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On finite Galois coverings of projective manifolds

Dedicated to Professor Ichiro Satake on his sixtieth birthday

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Introduction.

By a *projective manifold*, we mean a connected compact complex manifold which can be imbedded holomorphically into the complex projective space $\boldsymbol{P}^m$ for some $m \geqq 1$. Let M be a projective manifold. A *finite branched covering* (or simply a *finite covering*) of M is by definition an irreducible normal complex space X together with a surjective proper finite holomorphic mapping $\pi: X \to M$. In this case, X is also projective by Grauert [**3**]. A *morphism* (resp. an *isomorphism*) *of* $\pi: X \to M$ *to* another finite covering $\pi': X' \to M$ is by definition a holomorphic (resp. biholomorphic) mapping $\phi: X \to X'$ such that $\pi' \cdot \phi = \pi$. The set G_π of all *automorphisms of* $\pi: X \to M$ forms a group under composition and is called the *automorphism group of* $\pi: X \to M$. G_π acts on each fiber of π.

$\pi: X \to M$ is called a *Galois covering* if G_π acts transitively on every fiber of π. In this case, the quotient complex space X/G_π is holomorphically isomorphic to M.

There is a natural one-to-one correspondence between the set of all isomorphism classes of finite Galois coverings $\pi: X \to M$ of M and the set of all isomorphism classes of finite Galois extensions $K/\boldsymbol{C}(M)$, where $\boldsymbol{C}(M)$ is the field of all meromorphic functions on M. In fact, the correspondence is given by

$$\pi \longmapsto K = \boldsymbol{C}(X)\,,$$
$$K \longmapsto \text{the } K\text{-normalization of } M,$$

(see Mumford [**6**, p. 396]).

Thus, to study finite Galois coverings of projective manifolds is nothing but to study finite Galois extensions of algebraic function fields of several complex variables.

In Namba [**7**, Theorem 3.5.7], we described the set of all isomorphism classes of finite Galois coverings of M in terms of the Tannaka algebraic system of unitary flat generalized vector bundles on M.

In this paper, we treat the problem of realization and existence of finite Galois coverings. That is, we prove the following two theorems:

THEOREM 1. *Let M be a projective manifold. For any finite Galois covering $\pi: X\to M$ of M, there are a finite subgroup G with $G\cong G_{\pi}$ of the automorphism group* $\mathrm{Aut}(\boldsymbol{P}^r)$ *of $\boldsymbol{P}^r$ for some $r\geqq 1$, and a meromorphic mapping $f: M\to N=\boldsymbol{P}^r/G$ such that $\pi: X\to M$ is isomorphic to the $\boldsymbol{C}(M_0\times_N\boldsymbol{P}^r)$-normalization of M, where $f_0: M_0\to N$ is a resolution of indeterminacy of f and $M_0\times_N\boldsymbol{P}^r$ is the fiber product of M_0 and $\boldsymbol{P}^r$ over N. ($M_0\times_N\boldsymbol{P}^r$ is irreducible in this case.)*

THEOREM 2. *For any projective manifold M and any finite group G, there exists a finite Galois covering $\pi: X\to M$ such that $G_{\pi}\cong G$.*

It seems that Theorem 2 for the case $\dim M=1$ was known among experts.

1. Solution of Riemann-Hilbert problem.

In order to prove Theorem 1, we need the solution by Deligne-Kita on Riemann-Hilbert problem for (generalized) Fuchsian differential equations.

Let M be a projective manifold and $\Omega=(\omega_{jk})$ be an $(m\times m)$-matrix-valued meromorphic 1-form on M such that $d\Omega+\Omega\wedge\Omega=0$. Consider the differential equation (Pfaffian system)

$$dF=F\Omega\,, \qquad \cdots\cdots (*)$$

where F is an $(m\times m)$-matrix-valued unknown function. Let $\mathrm{Supp}(D_{\infty}(\omega_{jk}))$ be the support of the polar divisor $D_{\infty}(\omega_{jk})$ of ω_{jk}. Put

$$B(\Omega)=\bigcup_{j,k}\mathrm{Supp}(D_{\infty}(\omega_{jk}))\,.$$

Then $B(\Omega)$ is a hypersurface of M. Put

$$M'=M-B(\Omega)\,.$$

It is then well known that there is an $(m\times m)$-matrix-valued holomorphic function F on the universal covering space $\tilde{M}'$ of M' with non-vanishing $\det F$ such that F is a solution of the equation $(*)$. Moreover other solutions can be written as AF for $A\in GL(m, \boldsymbol{C})$. The fundamental group $\pi_1(M', *)$ acts naturally on $\tilde{M}'$. For $\sigma\in\pi_1(M', *)$, there is $R(\sigma)\in GL(m, \boldsymbol{C})$ such that

$$\sigma^*F=F\cdot\sigma=R(\sigma)F\,.$$

Then

$$R:\ \sigma\in\pi_1(M', *)\longmapsto R(\sigma)\in GL(m, \boldsymbol{C})$$

is a representation of $\pi_1(M', *)$ and is called the *monodromy representation for the equation* $(*)$.

The equation $(*)$ is said to be *Fuchsian* if Ω has *generically log pole along* $B(\Omega)$. That is, for any point p of the regular locus $\operatorname{Reg} B(\Omega)$ of $B(\Omega)$ and any local coordinate system $(z_1, \cdots, z_n)$ around p such that $B(\Omega)=\{(z_1, \cdots, z_n) \mid z_1=0\}$ locally, Ω can be locally written as

$$\Omega = A_1 \frac{dz_1}{z_1} + A_2 dz_2 + \cdots + A_n dz_n ,$$

where $A_j\,(1 \leqq j \leqq n)$ are $(m \times m)$-matrix-valued holomorphic functions around p.

The equation $(*)$ is said to be *generalized Fuchsian* if $B(\Omega)$ is a union of hypersurfaces B' and B'' such that Ω has generically log pole along B' and $(*)$ has *apparent singularity along* B''. That is, for any point $p \in B''-B'$, there are a neighborhood W of p in M and an $(m \times m)$-matrix-valued meromorphic function G on W such that (1) G is holomorphic and has non-vanishing $\det G$ on $W-B''$ and (2) $dG=G\Omega$.

THEOREM 3 (Deligne [**1**]-Kita [**5**]). *Let B be a hypersurface of M and $R: \pi_1(M-B, *) \to GL(m, \boldsymbol{C})$ be a representation. Then there are a (possibly empty) hypersurface B' of M which has no common irreducible component with B and a generalized Fuchsian differential equation $(*)$ such that* (1) *$B(\Omega) \subset B \cup B'$,* (2) *the equation $(*)$ has apparent singularity along B' and* (3) *the monodromy group for $(*)$ equals R.*

REMARK 1. The solution of Riemann-Hilbert problem in the form of Theorem 3 can be obtained by the vanishing of the first cohomology group of some coherent sheaf. See also Namba [**7**, Theorem 2.2.1].

2. Fixed points for group action.

Let Y be a projective manifold and G be a finite subgroup of the automorphism group $\operatorname{Aut}(Y)$ of Y. Let $N=Y/G$ and $\mu: Y \to N$ be the quotient complex space and the projection, respectively. N is irreducible, normal and projective (cf. Hartshorne [**4**, p. 84, Proposition 1.6]).

For $A \in G$, put

$$\operatorname{Fix}(A) = \{p \in Y \mid A(p)=p\}.$$

If $A \neq 1$, then $\operatorname{Fix}(A)$ is a proper closed analytic subset of Y. Put

$$\operatorname{Fix}(G) = \bigcup_{A \neq 1} \operatorname{Fix}(A).$$

Then $\operatorname{Fix}(G)$ is a proper closed analytic subset of Y. We have clearly

LEMMA 1. *$\operatorname{Fix}(G)$ is a G-invariant set.*

Now let $\pi: X \to M$ be a finite Galois covering of a projective manifold M

and

$$R: G_\pi \xrightarrow{\sim} G$$

be an isomorphism. Suppose that there is a holomorphic mapping $\hat{f}: X \to Y$ such that

$$\hat{f}(\sigma p) = R(\sigma)\hat{f}(p) \qquad \text{for all} \quad (\sigma, p) \in G_\pi \times X.$$

Then a holomorphic mapping $f: M \to N = Y/G$ is induced and satisfies $f \cdot \pi = \mu \cdot \hat{f}$.
By Lemma 1, we have clearly

LEMMA 2. *$\hat{f}(X) \subset \mathrm{Fix}(G)$ if and only if $f(M) \subset \mu(\mathrm{Fix}(G))$.*

For the proof of Theorem 1, we also need

LEMMA 3. *Suppose that $\hat{f}(X)$ is not contained in $\mathrm{Fix}(G)$. Then the fiber product $M \times_N Y$ is irreducible and $\pi: X \to M$ is isomorphic to*

$$\pi' : X' \xrightarrow{\alpha} M \times_N Y \xrightarrow{\beta} M,$$

where α is the normalization of $M \times_N Y$ and β is the projection.

PROOF. We define a holomorphic mapping $\Phi: X \to M \times_N Y$ by $\Phi(p) = (\pi(p), \hat{f}(p))$. Then Φ is surjective. In fact, for any point $(q, y) \in M \times_N Y$, we take $p' \in X$ such that $\pi(p') = q$. Then $\mu\hat{f}(p') = f\pi(p') = f(q) = \mu(y)$. Hence there is $\sigma \in G_\pi$ such that $R(\sigma)\hat{f}(p') = y$. Put $p = \sigma(p')$. Then

$$\Phi(p) = (\pi(p), \hat{f}(p)) = (\pi(p'), \hat{f}(\sigma p')) = (q, R(\sigma)\hat{f}(p')) = (q, y).$$

Hence Φ is surjective, so $M \times_N Y$ is irreducible.

Next, for distinct points p and p' of X, suppose $\Phi(p) = \Phi(p')$. Then $\pi(p) = \pi(p')$ and $\hat{f}(p) = \hat{f}(p')$. There is $\sigma \in G_\pi$ with $\sigma \neq 1$ such that $\sigma(p) = p'$. Then

$$\hat{f}(p) = \hat{f}(\sigma p) = R(\sigma)\hat{f}(p).$$

That is, $\hat{f}(p) \in \mathrm{Fix}(R(\sigma))$. Hence, for a given $p \in X$, there are at most $\mathrm{ord}(G)$-points p' such that $\Phi(p') = \Phi(p)$. Thus Φ is a finite mapping.

By the assumption $\hat{f}(X) \not\subset \mathrm{Fix}(G)$, the set $\hat{f}^{-1}(\mathrm{Fix}(G))$ is a proper closed analytic subset of X. By the above discussion, Φ is injective on $X - \hat{f}^{-1}(\mathrm{Fix}(G))$.

Note that $\beta\Phi = \pi$. Put

$$R_\pi = \{p \in X \mid \pi \text{ is not biholomorphic around } p\},$$

$$R_\beta = \{\xi \in M \times_N Y \mid \beta \text{ is not biholomorphic around } \xi\}.$$

Then R_π is a hypersurface of X and R_β is a proper closed analytic subset of $M \times_N Y$. We have easily

$$\Phi^{-1}(R_\beta) \subset \hat{f}^{-1}(\mathrm{Fix}(G)).$$

Now, since Φ is locally written as $\Phi=\beta^{-1}\pi$, Φ is biholomorphic on $X-\hat{f}^{-1}(\mathrm{Fix}(G))-R_{\pi}$.

Thus, by Zariski main theorem (see Ueno [8, p. 9]), Φ induces an isomorphism of π to

$$\pi' : X' \xrightarrow{\alpha} M\times_N Y \xrightarrow{\beta} M. \qquad \text{q.e.d.}$$

REMARK 2. R_{π} and $B_{\pi}=\pi(R_{\pi})$ are called the *ramification locus* and the *branch locus of* π, respectively. They are hypersurfaces of X and M, respectively, by the purity of branch loci (see Fischer [2, p. 170]). Moreover, $\pi: X-R_{\pi}\to M-B_{\pi}$ is an unbranched Galois covering, whose automorphism group (that is, the covering transformation group) can be naturally identified with G_{π}.

3. Proof of Theorem 1.

Let $\pi: X\to M$ be a finite Galois covering of a projective manifold M. Let

$$R : G_{\pi} \longrightarrow GL(m, \boldsymbol{C})$$

be an *injective* representation of G_{π}. (For example, let R be the regular representation of G_{π}.) Let V be the complex vector space of all $(m\times m)$-matrices. Put

$$r = \dim V = m^2 .$$

$GL(m, \boldsymbol{C})$ acts on V as follows:

$$A(S) = AS \qquad \text{(the product of matrices)}$$

for $(A, S)\in GL(m, \boldsymbol{C})\times V$. Since $A(1)=A$, the action is effective. Thus we have an injective representation

$$G_{\pi} \longrightarrow GL(r, \boldsymbol{C})$$

which we denote by R again by abuse of notation. We regard V as the finite affine part of the projective space $\boldsymbol{P}^r$ identifying $S\in V$ with $(1: S)\in\boldsymbol{P}^r$. $GL(m, \boldsymbol{C})$ then acts on $\boldsymbol{P}^r$ as follows:

$$A(1: S) = (1: AS), \qquad A(0: S) = (0: AS).$$

The action is again effective. Thus we have an injective homomorphism

$$\hat{R} : G_{\pi} \longrightarrow \mathrm{Aut}(\boldsymbol{P}^r).$$

Put $G=\hat{R}(G_{\pi})$.

By Theorem 3, there are a hypersurface B' and an $(m\times m)$-matrix-valued holomorphic function F with non-vanishing $\det F$ on the universal covering space $\tilde{M}'$ of $M'=M-B_{\pi}\cup B'$ such that

$$\sigma^*F = (R\cdot\psi)(\sigma)F \qquad \text{for } \sigma\in\pi_1(M', *),$$

where B_π is the branch locus of π (see Remark 2) and

$$\phi : \pi_1(M', *) \longrightarrow \pi_1(M-B_\pi, *) \longrightarrow G_\pi$$

is the natural homomorphism.

Consider the holomorphic mapping $\hat{f}: \tilde{M}' \to \boldsymbol{P}^r$ defined by $\hat{f}(p)=(1: F(p))$. Then, by the construction of F in Theorem 3, $\hat{f}$ can be easily extended to a meromorphic mapping

$$\hat{f} : X - \mathrm{Sing}\, R_\pi \longrightarrow \boldsymbol{P}^r,$$

where $\mathrm{Sing}\, R_\pi$ is the singular locus of the ramification locus R_π of π. By Levi's theorem (see Fischer [2, p. 185]), $\hat{f}$ can be extended again to a meromorphic mapping

$$\hat{f} : X \longrightarrow \boldsymbol{P}^r$$

which satisfies

$$\sigma^* \hat{f} = \hat{R}(\sigma)\hat{f} \qquad \text{for } \sigma \in G_\pi .$$

We show that $\hat{f}(X)$ is not contained in $\mathrm{Fix}(G)$. In fact, if $\hat{f}(X) \subset \mathrm{Fix}(G)$, then $\hat{f}(X) \subset \mathrm{Fix}(\hat{R}(\sigma))$ for some $\sigma \in G_\pi$ such that $\sigma \neq 1$, because $\hat{f}(X)$ is an irreducible closed analytic subset of $\boldsymbol{P}^r$. Then

$$\hat{f}(p) = \hat{R}(\sigma)\hat{f}(p)$$

for all $p \in X$ such that $\hat{f}(p)$ is defined. Hence

$$F(p) = R(\sigma)F(p)$$

for all $p \in X$ such that $F(p)$ is defined. Then

$$R(\sigma) = F(p)F(p)^{-1} = 1$$

for all $p \in X$ such that $F(p)$ is defined and $\det F(p) \neq 0$. Such a point $p \in X$ exists. Hence $\sigma = 1$, a contradiction. Thus $\hat{f}(X)$ is not contained in $\mathrm{Fix}(G)$.

Now $\hat{f}$ induces a meromorphic mapping $f: M \to N = \boldsymbol{P}^r/G$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}} & \boldsymbol{P}^r \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mu} \\ M & \xrightarrow{f} & N \end{array}$$

commutes. Let $f_0: M_0 \to N$ be a resolution of indeterminacy of f. There is a commutative diagram

$$
\begin{tikzcd}
& M_0 \arrow[ddl, "\rho"'] \arrow[d] \arrow[ddr, "f_0"] & \\
& \Gamma_f \arrow[dl] \arrow[dr] & \\
M \arrow[rr, "f"'] & & N
\end{tikzcd}
$$

where Γ_f $(\subset M\times N)$ is the graph of f and $M_0 \to \Gamma_f$ is a proper modification (see Ueno [8, p. 13]). Let $\Gamma_{\hat{f}}$ $(\subset X\times \boldsymbol{P}^r)$ be the graph of $\hat{f}$ and let

$$\pi_1 : \Gamma_{\hat{f}} \longrightarrow \Gamma_f$$

be the holomorphic mapping defined by $\pi_1(p, z)=(\pi(p), \mu(z))$. Let $X_0 \to M_0 \times_{\Gamma_f} \Gamma_{\hat{f}}$ be the normalization $M_0 \times_{\Gamma_f} \Gamma_{\hat{f}}$. Then we have the following commutative diagram:

$$
\begin{tikzcd}
& & X_0 \arrow[ddll, "\hat{\rho}"'] \arrow[d] \arrow[ddrr, "\hat{f}_0"] \arrow[ddd, bend right, "\pi_0"'] & & \\
& & M_0 \times_{\Gamma_f} \Gamma_{\hat{f}} \arrow[dl] \arrow[dd] & & \\
X \arrow[dd, "\pi"'] & \Gamma_{\hat{f}} \arrow[l] \arrow[rrr] \arrow[dd, "\pi_1"] & & & \boldsymbol{P}^r \arrow[dd, "\mu"] \\
& & M_0 \arrow[dll, "\rho"'] \arrow[drr, "f_0"] & & \\
M & \Gamma_f \arrow[l] \arrow[rrr] & & & N
\end{tikzcd}
$$

Note that the composite mapping $\hat{\rho}: X_0 \to X$ is a proper modification, since $\rho: M_0 \to M$ is so. In particular, X_0 is irreducible and the composite mapping

$$\pi_0 : X_0 \longrightarrow M_0$$

is a finite Galois covering such that $G_{\pi_0} \cong G_\pi$ naturally. We identify these groups through the isomorphism.

Note that the composite holomorphic mapping $\hat{f}_0: X_0 \to \boldsymbol{P}^r$ satisfies

$$\sigma^* \hat{f}_0 = \hat{R}(\sigma) \hat{f}_0 \qquad \text{for } \sigma \in G_{\pi_0},$$

and induces $f_0: M_0 \to N$. Note also that $f_0(M_0)=f(M)$ is not contained in $\mu(\mathrm{Fix}(G))$ by Lemma 2. Thus, by Lemma 3, $M_0 \times_N \boldsymbol{P}^r$ is irreducible and π_0 is isomorphic to

$$\pi' : X' \xrightarrow{\alpha} M_0 \times_N \boldsymbol{P}^r \xrightarrow{\beta} M_0 ,$$

where α is the normalization of $M_0 \times_N \boldsymbol{P}^r$ and β is the projection. Now

$$\boldsymbol{C}(M_0 \times_N \boldsymbol{P}^r) \cong \boldsymbol{C}(X') \cong \boldsymbol{C}(X_0) \cong \boldsymbol{C}(X)$$

over $\boldsymbol{C}(M_0)=\boldsymbol{C}(M)$ (identified). Thus $\pi : X \to M$ is isomorphic to the $\boldsymbol{C}(M_0 \times_N \boldsymbol{P}^r)$-normalization of M.

This proves Theorem 1.

REMARK 3. The above proof shows that Theorem 1 can be in some sense regarded as a geometric counterpart of Namba [7, Theorem 2.2.10].

4. G-indecomposable meromorphic mappings.

As in §2, let Y be a projective manifold, G be a finite subgroup of $\mathrm{Aut}(Y)$, $N=Y/G$ be the quotient complex space and $\mu : Y \to N$ be the projection. Let $f: M \to N$ be a meromorphic mapping such that $f(M)$ is not contained in $\mu(\mathrm{Fix}(G))$. Let $f_0 : M_0 \to N$ be a resolution of indeterminacy of f. G acts on $M_0 \times_N Y$ as follows:

$$A(p, y) = (p, Ay)$$

for $A \in G$ and $(p, y) \in M_0 \times_N Y$. By the assumption $f(M) \not\subset \mu(\mathrm{Fix}(G))$, we can easily show that the action is effective. Moreover, G acts transitively on every fiber of the projection $M_0 \times_N Y \to M_0$.

In general, $M_0 \times_N Y$ may not be irreducible. Suppose that $M_0 \times_N Y$ is irreducible. (As is easily seen, this assumption does not depend on the choice of resolutions f_0 of indeterminacy of f.) Then $\boldsymbol{C}(M_0 \times_N Y)$-normalization

$$\pi : X \longrightarrow M$$

of M is a finite Galois covering such that G_π is naturally isomorphic to G.

This is one method for the realization of finite Galois coverings of M. Theorem 1 asserts that every finite Galois covering of M can be obtained in this way. Thus the problem of realization is reduced to looking for meromorphic mappings $f: M \to N = Y/G$ such that (1) $f(M) \not\subset \mu(\mathrm{Fix}(G))$ and (2) $M_0 \times_N Y$ is irreducible.

Now we ask when $M_0 \times_N Y$ is irreducible.

DEFINITION. Let $f: M \to N = Y/G$ be a meromorphic mapping such that $f(M) \not\subset \mu(\mathrm{Fix}(G))$. f is said to be *G-decomposable* if there are a proper subgroup H of G and a meromorphic mapping $h: M \to Y/H$ such that $f = \nu \cdot h$, where $\nu : Y/H \to N = Y/G$ is the projection. Otherwise, f is said to be *G-indecomposable*.

PROPOSITION 1. *$M_0 \times_N Y$ is irreducible if and only if $f: M \to N = Y/G$ is G-indecomposable.*

PROOF. Suppose that $M_0 \times_N Y$ is not irreducible. Note that G acts transitively on the set of irreducible components of $M_0 \times_N Y$. Let W be an irreducible component of $M_0 \times_N Y$. Put

$$H = \{A \in G \mid A(W) = W\}.$$

Then H is a proper subgroup of G. Note that H acts effectively on W and transitively on every fiber of the projection $W \to M_0$. Let $X' \to W$ be the normalization of W. The composite mapping

$$\pi' : X' \longrightarrow M_0$$

is a finite Galois covering of M_0 such that $G_{\pi'} \cong H$ naturally. We denote this isomorphism by $R: G_{\pi'} \xrightarrow{\sim} H$. The composite mapping

$$\hat{h}_0 : X' \longrightarrow W \subset M_0 \times_N Y \longrightarrow Y$$

satisfies

$$\sigma^* \hat{h}_0 = R(\sigma) \hat{h}_0 \qquad \text{for } \sigma \in G_{\pi'} .$$

Hence $\hat{h}_0$ induces a holomorphic mapping $h_0: M_0 \to Y/H$ such that $f_0 = \nu \cdot h_0$. Now h_0 induces a meromorphic mapping $h: M \to Y/H$ such that $f = \nu \cdot h$. Thus f is G-decomposable.

Conversely, suppose that f is G-decomposable. Let H and $h: M \to Y/H$ be as in the above definition. Note that $h(M)$ is not contained in $\mu'(\mathrm{Fix}(H))$, where $\mu': Y \to Y/H$ is the projection. Let $h_0: M_0 \to Y/H$ be a resolution of indeterminacy of h. Then $f_0 = \nu \cdot h_0$ is that of f.

The fiber product $M_0 \times_{Y/H} Y$ can be regarded as a closed analytic subset of $M_0 \times_N Y$. Since G (resp. H) acts transitively on every fiber of the projection $M_0 \times_N Y \to M_0$ (resp. $M_0 \times_{Y/H} Y \to M_0$), $M_0 \times_{Y/H} Y$ is in fact a union of some irreducible components of $M_0 \times_N Y$, and $M_0 \times_{Y/H} Y$ does not equal $M_0 \times_N Y$. Hence $M_0 \times_N Y$ is not irreducible. q. e. d.

EXAMPLE 1. Put $Y = \boldsymbol{P}^2$ and $G = \{1, A, B, AB\}$, where

$$A : (x, y) \longrightarrow (-x, y),$$
$$B : (x, y) \longrightarrow (x, -y).$$

Here (x, y) is an inhomogeneous coordinate system on $\boldsymbol{P}^2$. Then $N = Y/G$ and $\mu: Y \to N$ can be identified with $\boldsymbol{P}^2$ and the holomorphic mapping

$$(x, y) \longmapsto (x^2, y^2),$$

respectively. A meromorphic mapping $f: M \to N$ is given by

$$f : p \in M \longmapsto (x, y) = (f_1(p), f_2(p)) \in N = \boldsymbol{P}^2 ,$$

where f_1 and f_2 are meromorphic functions on M.

It is clear that $f(M)$ is not contained in $\mu(\mathrm{Fix}(G))$ if and only if neither of f_j ($j=1, 2$) is a zero-function. In this case, we can easily show that f is G-decomposable if and only if there is a meromorphic function h on M such that one of the following equalities holds: (1) $f_1=h^2$, (2) $f_2=h^2$, (3) $f_1f_2=h^2$.

EXAMPLE 2. Put $Y=(\boldsymbol{P}^1)^3$ and $N=\boldsymbol{P}^3$. We regard $N=\boldsymbol{P}^3$ as the symmetric product $S^3\boldsymbol{P}^1$ of $\boldsymbol{P}^1$. Then $N=\boldsymbol{P}^3=Y/G$, where G is isomorphic to the symmetric group of degree 3. More precisely, $\mu: Y \to N$ is given by

$$(y_1, y_2, y_3) \longmapsto (a_0 : a_1 : a_2 : a_3) = (1 : y_1+y_2+y_3 : y_1y_2+y_2y_3+y_3y_1 : y_1y_2y_3) ,$$

where y_j (resp. $(a_0 : a_1 : a_2 : a_3)$) is an inhomogeneous (resp. homogeneous) coordinate system on $\boldsymbol{P}^1$ (resp. $\boldsymbol{P}^3$). Thus y_j ($1 \leqq j \leqq 3$) are the roots of the equation

$$a_0x^3 - a_1x^2 + a_2x - a_3 = 0 .$$

Let $D(a_0, a_1, a_2, a_3)$ be the discriminant of the equation. Then

$$\mu(\mathrm{Fix}(G)) = \{(a_0 : a_1 : a_2 : a_3) \in \boldsymbol{P}^3 \mid D(a_0, a_1, a_2, a_3)=0\}$$

is an irreducible hypersurface of degree 4 in $N=\boldsymbol{P}^3$. A meromorphic mapping $f: M \to N=\boldsymbol{P}^3$ is given by

$$f : p \longrightarrow (a_0 : a_1 : a_2 : a_3) = (1 : f_1(p) : f_2(p) : f_3(p)) ,$$

where f_j ($1 \leqq j \leqq 3$) are meromorphic functions on M.

$f(M)$ is not contained in $\mu(\mathrm{Fix}(G))$ if and only if $D(1, f_1, f_2, f_3)$ is a non-zero meromorphic function on M. In this case, we can easily show that f is G-decomposable if and only if there is a meromorphic function h on M such that one of the following equalities holds:

(1) $h^3-2f_1h^2+(f_1^2+f_2)h+(f_3-f_1f_2) = 0$,
(2) $D(1, f_1, f_2, f_3) = h^2$.

PROPOSITION 2. *Let $f: M \to N=Y/G$ be a surjective meromorphic mapping with connected fibers. Then f is G-indecomposable.*

PROOF. Suppose that f is G-decomposable. Let H, $h: M \to Y/H$, $h_0: M_0 \to Y/H$ and $f_0=\nu \cdot h_0$ be as in the proof of Proposition 1:

$$\begin{tikzcd} & Y/H \arrow[d, "\nu"] \\ M_0 \arrow[ur, "h_0"] \arrow[r, "f_0"'] & N=Y/G \end{tikzcd}$$

Then f_0 is surjective. We show that h_0 is surjective. In fact, since

$$\nu(h_0(M_0)) = f_0(M_0) = Y/G ,$$

the restriction

$$\nu:\ h_0(M_0) \longrightarrow Y/G$$

of ν to $h_0(M_0)$ is surjective and finite. Note that both $h_0(M_0)$ and Y/H are irreducible and

$$\dim h_0(M_0) = \dim Y/G = \dim Y/H .$$

Hence

$$h_0(M_0) = Y/H .$$

Now, since ν is a finite surjective mapping with the mapping degree greater than one, $f_0 = \nu\cdot h_0$ can not have connected fibers, a contradiction. q. e. d.

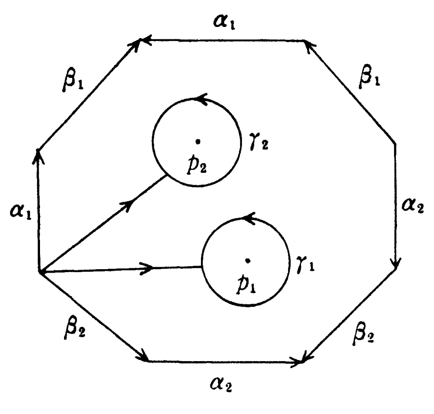
5. Proof of Theorem 2.

We divide the proof of Theorem 2 into two cases:

Case 1: $\dim M=1$. The following simple proof is due to Y. Morita: Let G be a finite group generated by s elements. Take points $p_1, \cdots, p_{m+1}$ on M. As is well known, $\pi_1(M-\{p_1, \cdots, p_{m+1}\}, *)$ is generated by $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_{m+1}$ with the unique relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1 \cdots \gamma_{m+1} = 1 ,$$

where g is the genus of M, $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ are generators of $\pi_1(M, *)$, and $\gamma_j\ (1 \leqq j \leqq m+1)$ is a loop rounding p_j once in the positive sense:



$$\begin{pmatrix} g=2 \\ m=1 \end{pmatrix}$$

Since

$$\gamma_{m+1} = \gamma_m^{-1} \cdots \gamma_1^{-1}\beta_g\alpha_g\beta_g^{-1}\alpha_g^{-1} \cdots \beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1} ,$$

the group $\pi_1(M-\{p_1, \cdots, p_{m+1}\}, *)$ can be regarded as the free group generated by $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_m$. Taking m so that $2g+m \geqq s$, there is a surjective homomorphism

$$\phi:\ \pi_1(M\text{-}\ \{p_1, \cdots, p_{m+1}\}, *) \longrightarrow G\,.$$

Corresponding to ϕ, there is a finite unramified Galois covering

$$\pi':\ X' \longrightarrow M-\{p_1, \cdots, p_{m+1}\}$$

such that $G_{\pi'}\cong G$. Let W_j be a neighborhood of p_j in M with a local coordinate w_j such that $w_j(p_j)=0$ and $W_j=\{w_j \mid |w_j|<1\}$. Consider the holomorphic mapping

$$\pi_j':U_j=\{z_j\in \boldsymbol{C} \mid |z_j|<1\} \longrightarrow W_j$$

defined by $z_j\to w_j=z_j^{\nu_j}$, where $\nu_j=\mathrm{ord}\,\phi(\gamma_j)$. We can patch up X' and $(\mathrm{ord}(G)/\nu_j)$-copies of U_j $(1\leqq j\leqq m+1)$ (resp. π' and π_j') and get a compact Riemann surface X (resp. a holomorphic mapping $\pi: X\to M$). Now

$$\pi:\ X \longrightarrow M$$

is clearly a finite Galois covering of M such that $G_\pi\cong G$.

Case 2: $\dim M\geqq 2$. Let Λ_0 be a fixed component free linear system on M such that $\dim \Phi_{\Lambda_0}(M)\geqq 2$, where

$$\Phi_{\Lambda_0}:M \longrightarrow \boldsymbol{P}^m$$

is the meromorphic mapping associated with Λ_0. (For example, let Λ_0 be very ample.) Let S be a general member of Λ_0. By Bertini's theorem (see Ueno [8, p. 45]), S is irreducible. Let Λ be a fixed component free linear pencil on M such that $S\in\Lambda$ and $\Lambda\subset\Lambda_0$. Then

$$f=\Phi_\Lambda: M \longrightarrow \boldsymbol{P}^1$$

is a surjective meromorphic mapping with connected fibers.

By Case 1, there is a finite Galois covering

$$\mu: Y \longrightarrow \boldsymbol{P}^1$$

such that $G_\mu\cong G$. We identify these groups through the isomorphism. Then we can identify $\boldsymbol{P}^1$ with Y/G.

By Proposition 2, f is G-indecomposable. By Proposition 1, $M_0\times_{P^1}Y$ is irreducible, where $f_0: M_0\to\boldsymbol{P}^1$ is a resolution of indeterminacy of f.

Now the $\boldsymbol{C}(M_0\times_{P^1}Y)$-normalization

$$\pi: X \longrightarrow M$$

of M is a finite Galois covering such that $G_\pi\cong G$. This proves Theorem 2.

REMARK 4. The above proof shows that there exist infinitely many non-isomorphic finite Galois coverings $\pi: X\to M$ of M such that $G_\pi\cong G$.

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