

## **$G$ -maps and tangential representations at $G$ -fixed points of $G$ -manifolds**

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

By Masaharu MORIMOTO

(Received May 23, 1984)

(Revised Oct. 29, 1984)

### **0. Introduction.**

Let  $G$  be a finite group. A  $G$ -manifold and a  $G$ -map are understood to be a *smooth*  $G$ -manifold and a *continuous*  $G$ -map respectively in this paper.

Smith equivalence has recently been studied by T. Petrie and others (e.g. [11], [19], [28]). The result in [11] is roughly described as follows. Let  $G$  be a finite cyclic group with at least four distinct primes dividing  $|G|$  (the order of  $G$ ). If complex  $G$ -modules  $V$  and  $W$  satisfy certain conditions concerning (i) dimension, (ii) restriction to Sylow subgroups and (iii)  $G$ -signature, then we have a rational homotopy sphere  $f: \Sigma \rightarrow S(\mathbf{R} \oplus V \oplus U)$  with  $G$ -action such that  $\Sigma^G = \{p, q\}$ ,  $T_p \Sigma \cong V \oplus U$  and  $T_q \Sigma \cong W \oplus U$  as real  $G$ -modules. Here  $\mathbf{R}$  is the 1-dimensional real  $G$ -module with trivial action and  $U$  a complex  $G$ -module.

The purpose of this paper is to consider a similar problem for an oriented  $G$ -manifold instead of  $S(\mathbf{R} \oplus V \oplus U)$  under several hypotheses.

Let  $X$  be a closed 1-connected oriented  $G$ -manifold with a finite number of  $G$ -fixed points  $x_1, \dots, x_n$ . We denote by  $V_i$  the oriented tangential  $G$ -representations at  $x_i$  respectively for  $i=1, \dots, n$ . Let  $A$  be the field of rational numbers  $\mathbf{Q}$  or the ring of integers  $\mathbf{Z}$ . Let  $W_1, \dots, W_n$  be oriented  $G$ -modules. Then we have the problem:

(PA) Give sufficient conditions on  $W_1, \dots, W_n$  for the existence of a closed 1-connected oriented  $G$ -manifold  $Y$  and a degree one  $G$ -map  $f: Y \rightarrow X$  satisfying the following properties,

(0.1) (i)  $f$  induces an isomorphism  $f_*: H_*(Y; A) \rightarrow H_*(X; A)$ . (ii)  $Y$  has  $G$ -fixed points  $y_1, \dots, y_n$ ,  $n = \text{card}(Y^G)$ , such that  $f(y_i) = x_i$  and  $T_{y_i} Y \cong W_i$  as oriented real  $G$ -modules for  $i=1, \dots, n$ .

---

This work was partially supported by Grant-in-Aid for Scientific Research (No. 59740034), Ministry of Education, Science and Culture.

In the present paper we give some sufficient conditions on  $W_1, \dots, W_n$  to get  $f: Y \rightarrow X$  satisfying above (0.1).

**THEOREM A.** *Let  $G$  be a finite abelian group of odd order and let  $X$  be a closed 1-connected oriented  $G$ -manifold with a finite number of  $G$ -fixed points  $x_1, \dots, x_n$ . Suppose  $|G|$  is divisible by at least two distinct primes and  $\dim X > 5$ , and further*

$$(0.2) \quad X^H = X^G \text{ for any subgroup } H \text{ not of prime power order, and}$$

$$(0.3) \quad (\text{Gap hypothesis}) \quad 2 \dim X^g < \dim X \text{ for any } g \in G - \{1\}.$$

*Let  $V_i$  be the oriented tangential  $G$ -representations at  $x_i$ . If oriented real  $G$ -modules  $W_1, \dots, W_n$  satisfy the conditions (0.4) and (0.5) below, then we have a closed 1-connected oriented  $G$ -manifold  $Y$  and a degree one  $G$ -map  $f: Y \rightarrow X$  satisfying (0.1) with coefficients  $A = \mathbb{Q}$ :*

(0.4) (Restriction condition)  $\text{res}_P W_i \cong \text{res}_P V_i$  as oriented real  $P$ -modules for all  $i=1, \dots, n$  and all Sylow subgroups  $P$  of  $G$ .

(0.5) (Signature condition)  $\sum_{i=1}^n \nu(W_i)(g) = \sum_{i=1}^n \nu(V_i)(g)$  for all elements  $g$  of  $G$  whose orders are divisible by at least two distinct primes. For the definition of  $\nu(-)(g)$  see Section 1.

This theorem is generalized to Theorem C in Section 1.

Next, we consider the case  $A = \mathbb{Z}$  under the assumption that  $G$  is a finite cyclic group of odd order and  $G$ -actions are semi-free. We shall be concerned with a special case of  $G$ -homotopy equivalences. If  $f: Y \rightarrow X$  is a  $G$ -homotopy equivalence, then by K. Kawakubo [17]  $S(T_y Y)$  is  $G$ -homotopy equivalent to  $S(T_{f(y)} X)$  for any  $y \in Y$ , and hence the Whitehead torsion  $\tau(T_y Y, T_{f(y)} X)$  is defined (see Section 7). We note that if the  $G$ -actions on  $S(V_i)$  and  $S(W_i)$  are free then (0.4) implies that  $S(W_i)$  is oriented  $G$ -homotopy equivalent to  $S(V_i)$ .

**THEOREM B.** *Let  $G$  be as above and let  $X$  be a closed 1-connected oriented  $G$ -manifold with semi-free action. Suppose  $\dim X > 5$  and  $X^G = \{x_1, \dots, x_n\}$ . Let  $V_i$  be the oriented tangential  $G$ -representations at  $x_i$ . If oriented real  $G$ -modules  $W_1, \dots, W_n$  satisfy (0.4), (0.5) and*

$$(0.6) \quad \sum_{i=1}^n [\tau(W_i, V_i)] = 0 \quad \text{in } Wh(G)/2Wh(G),$$

*then we have a closed 1-connected oriented  $G$ -manifold  $Y$  with semi-free action and a degree one  $G$ -homotopy equivalence  $f: Y \rightarrow X$  satisfying (0.1) with coefficients  $A = \mathbb{Z}$  and the further property:*

$$(0.7) \quad \text{The underlying manifold of } Y \text{ is diffeomorphic to that of } X.$$

For example we can use Theorem B to show the following assertion to which we do not refer in detail in the present paper.

Let  $G$  be a finite cyclic group whose order is odd and divisible by at least two distinct primes, and let  $W$  be a complex  $G$ -module of odd dimension  $n > 4$ . Suppose the corresponding  $G$ -structure  $P(W)$  on complex projective  $(n-1)$ -space  $CP^{n-1}$  is semi-free and has isolated fixed points  $x_1, \dots, x_n$ . Then we have a smooth semi-free  $G$ -structure  $Y$  on  $CP^{n-1}$  satisfying the properties:

- (i)  $Y/G$  is homotopy equivalent to  $P(W)/G$ .
- (ii)  $Y^G = \{y_1, \dots, y_n\}$  and  $T_{y_i}Y \cong T_{x_i}P(W)$  as oriented real  $G$ -modules.
- (iii)  $Y$  is not  $G$ -homotopy equivalent to  $P(W)$ .

Our main tool is the  $G$ -surgery theory employed in [11] and [27], with some modifications. In order to construct a quasi-normal map (for the definition see Section 2) we give an Atiyah-type theorem in Section 5 and introduce the notion of a resolving map in Section 6.

The author heartily expresses his gratitude to Professor T. Petrie for suggesting the problem and giving invaluable advice, and the author is also thankful to Professors M. Nakaoka, K. H. Dovermann, I. Hambleton, K. Kawakubo, M. Masuda for their cordial conversation.

## 1. Notations and statement of Main Theorem C.

We mean by a  $G$ -module a  $G$ -representation space of finite rank. For a complex  $G$ -module  $W$  we denote by  $r(W)$  its realification. If a real  $G$ -module is equipped with an orientation and if the  $G$ -action is orientation preserving, then we call the  $G$ -module (more precisely the pair of a real  $G$ -module and its orientation) an *oriented real  $G$ -module*. For a complex  $G$ -module  $W$  we denote by  $or(W)$  the oriented real  $G$ -module (the pair of  $r(W)$  and the orientation inherited from the complex structure). We shall use  $W$  instead of  $r(W)$  or  $or(W)$  if there is no fear of confusion. For oriented real  $G$ -modules  $W$  and  $W'$  we say that  $W$  is isomorphic to  $W'$  (and we write  $W \cong W'$ ) if there exists an orientation preserving isomorphism of real  $G$ -modules from  $W$  to  $W'$ .

Let  $G$  be a finite group of odd order and let  $Y$  be a closed oriented  $G$ -manifold with  $Y^G$  non-empty. For each  $y \in Y^G$  we obtain an oriented real  $G$ -module  $T_y Y$  whose orientation is given by that of  $Y$ .

For an oriented real  $G$ -module  $V$  and  $g \in G$  with  $V^g = \{0\}$ , we define  $\nu(V)(g)$  by

$$\nu(V)(g) = \prod_{i=1}^k \frac{1+e_i}{1-e_i}$$

if  $\text{res}_{\langle g \rangle} V \cong or(W)$  and  $e_i$  are the eigenvalues of  $g$  on  $W$ , where  $k = \dim V / 2$  and  $\langle g \rangle$  is the subgroup of  $G$  generated by  $g$ . If  $Y^g$  is finite,

$$\pm \text{Sign}(g, Y) = \sum_{y \in Y^g} \nu(T_y Y)(g).$$

Let  $\mathcal{Q}(G)$ ,  $\mathcal{S}(G)$  and  $\mathcal{S}_y(G)$  denote the Burnside ring, the set of subgroups of  $G$  and the set of Sylow subgroups of  $G$  respectively. For a subgroup  $H$  of  $G$  we have the homomorphism  $\chi_H: \mathcal{Q}(G) \rightarrow \mathbb{Z}$  given by  $\chi_H([A]) = \chi(A^H)$  for finite  $G$ -CW-complexes  $A$ , where  $\chi$  is the Euler characteristic.  $G$  acts on  $\mathcal{S}(G)$  by conjugation. We call a subset  $\mathcal{H}$  of  $\mathcal{S}(G)$  a *family* if  $\mathcal{H}$  is  $G$ -invariant (that is,  $\mathcal{H}$  is closed under conjugation). A family  $\mathcal{H}$  is said to be *closed* if, for  $H \in \mathcal{H}$ , any subgroup of  $H$  also belongs to  $\mathcal{H}$ . We define a pair  $(\mathcal{H}, \mathcal{H}')$  of closed families  $\mathcal{H}$  and  $\mathcal{H}'$  with  $\{1\} \in \mathcal{H}' \subset \mathcal{H} \neq \mathcal{S}(G)$  to be *good* if there exists an element  $\omega$  of  $\mathcal{Q}(G)$  such that  $\chi_H(\omega) = 0$  for any  $H \in \mathcal{H}'$  and  $\chi_H(\omega) = 1$  for any  $H \in \mathcal{S}(G) - \mathcal{H}$ .

EXAMPLE. In the following two examples (I) and (II) we suppose that  $G$  is nilpotent and  $|G|$  is divisible by at least two distinct primes.

(I) Let  $\mathcal{K}$  be the family consisting of subgroups of  $G$  not with prime power indices. Then  $(\mathcal{S}(G) - \{G\}, \mathcal{K})$  is a good pair.

(II) Let  $\mathcal{P}(G)$  be the set of subgroups  $P$  of  $G$  whose orders  $|P|$  are prime powers. Then  $(\mathcal{P}(G), \{1\})$  is good.

THEOREM C. Let  $G$  be a finite nilpotent group of odd order,  $(\mathcal{H}, \mathcal{H}')$  a good pair of families in  $\mathcal{S}(G)$  and  $X$  a closed 1-connected oriented  $G$ -manifold with  $X^G = \{x_1, \dots, x_n\}$ . Suppose  $\dim X > 5$ ,

$$(C.1) \quad X^H = X^G \text{ for all } H \in \mathcal{S}(G) - \mathcal{H}', \text{ and}$$

$$(C.2) \quad (\text{Gap hypothesis}) \quad 2 \dim X^g < \dim X \text{ for all } g \in G - \{1\}.$$

We put  $V_i = T_{x_i} X$  as oriented real  $G$ -modules for  $i = 1, \dots, n$ . If oriented real  $G$ -modules  $W_1, \dots, W_n$  satisfy (C.3) and (C.4) below, then we have a closed 1-connected oriented  $G$ -manifold  $Y$  and a degree one  $G$ -map  $f: Y \rightarrow X$  satisfying (0.1) with coefficients  $A = \mathbb{Q}$ :

$$(C.3) \quad (\text{Restriction condition}) \quad \text{res}_H W_i \cong \text{res}_H V_i \text{ for all } H \in \mathcal{S}_y(G) \cup \mathcal{H}.$$

$$(C.4) \quad (\text{Signature condition}) \quad \sum_{i=1}^n \nu(W_i)(g) = \sum_{i=1}^n \nu(V_i)(g) \text{ for all elements } g \text{ of } G \text{ with } \langle g \rangle \in \mathcal{S}(G) - \mathcal{H}.$$

## 2. Definition of a quasi-normal map.

Let  $G$  be a finite group and  $X$  a closed 1-connected  $G$ -manifold of dimension  $\geq 5$  in this section. We put  $d = \dim X$ .

In the following sections we need to deform a  $G$ -map  $f: Y \rightarrow X$  by  $G$ -surgery in  $Y - Y^s$ , where  $Y^s = \bigcup_{H \neq \{1\}} Y^H$ . Here  $G$ -surgery is a simple analogy of ordinary surgery (described in [33] for example).

If a real  $G$ -vector bundle is an oriented real vector bundle, then we call it an *oriented real  $G$ -vector bundle*. Here the  $G$ -action may not be orientation preserving. Since in the following sections we mainly treat groups of odd orders, we may not be too nervous on it.

It seems obvious that an analogy of ordinary surgery (so called  $G$ -surgery) works well for the following “quasi-normal maps” (see [9]).

DEFINITION 2.1. If  $Y$  is a closed oriented  $G$ -manifold,  $f: Y \rightarrow X$  a degree one  $G$ -map,  $Z$  a compact  $G$ -submanifold of  $Y$ ,  $\xi$  an oriented real  $G$ -vector bundle over  $X$  and  $b$  a stable isomorphism of oriented real  $G$ -vector bundles from  $(f^*\xi)|_Z$  to  $(TY)|_Z$  (i.e.,  $b: (f^*\xi)|_Z \oplus \underline{V} \rightarrow (TY)|_Z \oplus \underline{V}$  for some complex  $G$ -module  $V$ ) and if they satisfy the following (2.2)~(2.4), then we call the triple  $(f, \xi, b)$  a *quasi-normal map*. We often call such a  $G$ -map  $f$  also a quasi-normal map making an improper use of the term:

(2.2) Each connected component of  $Y$  and  $Z$  has dimension  $d$ .

(2.3) (Gap hypothesis)  $\dim Y^g < [d/2]$  for any  $g \in G - \{1\}$ .

(2.4) The inclusion map  $k: Z \rightarrow Y$  is  $[d/2]$ -connected, that is,  $k_*: \pi_0(Z) \rightarrow \pi_0(Y)$  is bijective,  $k_*: \pi_i(Z, z) \rightarrow \pi_i(Y, k(z))$  is bijective for all  $i < [d/2]$  and  $z \in Z$  and  $k_*: \pi_{[d/2]}(Z, z) \rightarrow \pi_{[d/2]}(Y, k(z))$  is surjective for any  $z \in Z$ .

Here  $[d/2]$  is the largest integer which does not exceed  $d/2$ .

As ordinary surgery, if a quasi-normal map  $f: Y \rightarrow X$  is given, then we can perform  $G$ -surgery detached from  $f^s = f|: Y^s \rightarrow X^s$  and get a new quasi-normal map  $f': Y' \rightarrow X$  which is  $[d/2]$ -connected. We have several types of  $G$ -surgery obstruction for a  $[d/2]$ -connected quasi-normal map  $f: Y \rightarrow X$ . For example:

(I) If  $\chi(Y^g) = \chi(X^g)$  for any  $g \in G - \{1\}$ , then we have the (rational)  $G$ -surgery obstruction  $\sigma(f; \mathbb{Q})$  in  $L_d^h(\mathbb{Q}[G], w)$  to converting  $f$  to a rational homology equivalence.

(II) If  $f^s: Y^s \rightarrow X^s$  is a  $G$ -homotopy equivalence, then we have the  $G$ -surgery obstruction  $\sigma(f) = \sigma(f; \mathbb{Z})$  in  $L_d^h(\mathbb{Z}[G], w)$  to converting  $f$  to a  $G$ -homotopy equivalence.

Although Definition 2.1 is enough for the following sections, we can define a quasi-normal map more generally as follows.

(2.5) We call a degree one  $G$ -map  $f: Y \rightarrow X$  with (2.2) and (2.3) a (generalized) *quasi-normal map* if there exist a compact oriented  $G$ -manifold  $W$  with boundary  $\partial W$ , a degree one  $G$ -map  $F: (W, \partial W) \rightarrow (I \times X, \partial(I \times X))$ , a finite  $G$ -CW-complex  $U$ , a  $[d/2]$ -connected  $G$ -map  $k: U \rightarrow W$ , an oriented real  $G$ -vector bundle  $\xi$  over  $I \times X$  and a stable isomorphism  $b: (F \circ k)^*\xi \rightarrow k^*TW$  of oriented real  $G$ -vector

bundles such that  $\partial W = Y_0 \amalg Y$  as oriented  $G$ -manifolds and  $F(y) = (1, f(y))$  for all  $y \in Y$ .

### 3. $G$ -surgery in rational coefficients.

The idea of this section is due to Dovermann and Petrie [11], and [12] and the mathematical tools are due to Alexander, Conner and Hamrick [1].

Let  $G$  be a finite group of odd order and  $n$  an even integer  $> 5$ ,  $n = 2m$  say. We denote by  $A$   $\mathbf{Z}$  or  $\mathbf{Q}$ , and by  $W_n(G, A)$  the equivariant Witt ring (denoted by  $W_n(A, G)$  in [1]). Roughly to say  $W_n(G, A)$  consists of equivalence classes of pairs  $(M, \phi)$  of  $M$   $A$ -torsion free  $A[G]$ -modules and  $\phi$   $G$ -invariant non-singular  $(-1)^m$ -symmetric  $A$ -valued bilinear forms defined on  $M \times M$ .

Let  $X$  be a closed oriented  $G$ -manifold of dimension  $n$ . We define  $w[G, X; A] \in W_n(G, A)$  as the equivalence class of  $(H^m(X; A)/\text{Torsion}, \phi_X)$ , where  $\phi_X$  is the cup product bilinear form on  $X$  (see [1; p. 98]). It is known that  $w[G, X; A]$  depends only on the oriented  $G$ -cobordism class of  $X$ . We have the natural homomorphism of changing rings  $\tau: W_n(G, \mathbf{Z}) \rightarrow W_n(G, \mathbf{Q})$ . Immediately we have  $\tau(w[G, X; \mathbf{Z}]) = w[G, X; \mathbf{Q}]$ .

Let  $g$  be an element of  $G$ . We denote by  $\langle g \rangle$  the subgroup of  $G$  generated by  $g$ , and by  $\mathcal{P}$  the set of positive prime integers. We set  $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$ .  $M(\mathcal{P}, \mathbf{Z}_2)$  stands for the ring consisting of all maps from  $\mathcal{P}$  to  $\mathbf{Z}_2$ . We define a map  $\varepsilon$  from  $G$  to  $M(\mathcal{P}, \mathbf{Z}_2)$  by putting  $\varepsilon(g)(p) = 1$ ,  $g \in G$  and  $p \in \mathcal{P}$ , if  $p$  divides  $|\langle g \rangle|$  and  $-1 \equiv p^h \pmod{|\langle g \rangle/P|}$  for some integer  $h$ , and  $\varepsilon(g)(p) = 0$  if not, where  $P$  is the Sylow  $p$ -subgroup of  $\langle g \rangle$ .

For a closed oriented  $G$ -manifold  $X$  of dimension  $n$ , we define the *torsion signature*  $f(G, X): G \rightarrow M(\mathcal{P}, \mathbf{Z}_2)$  by

$$f(g, X)(p) = \chi(F^2(P) \cap X^g) \varepsilon(g)(p),$$

where  $g \in G$ ,  $p \in \mathcal{P}$ ,  $f(g, X) = f(G, X)(g)$ ,  $\chi$  is the Euler characteristic,  $P$  is the Sylow  $p$ -subgroup of  $\langle g \rangle$  and  $F^2(P)$  is the  $P$ -fixed point set of  $X$  of codimension  $2 \pmod{4}$  (see [1; p. 142 (2.1) and p. 149 (3.5)]). If  $\dim X^g = 0$ , then

$$f(g, X)(p) = \sum_{x \in X^g} \frac{1}{2} (\dim T_x X - \dim T_x X^p) \varepsilon(g)(p).$$

We remark that  $f(G, X)$  depends only on the oriented  $G$ -cobordism class of  $X$ . Corollary (3.6) of [1; p. 151] gives

LEMMA 3.1. *Let  $X$  and  $Y$  be closed oriented  $G$ -manifolds of dimension  $n$ . Then  $w[G, X; \mathbf{Z}] = w[G, Y; \mathbf{Z}]$  if  $\text{Sign}(G, X) = \text{Sign}(G, Y)$  and  $f(G, X) = f(G, Y)$ .*

Let  $X$  and  $Y$  be closed 1-connected oriented  $G$ -manifolds of dimension  $n$ , and  $f: Y \rightarrow X$  an  $m$ -connected quasi-normal map wherein  $m = n/2$ . We put

$$K(f) = K_m(f; \mathbf{Q}) = \text{Ker}[f_*: H_m(Y; \mathbf{Q}) \rightarrow H_m(X; \mathbf{Q})].$$

We assume that  $\chi(X^g) = \chi(Y^g)$  for any  $g \in G - \{1\}$ . Then  $K(f)$  is  $\mathbf{Q}[G]$ -free. Since  $K(f) = \pi_{m+1}(f) \otimes \mathbf{Q}$  by the Hurewicz theorem and the universal coefficient theorem, we obtain the triple  $(K(f), \lambda, \mu)$  of the free  $\mathbf{Q}[G]$ -module  $K(f)$ , the intersection form  $\lambda: K(f) \times K(f) \rightarrow \mathbf{Q}[G]$  and the self-intersection map  $\mu: K(f) \rightarrow \Gamma$ , where  $\Gamma = \mathbf{Q}[G]/\{x - (-1)^m \bar{x} : x \in \mathbf{Q}[G]\}$  and the anti-automorphism  $-$  of  $\mathbf{Q}[G]$  is given by  $(\sum a_g g)^- = \sum a_g g^{-1}$  for  $g \in G$  and  $a_g \in \mathbf{Q}$  (see [33; p. 21]). Denote by  $\sigma(f; \mathbf{Q})$  the equivalence class of  $(K(f), \lambda, \mu)$  in the Wall group  $L_n^h(\mathbf{Q}[G], 1)$  of homotopy equivalences. It goes without saying that  $\sigma(f; \mathbf{Q})$  is equivariant surgery obstruction to converting  $f$  to a rational homology equivalence (see [9], [12]).

LEMMA 3.2. *The equivariant surgery obstruction  $\sigma(f; \mathbf{Q})$  vanishes if  $\text{Sign}(G, X) = \text{Sign}(G, Y)$  and  $f(G, X) = f(G, Y)$ .*

PROOF. An element of  $L_n^h(\mathbf{Q}[G], 1)$  is represented by  $(M', \lambda', \mu')$ , where  $M'$  is a free  $\mathbf{Q}[G]$ -module,  $\lambda': M' \times M' \rightarrow \mathbf{Q}[G]$  is a non-singular  $\mathbf{Q}$ -bilinear form and  $\mu': M' \rightarrow \Gamma$  is a map such that  $\lambda'(x, ay) = a\lambda'(x, y)$ ,  $\lambda'(x, y) = (-1)^m \overline{\lambda'(y, x)}$ ,  $\lambda'(x, x) = \mu'(x) + (-1)^m \mu'(x)^-$  in  $\mathbf{Q}[G]$ ,  $\mu'(x+y) - \mu'(x) - \mu'(y) = \lambda'(x, y)$  in  $\Gamma$ , and  $\mu'(ax) = a\mu'(x)\bar{a}$  for  $x, y \in M'$  and  $a \in \mathbf{Q}[G]$ . For such  $(M', \lambda', \mu')$  there uniquely exists a  $G$ -invariant non-singular  $\mathbf{Q}$ -valued  $(-1)^m$ -symmetric bilinear form  $\phi$  on  $M' \times M'$  such that

$$\lambda'(x, y) = \sum_{g \in G} \phi(x, g^{-1}y)g \quad \text{for } x, y \in M'.$$

The correspondence:  $(M', \lambda', \mu') \mapsto (M', \phi)$  induces a homomorphism  $\omega: L_n^h(\mathbf{Q}[G], 1) \rightarrow W_n(G, \mathbf{Q})$ . We have

$$\begin{aligned} \omega(\sigma(f; \mathbf{Q})) &= w[G, Y; \mathbf{Q}] - w[G, X; \mathbf{Q}] \\ &= \tau(w[G, Y; \mathbf{Z}] - w[G, X; \mathbf{Z}]). \end{aligned}$$

By Lemma 3.1 we get  $\omega(\sigma(f; \mathbf{Q})) = 0$ . Since  $1/2 \in \mathbf{Q}$ ,  $\omega$  is injective. Thus  $\sigma(f; \mathbf{Q})$  vanishes.

#### 4. Proof of Theorem C.

Let  $G, (\mathcal{H}, \mathcal{H}'), X, V_i$  and  $W_i, i=1, \dots, n$ , be as in Theorem C. We will show in Section 6 that there is a quasi-normal map  $f: Y \rightarrow X$  satisfying

$$(4.1) \quad Y^G = \{y_1, \dots, y_n\}, \quad f(y_i) = x_i \quad \text{and} \quad T_{y_i}Y \cong W_i,$$

$$(4.2) \quad \dim Y_\alpha = \dim X_{\tilde{f}(\alpha)} \quad \text{for all } \alpha \in \Pi(Y),$$

$$(4.3) \quad \chi(Y^g) = \chi(X^g) \quad \text{for any } g \in G - \{1\}, \quad \text{and}$$

$$(4.4) \quad \text{res}_H(f: Y \rightarrow X) \sim 0 \quad \text{for all } H \in \mathcal{H}'.$$

Here  $\Pi(X) = \coprod_H \pi_0(X^H)$ ,  $H$  running over all the subgroups of  $G$ , and  $\tilde{f}: \Pi(Y) \rightarrow \Pi(X)$  is given by  $\tilde{f} = \coprod_H f_H^\#$ ,  $(f_H^\#: \pi_0(Y^H) \rightarrow \pi_0(X^H))$ , and  $\text{res}_H(f: Y \rightarrow X) \sim 0$  means that there exist a compact oriented  $H$ -manifold  $W$  with boundary  $\partial W$  and a degree one  $H$ -map  $F: (W, \partial W) \rightarrow (I \times X, \partial(I \times X))$  such that  $\partial W = Y_0 \amalg Y$ ,  $F(Y) \subset \{1\} \times X$ ,  $f = F|: Y \rightarrow \{1\} \times X = X$ , and  $F|: Y_0 \rightarrow \{0\} \times X = (-X)$  is an  $H$ -homotopy equivalence.

LEMMA 4.5. *One has  $\text{Sign}(G, X) = \text{Sign}(G, Y)$  and  $f(G, X) = f(G, Y)$ .*

PROOF. We show  $\text{Sign}(G, X) = \text{Sign}(G, Y)$ . If  $g \in G$  generates a subgroup in  $\mathcal{H}'$ , then we have  $\text{Sign}(g, X) = \text{Sign}(g, Y)$  by (4.4). Suppose  $g$  to generate a subgroup in  $\mathcal{S}(G) - \mathcal{H}'$ . Then we have  $Y^g = Y^G$  by (C.1), (4.1), (4.2) and (4.3). Thus

$$\text{Sign}(g, Y) = \sum_{i=1}^n \nu(W_i)(g).$$

By (C.3) and (C.4) we have  $\text{Sign}(g, X) = \text{Sign}(g, Y)$ .

We can prove  $f(G, X) = f(G, Y)$  in a similar manner. That is,  $f(g, X) = f(g, Y)$  is shown in the two cases: one is that  $g$  generates a subgroup in  $\mathcal{H}'$ , and the other is that  $g$  generates a subgroup in  $\mathcal{S}(G) - \mathcal{H}'$ . We left the detail to the reader.

We remark that Lemma 4.5 is kept under  $G$ -surgery, that is, if  $f: Y \rightarrow X$  is converted by  $G$ -surgery to another quasi-normal map  $f': Y' \rightarrow X$ , then  $\text{Sign}(G, X) = \text{Sign}(G, Y')$ ,  $f(G, X) = f(G, Y')$ . Perform  $G$ -surgery of the above  $f: Y \rightarrow X$  below the middle dimension to get a quasi-normal map  $f': Y' \rightarrow X$  such that  $Y'$  is 1-connected and  $f'$  is  $m$ -connected, where  $2m = \dim X$ . Then  $K(f')$  is  $\mathbb{Q}[G]$ -free by (4.3). We get the  $G$ -surgery obstruction  $\sigma(f'; \mathbb{Q})$  in  $L_{2m}^h(\mathbb{Q}[G], 1)$ . By Lemmas 3.2 and 4.5 we have  $\sigma(f', \mathbb{Q}) = 0$ . Thus we can perform  $G$ -surgery of  $f'$  to get a quasi-normal map  $f'': Y'' \rightarrow X$  such that  $Y''$  is closed, 1-connected and oriented, and  $f''_*: H_*(Y''; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$  is bijective.

## 5. An Atiyah-type theorem.

Let  $X$  be a  $G$ -space and  $V$  a  $G$ -module. Then we get a  $G$ -vector bundle  $\underline{V} = X \times V$  over  $X$ . We ask for which  $G$ -modules  $V$  and  $W$ ,  $\underline{V}$  and  $\underline{W}$  are isomorphic. The following well known theorem partially answers the question.

THEOREM (Atiyah [2]). *Let  $X$  be a finite  $G$ -CW-complex with free  $G$ -action, and  $V$  and  $W$  complex  $G$ -modules such that  $\text{res}_P V \cong \text{res}_P W$  for all the Sylow subgroups  $P$  of  $G$ . Then there exists a complex  $G$ -module  $U$  such that  $\underline{V} \oplus \underline{U} \cong \underline{W} \oplus \underline{U}$ .*



For making a quasi-normal map we need certain information on isomorphisms:  $\underline{V} \oplus \underline{U} \rightarrow \underline{W} \oplus \underline{U}$ . We make a theorem including the information in the following. We have the theorem without the stable term  $U$  by placing restrictions on  $G$ , and it seems to be interesting independently of  $G$ -surgery.

Let  $\xi: E \rightarrow X$  and  $\xi': E' \rightarrow X$  be  $G$ -vector bundles and  $f_0, f_1: E' \rightarrow E$  isomorphisms of  $G$ -vector bundles (covering the identity map on  $X$ ). We call a  $G$ -vector bundle map  $F: E' \times I \rightarrow E$  a *regular  $G$ -homotopy* from  $f_0$  to  $f_1$  (and write  $F: f_0 \simeq f_1$ ) if  $F(z, 0) = f_0(z)$ ,  $F(z, 1) = f_1(z)$  and  $\xi(F(z, t)) = \xi'(z)$  for all  $z \in E'$  and  $t \in I$ .

**THEOREM 5.1** (an Atiyah-type theorem). *Let  $G$  be a finite nilpotent group and  $X$  a finite  $G$ -CW-complex with free  $G$ -action. If  $V$  and  $W$  are oriented real  $G$ -modules such that  $\text{res}_P V \cong \text{res}_P W$  as oriented real  $P$ -modules for all the Sylow subgroups  $P$  of  $G$ , then there exists an isomorphism  $f: \underline{V} \rightarrow \underline{W}$  of oriented real  $G$ -vector bundles over  $X$  with the property:*

*If an isomorphism  $h: \text{res}_K V \rightarrow \text{res}_K W$  of oriented real  $K$ -modules,  $K \in \mathcal{S}(G)$ , is arbitrarily given, then  $f$  is regularly  $K$ -homotopic to the induced isomorphism  $\bar{h} = \text{id} \times h: X \times V \rightarrow X \times W$  of oriented real  $K$ -vector bundles.*

The following corollary is a partial answer to the question raised at the beginning of this section.

**COROLLARY 5.2.** *Let  $G$ ,  $V$  and  $W$  be as above, and let  $Y$  be a finite  $G$ -CW-complex. Suppose  $Y$  to have the points  $y(1), \dots, y(m)$  such that  $Gy(i) \cap Gy(j) = \emptyset$  if  $i \neq j$ , and the  $G$ -action on  $Y - \bigcup \{Gy(i): i=1, \dots, m\}$  is free, where  $m$  is some integer. Further suppose that for  $K(i) = G_{y(i)}$  the isotropy subgroups at  $y(i)$ ,  $i=1, \dots, m$ , isomorphisms  $h_i: \text{res}_{K(i)} V \rightarrow \text{res}_{K(i)} W$  of oriented real  $K(i)$ -modules are given. Then there exists an isomorphism  $f: \underline{V} \rightarrow \underline{W}$  of oriented real  $G$ -vector bundles over  $Y$  such that  $f|_{(y(i) \times V)}: \text{res}_{K(i)} V \rightarrow \text{res}_{K(i)} W$  agree with  $h_i$  respectively for  $i=1, \dots, m$ .*

We prove the theorem in Section 9.

## 6. Construction of a quasi-normal map.

In this section we construct a quasi-normal map required in Section 4. Suppose  $G$  to be a finite nilpotent group of odd order and  $(\mathcal{H}, \mathcal{H}')$  to be a good pair of families in  $\mathcal{S}(G)$  (see Section 1 for definition).

Let  $X$  be a closed oriented  $G$ -manifold with  $X^G$  finite. Fix a point  $x$  in  $X^G$ . We call a degree one smooth  $G$ -map  $f: Y \rightarrow X$  an  $(\mathcal{H}, \mathcal{H}')$ -*resolving normal map* at  $x$  (or a *resolving map* for short) if

$$(6.1) \quad f^{-1}(X^G - \{x\}) = Y^G \text{ and } f|_{Y^G}: Y^G \rightarrow X^G - \{x\} \text{ is bijective,}$$

(6.2)  $G_y$  belongs to  $\mathcal{H}$  for all  $y \in f^{-1}(x)$ ,

(6.3)  $(df)_y: T_y Y \rightarrow T_{f(y)} X$  are bijective for all  $y \in f^{-1}(X^G)$ ,

(6.4) there exist an oriented real  $G$ -module  $M$  and an orientation preserving real  $G$ -vector bundle map  $\tilde{f}: TY \oplus \underline{M} \rightarrow TX \oplus \underline{M}$  covering  $f$ , and

(6.5)  $\text{res}_H(f: Y \rightarrow X) \sim 0$  for all  $H \in \mathcal{H}'$ .

Since  $f$  is of degree one, the conditions (6.1) and (6.3) imply  $T_y Y \cong T_{f(y)} X$  as oriented real  $G$ -modules for  $y \in Y^G$ . Given a resolving map  $f: Y \rightarrow X$  at  $x$ , we relate an element  $\omega(f)$  of  $\Omega(G)$  to  $f$  by

$$\omega(f) = \sum_{y \in f^{-1}(x)} (s((df)_y) / |G/G_y|) [G/G_y],$$

where  $s((df)_y) = 1$  (resp.  $-1$ ) if  $(df)_y$  is orientation preserving (resp. reversing). For  $u = \sum_H u(H) [G/H] \in \Omega(G)$ , where  $H$  is taken one for each element of  $\mathcal{S}(G)/G$ , and  $u(H) \in \mathbf{Z}$ , we define  $|u| \in \Omega(G)$  by  $|u| = \sum_H |u(H)| [G/H]$ . We say a resolving map  $f: Y \rightarrow X$  to be *nice* if

$$(6.6) \quad [f^{-1}(x)] = |\omega(f)| \quad \text{and}$$

$$(6.7) \quad [Y] = [X] + [f^{-1}(x)] - [G/G].$$

Let  $X$  be a finite  $G$ -CW-complex. For a real  $G$ -module  $M$ , we denote by  $[\underline{M}, \underline{M}]_G$  the set of proper  $G$ -homotopy classes of proper  $G$ -maps from  $\underline{M} = X \times M$  to  $\underline{M}$  itself covering the identity map on  $X$ . For another real  $G$ -module  $N$  with  $N \supset M$ , we have the suspension map from  $[\underline{M}, \underline{M}]_G$  to  $[\underline{N}, \underline{N}]_G$ . Roughly to say,  $\omega_G^0(X)$  is  $\varinjlim [\underline{M}, \underline{M}]_G$ , where  $M$  runs over a set of representatives chosen one for each isomorphism class of real  $G$ -modules (see [30]).

In the following we fix an element  $\omega$  of  $\Omega(G)$  such that  $\chi_H(\omega) = 0$  for  $H \in \mathcal{H}'$  and  $\chi_H(\omega) = 1$  for  $H \in \mathcal{S}(G) - \mathcal{H}$ .

LEMMA 6.8. *Given an oriented real  $G$ -module  $V$  with  $\text{Iso}(G, V - \{0\}) \subset \mathcal{H}'$  and a sufficiently large integer  $N$ , there exists a nice  $(\mathcal{H}, \mathcal{H}')$ -resolving normal map  $f: A \rightarrow S$  at  $(-1, 0)$  with  $\omega(f) = 1 - \omega^{N+1}$ , where  $S = S(\mathbf{R} \oplus V)$ ,  $-1 \in \mathbf{R}$  and  $0 \in V$ .*

PROOF. We put  $p = (1, 0)$  and  $q = (-1, 0)$ , where  $\pm 1 \in \mathbf{R}$  and  $0 \in V$ . We have  $\omega_G^0(S^G) = \omega_G^0(p) \oplus \omega_G^0(q)$ . Both  $\omega_G^0(p)$  and  $\omega_G^0(q)$  are identified with  $\Omega(G)$ . Denote by  $L$  the multiplicatively closed set generated by  $\omega$ , i.e.,  $L = \{\omega, \omega^2, \omega^3, \dots\}$ . The inclusion map  $j: S^G \rightarrow S$  induces the restriction homomorphism  $j^*: \omega_G^0(S) \rightarrow \omega_G^0(S^G)$ . Since  $\text{Iso}(G, V - \{0\}) \subset \mathcal{H}'$ ,  $L^{-1}j^*: L^{-1}\omega_G^0(S) \rightarrow L^{-1}\omega_G^0(S^G)$  is bijective. We suppose  $N$  to be so large that there exists an element  $u \in \omega_G^0(S)$  such that  $j^*(u) = (0, \omega^N) \in \omega_G^0(p) \oplus \omega_G^0(q)$ . We put  $v = 1 - \omega u \in \omega_G^0(S)$ . We remark that  $\text{res}_H v = 1 \in \omega_G^0(\text{res}_H S)$  for  $H \in \mathcal{H}'$ . Take a representative  $h: S \times M \rightarrow S \times M$  of  $v$ , where  $h$  is a proper

$G$ -map over the identity map on  $S$  and  $M$  is a complex  $G$ -module with  $M^G \neq \{0\}$ . By the equivariant transversality theorem ([26])  $h$  is properly  $G$ -homotopic to  $h': S \times M \rightarrow S \times M$  transverse to  $S = S \times \{0\}$  in  $S \times M$ . We put  $B = h'^{-1}(S)$  and  $h'' = h'|_B: B \rightarrow S$ . The  $G$ -manifold  $B$  has the orientation induced by  $h'$ ,  $(TB \oplus \nu(B, S \times M) = \{T(S \times M)\}|_B$  and  $\nu(B, S \times M) = h''^* \nu(S, S \times M)$ ). We can choose  $h'$  so that  $h'': B \rightarrow S$  is an  $(\mathcal{H}, \mathcal{H}')$ -resolving normal map at  $q = (-1, 0)$  with (5.6), by [14] or [29]. We have  $\omega(h'') = 1 - \omega^{N+1}$  and  $[h''^{-1}(q)] = |\omega(h'')|$ . We put  $\mathcal{K} = \{H \in \mathcal{S}(G) : \dim S^H \geq 2\}$ . Then  $\mathcal{K} \subset \mathcal{H}'$  and  $h''(B^H) \subset \{p, q\}$  if  $H \in \mathcal{S}(G) - \mathcal{K}$ . We have  $\chi_H([B]) = \chi_H([G/G] + [h''^{-1}(q)])$  and  $\chi_H([S]) = \chi_H(2[G/G])$  for  $H \in \mathcal{S}(G) - \mathcal{K}$ . Further we have  $\chi_H([B]) \equiv \chi_H([S]) \pmod{2}$  and  $\chi_H([h''^{-1}(q)]) = \chi_H(|1 - \omega^{N+1}|) \equiv \chi_H(1 - \omega^{N+1}) = 1 \pmod{2}$  for all  $H \in \mathcal{H}'$ . Since  $|G|$  is odd, we get

$$[B] - ([S] + [h''^{-1}(q)] - [G/G]) = \sum_{(H) \in \mathcal{K}/G} 2a(H)[G/H]$$

for some integers  $a(H)$ . We can perform  $G$ -surgery on trivial elements of  $\pi_1(h''^H)$  and  $\pi_2(h''^H)$  of  $h'': B \rightarrow S$  to get a nice  $(\mathcal{H}, \mathcal{H}')$ -resolving normal map  $f: A \rightarrow S$  at  $q$ .

We complete the proof of Lemma 6.8. We remark that the basic idea of the proof is due to Petrie [27].

Given an oriented real  $G$ -module  $V$ , we denote by  $V^-$  the oriented real  $G$ -module whose underlying real  $G$ -module is that of  $V$  and whose orientation is opposite to that of  $V$ .

LEMMA 6.9. *Let  $V$  be an oriented real  $G$ -module such that  $\dim V \geq 6$ ,*

$$(6.10) \quad 2\dim V^g < \dim V \quad \text{for all } g \in G - \{1\}, \text{ and}$$

$$(6.11) \quad \text{Iso}(G, V - \{0\}) \subset \mathcal{H}'.$$

*Let  $V'$  be another satisfying*

$$(6.12) \quad \text{res}_H V' \cong \text{res}_H V \quad \text{for all } H \in \mathcal{S}_Y(G) \cup \mathcal{H}.$$

*Then there exists a quasi-normal map  $f: Y \rightarrow S$ ,  $S = S(\mathbf{R} \oplus V)$ , with a  $(\dim V/2)$ -connected inclusion map  $k: Z \rightarrow Y$  and a stable isomorphism  $b: (f^*TS)|_Z \rightarrow (TY)|_Z$  of oriented real  $G$ -vector bundles over  $Z$  such that*

$$(6.13) \quad Y^G = \{u, v\}, \quad f(u) = p, \quad f(v) = q, \quad T_u Y \cong V' \text{ and } T_v Y \cong V^-, \text{ where } p = (1, 0) \text{ and } q = (-1, 0), \quad (\pm 1 \in \mathbf{R}, 0 \in V),$$

$$(6.14) \quad \dim Y_\alpha = \dim S_{\tilde{f}(\alpha)} \quad \text{for any } \alpha \in \Pi(Y),$$

$$(6.15) \quad [Y] = 2[G/G] \quad \text{in } \Omega(G),$$

$$(6.16) \quad \text{res}_H(f: Y \rightarrow S) \sim 0 \quad \text{for all } H \in \mathcal{H}', \text{ and}$$

(6.17)  $Z$  is a compact  $G$ -submanifold including an equivariant closed disk neighborhood of  $v$  in  $Y$ .

We note here that (6.11), (6.14) and (6.15) imply  $\text{Iso}(G, Y - Y^G) \subset \mathcal{H}'$ .

PROOF. We put  $p' = (1, 0) \in S(\mathbf{R} \oplus V')$ ,  $q' = (-1, 0) \in S(\mathbf{R} \oplus V')$  and  $S' = S(\mathbf{R} \oplus V')$ . There exist a positive integer  $N$  and nice  $(\mathcal{H}, \mathcal{H}')$ -resolving normal maps  $h: A \rightarrow S$  at  $p$  and  $h': A' \rightarrow S'$  at  $q'$  such that  $\omega(h) = 1 - \omega^{N+1} = \omega(h')$  by Lemma 6.8. We take a small closed equivariant tubular neighborhood  $D$  (resp.  $D'$ ) of  $h^{-1}(p)$  (resp.  $h'^{-1}(q')$ ) in  $A$  (resp.  $A'$ ). The condition (6.12) allows us to take an orientation reversing  $G$ -diffeomorphism  $\phi: D' \rightarrow D$ . We set  $A_0 = A - \text{Int } D$ ,  $A'_0 = A' - \text{Int } D'$  and  $\partial\phi = \phi|_{\partial D'}: \partial D' \rightarrow \partial D$ . We define  $Y$  by  $Y = A_0 \cup_{\partial\phi} A'_0$ .  $G$ -homotopically deform  $h: A \rightarrow S$  in a small neighborhood of  $D$  to  $h'': A \rightarrow S$  such that  $h''(D) = \{p\}$ . We define  $f: Y \rightarrow S$  by  $f|_{A_0} = h''|_{A_0}$  and  $f(A'_0) = \{p\}$ . Let  $B$  be a compact  $G$ -manifold. We denote by  $B^s$  the set of points in  $B$  at which the isotropy subgroups of  $G$  are non-trivial.  $N(B^s, B)$  denotes a closed equivariant regular neighborhood of  $B^s$  in  $B$  (which is a union of closed equivariant tubular neighborhoods of  $B^K$  in  $B$ ,  $\{1\} \neq K \subset G$ ). We define  $B^r$  by putting  $B^r = B - \text{Int } N(B^s, B)$ . We give  $Z$  by  $Z = A_0 \cup_{(\partial\phi)^r} A_0'^r$  and denote the inclusion map of  $Z$  into  $Y$  by  $k$ . By (6.10)  $k$  is  $(\dim V/2)$ -connected. We put  $u = h'^{-1}(p')$  and  $v = h^{-1}(q)$ . (6.13) and (6.7) are obvious. (6.14) follows from (6.4). (6.15) is obtained from (6.6), (6.7) and the construction. We have (6.16) by (6.5). It remains to make a stable isomorphism  $b: (f^*TS)|_Z \rightarrow (TY)|_Z$  of oriented real  $G$ -vector bundles over  $Z$ . Firstly we note  $\mathbf{R} \oplus TS \cong \mathbf{R} \oplus V$  and  $\mathbf{R} \oplus TS' \cong \mathbf{R} \oplus V'$ . By the construction we get stable isomorphisms  $f^*TS \rightarrow TA$  and  $h'^*TS' \rightarrow TA'$ . By taking the restriction of the first one we get a stable isomorphism  $c: (f^*TS)|_{A_0} \rightarrow (TA)|_{A_0}$ . We are going to make  $b$  with  $b|_{A_0} = c$ . By (6.12) and Theorem 5.1 (an Atiyah-type theorem) we have  $\mathbf{R} \oplus (TA')|_{A_0'^r} \oplus \underline{M} \cong \mathbf{R} \oplus (f^*TS)|_{A_0'^r} \oplus \underline{M}$  as oriented real  $G$ -vector bundles over  $A_0'^r$  for some complex  $G$ -module  $M$ . There are a lot of stable isomorphisms  $c': (f^*TS)|_{A_0'^r} \rightarrow (TA')|_{A_0'^r}$ . We wonder if we can choose  $c'$  to be combined with  $c$ . We put our eyes on the specially described property in Theorem 5.1 (an Atiyah-type theorem) and recall that  $c|_{\partial D}$  expands to a stable isomorphism over  $D$ . We see that there exists such a stable isomorphism  $c': (f^*TS)|_{A_0'^r} \rightarrow (TA')|_{A_0'^r}$  that is able to be combined with  $c$ ; we get  $b = c \cup c'$ .

In the rest of this section we let  $X$ ,  $V_i$  and  $W_i$ ,  $i=1, \dots, n$ , be as in Theorem C.

For each  $i$  we apply Lemma 6.9 to the case  $V = V_i$  and  $V' = W_i$ . We get a quasi-normal map  $f_i: Y_i \rightarrow S(\mathbf{R} \oplus V_i)$  with a  $(\dim X/2)$ -connected inclusion map  $k_i: Z_i \rightarrow Y_i$  and a stable isomorphism  $b_i: (f_i^*TS(\mathbf{R} \oplus V_i))|_{Z_i} \rightarrow (TY_i)|_{Z_i}$  of oriented real  $G$ -vector bundles over  $Z_i$  with the described properties in Lemma 6.9, for each  $i=1, \dots, n$ . We put  $p_i = (1, 0)$  and  $q_i = (-1, 0)$ ,  $(\pm 1 \in \mathbf{R}, 0 \in V_i)$ . We define

$u(i)$  and  $v(i)$  by  $Y_i^G = \{u(i), v(i)\}$ ,  $f_i(u(i)) = p_i$  and  $f_i(v(i)) = q_i$ . Then  $T_{u(i)}Y_i \cong W_i$ ,  $T_{v(i)}Y_i \cong V_i$ . We denote by  $k_0$  the identity map on  $X$  and by  $b_0$  the identity map on  $TX \oplus \underline{M}$ , where  $\underline{M}$  is a stable term. We have the quasi-normal map  $\text{id}_X: X \rightarrow X$  with data  $k_0$  and  $b_0$ . Take equivariant closed disk neighborhoods  $D_i$  of  $x_i$  in  $X$  and  $D'_i$  of  $v(i)$  in  $Z_i$  (see (6.17)), and take orientation reversing  $G$ -diffeomorphisms  $\phi_i: D'_i \rightarrow D_i$ . We put  $X_0 = X - \bigcup_i \text{Int } D_i$ ,  $Y_{i0} = Y_i - \text{Int } D'_i$ ,  $Z_{i0} = Z_i - \text{Int } D'_i$  and  $\partial\phi = \coprod_i (\phi_i|_{\partial D'_i}): \coprod_i \partial D'_i \rightarrow X_0$ ,  $i=1, \dots, n$ . We define  $Y$  and  $Z$  by  $Y = X_0 \cup_{\partial\phi} (\coprod_i Y_{i0})$  and  $Z = X_0 \cup_{\partial\phi} (\coprod_i Z_{i0})$ . We  $G$ -homotopically deform  $\text{id}_X: X \rightarrow X$  in a small neighborhood of  $\bigcup_i D_i$  to  $f_0: X \rightarrow X$  such that  $f_0(D_i) = \{x_i\}$ ,  $i=1, \dots, n$ . We define  $f: Y \rightarrow X$  by  $f|_{X_0} = f_0|_{X_0}$  and  $f(Y_{i0}) = \{x_i\}$ ,  $i=1, \dots, n$ . We put  $\xi = TX$  and denote by  $k$  the inclusion map of  $Z$  into  $Y$ . We can easily obtain a stable isomorphism  $b: (f^*\xi)|_Z \rightarrow (TY)|_Z$  of oriented real  $G$ -vector bundles over  $Z$  by combining  $b_i$  with  $b_0$ ,  $i=1, \dots, n$ . By the construction  $(f, \xi, b)$  is a quasi-normal map. (4.1)~(4.4) follow from (6.13)~(6.16) respectively.

## 7. Proof of Theorem B.

In this section let  $G$  be a finite cyclic group of odd order.

Let  $V$  and  $W$  be real  $G$ -modules. Suppose that the  $G$ -action on  $S(V)$  is free and  $S(V)$  is  $G$ -homotopy equivalent to  $S(W)$ . A  $G$ -homotopy equivalence  $f: S(V) \rightarrow S(W)$  determines the Whitehead torsion  $\tau(f)$  in  $Wh(G)$ . Since all  $G$ -homotopy equivalences from  $S(V)$  to  $S(W)$  are  $G$ -homotopic to one another,  $\tau(f)$  is independent of the choice of a  $G$ -homotopy equivalence  $f$ . Hence we denote the Whitehead torsion by  $\tau(V, W)$  instead of  $\tau(f)$ .

We do not use the following proposition in this paper, but that may be helpful to understand the condition (0.6). Since  $G$  is cyclic, of odd order, we have  $Wh(G) = U(\mathbb{Z}[G]) / \{\pm g | g \in G\}$ , where  $U(\mathbb{Z}[G])$  is the group of units in  $\mathbb{Z}[G]$ . Thus  $Wh(G)/2Wh(G) = U(\mathbb{Z}[G]) / \{\pm x^2 | x \in U(\mathbb{Z}[G])\}$ . Fix a generator  $g$  of  $G$ . Let  $t^a$ ,  $a \in \mathbb{Z}$ , be the complex  $G$ -module whose underlying space is  $\mathbb{C}$  and on which  $g$  acts as  $\exp(2a\pi\sqrt{-1}/|G|)$ .

**PROPOSITION.** *Let  $V$  and  $W$  be as above and suppose  $V \cong r(t^{a_1} \oplus \dots \oplus t^{a_k})$  and  $W \cong r(t^{b_1} \oplus \dots \oplus t^{b_k})$ . Then one has*

$$\tau(V, W) = \prod_{i=1}^k \frac{1 - g^{d_i}}{1 - g^{c_i}},$$

where  $c_i$  and  $d_i$  are given by  $a_i c_i \equiv 1 \equiv b_i d_i \pmod{|G|}$ .

**PROOF OF THEOREM B.** From the construction in Section 6 we obtain a quasi-normal map  $f: Y \rightarrow X$  satisfying

(7.1)  $Y$  is a closed 1-connected oriented  $G$ -manifold with semi-free action,

(7.2)  $Y^G = \{y_1, \dots, y_n\}$ ,  $f(y_i) = x_i$  and  $T_{y_i}Y \cong W_i$  for  $i=1, \dots, n$ , and

(7.3) if we forget  $G$ -action, then the bundle data of  $f$  expands over  $Y$  so that  $f$  is tangentially normally cobordant to the identity map on  $X$ .

(Here  $f$  may not be a  $G$ -homotopy equivalence.) We have the  $G$ -surgery obstruction  $\sigma(f)$  in  $L_{2m}^h(\mathbf{Z}[G], 1)$  to converting  $f$  to a  $G$ -homotopy equivalence, where  $2m = \dim X$ . There is an exact sequence (so called the Rothenberg exact sequence (see [31])):

$$\cdots \longrightarrow L_{2m}^s(\mathbf{Z}[G], 1) \longrightarrow L_{2m}^h(\mathbf{Z}[G], 1) \xrightarrow{a} H^{2m}(\mathbf{Z}_2; Wh(G)) \longrightarrow \cdots.$$

Since  $\mathbf{Z}_2$  acts trivially on  $Wh(G)$  ([21; Lemma 6.7]), we have  $H^{2m}(\mathbf{Z}_2; Wh(G)) = Wh(G)/2Wh(G)$ . Theorem 13.A.4 (ii) of [33] gives

LEMMA 7.4. *Let  $G$  be a finite cyclic group of odd order. Then the homomorphism*

$$a \oplus \text{trans.} \oplus \text{Sign} : L_{2m}^h(\mathbf{Z}[G], 1) \longrightarrow (Wh(G)/2Wh(G)) \oplus L_{2m}(\mathbf{Z}, 1) \oplus R(G)$$

*is injective.*

Since  $\text{trans.}(\sigma(f))=0$  and  $\text{Sign}(\sigma(f))=0$  by (7.3) and (0.5), it suffices to show  $a(\sigma(f))=0$ . Here we note that (0.7) is obtained from  $L_{2m+1}(\mathbf{Z}, 1)=0$  and (7.3).

Let  $N(X^G)$  be an equivariant closed tubular neighborhood of  $X^G$  in  $X$  and put  $X^r = X - \text{Int } N(X^G)$ . We can  $G$ -homotopically deform  $f$  to satisfy  $f(N(Y^G)) \subset N(X^G)$  and  $f(Y^r) \subset X^r$ . By observing the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*(\partial Y^r) & \longrightarrow & C_*(Y^r) & \longrightarrow & C_*(Y, N(Y^G)) \longrightarrow 0 \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ 0 & \longrightarrow & C_*(\partial X^r) & \longrightarrow & C_*(X^r) & \longrightarrow & C_*(X, N(X^G)) \longrightarrow 0 \end{array}$$

and the Poincaré duality map:  $H^m(Y; \mathbf{Z}) \rightarrow H_m(Y; \mathbf{Z})$ , we have

$$a(\sigma(f)) = \sum_{y \in Y^G} [\tau(T_y Y, T_{f(y)} X)] \quad \text{in } Wh(G)/2Wh(G)$$

(see [13]). This and (0.6) imply  $a(\sigma(f))=0$ . Hence we can convert  $f$  to a  $G$ -homotopy equivalence by  $G$ -surgery in the trivial orbit type as was required.

## 8. Examples.

This section gives examples of Theorem B in which the target  $G$ -manifolds are *equivariant complex projective spaces*.

For a positive integer  $k$ , we denote by  $\mathbf{Z}_k$  the group of  $k$ -th roots of 1 in  $\mathbf{C}$ , and we put  $\zeta = \exp(2\pi\sqrt{-1}/k)$ . For an integer  $a$  we have the complex  $\mathbf{Z}_k$ -module  $t(k)^a$  of dimension one on which  $\zeta$  has the eigenvalue  $\zeta^a$ . We fix  $k$  and put  $G = \mathbf{Z}_k$  and  $t^a = t(k)^a$ . Let  $V$  be a complex (more precisely unitary)  $G$ -module.  $S(V)$  stands for the unit sphere of  $V$  and  $S^1$  denotes the group consisting of

complex numbers of absolute value one. Since  $S^1$  acts on  $V$  by complex multiplication,  $V$  can be thought of as a complex  $(G \times S^1)$ -module and  $S(V)$  as a  $(G \times S^1)$ -manifold. We set  $P(V) = S(V)/S^1$  and call  $P(V)$  an *equivariant complex projective space*.  $P(V)$  is a closed 1-connected oriented  $G$ -manifold if  $\dim_{\mathbb{C}} V \geq 2$ .  $V$  is decomposed to irreducible complex  $G$ -modules of dimension one. Hence we put  $V = \bigoplus_i t^{a(i)}$ , where  $i$  ranges over the integers from 1 to  $n = \dim_{\mathbb{C}} V$  and  $a(i)$  are integers with  $0 \leq a(i) < |G|$ .  $P(V)^G$  is finite if and only if all  $a(i)$ ,  $i=1, \dots, n$ , are distinct. We assume  $P(V)^G$  to be finite. Each complex line  $t^{a(i)}$  in  $V$  corresponds to a  $G$ -fixed point of  $P(V)$ ,  $x_i$  say; we have  $P(V)^G = \{x_1, \dots, x_n\}$ .  $V_i = T_{x_i} P(V)$  is isomorphic to  $\bigoplus_j t^{a(j) - a(i)}$ , where the sum is taken over the integers  $j$  with  $1 \leq j \leq n$  and  $j \neq i$ . We see

$$\text{Sign}(g, P(V)) = \sum_{i=1}^n \nu(V_i)(g) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

for all generators  $g$  of  $G$ .

In the rest of this section let  $p$  and  $q$  be distinct primes  $\geq 5$ ,  $G = \mathbf{Z}_{pq}$  ( $= \mathbf{Z}_p \times \mathbf{Z}_q$ ),  $u = t(p)$  and  $v = t(q)$ . For integers  $a$  and  $b$ ,  $u^a \otimes v^b$  is regarded as a complex  $G$ -module.

EXAMPLE 8.1. Let  $a, b, x$  and  $y$  be integers satisfying  $0 < a < p$ ,  $0 < b < p$ ,  $a \not\equiv \pm b \pmod{p}$ ,  $0 < x < q$ ,  $0 < y < q$  and  $x \not\equiv \pm y \pmod{q}$ . Give a complex  $G$ -module  $V$  by

$$V = u^a \otimes v^x \oplus u^{-a} \otimes v^{-x} \oplus u^b \otimes v^y \oplus u^{-b} \otimes v^{-y}.$$

Then we have

$$V_1 = u^{-2a} \otimes v^{-2x} \oplus u^{b-a} \otimes v^{y-x} \oplus u^{-(a+b)} \otimes v^{-(x+y)},$$

$$V_2 = u^{2a} \otimes v^{2x} \oplus u^{a+b} \otimes v^{x+y} \oplus u^{a-b} \otimes v^{x-y},$$

$$V_3 = u^{a-b} \otimes v^{x-y} \oplus u^{-(a+b)} \otimes v^{-(x+y)} \oplus u^{-2b} \otimes v^{-2y}, \text{ and}$$

$$V_4 = u^{a+b} \otimes v^{x+y} \oplus u^{b-a} \otimes v^{y-x} \oplus u^{2b} \otimes v^{2y}.$$

We put

$$W_1 = u^{-2a} \otimes v^{-2x} \oplus u^{b-a} \otimes v^{-(x+y)} \oplus u^{-(a+b)} \otimes v^{y-x},$$

$$W_2 = u^{2a} \otimes v^{2x} \oplus u^{a+b} \otimes v^{x-y} \oplus u^{a-b} \otimes v^{x+y},$$

$W_3 = V_3$  and  $W_4 = V_4$ . These  $G$ ,  $X = P(V)$ ,  $V_i$  and  $W_i$ ,  $i=1, \dots, 4$ , satisfy the hypotheses of Theorem B. We remark that there is no complex  $G$ -module  $V'$  such that  $P(V')^G = \{y_1, \dots, y_4\}$ , and  $T_{y_i} P(V') \cong W_i$ ,  $i=1, \dots, 4$ , as real  $G$ -modules.

More generally we can get examples for equivariant projective spaces  $P(V)$  of type

$$V = \bigoplus_{i=1}^k (u^{a(i)} \otimes v^{b(i)} \oplus u^{-a(i)} \otimes v^{-b(i)}),$$

where  $a(i)$  and  $b(i)$ ,  $i=1, \dots, k$ , are some integers.

EXAMPLE 8.2. Let  $a, b, x$  and  $y$  be as in Example 8.1. Give complex  $G$ -modules  $V$  and  $V'$  by

$$V = u^a \otimes v^x \oplus u^{-a} \otimes v^{-x} \oplus u^b \otimes v^y \oplus u^{-b} \otimes v^{-y} \oplus u^0 \otimes v^0,$$

$$V' = u^a \otimes v^{-x} \oplus u^{-a} \otimes v^x \oplus u^b \otimes v^{-y} \oplus u^{-b} \otimes v^y \oplus u^0 \otimes v^0.$$

Then we have

$$V'_1 = u^{-2a} \otimes v^{2x} \oplus u^{b-a} \otimes v^{x-y} \oplus u^{-(a+b)} \otimes v^{x+y} \oplus u^{-a} \otimes v^x,$$

$$V'_2 = u^{2a} \otimes v^{-2x} \oplus u^{a+b} \otimes v^{-(x+y)} \oplus u^{a-b} \otimes v^{y-x} \oplus u^{-a} \otimes v^x,$$

$$V'_3 = u^{a-b} \otimes v^{y-x} \oplus u^{-(a+b)} \otimes v^{x+y} \oplus u^{-2b} \otimes v^{2y} \oplus u^{-b} \otimes v^y,$$

$$V'_4 = u^{a+b} \otimes v^{-(x+y)} \oplus u^{b-a} \otimes v^{x-y} \oplus u^{2b} \otimes v^{-2y} \oplus u^b \otimes v^{-y}, \quad \text{and}$$

$$V'_5 = u^a \otimes v^{-x} \oplus u^{-a} \otimes v^x \oplus u^b \otimes v^{-y} \oplus u^{-b} \otimes v^y.$$

By replacing  $x$  and  $y$  by  $-x$  and  $-y$ , we have the equalities for  $V_i, i=1, \dots, 5$ . We put  $W_i = V'_i$  for  $i=1, \dots, 5$ . These  $G, X=P(V), V_i$  and  $W_i, i=1, \dots, 5$ , satisfy the hypotheses of Theorem B. We note that  $P(V)$  is not  $G$ -homotopy equivalent to  $P(V')$  (see A. Liulevicius [18]).

## 9. Proof of Theorem 5.1.

Theorem 5.1 immediately follows from

THEOREM 9.1. *Let  $G$  be a finite nilpotent group and  $X$  a finite  $(n-1)$ -connected  $G$ -CW-complex of dimension  $n$  and with free  $G$ -action. If  $V$  and  $W$  are oriented real  $G$ -modules such that  $\text{res}_P V \cong \text{res}_P W$  as oriented real  $P$ -modules for all the Sylow subgroups  $P$  of  $G$ , then there exists an isomorphism  $f: \underline{V} \rightarrow \underline{W}$  of oriented real  $G$ -vector bundles over  $X$  with the property: if an isomorphism  $h_K: \text{res}_K V \rightarrow \text{res}_K W$  of oriented real  $K$ -modules,  $K \in \mathcal{S}(G)$ , is arbitrarily given, then  $f|X^{(n-1)}: X^{(n-1)} \times V \rightarrow X^{(n-1)} \times W$  is regularly  $K$ -homotopic to the induced isomorphism  $\bar{h}_K|X^{(n-1)} = \text{id} \times h_K: X^{(n-1)} \times V \rightarrow X^{(n-1)} \times W$  of oriented real  $K$ -vector bundles, where  $X^{(n-1)}$  is the  $(n-1)$ -skeleton of  $X$ .*

This theorem is a composition of the following two lemmas. In the two lemmas let  $G, X, V$  and  $W$  be as above. We number the Sylow subgroups of  $G: \mathcal{S}_y(G) = \{P(i): i=1, \dots, m\}$ ,  $m = |\mathcal{S}_y(G)|$ , and put  $Q(i) = P(i) \cap K$  for  $i=1, \dots, m$ . We fix isomorphisms  $h_i: \text{res}_{P(i)} V \rightarrow \text{res}_{P(i)} W$  of oriented real  $P(i)$ -modules and put  $\bar{h}_i = \text{id} \times h_i: X \times V \rightarrow X \times W$  for all  $i$ .

LEMMA 9.2. *There exist an isomorphism  $f: \underline{V} \rightarrow \underline{W}$  of oriented real  $G$ -vector bundles over  $X$  and regular  $P(i)$ -homotopies:  $f|X^{(n-1)} \simeq \bar{h}_i|X^{(n-1)}$  for all  $i$ .*

LEMMA 9.3. *Let  $K$  be a subgroup of  $G, f: \underline{V} \rightarrow \underline{W}$  an isomorphism of oriented*



real  $K$ -vector bundles over  $X$  and  $h_K: \text{res}_K V \rightarrow \text{res}_K W$  an isomorphism of oriented real  $K$ -modules. We put  $\bar{h}_K = \text{id} \times h_K: X \times V \rightarrow X \times W$ . If there exist regular  $Q(i)$ -homotopies:  $f|_{X^{(n-1)}} \simeq \bar{h}_i|_{X^{(n-1)}}$  for all  $i$ , then one has a regular  $K$ -homotopy:  $f|_{X^{(n-1)}} \simeq \bar{h}_K|_{X^{(n-1)}}$ .

Before starting the proofs of the two lemmas we prepare several words and notations.

Let  $\xi: E \rightarrow A$  be a fiber bundle and  $B$  a subspace of  $A$ . A continuous map  $s: B \rightarrow E$  is called a *section on  $B$  of  $\xi$*  if  $\xi \circ s(b) = b$  for all  $b \in B$ . Let  $s$  and  $s': B \rightarrow E$  be sections of  $\xi$ . A continuous map  $t: B \times I \rightarrow E$  is said to be a *homotopy* (or a *first homotopy*) *on  $B$  from  $s$  to  $s'$*  and written  $t: s \simeq s'$  if  $\xi \circ t(b, c) = b$ ,  $t(b, 0) = s(b)$  and  $t(b, 1) = s'(b)$  for all  $b \in B$  and  $c \in I$ . Let  $t$  and  $t'$  be first homotopies on  $B$  from  $s$  to  $s'$ . We call a continuous map  $u: B \times I \times I \rightarrow E$  a *second homotopy on  $B$  from  $t$  to  $t'$*  and write  $u: t \simeq t'$  if  $\xi \circ u(b, c, d) = b$ ,  $u(b, c, 0) = t(b, c)$ ,  $u(b, c, 1) = t'(b, c)$ ,  $u(b, 0, d) = s(b)$  and  $u(b, 1, d) = s'(b)$  for all  $b \in B$ ,  $c \in I$  and  $d \in I$ . We note that a second homotopy is relative to the corresponding sections.

Let  $G, X, V$  and  $W$  be as in Theorem 9.1. We equip  $\text{Hom}_R(V, W)$  with the  $G$ -action: for  $g \in G$ ,  $x \in \text{Hom}_R(V, W)$  and  $v \in V$ ,  $(gx)(v)$  is given as  $gx(g^{-1}v)$ . We put

$$M = \{x \in \text{Hom}_R(V, W) : x \text{ is bijective and orientation preserving}\}.$$

Then  $M$  is  $G$ -invariant and homeomorphic to  $GL^+(\dim V, R)$ . Hence  $M$  is connected and simple (i.e.,  $\pi_1(M)$  trivially acts on  $\pi_j(M)$ ), further  $\pi_j(M)$  is finitely generated abelian for each  $j \geq 1$ .

ASSERTION 9.4. *The induced  $G$ -action on  $\pi_j(M)$  is trivial.*

PROOF. By definition each  $g \in G$  preserves the orientations of  $V$  and  $W$ . If  $g \in G$  is fixed, the  $\langle g \rangle$ -actions on  $V$  and  $W$  expand to  $S^1$ -actions with  $\langle g \rangle \subset S^1$  respectively. Since  $S^1$  is connected,  $S^1$  acts on  $\pi_j(M)$  trivially. Hence  $g$  acts on  $\pi_j(M)$  trivially.

Let  $K$  be a subgroup of  $G$ ,  $P(i)$  and  $Q(i)$ ,  $i=1, \dots, m$ , subgroups of  $G$  defined before Lemma 9.2. Let  $p_i: X/P(i) \rightarrow X/G$ ,  $q_i: X \rightarrow X/P(i)$ ,  $p'_i: X/Q(i) \rightarrow X/K$ ,  $q'_i: X \rightarrow X/Q(i)$ ,  $r: X/K \rightarrow X/G$ ,  $r_i: X/P(i) \rightarrow X/G$  and  $q_0: X \rightarrow X/G$  and  $q'_0: X \rightarrow X/K$  be the canonical projections. We have the commutative diagram:

$$\begin{array}{ccccc}
 X/Q(i) & & \xrightarrow{p'_i} & & X/K \\
 & \nwarrow q'_i & & \nearrow q'_0 & \\
 & & X & & \\
 & \nearrow q_i & & \nwarrow q_0 & \\
 X/P(i) & & \xrightarrow{p_i} & & X/G
 \end{array}$$

$\downarrow r_i$    $\downarrow r$

We consider the bundles  $\xi: E = X \times M \rightarrow X$ ,  $\xi_i: E_i = X \times_{P(i)} M \rightarrow X/P(i)$ ,  $\xi'_i: E'_i = X \times_{Q(i)} M \rightarrow X/Q(i)$  for  $i=1, \dots, m$ ,  $\xi_0: E_0 = X \times_G M \rightarrow X/G$  and  $\xi'_0: E'_0 = X \times_K M \rightarrow X/K$ . Then we have the canonical identifications from  $E_i$  to  $p_i^* E_0$  for  $i=1, \dots, m$ , etc. The set of isomorphisms:  $\underline{V} \rightarrow \underline{W}$  of oriented real  $G$ -vector bundles is in one to one correspondence with the set of sections:  $X/G \rightarrow E_0$ . Actually, if  $f: \underline{V} \rightarrow \underline{W}$  is an isomorphism, we have the section  $s: X/G \rightarrow E_0$  corresponding to  $f$  by  $s([x]) = [x, f_x]$  for  $x \in X$ , where  $[ ]$  stand for the equivalence classes and  $f_x \in M$  is given by  $f_x(v) = f(x, v)$  for  $v \in V$ . Similarly the set of regular  $P(i)$ -homotopies between isomorphisms:  $\text{res}_{P(i)} \underline{V} \rightarrow \text{res}_{P(i)} \underline{W}$ , is in one to one correspondence with the set of homotopies between sections:  $X/P(i) \rightarrow E_i$  for each  $i=1, \dots, m$ . We denote by  $s_i$  the section:  $X/P(i) \rightarrow E_i$  corresponding to  $\bar{h}_i$ .

PROOF OF LEMMA 9.2. It suffices to prove that there exist a section  $s: X/G \rightarrow E_0$  and homotopies  $t_i: p_i^* s|(X^{(n-1)}/P(i)) \simeq s_i|(X^{(n-1)}/P(i))$  for all  $i=1, \dots, m$ . We construct  $s$  and  $t_i$  by an inductive method on the dimensions of skeletons of  $X$ .

Since  $M$  is connected, there exist a section  $s^1: X^{(1)}/G \rightarrow E_0$  and homotopies  $t_i^0: p_i^* s^1|(X^{(0)}/P(i)) \simeq s_i|(X^{(0)}/P(i))$  for all  $i$ . Thus, for a non-negative integer  $j \leq n-2$ , we suppose that there exist a section  $s^{j+1}: X^{(j+1)}/G \rightarrow E_0$  and homotopies  $t_i^j: p_i^* s^{j+1}|(X^{(j)}/P(i)) \simeq s_i|(X^{(j)}/P(i))$  for all  $i$ .

By the obstruction theory of Steenrod [32], the obstruction  $\sigma(s^{j+1})$  to extending  $s^{j+1}|(X^{(j)}/G)$  to a section from  $X^{(j+2)}$  lies in  $H^{j+2}(X/G; \pi_{j+1}(M))$ . Since  $p_i^* s^{j+1}|(X^{(j)}/P(i))$  are homotopic to  $s_i|(X^{(j)}/P(i))$ ,  $p_i^* s^{j+1}|(X^{(j)}/P(i))$  are extendible to sections from  $X^{(j+2)}/P(i)$  for all  $i$ . This implies  $p_i^*(\sigma(s^{j+1}))=0$ . Since

$$\bigoplus_{i=1}^m p_i^* : H^{j+2}(X/G; \pi_{j+1}(M)) \longrightarrow \bigoplus_{i=1}^m H^{j+2}(X/P(i); \pi_{j+1}(M))$$

is injective, we have  $\sigma(s^{j+1})=0$ . We can take an extension  ${}^{j+2}s: X^{(j+2)}/G \rightarrow E_0$  of  $s^{j+1}|(X^{(j)}/G)$ . The obstructions  $\sigma(p_i^* {}^{j+2}s, s_i)$  to finding homotopies:  $p_i^* {}^{j+2}s|(X^{(j+1)}/P(i)) \simeq s_i|(X^{(j+1)}/P(i))$ ,  $i=1, \dots, m$ , are in  $H^{j+1}(X/P(i); \pi_{j+1}(M))$ , where the obstructions are determined by  $p_i^* {}^{j+2}s, s_i$  and  $t_i^j$  respectively. Since

$$\bigoplus_{i=1}^m p_i^* : H^{j+1}(X/G; \pi_{j+1}(M)) \longrightarrow \bigoplus_{i=1}^m H^{j+1}(X/P(i); \pi_{j+1}(M))$$

is surjective, there is  $b \in H^{j+1}(X/G; \pi_{j+1}(M))$  with  $p_i^* b = \sigma(p_i^* {}^{j+2}s, s_i)$ . Take a cocycle  $b' \in C^{j+1}(X/G; \pi_{j+1}(M))$  which represents  $b$ . Lemma 33.9 of [32] allows us to take a section  ${}_{j+1}s: X^{(j+1)}/G \rightarrow E$  such that  ${}_{j+1}s|(X^{(j)}/G) = {}^{j+2}s|(X^{(j)}/G)$  ( $= s^{j+1}|(X^{(j)}/G)$ ) and  $d({}_{j+1}s, {}^{j+2}s|(X^{(j+1)}/G)) = -b'$ . By 33.5 of [32] we have

$$c({}_{j+1}s) = \delta d({}_{j+1}s, {}^{j+2}s|(X^{(j+1)}/G)) + c({}^{j+2}s|(X^{(j+1)}/G))$$

in  $C^{j+2}(X/G; \pi_{j+1}(M))$ . Thus  $c({}_{j+1}s)=0$ . This means that  ${}_{j+1}s$  extends to a section  $s^{j+2}: X^{(j+2)}/G \rightarrow E_0$ . Then we have  $\sigma(p_i^* s^{j+2}, s_i)=0$  in  $H^{j+1}(X/P(i); \pi_{j+1}(M))$

for all  $i$ . There exist homotopies  $t_i^{j+1}: p_i^* s^{j+2}|(X^{(j+1)}/P(i)) \simeq s_i|(X^{(j+1)}/P(i))$  for all  $i$ .

By induction we get a desired  $s: X/G \rightarrow E_0$  and homotopies  $t_i: p_i^* s|(X^{(n-1)}/P(i)) \simeq s_i|(X^{(n-1)}/P(i))$  for all  $i$ . We complete the proof of Lemma 9.2.

PROOF OF LEMMA 9.3. Let  $s$  and  $s_K: X/K \rightarrow E'_0$  be the sections corresponding to  $K$ -isomorphisms  $f$  and  $\bar{h}_K$  respectively. It suffices to give a homotopy:  $s|(X^{(n-1)}/K) \simeq s_K|(X^{(n-1)}/K)$ . For convenience' sake we suppose that  $Q(i)$  is non-trivial if and only if  $i \leq m'$ . In the following  $i$  ranges over the integers from one to  $m'$ . Since  $M^{Q(i)}$  is connected, there exist regular  $Q(i)$ -homotopies  $k_i: h_i \simeq h_K$ . Set  $\bar{k}_i = \text{id} \times k_i: X^{(n-1)} \times V \times I \rightarrow X^{(n-1)} \times W$ . Then  $\bar{k}_i$  give homotopies  $u_i: r_i^* s_i|(X^{(n-1)}/Q(i)) \simeq p_i^* s_K|(X^{(n-1)}/Q(i))$ . By the assumption in Lemma 9.3 we have homotopies  $t_i: p_i^* s|(X^{(n-1)}/Q(i)) \simeq r_i^* s_i|(X^{(n-1)}/Q(i))$ . Thus for all  $i$  we have the homotopies  $t'_i = t_i \vee u_i: p_i^* s|(X^{(n-1)}/Q(i)) \simeq p_i^* s_K|(X^{(n-1)}/Q(i))$ .

ASSERTION 9.5. One can choose  $k_i$  so that there exist second homotopies:  $q_i^* t'_i|(X^{(0)} \times I) \simeq q_j^* t'_j|(X^{(0)} \times I)$  for all  $j$  with  $2 \leq j \leq m'$ .

The proof is left to the reader. Note that the natural map from  $\pi_1(M^{Q(j)})$  to  $\pi_1(M)$  are surjective and  $q_j^*: H^0(X/Q(j); \pi_1(M)) \rightarrow H^0(X; \pi_1(M))$  are bijective for  $j$  with  $(2, |Q(j)|) = 1$ . Using the obstruction theory of Steenrod, the reader see the existence of required  $k_i$ . In the following we assume  $k_i$  to be chosen as in Assertion 9.5.

ASSERTION 9.6. There exist a (first) homotopy  $u^1: s|(X^{(1)}/K) \simeq s_K|(X^{(1)}/K)$  and second homotopies  $v_i^0: p_i^* u^1|(X^{(0)}/Q(i) \times I) \simeq t'_i|(X^{(0)}/Q(i) \times I)$  for all  $i$ .

PROOF. Since  $M$  is connected and  $q_0^*: H^1(X/K; \pi_1(M)) \rightarrow H^1(X; \pi_1(M))$  is injective, we get a homotopy  $^1u: s|(X^{(1)}/K) \simeq s_K|(X^{(1)}/K)$  by the obstruction theory. We have the obstruction  $\sigma \in H^0(X/Q(1); \pi_1(M))$  to finding a second homotopy:  $p_1^* u^1|(X^{(0)}/Q(1) \times I) \simeq t'_1|(X^{(0)}/Q(1) \times I)$ . Since  $p_1^*: H^0(X^{(n-1)}/K; \pi_1(M)) \rightarrow H^0(X^{(n-1)}/Q(1); \pi_1(M))$  is surjective, by the same technique used in the proof of Lemma 9.2 we get a homotopy  $u^1: s|(X^{(1)}/K) \simeq s_K|(X^{(1)}/K)$ , and a second homotopy  $v_1^0: p_1^* u^1|(X^{(0)}/Q(1) \times I) \simeq t'_1|(X^{(0)}/Q(1) \times I)$ . We denote by  $\sigma_i$  the obstructions in  $H^0(X^{(n-1)}/Q(i); \pi_1(M))$  to finding second homotopies  $v_i^0$  for  $i = 1, \dots, m'$ . Of course  $\sigma_1 = 0$ . Note  $q_i^*: H^0(X^{(n-1)}/Q(i); \pi_1(M)) \rightarrow H^0(X^{(n-1)}; \pi_1(M))$  being injective. Assertion 9.5 implies  $q_i^* \sigma_i = q_1^* \sigma_1 = 0$ . Hence we have  $\sigma_i = 0$  for all  $i$ . This completes the proof of Assertion 9.6.

For a non-negative integer  $j \leq n-3$  we suppose that there exist a homotopy  $u^{j+1}: s|(X^{(j+1)}/K) \simeq s_K|(X^{(j+1)}/K)$  and second homotopies  $v_i^j: p_i^* u^{j+1}|(X^{(j)}/Q(i) \times I) \simeq t'_i|(X^{(j)}/Q(i) \times I)$  for all  $i = 1, \dots, m'$ . We note that

$$\bigoplus_{i=1}^{m'} p_i^* : H^{j+2}(X/K; \pi_{j+2}(M)) \longrightarrow \bigoplus_{i=1}^{m'} H^{j+2}(X/Q(i); \pi_{j+2}(M))$$

is injective and

$$\bigoplus_{i=1}^{m'} p_i'^* : H^{j+1}(X/K; \pi_{j+2}(M)) \longrightarrow \bigoplus_{i=1}^{m'} H^{j+1}(X/Q(i); \pi_{j+2}(M))$$

is surjective. By the same technique used in the proof of Lemma 9.2 we get a homotopy  $u^{j+2} : s|(X^{(j+2)}/K) \simeq s_K|(X^{(j+2)}/K)$  and second homotopies  $v_i^{j+1} : p_i'^* u^{j+2}|(X^{(j+1)}/Q(i) \times I) \simeq t_i'(X^{(j+1)}/Q(i) \times I)$  for all  $i$ . By induction we get a homotopy  $u : s|(X^{(n-1)}/K) \simeq s_K|(X^{(n-1)}/K)$ .

### References

- [1] J. Alexander, P. Conner and G. Hamrick, Odd order group actions and Witt classification of inner products, *Lecture Notes in Math.*, **625**, Springer, 1977.
- [2] M. Atiyah, Characters and cohomology of finite groups, *Publ. Math. I.H.E.S.*, **9** (1961), 23-64.
- [3] M. Atiyah and R. Bott, The Lefschetz fixed-point theorem of elliptic complexes: II. Applications, *Ann. of Math.*, **88** (1968), 451-491.
- [4] M. Atiyah and I. Singer, The index of elliptic operators: III, *Ann. of Math.*, **87** (1968), 546-604.
- [5] M. Cohen, A course in simple-homotopy theory, *Graduate Texts in Math.*, **10**, Springer, Berlin - New York, 1973.
- [6] T. tom Dieck, Homotopy-equivalent group representations, *J. Reine Angew. Math.*, **298** (1978), 182-195.
- [7] T. tom Dieck, Transformation groups and representation theory, *Lecture Notes in Math.*, **766**, Springer, 1979.
- [8] T. tom Dieck and T. Petrie, Geometric modules over the Burnside ring, *Invent. Math.*, **47** (1978), 273-287.
- [9] K. H. Dovermann and T. Petrie, G-surgery II, *Mem. Amer. Math. Soc.*, **260**, 1982.
- [10] K. H. Dovermann and T. Petrie, An introduction theorem for equivariant surgery (G-surgery III), *Amer. J. Math.*, **105** (1983), 1369-1403.
- [11] (K. H. Dovermann and) T. Petrie, Rational Smith equivalence, *Res. Inst. Math. Sci. Kōkyōroku*, **501**, Kyoto Univ., Kyoto, 1983, 74-85.
- [12] K. H. Dovermann and M. Rothenberg, An equivariant surgery sequence and equivariant diffeomorphism and homeomorphism classification (A survey), *Lecture Notes in Math.*, **788**, Springer, 1980, 257-278.
- [13] K. H. Dovermann and M. Rothenberg, Poincaré duality and generalized Whitehead torsion, preprint.
- [14] H. Hauschild, Zerspaltung äquivarianter Homotopiemengen, *Math. Ann.*, **230** (1977), 279-292.
- [15] S. Illman, Smooth equivariant triangulations of  $G$ -manifolds for  $G$  a finite group, *Math. Ann.*, **233** (1978), 199-220.
- [16] K. Kawakubo, Equivariant homotopy equivalence of group representations, *J. Math. Soc. Japan*, **32** (1980), 105-118.
- [17] K. Kawakubo, Compact Lie group actions and fiber homotopy type, *J. Math. Soc. Japan*, **33** (1981), 295-321.
- [18] A. Liulevicius, Characters do not lie, *London Math. Soc. Lecture Note Series*, **26**, 1976, 139-146.
- [19] M. Masuda and T. Petrie, Lectures on transformation groups and Smith equivalence,

- Contemporary Math., **36**, Amer. Math. Soc., 1985, 191-242.
- [20] M. Masuda and Y.-D. Tsai, Tangential representations of cyclic group actions on homotopy complex projective spaces, *Osaka J. Math.*, **22** (1985), 907-919.
  - [21] J. Milnor, Whitehead torsion, *Bull. Amer. Math. Soc.*, **72** (1966), 358-426.
  - [22] M. Morimoto and K. Iizuka, Extendibility of  $G$ -maps to pseudo-equivalences to finite  $G$ -CW-complexes whose fundamental groups are finite, *Osaka J. Math.*, **21** (1984), 59-69.
  - [23] R. Oliver, Fixed point set of group actions on finite acyclic complexes, *Comment. Math. Helv.*, **50** (1975), 155-177.
  - [24] R. Oliver and T. Petrie,  $G$ -CW-surgery and  $K_0(\mathbb{Z}G)$ , *Math. Z.*, **179** (1982), 11-42.
  - [25] K. Pawałowski, Group actions with inequivalent representations at fixed points, *Aarhus Preprint Series*, 1983/84, No. 32, Aarhus Univ., Aarhus.
  - [26] T. Petrie, Pseudoequivalence of  $G$  manifolds, *Proc. of Symp. in Pure Math.*, **32** (1978), 169-210.
  - [27] T. Petrie, One fixed point actions on spheres, I, *Adv. in Math.*, **46** (1982), 3-14.
  - [28] T. Petrie, Smith equivalences of representations, *Math. Proc. Cambridge Philos. Soc.*, **94** (1983), 61-99.
  - [29] R. Rubinsztein, On the equivariant homotopy of spheres, *Dissertations Math. (Rozprawy Mat.)*, **134** (1976).
  - [30] G. Segal, Equivariant stable homotopy, *Actes Congrès intern. Math.*, **2** (1970), 59-63.
  - [31] J. Shaneson, Wall's surgery obstruction groups for  $\mathbb{Z} \times G$ , *Ann. of Math.*, **90** (1969), 296-334.
  - [32] N. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, 1951.
  - [33] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, London, 1970.

Masaharu MORIMOTO

Department of Mathematics  
 College of Liberal Arts and Sciences  
 Okayama University  
 Tsushima, Okayama 700  
 Japan