

Throughout the paper, we shall work in either the PL or smooth category. All manifolds are compact and orientable. The boundary of a manifold M is denoted by ∂M, and the closure and the interior of a space X are denoted by $\mathrm{cl}\,X$ and $\mathrm{int}\,X$, respectively.

A surface F in a 3-manifold M is properly embedded, that is, $F\cap\partial M=\partial F$. A surface in ∂M is a submanifold of ∂M. The reader is referred to [**5, 6, 7, 23**] for the definitions of incompressible and boundary-parallel surfaces and of irreducible, boundary-irreducible, and sufficiently large 3-manifolds.

A *3-manifold pair* (M, F) consists of a 3-manifold M and a surface F in ∂M. (M, F) is *irreducible* if M is irreducible and F is incompressible in M.

An irreducible, sufficiently large 3-manifold is called a *Haken manifold*. An irreducible, boundary-irreducible 3-manifold M is called *atoroidal* if each incompressible annulus and torus in M is boundary-parallel.

A compact 3-manifold M is *hyperbolic* if $\operatorname{int} M$ has a complete Riemannian metric with finite volume and constant negative curvature. A link is *hyperbolic* if its exterior is hyperbolic. Since a link exterior is sufficiently large, Thurston's main theorem [**13**, Theorem B], together with the torus theorem [**6, 7**], yields that a link is hyperbolic if its exterior is atoroidal.

We make use of the gluing lemmas due to Myers [**14, 15**], which give some sufficient conditions under which the union of two 3-manifolds along incompressible surfaces in their boundaries is simple and Haken.

DEFINITION. For a 3-manifold pair (M, F), we consider the following conditions:

(1) (M, F) and $(M, \mathrm{cl}(\partial M-F))$ are irreducible 3-manifold pairs.
(2) No component of F is a disk or a 2-sphere.
(3) Every disk D in M with $D\cap F$ a single arc is boundary-parallel.
(4) No component of F is an annulus or a torus.
(5) Every incompressible annulus A in M with $\partial A\cap\partial F=\emptyset$ is boundary-parallel.
(6) Every incompressible torus in M is boundary-parallel.
(7) Every disk D in M with $D\cap F$ a pair of disjoint arcs is boundary-parallel.

(M, F) has *Property* A, if it satisfies (1)-(3).
(M, F) has *Property* B′, if it satisfies (1)-(6).
(M, F) has *Property* C′, if it satisfies (1)-(7).

Now suppose that $M=M_0\cup M_1$, where M_0 and M_1 are 3-manifolds and $F=M_0\cap M_1=\partial M_0\cap\partial M_1$ is a 2-manifold.

PROPOSITION 1 ([**15**, Lemma 2.4]). *If (M_0, F) and (M_1, F) have Property* A, *then M is Haken.*

PROPOSITION 2 ([**15**, Lemma 2.5]). *If (M_0, F) has Property* B′ *and (M_1, F) has Property* C′, *then M is atoroidal and Haken.*

An *n-string tangle* is a pair (B, t) where B is a 3-ball and t is a set of n disjoint arcs, called *strings*, embedded in B with $t\cap\partial B=\partial t$. The *boundary* of (B, t) is $(\partial B, \partial t)$, which we denote by $\partial(B, t)$. An n-string tangle (B, t) is *trivial*, if there is a homeomorphism of pairs from (B, t) to $(D\times[0, 1], \{p_1, p_2, \cdots, p_n\}\times[0, 1])$, where D is a disk containing points $p_1, p_2, \cdots, p_n$ in its interior.

Let $t=\tau_1\cup\tau_2\cup\cdots\cup\tau_n$ and N_i be the disjoint tubular neighborhoods of τ_i in

B. A *tangle space* M is $\mathrm{cl}(B-N_1\cup N_2\cup\cdots\cup N_n)$. Let Δ_i and Δ_i' be two disks with $N_i\cap\partial B=\Delta_i\cup\Delta_i'$. Let D be a disk in ∂B such that $\mathrm{int}D\cap(N_1\cup N_2\cup\cdots\cup N_n)=\Delta_1\cup\Delta_2\cup\cdots\cup\Delta_k$, $2\leqq k\leqq n$. We consider the 3-manifold pairs (M, F) and (M, G), where $F=\mathrm{cl}(\partial B-N_1\cup N_2\cup\cdots\cup N_n)$ and $G=\mathrm{cl}(D-\Delta_1\cup\Delta_2\cup\cdots\cup\Delta_k)$.

LEMMA 1. *If M is boundary-irreducible, then (M, F) and (M, G) have Property* A.

PROOF. Clearly M is irreducible. Let $\pi_1(\partial M)=\langle x_1, y_1, x_2, y_2, \cdots, x_n, y_n \mid [x_1, y_1][x_2, y_2]\cdots[x_n, y_n]=1\rangle$, where x_i is represented by the loop $\partial\Delta_i$ and $[x_i, y_i]=x_i^{-1}y_i^{-1}x_iy_i$. Since the subgroup of $\pi_1(\partial M)$ generated by $x_1, x_2, \cdots, x_k$ is freely generated by them [**12**, Theorem 4.10], the inclusion homomorphism $\pi_1(G)\to\pi_1(\partial M)$ is injective. In the same way, the inclusion homomorphism $\pi_1(\mathrm{cl}(\partial M-G))\to\pi_1(\partial M)$ is injective. Thus (M, G) satisfies (1). For (M, F), (1) is proved similarly. Properties (2) and (3) are obvious, and so the proof is complete.

We call an n-string tangle *atoroidal* if its tangle space is atoroidal. Then from this lemma, it is easy to see

PROPOSITION 3. *If the n-string tangle (B, t) is atoroidal, then (M, F) and (M, G) have Property* C'.

3. 2-string tangles.

In this section, tangles are 2-string.

DEFINITION. A tangle (B, t) is *prime* if it has the following properties:

(i) Any 2-sphere in B, which meets t transversely in two points, bounds in B a ball meeting t in an unknotted spanning arc;

(ii) (B, t) is not trivial.

The condition (ii), in the presence of (i), is equivalent to the following property:

(ii') There is no disk in B which separates the two arcs of t.

DEFINITION. A prime tangle (B, t) is *simple* if it has the following property:

(iii) There is no incompressible torus embedded in $B-t$.

An atoroidal tangle is simple, but the converse is not valid. Let P be a simple tangle [**22**, § 2] with its tangle space as in Figure 1. The annulus A is incompressible, though not boundary-parallel.

Let $T=(B, \tau_1\cup\tau_2)$ be a tangle and N_1 and N_2 be the disjoint regular neighborhoods of τ_1 and τ_2 in B respectively. Let $M=\mathrm{cl}(B-N_1\cup N_2)$. Let U_i be an

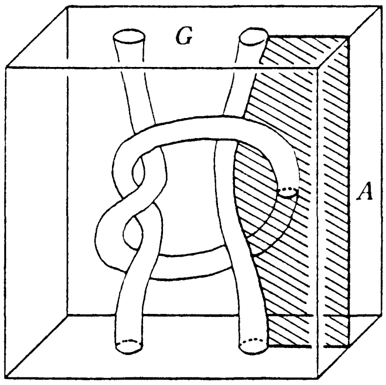

Figure 1.

annulus $\mathrm{cl}(\partial N_i-\partial B_i)$, $i=1, 2$, and a and b be the components of ∂U_1 and c and d be those of ∂U_2. Let C be a disk in ∂B containing a and c in its interior and $e=\partial C$. Let $G=\mathrm{cl}(C-N_1\cup N_2)$, $H=\mathrm{cl}(\partial B-C\cup N_1\cup N_2)$, and $F=G\cup H$.

The purpose of this section is to prove the following theorems:

THEOREM 1. *If T is a prime tangle, then (M, F) has Property* A.

THEOREM 2. *If T is a simple tangle, then (M, F) has Property* C′.

THEOREM 3. *If T is a simple tangle, then (M, G) has Property* C′.

By Proposition 2, we have a sharper version of [**22**, Theorem 1], which is already observed by Y. Nakanishi and A. Kawauchi (unpublished).

COROLLARY 3.1. *Given two simple tangles T_1 and T_2, let $f:\partial T_1\to\partial T_2$ be a homeomorphism. Let L be the link with one or two components in S^3 obtained by identifying the boundaries of the tangles via f. Then the exterior of L is (atoroidal) hyperbolic.*

PROOF OF THEOREM 1. Clearly $U_1\cup U_2$ is incompressible in M. Suppose that D is a disk in M such that $\partial D\subset F$ and that ∂D is not contractible in F. Then for homological reasons, ∂D divides $a\cup b$ from $c\cup d$ in ∂B. This means D separates τ_1 and τ_2, a contradiction. Thus (M, F) satisfies (1).

Let E be a disk in M with $\partial E\cap F$ a single arc α. There are two possibilities.

Case 1. α joins a to itself. E can be isotoped so that ∂E lies in F. By the incompressibility of F, E is boundary-parallel.

Case 2. α joins a to b. Let V be a 3-cell in B consisting of the regular neighborhood of E and N_1. Then the disk $\mathrm{cl}(\partial V-\partial B)$ separates the two arcs, a contradiction.

This establishes (3), completing the proof.

Now we may divide the proof of Theorem 2 into the next two lemmas:

LEMMA 2. *If T is a simple tangle, then (M, F) satisfies* (5).

PROOF. Suppose A is an incompressible annulus in M with $\partial A \cap \partial F = \emptyset$. Let J_1 and J_2 be the components of ∂A. If either J_1 or J_2 is contractible in ∂M, then by (1), A is compressible. If J_1 is parallel to a and J_2 to b in F, then isotop A, keeping J_1 fixed, so that both J_1 and J_2 are parallel to a in F. Thus we may assume that J_1 and J_2 are parallel in F for homological reasons. Then J_1 and J_2 cobound the annulus A' in F. Let U be the result of isotoping the torus $A \cup A'$ slightly into $\operatorname{int} M$. Since U is compressible in M, U bounds either a solid torus or a knot exterior Q. If U bounds Q, then the boundary of the compressing disk is a meridian of Q. Since A' lies on ∂B, J_1 is homotopic to a meridian of Q. This contradicts the incompressibility of A in M, cf. [**15**, Proof of Lemma 4.7]. Thus U bounds a solid torus, and so $A \cup A'$ bounds a solid torus V. Regard B as embedded in a 3-sphere S^3. Attach a 2-handle h^2 along A' in S^3. If A is not parallel to A' across V, then $V \cup h^2$ is a punctured lens space in S^3, contradicting the Schönflies theorem. This completes the proof.

LEMMA 3. *If T is a simple tangle, then (M, F) satisfies* (7).

PROOF. Suppose D is a disk in M which meets F in the disjoint arcs α_1 and α_2 and meets $U_1 \cup U_2$ in the disjoint arcs β_1 and β_2.

Case 1. Both α_1 and α_2 join a to itself. Isotop D so that ∂D lies on F. The incompressibility of F implies that D is boundary-parallel.

Case 2. α_1 joins a to itself and α_2 joins a to b. Isotop D so that $\partial D \cap F$ is a single arc joining a to b. This is impossible by (3).

Case 3. α_1 joins a to itself and α_2 joins b to itself. Let γ_1 (resp. γ_2) be an arc in a (resp. b) joining the points of α_1 (resp. α_2) such that the simple closed curve $\alpha_1 \cup \gamma_1$ (resp. $\alpha_2 \cup \gamma_2$) bounds a 2-disk Δ_1 (resp. Δ_2) in ∂B with $\Delta_i \cap (a \cup b) = \gamma_i$, $i=1, 2$. If either Δ_1 or Δ_2 contains neither c nor d, then isotop D so that ∂D lies on F. So we may assume that $c \subset \Delta_1$ and $d \subset \Delta_2$. Let E_1 and E_2 be the disks with $E_1 \cup E_2 = U_1$, $E_1 \cap E_2 = \beta_1 \cup \beta_2$, and $\partial E_1 = \beta_1 \cup \beta_2 \cup \gamma_1 \cup \gamma_2$. Let S be the result of isotoping the sphere $E_1 \cup D \cup \Delta_1 \cup \Delta_2$ slightly into $\operatorname{int} M$. Then S bounds a ball in B meeting τ_2 in an unknotted spanning arc. Thus τ_1 and τ_2 are parallel. If they are unknotted, then T is trivial, contradicting (ii); if knotted, then the result of isotoping the torus $E_2 \cup D \cup (F - \Delta_1 \cup \Delta_2)$ slightly into $\operatorname{int} M$ is incompressible, contradicting (iii).

Case 4. Both α_1 and α_2 join a to b. If β_1 joins a to itself and β_2 joins b to itself, then isotop D so that ∂D lies on F. If both β_1 and β_2 join a to b, then ∂D is not homologically trivial in $M \cup N_2$.

Case 5. Both α_1 and α_2 join a to c. Isotop D so that ∂D lies on F.

Case 6. α_1 joins a to c and α_2 joins b to d. Then two arcs are parallel, which is impossible as proved in Case 3.

This completes the proof.

PROOF OF THEOREM 3. Since (M, F) satisfies (5), so does (M, G). Thus the result follows from the following lemmas:

LEMMA 4. *If T is a prime tangle, then (M, G) satisfies* (1).

PROOF. By the proof of Lemma 1, the inclusion homomorphisms $\pi_1(G)\to\pi_1(F)$ and $\pi_1(\mathrm{cl}(\partial M-G))=\pi_1(H\cup U_1\cup U_2)\to\pi_1(F)$ are injective. By the incompressibility of F, both G and $\mathrm{cl}(\partial M-G)$ are incompressible.

LEMMA 5. *If T is a simple tangle, then (M, G) satisfies* (3).

PROOF. Let D be a disk in M with $\partial D\cap G$ a single arc α. We have four cases.

Case 1. α joins a to itself. This is reduced to Case 3 of the proof of Lemma 3.

Case 2. α joins a to c. This is reduced to Case 6 of the proof of Lemma 3.

Case 3. α joins a to e. This is reduced to Case 2 in the proof of (3) for (M, F) in Theorem 1.

Case 4. α joins e to itself. This is reduced to the proof of the incompressibility of F.

This completes the proof.

LEMMA 6. *If T is a simple tangle, then (M, G) satisfies* (7).

PROOF. Suppose D is a disk in M which meets G in the disjoint arcs α_1 and α_2. We have the cases:

Case 1. Both α_1 and α_2 join a to itself.
Case 2. α_1 joins a to itself and α_2 joins a to c.
Case 3. α_1 joins a to itself and α_2 joins a to e.
Case 4. α_1 joins a to itself and α_2 joins c to itself.
Case 5. α_1 joins a to itself and α_2 joins c to e.
Case 6. α_1 joins a to itself and α_2 joins e to itself.
Case 7. Both α_1 and α_2 join a to c.
Case 8. α_1 joins a to c and α_2 joins a to e.
Case 9. α_1 joins a to c and α_2 joins e to itself.
Case 10. Both α_1 and α_2 join a to e.
Case 11. α_1 joins a to e and α_2 joins c to e.
Case 12. α_1 joins a to e and α_2 joins e to itself.
Case 13. Both α_1 and α_2 join e to itself.

For Cases 4, 5, 6 and 9, we can isotop D so that $\partial D\cap G$ is a single arc, reducing to (3) for (M, G). For other cases we suppose that D cannot be isotoped so that neither $\partial D \cap G$ nor $\partial D \cap H$ is a single arc.

For Cases 1 and 8, ∂D is not homologically trivial in $M \cup N_2$, for Cases 2 and 3, ∂D is not homologically trivial in $M \cup N_1$, and for Case 7, ∂D is not homologically trivial in either $M \cup N_1$ or $M \cup N_2$. These are impossible.

For Cases 10 and 11, we can isotop D so that D meets F in a pair of disjoint arcs, reducing to (7) for (M, F).

For Case 12, we can isotop D so that D meets F in a single arc, reducing to (3) for (M, F).

For Case 13, we can isotop D so that ∂D lies on F, reducing to the incompressibility of F.

This completes the proof.

4. Links with Brunnian properties.

For an n-string tangle T as in Figure 2, where two strands α_i and β_i emerge from the same string for each i, we define the following: The numerator of T is the link with n components as shown in Figure 3. We call the n-string tangle as shown in Figure 4 *strongly trivial*. Given two n-string tangles T_1 and T_2, an n-string tangle obtained by juxtaposition as shown in Figure 5 is called the *tangle sum* of T_1 and T_2, which we denote by T_1+T_2. We define the *join* of T_1 and T_2 as the numerator of the tangle sum T_1+T_2.

Let $S=\{i_1, i_2, \cdots, i_k\}$ with $1 \leqq i = i_1 < i_2 < \cdots < i_k \leqq n$. We define the trivial n-string tangle $\Lambda S=(B, \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_n)$, which is an n-braid, in the following

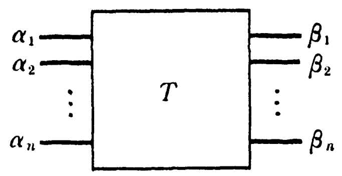

Figure 2.

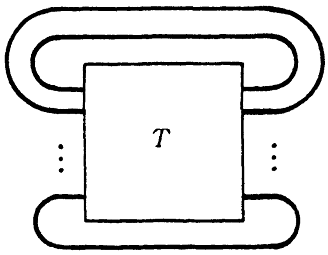

Figure 3.

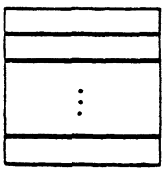

Figure 4.

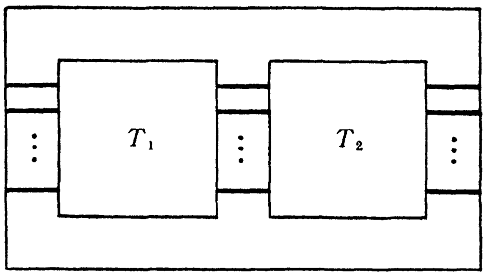

Figure 5.

way: We set the subtangle $(B, \lambda_1 \cup \cdots \cup \lambda_{i-1} \cup \lambda_{i+1} \cup \cdots \cup \lambda_n)$ to be the strongly trivial $(n-1)$-tangle. Let $L = L_1 \cup L_2 \cup \cdots \cup L_n$ be the numerator of ΛS, where L_j corresponds to λ_j. We set the string λ_i so that L_i represents the commutator $[x_{i_2}, x_{i_3}, \cdots, x_{i_k}]$ if $k \geqq 3$ and x_{i_2} if $k=2$ in $\pi_1(S^3 - L_1 \cup \cdots \cup L_{i-1} \cup L_{i+1} \cup \cdots \cup L_n)$, which is the free group with basis $\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}$ with x_j a meridian of L_j. Here $[x_{i_2}, \cdots, x_{i_{j-1}}, x_{i_j}] = [[x_{i_2}, \cdots, x_{i_{j-1}}], x_{i_j}]$. For example, Figure 6 is the trivial 6-string tangle $\Lambda\{1, 3, 4, 6\}$. Note that if $S = I_n$, then L is the Brunnian link constructed by H. Brunn [3], see [21, p. 67].

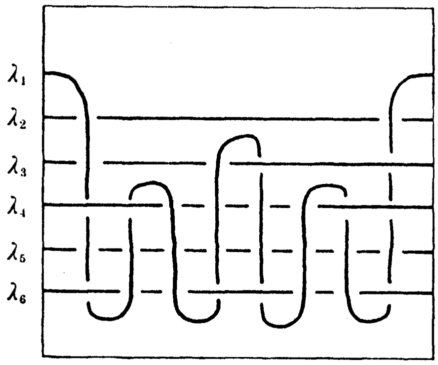

Figure 6.

Let ΣS be the n-string tangle illustrated in Figure 7, where P is the 2-string simple tangle given in Figure 1 and the string σ_i corresponds to λ_i for each i.

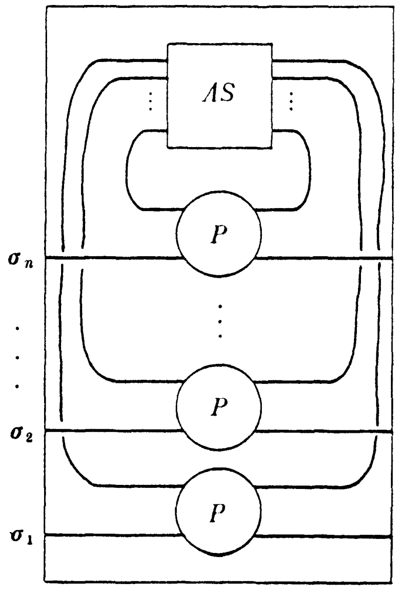

Figure 7.

PROPOSITION 4. *If $n\geqq 2$, then the tangle ΣI_n is atoroidal.*

PROOF. We consider the tangle $\Lambda=(A, \mu_1\cup\mu_2\cup\cdots\cup\mu_n)$ and the disks $D_1, D_2, \cdots, D_n$ in ∂A as shown in Figure 8. Let N_i be the disjoint tubular neighborhoods of μ_i such that $N_i\cap\partial A\subset \mathrm{int} D_i$. Let $M=\mathrm{cl}(A-N_1\cup N_2\cup\cdots\cup N_n)$, $F_i=\mathrm{cl}(D_i-N_i)$, and $F=F_1\cup F_2\cup\cdots\cup F_n$. Then the tangle space of ΣI_n is constructed from (M, F) and $\bigcup_{i=1}^{n}(X_i, G_i)$ by identifying F_i and G_i, where (X_i, G_i) is the copy of the 3-manifold pair (X, G) given in Figure 1.

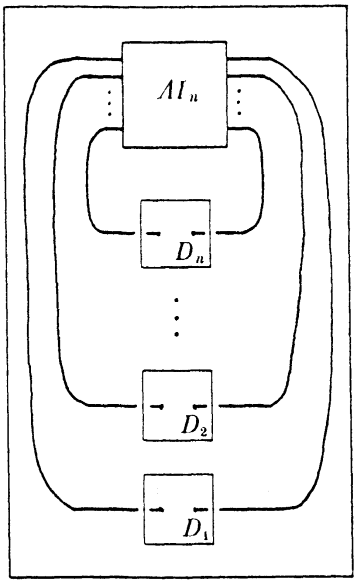

Figure 8.

Since the tangle P is simple, (X_i, G_i) has Property C′ by Theorem 3. Thus by Proposition 2, it is sufficient to prove

LEMMA 7. *(M, F) has Property* B′.

PROOF. Let $U_i=\mathrm{cl}(\partial N_i-D_i)$ and a_i and b_i be the components of ∂U_i. Let $c_i=\partial D_i$. Let $H=\mathrm{cl}(\partial A-F)$. Add a 2-handle h_i^2 to M along a collar C_i on ∂D_i in F_i for each i. Further add a 3-handle h^3 to a component of the boundary of $M\cup h_1^2\cup h_2^2\cup\cdots\cup h_n^2$. Then $W=M\cup h_1^2\cup h_2^2\cup\cdots\cup h_n^2\cup h^3$ is the exterior of the numerator of ΛI_n, which is the Brunnian link. We break the proof into steps.

1. *$F_i\cup U_i$ is incompressible.* Suppose that D is a disk in M such that $\partial D\subset F_i\cup U_i$ and that ∂D is not contractible in $F_i\cup U_i$. Isotop D so that ∂D misses the collar C_i. For homological reasons, ∂D is a longitude of L. This means that the link L is splittable, a contradiction.

2. *H is incompressible.* Suppose that D is a disk in M such that $\partial D\subset H$. There is a 2-disk D' in h^3 such that $D\cup D'$ is a 2-sphere in W. If ∂D is not contractible, then the link L is splittable, a contradiction.

3. (M, F) *has Property* A. By Steps 1 and 2, (M, F) satisfies (1), and (2) is obvious.

Let D be a disk in M with $\partial D\cap F_i$ is a single arc α. By the incompressibility of $F_i\cup U_i$, α joins c_i to itself. For homological reasons, α does not separate a_i from b_i in F_i. So we can isotop D so that ∂D lies on H. By the incompressibility of H, D is boundary-parallel.

4. (M, F) *satisfies* (5). Suppose A is an incompressible annulus in M with $\partial A\cap\partial F=\emptyset$. let J_1 and J_2 be the components of ∂A. If either J_1 or J_2 is contractible in ∂M, then by (1), A is compressible.

Case 1. J_1 is parallel to a_i in $F_i\cup U_i$. For homological reasons, J_2 is parallel to a_i or b_i. A can be isotoped so that both J_1 and J_2 are parallel to a_i in F_i.

Case 2. J_1 lies on H. For homological reasons, we may assume that J_2 lies on H. Let E_1 and E_2 be the disjoint disks in h^3 with $\partial E_i=J_i$. If J_1 and J_2 are not parallel, then the link L is splitted by the 2-sphere $A\cup E_1\cup E_2$, a contradiction.

Therefore in any case, J_1 and J_2 may be assumed to be parallel in the same component of $\partial M-\partial F$. Since M contains no incompressible tori, by a similar proof to that of Lemma 2, A is boundary-parallel. This completes the proof.

THEOREM 4. *Let* $\mathfrak{A}=\{S_1, S_2, \cdots, S_r\}$ *be a family of subsets of* I_n *satisfying the condition* (*). *Let* $L=L_1\cup L_2\cup\cdots\cup L_n$ *be the join of two copies of the tangle sum* $\Sigma=\Sigma S_1+\Sigma S_2+\cdots+\Sigma S_r$. *Then* L *has the Brunnian property of type* $\mathfrak{A}$ *such that the exterior of each sublink* L_{S_i} *is (atoroidal) hyperbolic.*

PROOF. Let $\Sigma S_j=(B_j, \sigma_1^j\cup\sigma_2^j\cup\cdots\cup\sigma_n^j)$ and $\Sigma=(B, \sigma_1\cup\sigma_2\cup\cdots\cup\sigma_n)$, where σ_i^j and σ_i correspond to L_i for each i. For a subset $S=\{i_1, i_2, \cdots, i_k\}$ of I_n, $i_1<i_2<\cdots<i_k$, we denote the subtangles $(B_j, \sigma_{i_1}^j\cup\sigma_{i_2}^j\cup\cdots\cup\sigma_{i_k}^j)$ and $(B, \sigma_{i_1}\cup\sigma_{i_2}\cup\cdots\cup\sigma_{i_k})$ by $(\Sigma S_j)_S$ and Σ_S, respectively. If $S_j\supsetneqq S\cap S_j$, then the subtangle $(\Sigma S_j)_S$ is strongly trivial. So the subtangle Σ_S is the sum of those subtangles $(\Sigma S_j)_S$ for which $S\supset S_j$. Thus if $S\notin\mathfrak{A}$, then L_S splits. Conversely in order to

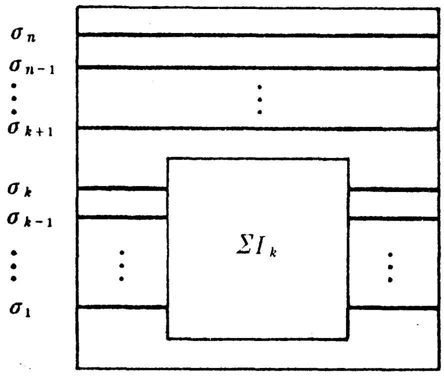

Figure 9.

prove that the exterior of the sublink L_{S_j} is atoroidal, we have only to prove that the exterior of L is atoroidal assuming $S_1=I_n$.

Since ΣI_n is atoroidal, by induction, it is sufficient to prove that the tangle sum $T+\Sigma S_j$ is atoroidal, where T is an atoroidal n-string tangle. We may suppose that $S_j=\{1, 2, \cdots, k\}=I_k$, $2\leqq k\leqq n-1$. Then ΣS_j can be illustrated as in Figure 9, and so $T+\Sigma S_j$ is atoroidal by Propositions 2 and 3. Now Σ is atoroidal again by Propositions 2 and 3, the exterior of L, the join of two copies of Σ, is atoroidal. This completes the proof.

REMARK. In the above theorem, each component L_i is unknotted, and thus not hyperbolic. For any subset S of I_n, we can construct a link L satisfying the following property in addition to those of the theorem: A component L_i is hyperbolic if $i\in S$ and is unknotted if $i\notin S$.

Now we come upon a question. For what family of subsets $\mathfrak{A}$ of I_n with $\emptyset\notin\mathfrak{A}$, does there exist an n-component link L with the property that the sublink L_S is hyperbolic if and only if $S\in\mathfrak{A}$?

5. Alexander invariants of links.

We use the notation of Section 4.

THEOREM 5. *Every link in S^3 is concordant to a link with the same Alexander invariant whose exterior is (atoroidal) hyperbolic.*

PROOF. Let L_0 be a link in S^3. We may suppose that L_0 has bridge number $n\geqq 2$. It is known ([**1**, Section 5.1] or [**21**, p. 115]) that L_0 is constructed as follows: Let T_1 and T_2 be trivial n-string tangles. Then there is a homeomorphism $f:\partial T_1\to\partial T_2$ such that the link created by identifying the boundaries of the tangles via f is L_0. Let Λ and Ω be the n-string tangles obtained from ΣI_n by substituting the trivial 2-string tangle and the K-T grabber of Figure 10 respectively, for each tangle P. Let Λ_1 and Λ_2 be the copies of Λ, and Ω_1 and Ω_2 those of Ω. Since Λ is trivial, we can express (S^3, L_0) as $\Lambda_1\cup_h\Lambda_2$, where h is a homeomorphism $\partial\Lambda_1\to\partial\Lambda_2$. Then the link L_1

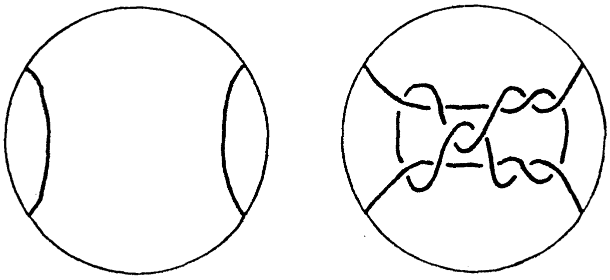

Figure 10.

obtained by $\Omega_1 \cup_h \Omega_2$ is the desired link, where we identify $\partial\Lambda_i$ and $\partial\Omega_i$ in the obvious manner. Since the K-T grabber is simple by [**22**, Lemma 3], we can show that Ω is atoroidal in the same way as in the proof of Proposition 4, so L_1 has an atoroidal exterior by Propositions 2 and 3. On the other hand, L_1 is obtained from L_0 by removing $2n$ trivial 2-string tangles and sewing back $2n$ K-T grabbers. Thus by [**17**, Lemma 3.3], L_1 is concordant to L_0, and L_0 and L_1 have the same Alexander invariant. This completes the proof.

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