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The existence and uniqueness of nonstationary ideal incompressible flow in exterior domains in $\boldsymbol{R}^3$

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Introduction.

We consider the motion of an ideal incompressible fluid past a finite number of isolated rigid bodies $O_1, \cdots, O_m$ in $\boldsymbol{R}^3$. The velocity $\boldsymbol{v}=(v^1(x,t), v^2(x,t), v^3(x,t))$ and the (scalar) pressure $p=p(x,t)$ of the fluid motion are governed by the Euler equation in $\Omega=\boldsymbol{R}^3\setminus(O_1\cup\cdots\cup O_m)$

$$(1)\qquad \frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\nabla p=\boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{v}=0 \qquad t\in[0, T]$$

subject to the following conditions at infinity and on the boundary $S=\partial\Omega$

$$(2)\qquad \lim_{|x|\to\infty}\boldsymbol{v}(x,t)=\boldsymbol{v}_\infty, \qquad \boldsymbol{v}\cdot\boldsymbol{n}|_S=0 \qquad t\in[0, T]$$

satisfying the initial condition

$$(3)\qquad \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x),$$

where $\boldsymbol{f}=\boldsymbol{f}(x,t)$ is a given external force vector, $\boldsymbol{v}_0(x)$ is a given initial velocity and $\boldsymbol{v}_\infty$ is a given constant vector.

The purpose of the present paper is to prove the existence of a solution $\{\boldsymbol{v}, p\}$ of (1), (2) and (3) which satisfies the asymptotic condition at infinity that $\boldsymbol{v}$ converges to $\boldsymbol{v}_\infty$ faster than $|x|^{-(1+\delta)}$ for a certain $\delta\geqq 0$.

The problem of existence and uniqueness of solutions of the Euler equation has been considered by several authors. Recently, when $\Omega=\boldsymbol{R}^3$, this problem was studied by Swann [20], Kato [15], Bardos and Frisch [2] and Cantor [5]. When Ω is bounded in $\boldsymbol{R}^3$, the problem was studied by Ebin and Marsden [8], Swann [21], Bourguignon and Brézis [4], Temam [22] and Kato and Lai [16]. In the two-dimensional case, the existence of a global solution was studied by Judovič [13] and Kato [14] (in a bounded case), and by Kikuchi [17] (in an unbounded case). Among the above quoted papers, the works of Cantor [5] and Swann [21] especially inspire the idea of our proof. In [5], Cantor constructed solutions introducing the weighted Sobolev space over $\boldsymbol{R}^3$, on which the Laplacian is an isomorphism. This work suggested to the author to use the corresponding space on the exterior domains; this space makes possible the construction of solutions of linear elliptic system (see Lemma 2.10 and Proposition 2.11). Our proof is essentially a modified form of that of Swann [21], who constructed solutions by considering the vorticity equation, with suitable treatment of the exterior problem (see Proposition 3.2).

The contents of this paper are as follows. In Section 1 we introduce the notations and state the main theorem. In Section 2 we study some properties of $M^p_{s,\delta}$ and study linear elliptic system. In Section 3 we construct a solution of (1), (2) and (3) by considering the vorticity equation.

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§1. Theorem.

We begin with the assumption on the domain Ω.

(A)
- (i) Ω is simply connected and $\boldsymbol{R}^3\setminus\Omega$ consists of m numbers of compact components $O_1, \cdots, O_m$.
- (ii) The boundary $S=S_1+\cdots+S_m$ ($S_j=\partial O_j$) is sufficiently smooth.

Before stating our results more precisely, we introduce some notations. In what follows we consider real scalar or vector functions defined on Ω (or $\Omega\times[0, T]$) or on their closures $\bar{\Omega}$ (or $\bar{\Omega}\times[0, T]$). In this paper we use the same notations for scalar and vector functions as there will be no fear of ambiguity.

The length of the vector $\boldsymbol{u}=(u^1, u^2, u^3)$ is denoted by $|\boldsymbol{u}|=(\sum_{j=1}^3(u^j)^2)^{1/2}$. For $1\leqq p<\infty$, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$. The scalar product in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$. For $s\geqq 0$ (integer) and $1\leqq p<\infty$, $W^{s,p}(\Omega)$ is the Sobolev space of L^p-functions on Ω such that all their derivatives up to order s belong to $L^p(\Omega)$. The norm in $W^{s,p}(\Omega)$ is denoted by $|\cdot|_{s,p}$. For $s\geqq 0$ (integer), $C^s(\Omega)$ (resp. $C^s(\bar{\Omega})$) is the set of all s times continuously differentiable functions on Ω (resp. $\bar{\Omega}$) and $C^s_0(\bar{\Omega})$ is the set of all $u\in C^s(\bar{\Omega})$ such that all their derivatives up to order s are bounded. The norm in $C^s_0(\bar{\Omega})$ is denoted by

$$|u|_{s,\infty}=\sum_{|\alpha|\leqq s}\sup_{x\in\Omega}|D^\alpha u(x)| \qquad (|\cdot|_{0,\infty}=|\cdot|_\infty).$$

$C^s_0(\bar{\Omega})$, s being non-negative integer, is the set of all functions that belong to $C^s(\bar{\Omega})$ and have compact support in $\bar{\Omega}$.

Let $\sigma(x)=(1+|x|^2)^{1/2}$. For $p\geqq 1$, $\lambda\geqq 0$ and $s\geqq 0$ (integer), $M^p_{s,\lambda}$ denotes the closure of $C_0^s(\bar{\Omega})$ with respect to the norm

$$|u|_{p,s,\lambda} = \sum_{|\alpha|\leqq s} |\sigma^{\lambda+|\alpha|} D^\alpha u|_p .$$

If X is a Banach space, then $L^p(0, T; X)$ $(1\leqq p\leqq\infty)$ denotes the set of all L^p-functions of $t\in(0, T)$ with values in X. Let $C([0, T]; X)$ (resp. $C^1([0, T]; X)$) denotes the set of all X-valued continuous (resp. continuously differentiable) functions of t. We write for fixed $t\in[0, T]$ $\|u\|_{(\cdot),t}=\sup_{s\in[0,t]}|u(\cdot, t)|_{(\cdot)}$ for various norms $(\cdot)$.

For a vector-valued function $\boldsymbol{u}$, $\boldsymbol{u}\cdot\boldsymbol{n}|_S$ is the outward normal component of $\boldsymbol{u}$ on S and $\boldsymbol{u}_\tau|_S$ is the tangential component of $\boldsymbol{u}$ on S, and also we write

$$D\boldsymbol{u} = \left(\frac{\partial \boldsymbol{u}}{\partial x_j}\ ;\ j=1, 2, 3\right), \qquad D^2\boldsymbol{u} = \left(\frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k}\ ;\ \ j, k=1, 2, 3\right).$$

Now we can state our main result.

THEOREM 1.1. *Let $p>3$ and $0\leqq\delta<1-3/p$. Assume that*

(i) $\boldsymbol{v}_0$ *belongs to* $C_b(\bar{\Omega})\cap C^1(\Omega)$, $\quad \mathrm{rot}\,\boldsymbol{v}_0 \in M^p_{1,\delta+2}$;
$\quad \mathrm{div}\,\boldsymbol{v}_0 = 0$; $\quad \boldsymbol{v}_0\cdot\boldsymbol{n}|_S = 0$; $\quad \lim_{|x|\to\infty} \boldsymbol{v}_0 = \boldsymbol{v}_\infty$,

(ii) $\boldsymbol{f}$ *belongs to* $C([0, T]; C_b(\bar{\Omega})\cap C^1(\Omega))$, $\quad \mathrm{rot}\boldsymbol{f} \in L^\infty(0, T; M^p_{1,\delta+2})$.

Then there exists $T_0>0$, $T_0\leqq T$, depending only on $\mathrm{rot}\,\boldsymbol{v}_0$, $\boldsymbol{v}_\infty$, rot$\boldsymbol{f}$ *and Ω, such that* (1), (2) *and* (3) *have a solution $\{\boldsymbol{v}, p\}$ on $[0, T_0]$ satisfying*

$$\begin{aligned} &\boldsymbol{v}-\boldsymbol{v}_\infty \in C([0, T_0]; M^p_{1,\delta+1})\cap L^\infty(0, T_0; M^p_{2,\delta+1}), \\ &\frac{\partial \boldsymbol{v}}{\partial t} \in L^\infty(0, T_0; M^p_{1,\delta+1}), \qquad \nabla p \in L^\infty(0, T_0; C_b(\bar{\Omega})). \end{aligned} \tag{1.1}$$

Such a solution is unique up to an arbitrary function of t which may be added to p.

§2. Preliminaries.

2.1. Properties of $M^p_{s,\lambda}$.

LEMMA 2.1. *Let $\lambda>\mu\geqq 0$. If $\sigma^\lambda u\in L^p(\Omega)$, then $\sigma^\mu u\in L^r(\Omega)$ provided that $3p/\{(\lambda-\mu)p+3\}<r\leqq p$.*

PROOF. $p=r$ is a trivial case. If $p>r$, then by the Hölder inequality

$$\int_\Omega |\sigma^\mu u|^r dx \leqq \left(\int_\Omega |\sigma^\lambda u|^p dx\right)^{r/p} \left(\int_\Omega \sigma^{-(\lambda-\mu)rq} dx\right)^{1/q},$$

where $q=p/(p-r)$ and $(\lambda-\mu)rq>3$.

REMARKS. (i) If $s\leqq s^*$ and $\lambda\leqq\lambda^*$, then

$$M^p_{s^*,\lambda^*} \subset M^p_{s,\lambda}. \tag{2.1}$$

(ii) Let $3<p<6$ and let δ be as in Theorem 1.1. Then

$$M^p_{s,\delta+1} \subset W^{s,2}(\Omega). \tag{2.2}$$

LEMMA 2.2 (The Sobolev imbedding theorem). *Let* $s\geqq 1$, $p>3$ *and* $\lambda\geqq 0$. *Then* $M^p_{s,\lambda}\subset C^{s-1}_b(\bar{\Omega})$ *and*

$$|u|_{s-1,\infty} \leqq c_1|u|_{s,p} \leqq c_1|u|_{p,s,\lambda}. \tag{2.3}$$

In particular, if $\lambda>0$, *then*

$$|u(x)| = O(|x|^{-\lambda}) \qquad \text{as } |x|\to\infty. \tag{2.4}$$

PROOF. $|u(x)| \leqq \sigma(x)^{-\lambda}|\sigma^\lambda u|_\infty \leqq c_1(1+|x|^2)^{-\lambda/2}|u|_{p,s,\lambda}$.

LEMMA 2.3 (Cantor [**5**, Proposition 1.1]). *If* $p>1$, $s>3/p$, $\delta\geqq 0$ *and* $0\leqq k\leqq s$, *then the pointwise multiplication of functions* : $M^p_{s,\delta}\times M^p_{s-k,\delta+k}\ni(\varphi, \psi)\mapsto\varphi\cdot\psi\in M^p_{s-k,\delta+k}$ *induces a continuous map.*

LEMMA 2.4. (Cantor [**6**, Theorem 2]. Also see Nirenberg and Walker [**19**, Theorem 2.1].) *Let* $p>3$, $s\geqq 0$ *and* $0\leqq\delta<1-3/p$. *Then the Laplacian* $\Delta: M^p_{s+2,\delta}(\boldsymbol{R}^3)\to M^p_{s,\delta+2}(\boldsymbol{R}^3)$ *is an isomorphism. Moreover, there is a positive constant c depending only on s such that*

$$|\varphi|_{p,s+2,\delta,\boldsymbol{R}^3} \leqq c|\Delta\varphi|_{p,s,\delta+2,\boldsymbol{R}^3} \qquad \textit{for all } \varphi\in M^p_{s+2,\delta}(\boldsymbol{R}^3). \tag{2.5}$$

COROLLARY 2.5. *Let p, s, δ and c be as in the above lemma. Then the following inequality holds: for all* $\boldsymbol{v}\in M^p_{s+1,\delta+1}(\boldsymbol{R}^3)$

$$|\boldsymbol{v}|_{p,s+1,\delta+1,\boldsymbol{R}^3} \leqq c(|\mathrm{rot}\,\boldsymbol{v}|_{p,s,\delta+2,\boldsymbol{R}^3}+|\mathrm{div}\,\boldsymbol{v}|_{p,s,\delta+2,\boldsymbol{R}^3}). \tag{2.6}$$

PROOF. The above lemma implies that there is a solution $\boldsymbol{u}\in M^p_{s+2,\delta}$ of $-\Delta\boldsymbol{u}=\mathrm{rot}\,\boldsymbol{v}$ such that

$$|\boldsymbol{u}|_{p,s+2,\delta,\boldsymbol{R}^3} \leqq c|\mathrm{rot}\,\boldsymbol{v}|_{p,s,\delta+2,\boldsymbol{R}^3}. \tag{2.7}$$

Furthermore, there is a solution $q\in M^p_{s+2,\delta}$ of $-\Delta q=\mathrm{div}\,\boldsymbol{v}$ such that

$$|q|_{p,s+2,\delta,\boldsymbol{R}^3} \leqq c|\mathrm{div}\,\boldsymbol{v}|_{p,s,\delta+2,\boldsymbol{R}^3}. \tag{2.8}$$

We can easily see that $\boldsymbol{v}$ is represented as $\boldsymbol{v}=\mathrm{rot}\,\boldsymbol{u}-\nabla q$ (see Lemma 2.9 (i) in this paper). Hence by (2.7) and (2.8) we have (2.6).

LEMMA 2.6 (Cantor [**7**, Theorem 2.1]). *Let p, s and δ be as in Lemma* 2.4. *Then the map N defined by*

$$N(u) = \left(-\Delta u, \left.\frac{\partial u}{\partial n}\right|_S\right) \;:\; M^p_{s+2,\delta} \longrightarrow M^p_{s,\delta+2}\times W^{s+1-1/p,p}(S)$$

is an isomorphism.

2.2. The boundary value problem.

LEMMA 2.7. *Let* $\boldsymbol{u}\in C(\Omega)$ *be a vector function such that* $\operatorname{rot}\boldsymbol{u}=0$ *(generalized). Then there is a scalar function* $q\in C^1(\Omega)$ *such that* $\boldsymbol{u}=\nabla q$. *If, in addition,* $\operatorname{div}\boldsymbol{u}=0$ *(generalized) and* $\lim_{|x|\to\infty}\boldsymbol{u}(x)=0$ *then* q *satisfies*

$$\lim_{|x|\to\infty} q(x) = \text{const.} \tag{2.9}$$

REMARK. By virtue of (2.11) below we see that the constant in (2.9) is independent of the direction in which x tends to infinity.

PROOF. As is well known, if $\boldsymbol{u}\in C^1(\Omega)$, then (A) and $\operatorname{rot}\boldsymbol{u}=0$, together with the Stokes theorem, imply that $q(x)$ given by the following equation is well-defined.

$$q(x) = \int_{x_0}^{x} \boldsymbol{u}(y)\cdot\boldsymbol{\tau}(y)d_y\Gamma+q(x_0), \tag{2.10}$$

where the integral of $\boldsymbol{u}$ is along any path in Ω from a fixed point x_0 to x, and $q(x_0)$ is an arbitrarily given constant. This q satisfies $\nabla q=\boldsymbol{u}$. For $\boldsymbol{u}\in C(\Omega)$ with $\operatorname{rot}\boldsymbol{u}=0$ (generalized), $q(x)$ is still well-defined by (2.10), as can be seen by approximating it with smooth flows (see e.g. [**17**, Lemma 2.13]).

To prove (2.9), by virtue of (2.10) it suffices to show that

$$\nabla q(x) = O(|x|^{-2}) \qquad as\ |x|\to\infty. \tag{2.11}$$

Let $R>0$ such that $B(0, R)\supset S$ ($B(0, R)$ denotes the ball with center 0 and radius R). Let a cut-off function $\eta_R^*(x)$ be

$$\eta_R^*(x) = \eta_R(|x|), \tag{2.12}$$

where $\eta_R(r)\in C^\infty[0, \infty)$ such that $\eta_R(r)=1$ if $r\geqq 2R$, $\eta_R(r)=0$ if $r\leqq R$. We may regard $q^*(x)=\eta_R^*(x)q(x)$ as a function on $\boldsymbol{R}^3$. Define $\bar{q}(x)$ on $\boldsymbol{R}^3$ by

$$\begin{aligned}\frac{\partial}{\partial x_j}\bar{q}(x) &= -\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{x_j-y_j}{|x-y|^3}\Delta q^*(y)dy \\ &= -\frac{1}{4\pi}\int_{B(0,2R)\setminus B(0,R)}\frac{x_j-y_j}{|x-y|^3}(\Delta\eta_R^*\cdot q+2\nabla\eta_R^*\cdot\nabla q)dy \qquad (j=1, 2, 3).\end{aligned}$$

(We note that $\Delta q=\operatorname{div}(\nabla q)=\operatorname{div}\boldsymbol{u}=0$.) Then $\partial\bar{q}/\partial x_j-\partial q^*/\partial x_j$ is harmonic on $\boldsymbol{R}^3$ and

$$\frac{\partial}{\partial x_j}\bar{q}(x) = O(|x|^{-2}) \qquad \text{as } |x|\to\infty. \tag{2.13}$$

On the other hand, we see that $\lim_{|x|\to\infty}(\partial/\partial x_j)q^*(x)=0$ because $\lim_{|x|\to\infty}\boldsymbol{u}(x)=0$. This implies by virtue of the Liouville theorem for harmonic functions that

$\partial\bar{q}/\partial x_j-\partial q^*/\partial x_j\equiv 0$ $(j=1, 2, 3)$. Hence (2.11) follows from (2.13).

COROLLARY 2.8. *Let* $\boldsymbol{u}$ *be a harmonic field* (*i.e.* $\operatorname{rot}\boldsymbol{u}=0$, $\operatorname{div}\boldsymbol{u}=0$) *satisfying* $\boldsymbol{u}\cdot\boldsymbol{n}|_S=0$ *and tending to zero at infinity. Then* $\boldsymbol{u}=0$.

PROOF. It follows from the above lemma that there is a scalar function q such that $\nabla q=\boldsymbol{u}$ and

$$\Delta q=\operatorname{div}\nabla q=0\,; \qquad \left.\frac{\partial q}{\partial n}\right|_S=\boldsymbol{u}\cdot\boldsymbol{n}|_S=0\,; \qquad \lim_{|x|\to\infty} q(x)=\text{const.}$$

Therefore we see that $q\equiv$const. by the uniqueness theorem for the exterior Neumann problem. Hence $\boldsymbol{u}=0$.

Similar proof applies to the following cases.

LEMMA 2.9. (i) *If a harmonic field* $\boldsymbol{u}$ *on* $\boldsymbol{R}^3$ *tends to zero at infinity, then* $\boldsymbol{u}=0$. (ii) *When* Ω *is a bounded and simply connected domain, a harmonic field* $\boldsymbol{u}$ *such that* $\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega}=0$ *satisfies that* $\boldsymbol{u}=0$ *in* Ω.

Next we shall study the following elliptic system:

$$\operatorname{rot}\boldsymbol{v}=\boldsymbol{w}, \qquad \operatorname{div}\boldsymbol{v}=0, \qquad \boldsymbol{v}\cdot\boldsymbol{n}|_S=0. \tag{2.14}$$

To this end we introduce subspaces of $M^p_{s,\lambda}$.

DEFINITION. Let p and δ be as in Theorem 1.1 and $s\geqq 1$. We define

$$X^p_{s,\delta+1}=\{\boldsymbol{v}\in M^p_{s,\delta+1} : \operatorname{div}\boldsymbol{v}=0,\ \boldsymbol{v}\cdot\boldsymbol{n}|_S=0\},$$
$$Y^p_{1,\delta+2}=\left\{\boldsymbol{w}\in M^p_{1,\delta+2} : \operatorname{div}\boldsymbol{w}=0,\ \int_{S_j}(\boldsymbol{w}\cdot\boldsymbol{n})dS=0\ (j=1,\cdots,m)\right\}.$$

REMARK. We can easily see that

$$\{\operatorname{rot}\boldsymbol{v} : \operatorname{rot}\boldsymbol{v}\in M^p_{1,\delta+2}\}\subset Y^p_{1,\delta+2}. \tag{2.15}$$

We have the following inequality.

LEMMA 2.10. *There is a positive constant* c_2 *depending only on* Ω *and* s $(s\geqq 1)$ *such that*

$$|\boldsymbol{v}|_{p,s,\delta+1}\leqq c_2|\operatorname{rot}\boldsymbol{v}|_{p,s-1,\delta+2} \qquad \textit{for all } \boldsymbol{v}\in X^p_{s,\delta+1}. \tag{2.16}$$

(The proof is given in Subsection 2.3.)

PROPOSITION 2.11. *Let* $\boldsymbol{w}\in Y^p_{1,\delta+2}$. *Then there exists a unique solution* $\boldsymbol{v}\in X^p_{2,\delta+1}$ *of*

$$\operatorname{rot}\boldsymbol{v}=\boldsymbol{w}. \tag{2.17}$$

PROOF. The uniqueness follows from (2.4) and Corollary 2.8. To prove the existence, we need to show that $\boldsymbol{w}$ can be extended to a solenoidal vector $\tilde{\boldsymbol{w}}\in M^p_{1,\delta+2}(\boldsymbol{R}^3)$. Following Itô [12, Lemma 2.9] we shall construct vector functions $\boldsymbol{w}_j$ $(j=1, \cdots, m)$ such that

$$\boldsymbol{w}_j \in W^{1,p}(O_j), \qquad \operatorname{div}\boldsymbol{w}_j = 0; \qquad \boldsymbol{w}_j|_{S_j} = \boldsymbol{w}|_{S_j}. \tag{2.18}$$

It follows from the definition of $Y^p_{1,\delta+2}$ that there is a solution $q_j\in W^{2,p}(O_j)$ of the Neumann problem:

$$-\Delta q_j = 0 \quad \text{on } O_j, \qquad \left.\frac{\partial q_j}{\partial n}\right|_{S_j} = (\boldsymbol{w}|_{S_j})\cdot\boldsymbol{n}|_{S_j} \qquad \text{for each } j \tag{2.19}$$

(see Agmon, Douglis and Nirenberg [1]). Using the inverse theorem on traces (see Besov, Il'in and Nikol'skii [3, Theorem 25.2]), we have $\boldsymbol{u}_j\in W^{2,p}(O_j)$ $(j=1, \cdots, m)$ satisfying the boundary condition:

$$\left.\frac{\partial \boldsymbol{u}_j}{\partial n}\right|_{S_j} = \boldsymbol{w}|_{S_j}-(\nabla q_j)|_{S_j}; \qquad \boldsymbol{u}_j|_{S_j} = 0. \tag{2.20}$$

Furthermore, there exist smooth scalar functions φ_j such that

$$\nabla\varphi_j = \boldsymbol{n} \qquad \text{on } S_j \quad (j=1, \cdots, m). \tag{2.21}$$

We put

$$\boldsymbol{w}_j = \operatorname{rot}(\nabla\varphi_j\wedge\boldsymbol{u}_j)+\nabla q_j, \tag{2.22}$$

where $\boldsymbol{u}\wedge\boldsymbol{v}$ denotes the 'vector product' of $\boldsymbol{u}$ with $\boldsymbol{v}$. Then we have

$$\boldsymbol{w}_j = (\nabla\varphi_j\cdot\nabla)\boldsymbol{u}_j-(\boldsymbol{u}_j\cdot\nabla)\nabla\varphi_j+(\operatorname{div}\boldsymbol{u}_j)\nabla\varphi_j-(\Delta\varphi_j)\boldsymbol{u}_j+\nabla q_j \qquad \text{(Itô's identity)}.$$

Hence, since it follows from (2.19) and (2.20) that $(\operatorname{div}\boldsymbol{u}_j)|_{S_j}=0$, we can easily see that (2.18) holds.

Now we define a vector function $\tilde{\boldsymbol{w}}$ on $\boldsymbol{R}^3$ by

$$\tilde{\boldsymbol{w}} = \begin{cases} \boldsymbol{w} & \text{in } \Omega \\ \boldsymbol{w}_j & \text{on } O_j \quad (j=1, \cdots, m). \end{cases} \tag{2.23}$$

Then (2.18) implies that $\tilde{\boldsymbol{w}}\in M^p_{1,\delta+2}(\boldsymbol{R}^3)$ and $\operatorname{div}\tilde{\boldsymbol{w}}=0$. From Lemma 2.4, it follows that there exists a solution $\boldsymbol{u}\in M^p_{3,\delta}(\boldsymbol{R}^3)$ of $-\Delta\boldsymbol{u}=\tilde{\boldsymbol{w}}$. In addition, we can see that $\operatorname{div}\boldsymbol{u}=0$ since $\tilde{\boldsymbol{w}}$ is solenoidal. Writing $\bar{\boldsymbol{v}}=\operatorname{rot}(\boldsymbol{u}|_\Omega)$, we have

$$\bar{\boldsymbol{v}} \in M^p_{2,\delta+1}, \qquad \operatorname{rot}\bar{\boldsymbol{v}} = \boldsymbol{w}; \qquad \operatorname{div}\bar{\boldsymbol{v}} = 0. \tag{2.24}$$

Lemma 2.6 yields that there is a solution q of the Neumann problem:

$$q \in M^p_{3,\delta}, \qquad -\Delta q = 0; \qquad \left.\frac{\partial q}{\partial n}\right|_S = \bar{\boldsymbol{v}}\cdot\boldsymbol{n}|_S. \tag{2.25}$$

We put

(2.26) $$v=\bar{v}-\nabla q.$$

Then it follows from (2.24) and (2.25) that $\boldsymbol{v}$ belongs to $X^p_{2,\delta+1}$ and is a solution of (2.17).

DEFINITION. Let $\boldsymbol{w}\in Y^p_{1,\delta+2}$. Let $\boldsymbol{v}\in X^p_{2,\delta+1}$ be the solution of (2.17). We define the operator F as

(2.27) $$F(\boldsymbol{w})=\boldsymbol{v}.$$

Then Lemma 2.10 implies that

(2.28) $$|F(\boldsymbol{w})|_{p,2,\delta+1}\leqq c_2|\boldsymbol{w}|_{p,1,\delta+2}.$$

Furthermore, we have the following corollary.

COROLLARY 2.12. *The operator F is a continuous map from $C([0,T];Y^p_{1,\delta+2})$ to $C([0,T];X^p_{2,\delta+1})$ with*

(2.29) $$\|F(\boldsymbol{w})\|_{p,2,\delta+1,T}\leqq c_2\|\boldsymbol{w}\|_{p,1,\delta+2,T}\qquad \textit{for any } \boldsymbol{w}\in C([0,T];Y^p_{1,\delta+2}).$$

Using F we shall rewrite the initial condition in the form that is easier to handle.

LEMMA 2.13. *The initial velocity $\boldsymbol{v}_0$ satisfies*

(2.30) $$\boldsymbol{v}_0-\boldsymbol{v}_\infty\in M^p_{2,\delta+1}.$$

In addition, there is a constant c_3 depending only on $\boldsymbol{v}_\infty$ and Ω such that

(2.31) $$|\boldsymbol{v}_0|_{1,\infty}+c_1|D^2\boldsymbol{v}_0|_{p,0,\delta+3}\leqq 2c_1c_2|\mathrm{rot}\,\boldsymbol{v}_0|_{p,1,\delta+2}+c_3.$$

PROOF. Let V be a harmonic function satisfying

(2.32) $$\left.\frac{\partial V}{\partial n}\right|_S=\boldsymbol{v}_\infty\cdot\boldsymbol{n}|_S,\qquad \lim_{|x|\to\infty}V(x)=0.$$

Then $\boldsymbol{v}_0$ is represented as

(2.33) $$\boldsymbol{v}_0=\boldsymbol{v}_\infty-\nabla V+F(\mathrm{rot}\,\boldsymbol{v}_0).$$

Indeed, putting $\boldsymbol{u}=\boldsymbol{v}_0-\boldsymbol{v}_\infty+\nabla V-F(\mathrm{rot}\,\boldsymbol{v}_0)$ and using (2.4), (2.27) and (2.32), we see that $\boldsymbol{u}$ is a harmonic field satisfying $\boldsymbol{u}\cdot\boldsymbol{n}|_S=0$ and $\lim_{|x|\to\infty}\boldsymbol{u}=0$. Hence $\boldsymbol{u}=0$ follows from Corollary 2.8. This yields (2.33). As is well known, the harmonic function V satisfying (2.32) is given by the single layer potential. This implies that $D^\alpha V(x)=O(|x|^{-(|\alpha|+1)})$ as $|x|\to\infty$ ($|\alpha|=0,1,\cdots$), which yields $\nabla V\in M^p_{2,\delta+1}$. Hence we obtain (2.30). It follows from (2.33), (2.3) and (2.28) that

$$\begin{aligned}&|\boldsymbol{v}_0|_{1,\infty}+c_1|D^2\boldsymbol{v}_0|_{p,0,\delta+3}\\&\leqq 2c_1c_2|\mathrm{rot}\,\boldsymbol{v}_0|_{p,1,\delta+2}+|\nabla V|_{1,\infty}+c_1|D^2(\nabla V)|_{p,0,\delta+3}+\boldsymbol{v}_\infty.\end{aligned}$$

Noting that V depends only on $\boldsymbol{v}_\infty$ and Ω we have (2.31).

2.3. Proof of Lemma 2.10. In order to obtain the inequality (2.16) we need estimates in bounded domains.

LEMMA 2.14. *Let D be a bounded and simply connected domain with smooth boundary. Let s be a non-negative integer and let $p\geqq 2$ for $s=0$ and $p>1$ for $s\geqq 1$. Then there is a positive constant c depending only on s and D such that*

$$|\boldsymbol{v}|_{s+1,p}\leqq c(|\mathrm{rot}\,\boldsymbol{v}|_{s,p}+|\mathrm{div}\,\boldsymbol{v}|_{s,p}+|\boldsymbol{v}\cdot\boldsymbol{n}|_{W^{s+1-1/p,p}(\partial D)}) \quad \textit{for all } \boldsymbol{v}\in W^{s+1,p}(D). \tag{2.34}$$

REMARK. When $p=2$, see Foias and Temam ([**9**, Proposition 1.4]).

PROOF. For $s\geqq 1$, we know by [**4**, Lemma 5] that

$$|\boldsymbol{v}|_{s+1,p}\leqq c(|\mathrm{rot}\,\boldsymbol{v}|_{s,p}+|\mathrm{div}\,\boldsymbol{v}|_{s,p}+|\boldsymbol{v}|_{s,p}+|\boldsymbol{v}\cdot\boldsymbol{n}|_{W^{s+1-1/p,p}(\partial D)}) \quad \text{for all } \boldsymbol{v}\in W^{s+1,p}(D). \tag{2.35}$$

Hence since the imbedding of $W^{s+1,p}(D)\to W^{s,p}(D)$ is compact, it follows from Lemma 2.9 that (2.35) can be replaced by (2.34).

We shall prove (2.34) for $s=0$. Put

$$Y^p=\left\{\boldsymbol{w}\in L^p(D)\ :\ \mathrm{div}\,\boldsymbol{w}=0,\ \int_{\partial D_j}(\boldsymbol{w}\cdot\boldsymbol{n})dS=0 \text{ for all } j\right\},$$

where ∂D_j denotes a connected component of ∂D. The results of Fujiwara and Morimoto ([**10**, Lemmas 1 and 3]) yield that Y^p is a closed subspace of $L^p(D)$. Moreover, any $\boldsymbol{w}\in Y^p$ satisfies $(\boldsymbol{w},\boldsymbol{h})=0$ for all $\boldsymbol{h}\in\{\mathrm{rot}\,\boldsymbol{h}=0,\ \mathrm{div}\,\boldsymbol{h}=0,\ \boldsymbol{h}_\tau|_{\partial D}=0\}$ (see Foias and Temam [**9**, Propositions 2 and 3]). Hence we can use the result of Morrey ([**18**, Theorem 7.7.4]): for any $\boldsymbol{w}\in Y^p$, there is a solution $\boldsymbol{u}$ of $-\Delta\boldsymbol{u}=\boldsymbol{w}$; $\boldsymbol{u}_\tau|_{\partial D}=0$; $(\mathrm{div}\,\boldsymbol{u})|_{\partial D}=0$ such that $\boldsymbol{u}$, $\mathrm{rot}\,\boldsymbol{u}$ and $\mathrm{div}\,\boldsymbol{u}$ belong to $W^{1,p}(D)$. Writing $F_D(\boldsymbol{w})=\mathrm{rot}\,\boldsymbol{u}$, we can easily see that $F_D(\boldsymbol{w})$ satisfies

$$\mathrm{rot}\,F_D(\boldsymbol{w})=\boldsymbol{w},\qquad \mathrm{div}\,F_D(\boldsymbol{w})=0,\qquad F_D(\boldsymbol{w})\cdot\boldsymbol{n}|_{\partial D}=0. \tag{2.36}$$

It follows from Lemma 2.9 (ii) that F_D is a closed operator from Y^p to $\{\boldsymbol{v}\in W^{1,p}(D)\ :\ \mathrm{div}\,\boldsymbol{v}=0,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D}=0\}$ (a closed subspace of $W^{1,p}(D)$). Hence by the closed graph theorem we have

$$|F_D(\boldsymbol{w})|_{1,p}\leqq c|\boldsymbol{w}|_p. \tag{2.37}$$

Given any $\boldsymbol{v}\in W^{1,p}(D)$, there is a solution $\pi_v\in W^{2,p}(D)$ of the Neumann problem:

$$\Delta\pi_v=\mathrm{div}\,\boldsymbol{v},\qquad \left.\frac{\partial\pi_v}{\partial n}\right|_{\partial D}=\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D} \tag{2.38}$$

with

(2.39) $$|\nabla\pi_v|_{1,p} \leqq c(|\mathrm{div}\,\boldsymbol{v}|_p + |\boldsymbol{v}\cdot\boldsymbol{n}|_{W^{1-1/p,p}(\partial D)})$$

(see Agmon, Douglis and Nirenberg [**1**]). Since $\mathrm{rot}\,\boldsymbol{v}\in Y^p$ holds for any $\boldsymbol{v}\in W^{1,p}(D)$ (see [**9**, Proposition 1.3]), it follows from (2.36), (2.38) and Lemma 2.9 (ii) that $\boldsymbol{v}=F_D(\mathrm{rot}\,\boldsymbol{v})+\nabla\pi_v$. Hence combining (2.37) and (2.39) we obtain (2.34) for $s=0$.

PROOF OF LEMMA 2.10. Assume that the inequality (2.16) is false. Then there is a sequence $\{\boldsymbol{v}_n\}\subset X^p_{s,\delta+1}$ such that

(2.40) $$\begin{aligned} &\text{(i)} && |\boldsymbol{v}_n|_{p,s,\delta+1} = 1, \\ &\text{(ii)} && |\mathrm{rot}\,\boldsymbol{v}_n|_{p,s-1,\delta+2} \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Letting η^*_R be as (2.12), we may regard $\eta^*_R\cdot\boldsymbol{v}_n$ as a function on $\boldsymbol{R}^3$. We have $\mathrm{rot}(\eta^*_R\cdot\boldsymbol{v}_n)=\eta^*_R\cdot\mathrm{rot}\,\boldsymbol{v}_n+(\nabla\eta^*_R)\wedge\boldsymbol{v}_n$ and $\mathrm{div}(\eta^*_R\cdot\boldsymbol{v}_n)=\nabla\eta^*_R\cdot\boldsymbol{v}_n$ ($\mathrm{div}\,\boldsymbol{v}_n=0$). Noting that $\boldsymbol{u}_n\equiv(\nabla\eta^*_R)\wedge\boldsymbol{v}_n$ and $\varphi_n\equiv\nabla\eta^*_R\cdot\boldsymbol{v}_n$ have the same compact support in $B_{2R\setminus R}\equiv B(0, 2R)\setminus B(0, R)$, we may regard $\boldsymbol{u}_n$ and φ_n as functions on $B_{2R\setminus R}$. Then Corollary 2.5 implies that

(2.41) $$\begin{aligned} |\eta^*_R\cdot\boldsymbol{v}_n|_{p,s,\delta+1,\boldsymbol{R}^3} \leqq c(&|\eta^*_R\cdot\mathrm{rot}\,\boldsymbol{v}_n|_{p,s-1,\delta+2,\boldsymbol{R}^3} \\ &+|\boldsymbol{u}_n|_{W^{s-1,p}(B_{2R\setminus R})}+|\varphi_n|_{W^{s-1,p}(B_{2R\setminus R})}). \end{aligned}$$

Since (2.40) (i) implies that $\{\boldsymbol{u}_n\}$ and $\{\varphi_n\}$ are uniformly bounded in $W^{s,p}(B_{2R\setminus R})$, it follows from the Rellich theorem that there are subsequences $\{\boldsymbol{u}_{n_k}\}$ and $\{\varphi_{n_k}\}$ which are Cauchy sequences in $W^{s-1,p}(B_{2R\setminus R})$. On the other hand, (2.40) (ii) implies that

$$|\eta^*_R\cdot\mathrm{rot}\,\boldsymbol{v}_n|_{p,s-1,\delta+2,\boldsymbol{R}^3} \leqq c|\mathrm{rot}\,\boldsymbol{v}_n|_{p,s-1,\delta+2} \to 0 \quad \text{as } n\to\infty.$$

Hence it follows from (2.41) that $\{\eta^*_R\cdot\boldsymbol{v}_{n_k}\}$ is a Cauchy sequence in $M^p_{s,\delta+1}$.

Using Lemma 2.14 and by the similar argument to the above we can prove that the sequence $\{(1-\eta^*_R)\boldsymbol{v}_n\}$ has a subsequence which converges strongly in $M^p_{s,\delta+1}$. Hence we see that there is a subsequence which converges strongly to $\boldsymbol{v}^*$ in $M^p_{s,\delta+1}$. This, together with (2.40) (i), implies that $|\boldsymbol{v}^*|_{p,s,\delta+1}=1$. Furthermore, since $X^p_{s,\delta+1}$ is closed in $M^p_{s,\delta+1}$, we have $\mathrm{rot}\,\boldsymbol{v}^*=0$, $\mathrm{div}\,\boldsymbol{v}^*=0$, $\boldsymbol{v}^*\cdot\boldsymbol{n}|_S=0$ and $\lim_{|x|\to\infty}\boldsymbol{v}^*=0$. This contradicts Corollary 2.8. This completes the proof.

§3. Construction of solutions.

In this section we shall prove Theorem 1.1. To this end we consider the vorticity equation obtained by taking the rotation of the first equation of (1) and using $\mathrm{div}\,\boldsymbol{v}=0$. We note that the following identity holds:

(3.1) $$\mathrm{rot}[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] = (\boldsymbol{v}\cdot\nabla)\mathrm{rot}\,\boldsymbol{v}-(\mathrm{rot}\,\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\mathrm{div}\,\boldsymbol{v}\cdot\mathrm{rot}\,\boldsymbol{v}.$$

Precisely, in Subsections 3.1 and 3.2 we shall construct a solution $\{v, w\}$ of the following equations:

$$\mathrm{rot}\, v = w, \tag{3.2}$$

$$\mathrm{div}\, v = 0, \tag{3.3}$$

$$\frac{\partial w}{\partial t}+(v\cdot\nabla)w-(w\cdot\nabla)v = \mathrm{rot} f \tag{3.4}$$

with the boundary conditions (2) for v and the initial condition

$$w|_{t=0} = \mathrm{rot}\, v_0 \tag{3.5}$$

for w. (Note that the initial condition (3) for v is not required in the construction, but will be satisfied automatically (see Subsection 3.3).)

Theorem 1.1 will be proved in Subsection 3.3.

3.1. Preliminaries for iteration. A solution $\{v, w\}$ of the above equations (3.2)-(3.5), (2) will be constructed by means of iteration (Subsection 3.2). In the iteration these equations will be split into two systems of linear equations (one is elliptic and the other hyperbolic) and these systems will be solved successively in alternation. In this subsection we deal with these linear systems.

The first system consists of (3.2) and (3.3) with the boundary condition (2). Here we regard $w=w(\cdot, t)$ as given.

PROPOSITION 3.1. *Suppose that* $w\in C([0, T]; Y^p_{1,\delta+2})$. *Let* F *be the operator defined by* (2.27). *Then*

$$v = F(w)+v_0-F(\mathrm{rot}\, v_0) \tag{3.6}$$

gives a unique solution of (3.2), (3.3) *under the boundary condition* (2). *The solution* v *satisfies*

$$v-v_\infty \in C([0, T]; M^p_{2,\delta+1}), \tag{3.7}$$

$$\|v-v_\infty\|_{p,2,\delta+1,T} \leqq c_2(\|w\|_{p,1,\delta+2,T}+|\mathrm{rot}\, v_0|_{p,1,\delta+2})+|v_0-v_\infty|_{p,2,\delta+1}, \tag{3.8}$$

$$\|v\|_{\infty,T}+\|Dv\|_{\infty,T}+c_1\|D^2v\|_{p,0,\delta+3,T} \leqq c_1c_2(3\|w\|_{p,1,\delta+2,T}+4|\mathrm{rot}\, v_0|_{p,1,\delta+2})+c_3, \tag{3.9}$$

where $v_0=v(\cdot, 0)$ *is the initial velocity.*

PROOF. The uniqueness follows from Corollary 2.8. It follows from (2.27) that (3.6) gives a solution of (3.2), (3.3) and (2). Using (2.3) and (2.29) we have

$$\begin{aligned}\|v\|_{\infty,T}+\|Dv\|_{\infty,T} &\leqq c_1(2\|F(w)\|_{p,2,\delta+1,T}+|F(\mathrm{rot}\, v_0)|_{p,2,\delta+1})+|v_0|_{1,\infty}\\ &\leqq c_1c_2(2\|w\|_{p,1,\delta+2,T}+|\mathrm{rot}\, v_0|_{p,1,\delta+2})+|v_0|_{1,\infty}.\end{aligned}$$

Similarly (2.29) implies that

$$\|D^2 \boldsymbol{v}\|_{p,0,\delta+3,T} \leqq c_2(\|\boldsymbol{w}\|_{p,1,\delta+2,T} + |\mathrm{rot}\, \boldsymbol{v}_0|_{p,1,\delta+2}) + |D^2 \boldsymbol{v}_0|_{p,0,\delta+3}.$$

Hence by (2.31) we have (3.9). Similarly (3.8) follows from (2.29).

The second system is the vorticity equation (3.4) with the initial condition (3.5). Here we regard $\boldsymbol{v}$ and $\boldsymbol{f}$ as given. We construct a solution following Swann's argument (see [**21**], also see Kato [**14**] and Judovič [**13**]) and obtain the following result.

PROPOSITION 3.2. *Let* $\boldsymbol{v} \in \boldsymbol{v}_\infty + C([0, T]; M^p_{2,\delta+1})$ *be such that* $\mathrm{div}\, \boldsymbol{v} = 0$ *and* $\boldsymbol{v} \cdot \boldsymbol{n}|_S = 0$. *Then there exists a unique solution* $\boldsymbol{w} \in C([0, T]; Y^p_{1,\delta+2})$ *of* (3.4) *and* (3.5) *satisfying*

$$(3.10)\quad |\boldsymbol{w}(t)|_{p,0,\delta+2} \leqq 2^{\delta+2-1/p}\{1+(T\|\boldsymbol{v}\|_{\infty,T})^{\delta+2}\}\Big(|\mathrm{rot}\, \boldsymbol{v}_0|_{p,0,\delta+2} + \int_0^t |\mathrm{rot}\, \boldsymbol{f}(s)|_{p,0,\delta+2} ds + \|D\boldsymbol{v}\|_{\infty,T} \int_0^t |\boldsymbol{w}(s)|_{p,0,\delta+2} ds\Big) \qquad t \in [0, T].$$

This solution also satisfies

$$(3.11)\qquad |D\boldsymbol{w}(t)|_{p,0,\delta+3} \leqq 2^{\delta+3-1/p} \cdot \exp(3T\|D\boldsymbol{v}\|_{\infty,T}) \cdot \{1+(T\|\boldsymbol{v}\|_{\infty,T})^{\delta+3}\}$$
$$\times\Big(|D(\mathrm{rot}\, \boldsymbol{v}_0)|_{p,0,\delta+3} + \int_0^t |D(\mathrm{rot}\, \boldsymbol{f}(s))|_{p,0,\delta+3} ds$$
$$+ \|D\boldsymbol{v}\|_{\infty,T} \int_0^t |D\boldsymbol{w}(s)|_{p,0,\delta+3} ds$$
$$+ c_1 \int_0^t |D^2 \boldsymbol{v}(s)|_{p,0,\delta+3} ds \cdot \|\boldsymbol{w}\|_{p,1,\delta+2,t}\Big) \qquad t \in [0, T].$$

The proof of this proposition is divided into several parts.

(I) Define a family of curves $(X(x, t; s), s)$ in $\bar{\Omega} \times [0, T]$ by

$$(3.12)\qquad \begin{cases} \dfrac{d}{ds} X(x, t; s) = \boldsymbol{v}(X(x, t; s), s) & 0 \leqq s,\ t \leqq T \\ X(x, t; t) = x. \end{cases}$$

Then, since $\boldsymbol{v} \in \boldsymbol{v}_\infty + C([0, T]; M^p_{2,\delta+1}) \subset C([0, T]; C^{1+\theta}(\bar{\Omega}))$,

$$(3.13)\qquad X(x, t; s) \in C^1([0, T]; C^{1+\theta}(\bar{\Omega})) \qquad (0 < \theta < 1 - 3/p).$$

Accordingly, (3.12) gives a unique local curve in $\bar{\Omega} \times [0, T]$ for each $(x, t) \in \bar{\Omega} \times [0, T]$. It follows from $\boldsymbol{v} \cdot \boldsymbol{n}|_S = 0$ that

$$(3.14)\qquad (X(x, t; s), s) \text{ in } \Omega \times [0, T] \text{ cannot reach } S \times [0, T].$$

Hence all solutions of (3.12) in $\bar{\Omega} \times [0, T]$ exist globally (see Kato [**14**, Lemma 2.2]). Furthermore the definition (3.12) and the uniqueness of stream lines imply that

$$(3.15)\qquad X(X(x, t; s), s; \tau) = X(x, t; \tau) \qquad \text{for } 0 \leqq s,\ t,\ \tau \leqq T.$$

We put $G(x, t; s)=D(X(x, t; s))/D(x)$, where $D(\boldsymbol{u}(x))/D(x)$ denotes the Jacobi matrix of $\boldsymbol{u}(x)$. Then differentiating (3.12) with respect to x_j $(j=1, 2, 3)$, we see that $G(x, t; s)$ satisfies the following equation:

$$\begin{cases} \dfrac{d}{ds}G(x, t; s) = \dfrac{D(\boldsymbol{v}(y, s))}{D(y)}\Big|_{y=X(x,t;s)} \cdot G(x, t; s) \\ G(x, t; t) = E \text{ (identity matrix)}. \end{cases} \tag{3.16}$$

Since $\operatorname{div}\boldsymbol{v}=0$ on $\Omega\times[0, T]$, (3.16) implies that

$$\det G(x, t; s) = 1. \tag{3.17}$$

We now define

$$\boldsymbol{w}(x, t) = G(x, t; 0)^{-1}\boldsymbol{a}(X(x, t; 0))+\int_0^t G(x, t; \tau)^{-1}\boldsymbol{b}(X(x, t; \tau), \tau)d\tau, \tag{3.18}$$

where $\boldsymbol{a}\equiv\operatorname{rot}\boldsymbol{v}_0$ and $\boldsymbol{b}\equiv\operatorname{rot}\boldsymbol{f}$.

In the following parts we shall prove that $\boldsymbol{w}$ defined by (3.18) satisfies the required properties.

(II) *Let* $u(x)\in M^p_{1,\delta+2}$. *Then* $u(X(x, t; s))$ *belongs to* $M^p_{1,\delta+2}$ *for fixed* $s, t\in [0, T]$ *and satisfies*

$$|u(X(\cdot, t; s))|_{p,0,\delta+2} \leqq 2^{\delta+2-1/p}\{1+(T\|\boldsymbol{v}\|_{\infty,T})^{\delta+2}\}|u|_{p,0,\delta+2}, \tag{3.19}$$

$$|Du(X(\cdot, t; s))|_{p,0,\delta+3} \leqq 2^{\delta+3-1/p}\cdot\exp(3T\|D\boldsymbol{v}\|_{\infty,T}) \cdot\{1+(T\|\boldsymbol{v}\|_{\infty,T})^{\delta+3}\}|Du|_{p,0,\delta+3}. \tag{3.20}$$

PROOF. It follows from (3.12) that

$$|X(x, s; t)| \leqq |x|+T\|\boldsymbol{v}\|_{\infty,T}. \tag{3.21}$$

This implies that

$$\sigma(X(x, s; t))^{\lambda p} \leqq 2^{\lambda p-1}\{\sigma(x)^{\lambda p}+(T\|\boldsymbol{v}\|_{\infty,T})^{\lambda p}\} \qquad (\lambda p\geqq 2). \tag{3.22}$$

Therefore, using the change of variables $y=X(x, t; s)$ and using (3.15) and (3.17) we have

$$\begin{aligned} |\sigma(\cdot)^{\delta+2}u(X(\cdot, t; s))|_p &= |\sigma(X(\cdot, s; t))^{\delta+2}u(\cdot)|_p \\ &\leqq 2^{\delta+2-1/p}\{1+(T\|\boldsymbol{v}\|_{\infty,T})^{\delta+2}\}|u|_{p,0,\delta+2}. \end{aligned}$$

This proves (3.19). By the chain rule we have

$$\frac{\partial u}{\partial x_j}(X(x, t; s)) = \sum_{k=1}^{3} u_{x_k}(X(x, t; s))\cdot\frac{\partial X^k}{\partial x_j}(x, t; s) \qquad (j=1, 2, 3),$$

where $u_{x_k}=\partial u/\partial x_k$. By the Gronwall inequality we deduce from (3.16)

$$\left|\frac{\partial X}{\partial x_j}(\cdot, t; s)\right|_\infty \leqq \exp(3T\|D\boldsymbol{v}\|_{\infty,T}) \qquad (j=1, 2, 3) \qquad \text{for } s, t\in[0, T].$$

Using (3.22) we can prove that

$$|u_{x_k}(X(\cdot, t; s))|_{p,0,\delta+3} \leqq 2^{\delta+3-1/p}\{1+(T\|v\|_{\infty,T})^{\delta+3}\}|u_{x_k}(\cdot)|_{p,0,\delta+3}$$

in a similar way to the proof of (3.19). Hence we have (3.20).

(III) *Let* $w(x, t)$ *be defined by* (3.17). *Then*

$$w \in C([0, T]; M^p_{1,\delta+2}), \tag{3.23}$$

and w *satisfies the estimates* (3.10) *and* (3.11). *Moreover,* w *is a weak solution of* (3.4), *i.e. for any vector function* $u \in C^1_0(\bar{\Omega})$

$$\frac{d}{dt}(w, u) = (w, (v\cdot\nabla)u)+((w\cdot\nabla)v, u)+(b, u). \tag{3.24}$$

PROOF. We shall first show that

$$G(x, t; s)^{-1}-E \in C^1([0, T]; M^p_{1,\delta+2}). \tag{3.25}$$

It follows from (II) that $Dv(y, s)|_{y=X(x,t;s)} \in C([0, T]; M^p_{1,\delta+2})$. Accordingly, Lemma 2.3 implies that the operator: $z \mapsto (D(v(y, s))/D(y))|_{y=X(x,t;s)}\cdot z$ is bounded from $M^p_{1,\delta+2}$ into itself. Hence by (3.16) we have $G(x, t; s)-E \in C^1([0, T]; M^p_{1,\delta+2})$. On the other hand (3.17) implies that $G(x, t; s)^{-1}$ is the cofactor matrix of $G(x, t; s)$. Furthermore, a simple calculation yields that for any 3×3 matrix, $\mathrm{cof}(G)-E=\mathrm{cof}(G-E)-(G-E)+(\mathrm{trace}(G-E))E$, where $\mathrm{cof}(A)$ denotes the cofactor matrix of A. Hence by Lemma 2.3 we obtain (3.25). $w(x, t)$ is written as follows:

$$w = (G^{-1}-E)a(X)+a(X)+\int_0^t \{(G^{-1}-E)b(X)+b(X)\}d\tau.$$

Hence from (II), (3.25) and Lemma 2.3 we deduce (3.23).

We shall show that $w(X(x, t; s), s)$ satisfies the inhomogeneous ordinary differential equation:

$$\begin{cases} \dfrac{d}{ds}w(X(x, t; s), s) = (w(X(x, t; s), s)\cdot\nabla_y)v(y, s)|_{y=X(x,t;s)} \\ \qquad\qquad\qquad\qquad\qquad +b(X(x, t; s), s) \equiv I(x, t; s), \\ w(X(x, t; 0), 0) = a(X(x, t; 0)). \end{cases} \tag{3.26}$$

Since it follows from (3.15) and the chain rule that $G(X(x, t; s), s; \tau)G(x, t; s) = G(x, t; \tau)$, we have

$$G(X(x, t; s), s; \tau)^{-1} = G(x, t; s)G(x, t; \tau)^{-1}. \tag{3.27}$$

This, together with (3.15), implies that $w(X(x, t; s), s)$ can be written as

$$\begin{aligned} w(X(x, t; s), s) &= G(x, t; s)G(x, t; 0)^{-1}a(X(x, t; 0)) \\ &\quad +G(x, t; s)\int_0^s G(x, t; \tau)^{-1}b(X(x, t; \tau), \tau)d\tau. \end{aligned} \tag{3.28}$$

On the other hand, (3.16) means that $G(x, t; s)$ is a fundamental matrix solution of the homogeneous ordinary differential equation: $dz(s)/ds = (z(s)\cdot\nabla_y)v(y, s)|_{y=X(x,t;s)}$. Hence a well-known result from ordinary differential equations yields that $w(X(x, t; s), s)$ given by (3.28) is a solution of (3.26). Then (3.26) and the second equation of (3.12) imply that $w(x, t)$ is represented as

$$(3.29)\qquad w(x, t)-a(X(x, t; 0)) = \int_0^t \frac{d}{ds} w(X(x, t; s), s)ds = \int_0^t I(x, t; s)ds.$$

We shall now prove the estimates (3.10) and (3.11). (3.29) gives

$$(3.30)\qquad |w(\cdot, t)|_{p,0,\delta+2} \leqq |a(X(\cdot, t; 0))|_{p,0,\delta+2} + \left|\int_0^t I(\cdot, t; s)ds\right|_{p,0,\delta+2}.$$

It follows from (3.19) that

$$(3.31)\qquad |a(X(\cdot, t; 0))|_{p,0,\delta+2} \leqq 2^{\delta+2-1/p}\{1+(T\|v\|_{\infty,T})^{\delta+2}\}|a|_{p,0,\delta+2}.$$

Using the extended Minkowski inequality and (3.19) we have

$$\begin{aligned}(3.32)\qquad \left|\int_0^t I(\cdot, t; s)ds\right|_{p,0,\delta+2} &\leqq \int_0^t |I(\cdot, t; s)|_{p,0,\delta+2}ds \\ &\leqq 2^{\delta+2-1/p}\{1+(T\|v\|_{\infty,T})^{\delta+2}\}\Big(\int_0^t |b(\cdot, s)|_{p,0,\delta+2}ds \\ &\qquad + \|Dv\|_{\infty,T}\int_0^t |w(\cdot, s)|_{p,0,\delta+2}ds\Big).\end{aligned}$$

Combining (3.31) and (3.32) with (3.30), we have (3.10). Applying (3.20) to (3.29), (3.11) can be proved in a way similar to the proof of (3.10).

Finally we shall prove (3.24). Let $J_h = (1/h)(w(\cdot, t+h) - w(\cdot, t), u)$. We divide J_h into two parts:

$$\begin{aligned}(3.33)\qquad J_h &= \Big(\frac{1}{h}\{w(X(\cdot, t+h; t+h), t+h) - w(X(\cdot, t+h; t), t)\}, u\Big) \\ &\qquad + \frac{1}{h}(w(X(\cdot, t+h; t), t) - w(\cdot, t), u) \equiv J_{1h} + J_{2h}.\end{aligned}$$

(Note that $X(x, t+h; t+h) = x$.) It follows from (3.26) that

$$\begin{aligned}(3.34)\qquad \lim_{h\to 0} J_{1h} &= \Big(\Big(\frac{d}{ds} w(X(\cdot, t; s), s)\Big)\Big|_{s=t}, u\Big) \\ &= ((w\cdot\nabla)v + b, u).\end{aligned}$$

Changing variables $y = X(x, t+h; t)$ and using (3.15) and (3.17) we have $(w(X(\cdot, t+h; t), t), u(\cdot)) = (w(\cdot, t), u(X(\cdot, t; t+h)))$. Hence from (3.12) we see that

$$\begin{aligned}(3.35)\qquad \lim_{h\to 0} J_{2h} &= \lim_{h\to 0}\Big(w(\cdot, t), \frac{1}{h}\{u(X(\cdot, t; t+h)) - u(X(\cdot, t; t))\}\Big) \\ &= (w, (v\cdot\nabla)u).\end{aligned}$$

Combining (3.33), (3.34) and (3.35) we obtain

$$\frac{d}{dt}(\boldsymbol{w}, \boldsymbol{u}) = \lim_{h\to 0} J_h = ((\boldsymbol{w}\cdot\nabla)\boldsymbol{v}, \boldsymbol{u}) + (\boldsymbol{w}, (\boldsymbol{v}\cdot\nabla)\boldsymbol{u}) + (\boldsymbol{b}, \boldsymbol{u}).$$

This proves (3.24).

(IV) *For any scalar function* $\varphi \in C_0^1(\Omega)$,

$$(\boldsymbol{w}, \nabla\varphi) = 0. \tag{3.36}$$

PROOF. We shall prove that

$$I \equiv (G(\cdot, t; 0)^{-1}\boldsymbol{a}(X(\cdot, t; 0)), \nabla\varphi) = 0. \tag{3.37}$$

Using the change of variables $y = X(x, t; 0)$ and using (3.17), (3.27) and the second equation of (3.16), we have

$$\begin{aligned} I &= \int_\Omega G(X(x, 0; t), t; 0)^{-1}\boldsymbol{a}(x)\cdot\nabla_y\varphi(y)|_{y=X(x,0;t)}dx \\ &= \int_\Omega G(x, 0; t)\boldsymbol{a}(x)\cdot\nabla_y\varphi(y)|_{y=X(x,0;t)}dx \\ &= \int_\Omega \boldsymbol{a}(x)\cdot\nabla_x\varphi(X(x, 0; t))dx. \end{aligned} \tag{3.38}$$

By virtue of (3.13), (3.14) and (3.21) we note that $\varphi(X(x, 0; t)) \in C_0^1(\bar{\Omega})$ and $\varphi(X(x, 0; t))|_S = 0$. Hence, since $\operatorname{div}\boldsymbol{a} = \operatorname{div}(\operatorname{rot}\boldsymbol{v}_0) \equiv 0$, from an integration by parts we see that the right-hand side of (3.38) vanishes. This proves (3.37). Similarly we can prove that

$$\left(\int_0^t G(\cdot, t; s)^{-1}\boldsymbol{b}(X(\cdot, t; s), s)ds, \nabla\varphi\right) = 0.$$

Hence we obtain (3.36).

(V) *For all* $j = 1, \cdots, m$

$$\int_{S_j} (\boldsymbol{w}(\cdot, t)\cdot\boldsymbol{n})dS = 0 \qquad \textit{for all } t \in [0, T]. \tag{3.39}$$

PROOF. Let g_j $(j = 1, \cdots, m)$ be scalar functions on the boundary S such that

$$\begin{aligned} &\text{(i)} \quad \left(\int_S g_j(y)\frac{1}{|x-y|}d_yS\right)\Big|_{x\in S_k} = c_{jk} \quad (j, k = 1, \cdots, m), \\ &\qquad\qquad\qquad\qquad\qquad \text{where } c_{jk} \text{ are constants,} \\ &\text{(ii)} \quad \det(c_{jk})_{j,k=1,\cdots,m} \neq 0. \end{aligned} \tag{3.40}$$

(For the existence of such functions g_j, see e.g. Günter [**11**, Sections 11 and 13 in Chapter IV].) We put $e_j(x) = \int_S g_j(y)\frac{1}{|x-y|}d_yS$. Then we have

(3.41) $$(\mathrm{rot}\,\boldsymbol{v}_0,\ \nabla e_j)=0 \qquad \text{for all } j=1, \cdots, m.$$

Indeed, (2.15), (3.40) (i) and an integration by parts implies that

$$I_R \equiv \int_{B(0,R)} (\mathrm{rot}\,\boldsymbol{v}_0\cdot\nabla e_j)dx = \int_{|x|=R} e_j((\mathrm{rot}\,\boldsymbol{v}_0)\cdot\boldsymbol{n})dS.$$

Since it follows from (2.4) that $\mathrm{rot}\,\boldsymbol{v}_0(x)=O(|x|^{-(\delta+2)})$ as $|x|\to\infty$ and since $e_j(x)=O(|x|^{-1})$ as $|x|\to\infty$, we have $(\mathrm{rot}\,\boldsymbol{v}_0,\ \nabla e_j)=\lim_{R\to\infty}I_R=0$. Similarly we can prove that

(3.42) $$(\mathrm{rot}\,\boldsymbol{f}(\cdot,\ t),\ \nabla e_j)=0 \qquad (j=1, \cdots, m) \qquad \text{for all } t\in[0,\ T].$$

Since $\nabla e_j(x)=O(|x|^{-2})$ as $|x|\to\infty$, we note that $\boldsymbol{u}\in C_0^1(\bar{\Omega})$ in (3.24) can be replaced by ∇e_j. Hence (3.42) implies that

(3.43) $$\frac{d}{dt}(\boldsymbol{w},\ \nabla e_j)=(\boldsymbol{w},\ (\boldsymbol{v}\cdot\nabla)\nabla e_j)+((\boldsymbol{w}\cdot\nabla)\boldsymbol{v},\ \nabla e_j) \qquad \text{for all } j=1, \cdots, m.$$

We shall show that the right-hand side of (3.43) vanishes. Since it follows from $\boldsymbol{v}\cdot\boldsymbol{n}|_S=0$ and $(\nabla e_j)_\tau|_S=0$ that $(\boldsymbol{v}\cdot\nabla e_j)|_S=0$, an integration by parts and $\mathrm{div}\,\boldsymbol{w}=0$ yield

(3.44) $$((\boldsymbol{w}\cdot\nabla)\boldsymbol{v},\ \nabla e_j)=-(\boldsymbol{v},\ (\boldsymbol{w}\cdot\nabla)\nabla e_j).$$

(Note that $(\boldsymbol{w}(x,\ \cdot)\cdot\boldsymbol{v}(x,\ \cdot))\nabla e_j(x)=O(|x|^{-(\delta+4)})$ as $|x|\to\infty$ by virtue of (2.4).) Rewriting the right-hand side of (3.43) by (3.44), we have

(3.45) $$\frac{d}{dt}(\boldsymbol{w},\ \nabla e_j)=\int_\Omega \sum_{k,i} w^k v^i\Big(\frac{\partial^2 e_j}{\partial x_i \partial x_k}-\frac{\partial^2 e_j}{\partial x_k \partial x_i}\Big)dx=0 \qquad (j=1, \cdots, m).$$

(3.41) and (3.45) imply that $(\boldsymbol{w},\ \nabla e_j)=0$ for all $j=1, \cdots, m$. Hence noting that $e_j(x)\boldsymbol{w}(x,\ \cdot)=O(|x|^{-(\delta+3)})$ as $|x|\to\infty$ and using an integration by parts and $\mathrm{div}\,\boldsymbol{w}=0$, we have

$$0=(\boldsymbol{w},\ \nabla e_j)=\sum_{k=1}^m \int_{S_k} e_j(\boldsymbol{w}\cdot\boldsymbol{n})dS.$$

Hence we easily see that (3.40) (i) and (ii) imply (3.39).

(VI) We shall finally prove the uniqueness. Suppose that there is another solution $\bar{\boldsymbol{w}}\in C([0,\ T]\,;M_{0,\delta+2}^{p})$ of (3.4) and (3.5). Writing $\boldsymbol{w}^*=\boldsymbol{w}-\bar{\boldsymbol{w}}$ and using the stream line $X(x,\ t\,;s)$ defined by (3.12), we see that $\boldsymbol{w}^*(X(x,\ t\,;s),\ s)$ satisfies the following equation:

$$\frac{d}{ds}\boldsymbol{w}^*(X(x,\ t\,;s),\ s)=(\boldsymbol{w}^*(X(x,\ t\,;s),\ s)\cdot\nabla_y)\boldsymbol{v}(y,\ s)|_{y=X(x,t;s)}.$$

Hence repeating the arguments used to deduce (3.10) from (3.26) (in this case $\boldsymbol{a}\equiv 0$ and $\boldsymbol{b}\equiv 0$), the following inequality can be obtained:

$$|\boldsymbol{w}^*(t)|_{p,0,\delta+2} \leqq 2^{\delta+2-1/p}\{1+(T\|\boldsymbol{v}\|_{\infty,T})^{\delta+2}\}\|D\boldsymbol{v}\|_{\infty,T}\int_0^t |\boldsymbol{w}^*(s)|_{p,0,\delta+2}ds. \tag{3.46}$$

This, together with $\boldsymbol{w}^*(0)=0$, implies that $\boldsymbol{w}^*(\cdot, t)=0$.

3.2. Existence of solutions of the vorticity equation. We construct solutions of (3.2)-(3.5) and (2) by means of the following iterative process. The vectors $\boldsymbol{v}_0(x)$ and $\boldsymbol{w}_0(x)=\mathrm{rot}\,\boldsymbol{v}_0(x)$, which are the initial velocity and the initial vorticity respectively, are taken as the zeroth approximations. When the n-th approximation for the vorticity $\boldsymbol{w}_n(x, t)$ is known, then the n-th approximation for the velocity $\boldsymbol{v}_n(x, t)$ is determined as follows:

$$\mathrm{rot}\,\boldsymbol{v}_n = \boldsymbol{w}_n, \qquad \mathrm{div}\,\boldsymbol{v}_n = 0, \qquad \boldsymbol{v}_n\cdot\boldsymbol{n}|_S = 0, \qquad \lim_{|x|\to\infty} \boldsymbol{v}_n = \boldsymbol{v}_\infty. \tag{3.47}$$

(For the zeroth approximation (3.47) is automatically satisfied.) And when the n-th approximation for the velocity $\boldsymbol{v}_n(x, t)$ is known, the $(n+1)$-th approximation for the vorticity $\boldsymbol{w}_{n+1}(x, t)$ is a solution of the following equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{w}_{n+1}+(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{w}_{n+1}-(\boldsymbol{w}_{n+1}\cdot\nabla)\boldsymbol{v}_n = \mathrm{rot}\,\boldsymbol{f}, \\ \mathrm{div}\,\boldsymbol{w}_{n+1} = 0, \qquad \boldsymbol{w}_{n+1}|_{t=0} = \mathrm{rot}\,\boldsymbol{v}_0. \end{cases} \tag{3.48}$$

For all n ($n=0, 1, 2, \cdots$), (3.47) and (3.48) are solvable on $[0, T]$ by Propositions 3.1 and 3.2.

First we shall give an estimate for $\boldsymbol{w}_n$ which is uniform in n.

LEMMA 3.3. *Let p and δ be as in Theorem* 1.1. *Let*

$$K = 2^{\delta+5-1/p}e\Big(|\mathrm{rot}\,\boldsymbol{v}_0|_{p,1,\delta+2}+\int_0^T |\mathrm{rot}\,\boldsymbol{f}(s)|_{p,1,\delta+2}ds\Big), \tag{3.49}$$

$$T_1 = \min\{T,\ 1/[2^{\delta+5-1/p}e\{c_1c_2(3K+4|\mathrm{rot}\,\boldsymbol{v}_0|_{p,1,\delta+2})+c_3\}]\}. \tag{3.50}$$

Then we have

$$\|\boldsymbol{w}_n\|_{p,1,\delta+2,T_1} \leqq K \qquad \textit{for all } n. \tag{3.51}$$

PROOF. We shall prove (3.51) by induction. It is clear that (3.51) holds for $\boldsymbol{w}_0$. Assume that (3.51) holds for $\boldsymbol{w}_n$. Then (3.47) implies that $\boldsymbol{v}_n$ satisfies (3.9). (We note that T in the estimates (3.8)-(3.11) can be replaced by T_1.) Hence by (3.50) we have

$$\begin{aligned} &T_1(\|\boldsymbol{v}_n\|_{\infty,T_1}+\|D\boldsymbol{v}_n\|_{\infty,T_1}+c_1\|D^2\boldsymbol{v}_n\|_{p,0,\delta+3,T_1}) \\ &\leqq T_1\{c_1c_2(3K+4|\mathrm{rot}\,\boldsymbol{v}_0|_{p,1,\delta+2})+c_3\} \leqq 1/(2^{\delta+5-1/p}e). \end{aligned} \tag{3.52}$$

On the other hand, (3.48) implies that $\boldsymbol{w}_{n+1}$ satisfies (3.10) and (3.11). It follows from (3.52) that $\exp(3T_1\|D\boldsymbol{v}_n\|_{\infty,T_1})\leqq e$. Hence using (3.49), (3.50) and (3.51) we have for any $t\in[0, T_1]$

$$\begin{aligned}
|w_{n+1}(t)|_{p,1,\delta+2} &= |w_{n+1}(t)|_{p,0,\delta+2} + |Dw_{n+1}(t)|_{p,0,\delta+3} \\
&\leqq 2^{\delta+4-1/p} e\Big(|\mathrm{rot}\, v_0|_{p,1,\delta+2} + \int_0^t |\mathrm{rot} f(s)|_{p,1,\delta+2} ds \\
&\quad + \|Dv_n\|_{\infty,T_1} \cdot T_1 \|w_{n+1}\|_{p,1,\delta+2,T_1} + c_1 T_1 \|D^2 v_n\|_{p,0,\delta+3,T_1} \cdot \|w_{n+1}\|_{p,1,\delta+2,t}\Big) \\
&\leqq 2^{\delta+4-1/p} e\Big(|\mathrm{rot}\, v_0|_{p,1,\delta+2} + \int_0^T |\mathrm{rot} f(s)|_{p,1,\delta+2} ds\Big) \\
&\quad + 2^{\delta+4-1/p} e \cdot T_1 (\|Dv_n\|_{\infty,T_1} + c_1 \|D^2 v_n\|_{p,0,\delta+3,T_1}) \cdot \|w_{n+1}\|_{p,1,\delta+2,T_1} \\
&\leqq \frac{1}{2} K + \frac{1}{2} \|w_{n+1}\|_{p,1,\delta+2,T_1} .
\end{aligned}$$

This proves $\|w_{n+1}\|_{p,1,\delta+2,T_1} \leqq K$. This completes the proof.

From (3.8) and the above lemma immediately follows the following

COROLLARY 3.4. v_n $(n=0, 1, 2, \cdots)$ *satisfies*

$$\|v_n - v_\infty\|_{p,2,\delta+1,T_1} \leqq c_2(K + |\mathrm{rot}\, v_0|_{p,1,\delta+2}) + |v_0 - v_\infty|_{p,2,\delta+1} . \tag{3.53}$$

PROPOSITION 3.5. *Let p and δ be as in Theorem* 1.1 *and let T_1 be as in Lemma* 3.3. *Then there exists a unique solution $\{v, w\}$ of* (3.2)-(3.5) *and* (2) *on* $[0, T_1]$, *which satisfies*

$$\begin{aligned}
&w \in C([0, T_1]; M^p_{0,\delta+2}) \cap L^\infty(0, T_1; M^p_{1,\delta+2}), \\
&v - v_\infty \in C([0, T_1]; M^p_{1,\delta+1}) \cap L^\infty(0, T_1; M^p_{2,\delta+1}).
\end{aligned} \tag{3.54}$$

PROOF. We shall first show that as $n \to \infty$

(3.55) $\quad w_n(\cdot, t)$ converges in $M^p_{0,\delta+2}$ uniformly in $t \in [0, T_1]$,

(3.56) $\quad v_n(\cdot, t)$ converges in $M^p_{1,\delta+1}$ uniformly in $t \in [0, T_1]$.

Let $w_n^* = w_{n+1} - w_n$ and $u_n = v_{n+1} - v_n$. Then it follows from Propositions 3.1 and 3.2 that

$$w_n^* \in C([0, T_1]; Y^p_{1,\delta+2}), \qquad u_n \in C([0, T_1]; X^p_{2,\delta+1}). \tag{3.57}$$

Furthermore, from (3.47) and (3.48) we deduce

$$\mathrm{rot}\, u_n = w_n^* , \tag{3.58}$$

$$\frac{\partial}{\partial t} w_n^* + \Psi(v_n, w_n^*) = \Psi(w_n, u_{n-1}); \qquad w_n^*(x, 0) = 0, \tag{3.59}$$

where $\Psi(u, w) = (u \cdot \nabla) w - (w \cdot \nabla) u$. Since (3.59) means that w_n^* satisfies the equation (3.4) in which $\mathrm{rot} f$ is replaced by $\Psi(w_n, u_{n-1})$, the estimate (3.10) holds for w_n^*. Therefore, noting $w_n^*(x, 0) = 0$ and using (3.52) we have

$$|w_n^*(t)|_{p,0,\delta+2} \leqq c_4 \int_0^t |\Psi(w_n, u_{n-1})(\cdot, s)|_{p,0,\delta+2} ds \qquad \text{for any } t \in [0, T_1], \tag{3.60}$$

where $c_4=2^{\delta+5-1/p}e/(4e-1)$. On the other hand, it follows from (3.57), (3.58) and Lemma 2.10 that for all n

$$|\boldsymbol{u}_n(t)|_{p,1,\delta+1} \leqq c_2|\boldsymbol{w}_n^*(t)|_{p,0,\delta+2} \quad \text{for any } t\in[0, T_1]. \tag{3.61}$$

This, together with (2.1), (2.3) and (3.51), yields that

$$\begin{aligned}&|\Psi(\boldsymbol{w}_n, \boldsymbol{u}_{n-1})(\cdot, t)|_{p,0,\delta+2}\\ &\leqq \|\boldsymbol{w}_n\|_{\infty,T_1}\cdot|D\boldsymbol{u}_{n-1}(t)|_{p,0,\delta+2}+|\boldsymbol{u}_{n-1}(t)|_\infty\cdot\|D\boldsymbol{w}_n\|_{p,0,\delta+3,T_1}\\ &\leqq 2c_1c_2K\cdot|\boldsymbol{w}_{n-1}^*(t)|_{p,0,\delta+2} \qquad \text{for any } t\in[0, T_1].\end{aligned} \tag{3.62}$$

Combining (3.60) and (3.62) we have for all n

$$|\boldsymbol{w}_n^*(t)|_{p,0,\delta+2} \leqq c_5\int_0^t|\boldsymbol{w}_{n-1}^*(s)|_{p,0,\delta+2}ds \quad \text{for any } t\in[0, T_1], \tag{3.63}$$

where $c_5=2c_1c_2c_4K$. Furthermore, it follows from (3.51) that $\|\boldsymbol{w}_0^*\|_{p,0,\delta+2}\leqq 2K$. Hence repeating (3.63) successively, we obtain

$$\begin{aligned}&|\boldsymbol{w}_1^*(t)|_{p,0,\delta+2}\leqq 2Kc_5t, \quad |\boldsymbol{w}_2^*(t)|_{p,0,\delta+2}\leqq 2K\frac{(c_5t)^2}{2!}, \quad \cdots\cdots,\\ &|\boldsymbol{w}_n^*(t)|_{p,0,\delta+2}\leqq 2K\frac{(c_5t)^n}{n!} \qquad \text{for any } t\in[0, T_1].\end{aligned} \tag{3.64}$$

From the above estimate we see that $\boldsymbol{w}_n=\boldsymbol{w}_0+\sum_{k=0}^{n-1}\boldsymbol{w}_k^*$ satisfies (3.55). Combining (3.61) and (3.64), we also see that $\boldsymbol{v}_n=\boldsymbol{v}_0+\sum_{k=0}^{n-1}\boldsymbol{u}_k$ satisfies (3.56).

Let $\boldsymbol{w}=\lim_{n\to\infty}\boldsymbol{w}_n$ and $\boldsymbol{v}=\lim_{n\to\infty}\boldsymbol{v}_n$. Then (3.54) follows from (3.55), (3.56), Lemma 3.3 and Corollary 3.4. In addition, taking the limit $n\to\infty$ in (3.47) and (3.48) we conclude that $\{\boldsymbol{v}, \boldsymbol{w}\}$ is a solution of the vorticity equation (in the sense of (3.24)) satisfying (3.54).

In order to show the uniqueness, suppose that there is another solution $\{\bar{\boldsymbol{v}}, \bar{\boldsymbol{w}}\}$ satisfying

$$\bar{\boldsymbol{v}}\in \boldsymbol{v}_\infty+C([0, T_1]; M_{1,\delta+1}^p), \qquad \bar{\boldsymbol{w}}\in C([0, T_1]; M_{0,\delta+2}^p). \tag{3.65}$$

Writing $\boldsymbol{w}^*=\boldsymbol{w}-\bar{\boldsymbol{w}}$ and $\boldsymbol{u}=\boldsymbol{v}-\bar{\boldsymbol{v}}$ and using the stream line $X(x, t; s)$ of $\boldsymbol{v}$ defined by (3.12), we have from (3.4) and (3.5)

$$\begin{cases}\dfrac{d}{ds}\boldsymbol{w}^*(X(x, t; s), s)=(\boldsymbol{w}^*(X(x, t; s), s)\cdot\nabla_y)\boldsymbol{v}(y, s)|_{y=X(x,t;s)}\\ \qquad\qquad\qquad\qquad\qquad\qquad +\Psi(\boldsymbol{w}, \boldsymbol{u})(X(x, t; s), s),\\ \boldsymbol{w}^*(X(x, t; 0), 0)=0.\end{cases} \tag{3.66}$$

Hence repeating the arguments used to deduce (3.10) from (3.26) and noting that (3.52) holds for $\boldsymbol{v}$ in place of $\boldsymbol{v}_n$, we have

$$|\boldsymbol{w}^*(t)|_{p,0,\delta+2}\leqq c_4\int_0^t|\Psi(\boldsymbol{w}, \boldsymbol{u})(\cdot, s)|_{p,0,\delta+2}ds \quad \text{for any } t\in[0, T_1]. \tag{3.67}$$

Furthermore, from (3.2), (3.3) and (2),

$$u \in C([0, T_1]; X^p_{1,\delta+1}), \qquad \mathrm{rot}\, u = w^*. \tag{3.68}$$

Hence repeating the arguments used to deduce (3.61) and (3.62) from (3.57) and (3.58) we have for any $t \in [0, T_1]$

$$|u(t)|_{p,1,\delta+1} \leqq c_2 |w^*(t)|_{p,0,\delta+2}, \tag{3.69}$$

$$|\Psi(w, u)(\cdot, t)|_{p,0,\delta+2} \leqq 2c_1c_2 \|w\|_{p,1,\delta+2,T_1} \cdot |w^*(t)|_{p,0,\delta+2}. \tag{3.70}$$

It follows from (3.67) and (3.70) that for any $t \in [0, T_1]$

$$|w^*(t)|_{p,0,\delta+2} \leqq 2c_1c_2c_4 \|w\|_{p,1,\delta+2,T_1} \cdot \int_0^t |w^*(s)|_{p,0,\delta+2} ds. \tag{3.71}$$

This, together with $w^*(\cdot, 0)=0$, proves that $w^*(\cdot, t) \equiv 0$. From (3.69) we also see that $u(\cdot, t) \equiv 0$. This completes the proof.

3.3. Proof of Theorem 1.1. Let $\{v, w\}$ be as in Proposition 3.5 and let $T_0 = T_1$. Then the velocity v satisfies (2). We shall prove that v satisfies the other conditions. It follows from (3.6) that v is represented in the form:

$$v(t) = F(w(t) - \mathrm{rot}\, v_0) + v_0. \tag{3.72}$$

Hence from (3.5) and (2.28) we see that $\lim_{t \downarrow 0} v(t) = v_0$.

LEMMA 3.6. *$\partial v/\partial t$ exists and belongs to $L^\infty(0, T_0; M^p_{1,\delta+1})$.*

PROOF. Using (3.54) and Lemma 2.3 and noting $M^p_{1,\delta+2} \subset M^p_{1,\delta+1}$ (see (2.1)), we have

$$(v \cdot \nabla) v \in L^\infty(0, T_0; M^p_{1,\delta+1}). \tag{3.73}$$

This implies that $\mathrm{div}((v \cdot \nabla) v) \in L^\infty(0, T_0; M^p_{0,\delta+2})$ and $((v \cdot \nabla) v) \cdot n|_S \in L^\infty(0, T_0; W^{1-1/p, p}(S))$. Hence applying Lemma 2.6 to the Neumann problem:

$$-\Delta q_v = \mathrm{div}((v \cdot \nabla) v); \qquad \left.\frac{\partial q_v}{\partial n}\right|_S = -((v \cdot \nabla) v) \cdot n|_S, \tag{3.74}$$

we have

$$q_v \in L^\infty(0, T_0; M^p_{2,\delta}). \tag{3.75}$$

We put

$$u(t) = v(t) - v_0 + \int_0^t \{(v \cdot \nabla) v + \nabla q_v - F(\mathrm{rot} f)\} ds. \tag{3.76}$$

Then it follows from (2.27), (3.73)-(3.75) and Proposition 3.5 that

$$u \in C([0, T_0]; M^p_{1,\delta+1}), \qquad \mathrm{div}\, u = 0; \qquad u \cdot n|_S = 0. \tag{3.77}$$

Furthermore, u satisfies

$$\operatorname{rot} \boldsymbol{u} = 0. \tag{3.78}$$

Indeed, by virtue of (2.27), (3.1) and (3.3), an integration by parts implies that for any $\boldsymbol{\eta} \in C_0^\infty(\Omega)$

$$(\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\nabla q_v-F(\operatorname{rot}\boldsymbol{f})\},\ \operatorname{rot}\boldsymbol{\eta}) = -(\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}-(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}-\operatorname{rot}\boldsymbol{f}\},\ \boldsymbol{\eta}).$$

Hence from Proposition 3.5, we see that for any $\boldsymbol{\eta} \in C_0^\infty(\Omega)$

$$\begin{aligned} &\frac{d}{dt}(\boldsymbol{u},\ \operatorname{rot}\boldsymbol{\eta}) \\ &= \frac{d}{dt}(\boldsymbol{v},\ \operatorname{rot}\boldsymbol{\eta})+(\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\nabla q_v-F(\operatorname{rot}\boldsymbol{f})\},\ \operatorname{rot}\boldsymbol{\eta}) \\ &= -\frac{d}{dt}(\boldsymbol{w},\ \boldsymbol{\eta})-(\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}-(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}-\operatorname{rot}\boldsymbol{f}\},\ \boldsymbol{\eta}) = 0. \end{aligned}$$

This, together with $(\operatorname{rot}\boldsymbol{u})|_{t=0}=0$, proves (3.78). Hence combining (3.77) and (3.78) with Corollary 2.8 we have $\boldsymbol{u}=0$. This yields that

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\nabla q_v-F(\operatorname{rot}\boldsymbol{f})\}. \tag{3.79}$$

Noting the right-hand side of (3.79) belongs to $L^\infty(0,\ T_0; M^p_{1,\delta+1})$ we obtain the desired result.

Let $U=\partial \boldsymbol{v}/\partial t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}-\boldsymbol{f}$. Then we have $\operatorname{rot} U=0$ since $\boldsymbol{v}$ satisfies (3.2)-(3.4). In addition, since it follows from Lemma 3.6 and (3.73) that $U+\boldsymbol{f} \in L^\infty(0,\ T_0; M^p_{1,\delta+1})$, Lemma 2.2 implies that $U \in L^\infty(0,\ T_0; C_b(\bar{\Omega}))$. Hence from Lemma 2.7 there is a scalar function p such that

$$U = -\nabla p \in L^\infty(0,\ T_0; C_b(\bar{\Omega})). \tag{3.80}$$

This proves that $\{\boldsymbol{v},\ p\}$ satisfies the first equation of (1).

Suppose that there is another solution $\{\bar{\boldsymbol{v}},\ \bar{p}\}$ of (1)-(3) satisfying the conditions of the theorem. Then we easily see that $\boldsymbol{u}\equiv\boldsymbol{v}-\bar{\boldsymbol{v}}$ and $\boldsymbol{w}^*\equiv\boldsymbol{w}-\operatorname{rot}\bar{\boldsymbol{v}}$ satisfy (3.66). Hence we can prove the uniqueness in a similar way to the proof of the uniqueness in Proposition 3.5.

REMARKS. (i) From (3.65) we see that the uniqueness is valid for $\boldsymbol{v} \in \boldsymbol{v}_\infty+C([0,\ T_0]; M^p_{1,\delta+1})$.

(ii) If $3<p<6$, then from (2.2) we easily see that $\partial \boldsymbol{v}/\partial t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \in L^\infty(0,\ T_0; W^{1,2}(\Omega))$. If, in addition to the assumption of the theorem, $\boldsymbol{f} \in L^2(0,\ T_0; L^2(\Omega))$, then 'the pressure term' ∇p belongs to $L^2(0,\ T_0; L^2(\Omega))$. Hence the uniqueness holds in the set of all $\bar{\boldsymbol{v}}$ such that the following energy identity holds:

$$|\boldsymbol{v}(t)-\bar{\boldsymbol{v}}(t)|_2^2+\int_0^t(\boldsymbol{v}(s)-\bar{\boldsymbol{v}}(s),\ [(\boldsymbol{v}(s)-\bar{\boldsymbol{v}}(s))\cdot\nabla]\boldsymbol{v}(s))ds = 0.$$

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math., **12** (1959), 623-727.

[2] C. Bardos and U. Frisch, Finite-time regularity for bounded and unbounded ideal incompressible fluid using Hölder estimates, Turbulence and Navier-Stokes Equation, Lecture Notes in Math., **565**, Springer, 1976, pp. 1-13.

[3] O. V. Besov, V. P. Il'in and S. M. Nikol'skii, Integral Representations of Functions and Imbedding Theorems, Vol. II, Winston-Wiley, 1979.

[4] J. P. Bourguignon and H. Brézis, Remark on the Euler equation, J. Funct. Anal., **15** (1974), 341-363.

[5] M. Cantor, Perfect fluid flows over $\boldsymbol{R}^n$ with asymptotic conditions, J. Funct. Anal., **18** (1975), 73-84.

[6] M. Cantor, Spaces of functions with asymptotic conditions on $\boldsymbol{R}^n$, Indiana Univ. Math. J., **24** (1975), 897-902.

[7] M. Cantor, Boundary value problems for asymptotically homogeneous elliptic second order operators, J. Differential Equations, **34** (1979), 102-113.

[8] D. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math., **92** (1970), 102-153.

[9] C. Foias and R. Temam, Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation, Ann. Scuola Norm. Sup. Pisa Ser. 4, **5** (1978), 29-63.

[10] D. Fujiwara and H. Morimoto, An L_r-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **24** (1977), 685-700.

[11] N. M. Günter, Potential Theory and its Applications to Basic Problems of Mathematical Physics, Frederick Unger Publ., 1967.

[12] S. Itô, The existence and the uniqueness of regular solution of non-stationary Navier-Stokes equation, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **9** (1961), 103-140.

[13] V. Judovič, Two-dimensional nonstationary problem of the flow of an ideal incompressible fluid through a given region, Mat. Sb. (N. S.), **64** (1964), 562-588 (Russian) = Amer. Math. Soc. Transl. (2), **57** (1966), 277-304.

[14] T. Kato, On classical solutions of the two-dimensional non-stationary Euler equation, Arch. Rat. Mech. Anal., **25** (1967), 188-200.

[15] T. Kato, Nonstationary flows of viscous and ideal fluids in $\boldsymbol{R}^3$, J. Funct. Anal., **9** (1972), 296-305.

[16] T. Kato and C. Y. Lai, Nonlinear evolution equations and the Euler flow, J. Funct. Anal., **56** (1984), 15-28.

[17] K. Kikuchi, Exterior problem for the two-dimensional Euler equation, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **30** (1983), 63-92.

[18] C. B. Morrey, Jr., Multiple Integrals in the Calculus of Variations, Grundlehren Math. Wiss., **130**, Springer, 1966.

[19] L. Nirenberg and H. Walker, The null spaces of elliptic partial differential operator in $\boldsymbol{R}^n$, J. Math. Anal. Appl., **42** (1973), 271-301.

[20] H. S. G. Swann, The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in $\boldsymbol{R}_3$, Trans. Amer. Math. Soc., **157** (1971), 373-397.

[21] H. S. G. Swann, The existence and uniqueness of nonstationary ideal incompressible flow in bounded domains in $\boldsymbol{R}_3$, Trans. Amer. Math. Soc., **179** (1973), 167-180.

[22] R. Temam, On the Euler equations of incompressible perfect fluids, J. Funct. Anal., 20 (1975), 32-43.

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