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A decision method for a set of first order classical formulas and its applications to decision problems for non-classical propositional logics

Dedicated to Professor Shôji Maehara for his sixtieth birthday

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I. Main theorem.

Let L be the first order classical predicate logic without equality. We assume that L has a fixed binary predicate symbol R, unary predicate symbols $P_1, \cdots, P_N$ and no other non-logical constant symbols. R-free formulas are formulas in L which has no occurrences of R. R-positive formulas are formulas in L which has no negative occurrences of R. R-formulas are formulas defined inductively as follows:

(1) All R-free formulas are R-formulas;

(2) If A and B are R-formulas, then $\neg A$, $A \wedge B$, $A \vee B$, $A \supset B$ are all R-formulas;

(3) If $A(x)$ is an R-formula and x is a free variable not occurring in $A(v)$, then $\forall v A(v)$, $\forall v(R(x, v) \supset A(v))$, $\forall v(R(v, x) \supset A(v))$, $\exists v A(v)$, $\exists v(R(x, v) \wedge A(v))$, $\exists v(R(v, x) \wedge A(v))$ are all R-formulas.

By R-quantifiers, we denote the quantifiers of the form:

$$\forall v(R(x, v) \supset \cdots v \cdots), \qquad \forall v(R(v, x) \supset \cdots v \cdots),$$
$$\exists v(R(x, v) \wedge \cdots v \cdots), \qquad \exists v(R(v, x) \wedge \cdots v \cdots),$$

where $\cdots v \cdots$ has no occurrences of the free variable x. Then, R-formulas are formulas obtained from R-free formulas by applying propositional connectives, quantifiers and R-quantifiers.

For each R-formula A, let R-deg(A) be the non-negative integer, called the R-degree of A, defined as follows:

(1) $R\text{-deg}(A) = 0$ if A is R-free.

(2) $R\text{-deg}(\neg A) = R\text{-deg}(A)$,
$R\text{-deg}(A \wedge B) = R\text{-deg}(A \vee B) = R\text{-deg}(A \supset B) = \max\{R\text{-deg}(A), R\text{-deg}(B)\}$,

(3) $R\text{-deg}(\forall v A(v)) = R\text{-deg}(\exists v A(v)) = R\text{-deg}(A(x))$, and

R-deg$(\forall v(R(x, v)\supset A(v)))$ = R-deg$(\forall v(R(v, x)\supset A(v)))$
= R-deg$(\exists v(R(x, v)\wedge A(v)))$ = R-deg$(\exists v(R(v, x)\wedge A(v)))$ = R-deg$(A(x))+1$.

Also, Tr is the sentence $\forall u\forall v\forall w(R(u, v)\wedge R(v, w).\supset R(u, w))$ and Sym is the sentence $\forall u\forall v(R(u, v)\supset R(v, u))$. Let F be the set of finite conjunctions of sentences: R-sentences, R-positive sentences, Tr and Sym. For each sentence A in F, let R-deg(A) be $\max\{R\text{-deg}(A_i);\, A$ is $A_1\wedge A_2\wedge\cdots\wedge A_m$ and A_i is an R-sentence$\}$. For each non-negative integer n, let K_n be the integer defined by; $K_0=2^N$, $K_{n+1}=K_n\times(2^{K_n})\times(2^{K_n})$. Then, our main theorem is:

MAIN THEOREM. *For each sentence A in F, if A has a model, then it has a model whose cardinality is at most K_n, where $n=R$-deg(A).*

Suppose that X is a set of sentences in L. Then, a decision method for X is a method by which, given a sentence in X, we can decide in a finite number of steps whether or not it has a model. X is said to be decidable if there is a decision method for X. It is well-known that the set of all R-free sentences is decidable, but the set of all sentences in L is not. Our main theorem clearly implies:

COROLLARY. *F is decidable.*

In II below, we shall give some applications of our main theorem to decision problems of non-classical propositional logics. In III below, we shall give a proof of our main theorem.

II. Applications.

Suppose that L' is a formal logic. Then a decision method for L' is a method by which, given a formula of L', we can decide in a finite number of steps whether or not it is provable in L'.

1) Intuitionistic propositional logic. Let IPL be the intuitionistic propositional logic whose propositional variables are $p_1, p_2, \cdots, p_N$. For each formula A in IPL, and each free variable x in L, let (A, x) be the formula in L defined by;

(p_i, x) is $P_i(x)$, $(\neg A, x)$ is $\forall v(R(x, v)\supset\neg(A, v))$,
$(A\wedge B, x)$ is $(A, x)\wedge(B, x)$, $(A\vee B, x)$ is $(A, x)\vee(B, x)$, and
$(A\supset B, x)$ is $\forall v(R(x, v)\supset((A, v)\supset(B, v)))$.

Then, by Kripke's completeness theorem, we have:

COMPLETENESS THEOREM FOR IPL ([2]). *A is provable in IPL iff the sentence $\mathrm{Tr}\wedge\mathrm{Tr}(P_1)\wedge\cdots\wedge\mathrm{Tr}(P_N)\wedge\exists v\neg(A, v)$ has no models, where $\mathrm{Tr}(P_i)$ is the R-sentence $\forall u(P_i(u)\supset\forall v(R(u, v)\supset P_i(v)))$, for each formula A in IPL.*

Since $\mathrm{Tr} \wedge \mathrm{Tr}(P_1) \wedge \cdots \wedge \mathrm{Tr}(P_N) \wedge \exists v \neg(A, v)$ belongs to F, our main theorem clearly implies that the logic IPL is decidable.

2) **Modal propositional logics.** Let MPL be the modal propositional language whose logical constants are $\neg$, $\wedge$, $\vee$, $\supset$ and $\Box$, and whose propositional variables are $p_1, p_2, \cdots, p_N$. For each formula A in MPL, and each free variable x in L, let (A, x) be the formula in L defined by; (p_i, x) is $P_i(x)$, $(\neg A, x)$ is $\neg(A, x)$, $(A \wedge B, x)$ is $(A, x) \wedge (B, x)$, $(A \vee B, x)$ is $(A, x) \vee (B, x)$, $(A \supset B, x)$ is $(A, x) \supset (B, x)$, and $(\Box A, x)$ is $\forall v(R(x, v) \supset (A, v))$. Let M, S4, B, S5 be modal propositional logics in Kripke [**1**], whose language is MPL. Then, by Kripke's completeness theorem for modal logics, we have:

COMPLETENESS THEOREM FOR MODAL LOGICS ([**1**]). *For any formula A in* MPL,

(i) *A is provable in* M *iff $\forall u R(u, u) \wedge \exists v \neg(A, v)$ has no models,*

(ii) *A is provable in* S4 *iff $\forall u R(u, u) \wedge \mathrm{Tr} \wedge \exists v \neg(A, v)$ has no models,*

(iii) *A is provable in* B *iff $\forall u R(u, u) \wedge \mathrm{Sym} \wedge \exists v \neg(A, v)$ has no models,*

(iv) *A is provable in* S5 *iff $\forall u R(u, u) \wedge \mathrm{Tr} \wedge \mathrm{Sym} \wedge \exists v \neg(A, v)$ has no models.*

Since $\forall u R(u, u)$, Tr, Sym, $\exists v \neg(A, v)$ belong to F, our main theorem clearly implies that four logics M, S4, B, S5 are all decidable.

III. A proof.

For each non-negative integer n, let Σ_n be the set defined as follows: $\Sigma_0 = \mathrm{Pow}(\{1, 2, \cdots, N\})$, and $\Sigma_{n+1} = \Sigma_n \times \mathrm{Pow}(\Sigma_n) \times \mathrm{Pow}(\Sigma_n)$, where $\mathrm{Pow}(Z)$ is the power set of Z. Let $\Sigma = \bigcup \{\Sigma_n ; n < \omega\}$. Then the cardinality of Σ_n is K_n. For each σ in Σ, let $A(\sigma, x)$ be the unary formula defined as follows:

If σ belongs to Σ_0, $A(\sigma, x)$ is $\wedge\{P_i(x) ; i \in \sigma\} \wedge \wedge\{\neg P_i(x) ; i \notin \sigma\}$ and if $\sigma = \langle \nu, l, r \rangle \in \Sigma_{n+1}$,

$A(\sigma, x)$ is

$$A(\nu, x) \wedge \wedge\{\exists v(R(v, x) \wedge A(\alpha, v)) ; \alpha \in l\} \wedge \wedge\{\neg \exists v(R(v, x) \wedge A(\alpha, v)) ; \alpha \notin l\}$$
$$\wedge \wedge\{\exists v(R(x, v) \wedge A(\alpha, v)) ; \alpha \in r\} \wedge \wedge\{\neg \exists v(R(x, v) \wedge A(\alpha, v)) ; \alpha \notin r\}.$$

Then $A(\sigma, x)$ is an R-formula whose R-degree is n if σ belongs to Σ_n. From this definition we have:

COROLLARY 1. (i) *Suppose that σ belongs to Σ_n. Then, $A(\sigma, x)$ is equivalent to the disjunction of the formulas: $A(\langle \sigma, l, r \rangle, x)$, where $l \subseteq \Sigma_n$ and $r \subseteq \Sigma_n$, in L.*

(ii) *The disjunction of the formulas: $A(\sigma, x)$, $\sigma \in \Sigma_n$, is provable in L for each non-negative integer n.*

(iii) *If σ and ν are distinct elements of Σ_n, then the sentence $\neg \exists v(A(\sigma, v)$*

$\wedge A(\nu, v))$ *is provable in* L.

LEMMA 2. *Every R-formula* $A(x, \cdots, y)$ *of R-degree*$\leqq n$, *whose free variables are among* $x, \cdots, y$, *is equivalent to a Boolean combination* $B(x, \cdots, y)$ *of formulas of the forms:* $\exists v(A(\sigma, v))$, $A(\sigma, x), \cdots, A(\sigma, y)$, $\sigma \in \Sigma_n$. *Moreover* B *is obtained from* A, *concretely. Therefore every R-sentence of R-degree*$\leqq n$, *is equivalent to a Boolean combination of sentences of the forms:* $\exists v(A(\sigma, v))$, $\sigma \in \Sigma_n$.

Suppose that $\mathfrak{N}$ and $\mathfrak{B}$ are L-structures and f is a homomorphism of $\mathfrak{N}$ onto $\mathfrak{B}$. Then f is said to be a strong n-homomorphism of $\mathfrak{N}$ to $\mathfrak{B}$ if the following two conditions (a) and (b) hold: (a) For any elements a, b in $\mathfrak{B}$, if $\mathfrak{B} \models R(a, b)$, then there are a', b' in $\mathfrak{N}$ such that $f(a')=a$, $f(b')=b$ and $\mathfrak{N} \models R(a', b')$. (b) For any $\sigma \in \Sigma_n$ and a in $\mathfrak{N}$, $\mathfrak{N} \models A(\sigma, a)$ iff $\mathfrak{B} \models A(\sigma, f(a))$.

From this definition and Lemma 2, we have:

COROLLARY 3. *Suppose that* f *is a strong n-homomorphism of* $\mathfrak{N}$ *to* $\mathfrak{B}$.

(iv) *For each R-sentence* A *of R-degree*$\leqq n$, *if* $\mathfrak{N}$ *is a model of* A, *then* $\mathfrak{B}$ *is also a model of it.*

(v) *For each R-positive sentence* A, *if* $\mathfrak{N}$ *is a model of* A, *then* $\mathfrak{B}$ *is also a model of it.*

(vi) *If* $\mathfrak{N}$ *is a model of* Sym, *then* $\mathfrak{B}$ *is also a model of it.*

But it is not generally true that if $\mathfrak{N}$ is a model of Tr, then $\mathfrak{B}$ is also a model of it.

For each L-structure $\mathfrak{N}$, let $\mathrm{tr}(\mathfrak{N})$ be the L-structure defined by:

$$|\mathrm{tr}(\mathfrak{N})| = |\mathfrak{N}|, \quad \mathrm{tr}(\mathfrak{N})(P_i) = \mathfrak{N}(P_i), \quad i=1, \cdots, N, \quad \text{and}$$

$$\mathrm{tr}(\mathfrak{N})(R) = \{\langle a, b\rangle ; \text{there is a finite sequence } \langle a_1, a_2, \cdots, a_m\rangle \text{ such that } a_1=a,\ a_m=b \text{ and } \langle a_i, a_{i+1}\rangle \in \mathfrak{N}(R) \text{ for each } i=1, \cdots, m-1\}.$$

Then, we have:

COROLLARY 4. (vii) $\mathrm{tr}(\mathfrak{N})$ *is a model of* Tr.

(viii) *If* $\mathfrak{N}$ *is a model of* Sym, *then* $\mathrm{tr}(\mathfrak{N})$ *is also a model of it.*

(ix) *For any R-positive sentence* A, *if* $\mathfrak{N}$ *is a model of* A, *then* $\mathrm{tr}(\mathfrak{N})$ *is also a model of it.*

But it is not generally true that if $\mathfrak{N}$ is a model of A, then $\mathrm{tr}(\mathfrak{N})$ is also a model of it, for each R-sentence A. For each L-structure $\mathfrak{N}$, each element a of $\mathfrak{N}$ and each non-negative integer n, let $\mathrm{LI}(\mathfrak{N}, a, n)$ (resp. $\mathrm{RI}(\mathfrak{N}, a, n)$) be the set of the elements σ in Σ_n such that $\mathfrak{N}$ is a model of $\exists v(R(v, a) \wedge A(\sigma, v))$ (resp. $\exists v(R(a, v) \wedge A(\sigma, v))$). $\mathfrak{N}$ has the n-weak transitive property (abbreviated by n-w. t. p.) if $\mathrm{LI}(\mathfrak{N}, a, k)$ is a subset of $\mathrm{LI}(\mathfrak{N}, b, k)$ and $\mathrm{RI}(\mathfrak{N}, b, k)$ is a subset of $\mathrm{RI}(\mathfrak{N}, a, k)$, for each a and b in $\mathfrak{N}$ such that $\mathfrak{N}$ is a model of $R(a, b)$ and each

$k<n$. Then clearly if $\mathfrak{N}$ is a model of Tr, then it has the n-w.t.p. for each n and every L-structure has 0-w.t.p. On the other hand, we have:

LEMMA 5. *Suppose that $\mathfrak{N}$ has the* n-w.t.p. *Then;*

(x) *For each element a in $\mathfrak{N}$ and σ in Σ_n, $\mathfrak{N}$ is a model of $A(\sigma, a)$ iff* tr($\mathfrak{N}$) *is a model of it.*

(xi) *For each R-sentence A of R-degree$\leqq n$, $\mathfrak{N}$ is a model of A iff* tr($\mathfrak{N}$) *is a model of it.*

PROOF. By Lemma 2, it is obvious that (xi) follows from (x) immediately. So we prove (x) by induction on n.

If $n=0$, then (x) is trivial by the definition of tr($\mathfrak{N}$). Assume that (x) is true for n. We shall show that (x) is also true for $n+1$. By the definition of $A(\sigma, x)$, $\sigma\in\Sigma_{n+1}$, it is sufficient to prove the following two facts:

(a) $\mathfrak{N}$ is a model of $\exists v(R(v, a)\wedge A(\nu, v))$ iff tr($\mathfrak{N}$) is a model of it, for each element a in $\mathfrak{N}$ and each ν in Σ_n.

(b) $\mathfrak{N}$ is a model of $\exists v(R(a, v)\wedge A(\nu, v))$ iff tr($\mathfrak{N}$) is a model of it, for each element a in $\mathfrak{N}$ and each ν in Σ_n.

Since "only if" parts of (a) and (b) above are obvious, we prove "if" parts of them. Assume that tr($\mathfrak{N}$) is a model of $\exists v(R(v, a)\wedge A(\nu, v))$. Then there is an element b in tr($\mathfrak{N}$) such that tr($\mathfrak{N}$) is a model of $R(b, a)\wedge A(\nu, b)$. By the definition of tr($\mathfrak{N}$), there is a finite sequence $\langle a_1, a_2, \cdots, a_m\rangle$ such that $a_1=b$, $a_m=a$ and $\langle a_i, a_{i+1}\rangle\in\mathfrak{N}(R)$ for each $i=1, \cdots, m-1$. Since $\mathfrak{N}$ has the $(n+1)$-w.t.p.,

$$\nu\in \mathrm{LI}(\mathfrak{N}, a_2, n)\subseteqq \mathrm{LI}(\mathfrak{N}, a_3, n)\subseteqq \cdots \subseteqq \mathrm{LI}(\mathfrak{N}, a_m, n)=\mathrm{LI}(\mathfrak{N}, a, n).$$

Hence we have that $\nu\in\mathrm{LI}(\mathfrak{N}, a, n)$. This means that $\mathfrak{N}$ is a model of $\exists v(R(v, a)\wedge A(\nu, v))$. Therefore (a) holds. Similarly (b) holds. (q.e.d.)

On the other hand, we have the following:

LEMMA 6. *If $\mathfrak{N}$ has the* n-w.t.p. *and there is a strong n-homomorphism of $\mathfrak{N}$ to $\mathfrak{B}$, then $\mathfrak{B}$ has also the* n-w.t.p.

PROOF. By induction n. If $n=0$, then this lemma is obvious. Assume that this lemma holds for n, and $\mathfrak{N}$ has the $(n+1)$-w.t.p. Let f be a strong $(n+1)$-homomorphism of $\mathfrak{N}$ to $\mathfrak{B}$ and a, b be two elements of $\mathfrak{B}$ such that $\mathfrak{B}\models R(a, b)$. By the hypothesis of induction, it is sufficient to prove that

$$\text{(a)}\quad \mathrm{LI}(\mathfrak{B}, a, n)\subseteqq \mathrm{LI}(\mathfrak{B}, b, n); \qquad \text{(b)}\quad \mathrm{RI}(\mathfrak{B}, a, n)\supseteqq \mathrm{RI}(\mathfrak{B}, b, n).$$

Let $\sigma\in\mathrm{LI}(\mathfrak{B}, a, n)$. Then, $\mathfrak{B}$ is a model of $\exists v(R(v, a)\wedge A(\sigma, v))$. Let a', b' be two elements of $\mathfrak{N}$ such that $f(a')=a$, $f(b')=b$ and $\mathfrak{N}\models R(a', b')$. Since, f is a strong $(n+1)$-homomorphism of $\mathfrak{N}$ to $\mathfrak{B}$ and $\exists v(R(v, x)\wedge A(\sigma, v))$ is a formula of R-deg$\leqq n+1$, $\mathfrak{N}$ is a model of $\exists v(R(v, a')\wedge A(\sigma, v))$. This means that $\sigma\in\mathrm{LI}(\mathfrak{N}, a', n)$. Since $\mathfrak{N}$ has the $(n+1)$-w.t.p., $\sigma\in\mathrm{LI}(\mathfrak{N}, b', n)$. Hence $\mathfrak{N}$ is a model of $\exists v(R(v, b')$

$\wedge A(\sigma, v))$. Therefore, $\mathfrak{B}$ is a model of $\exists v(R(v, b)\wedge A(\sigma, v))$. This means that $\sigma\in \mathrm{LI}(\mathfrak{B}, b, n)$. This shows that (a) above holds. Similarly (b) above holds.

(q. e. d.)

For each L-structure $\mathfrak{N}$ and non-negative integer n, let f_n be the mapping from $\mathfrak{N}$ to Σ_n defined by: $f_n(a)=\sigma$ such that $\mathfrak{N}$ is a model of $A(\sigma, a)$ for each element a in $\mathfrak{N}$. By Corollary 1, there exists such σ uniquely for each a. Using this mapping, we define a new L-structure $\mathfrak{N}_n$ as follows: The universe of $\mathfrak{N}_n$ is the range of the mapping f_n and, $\mathfrak{N}_n(P_i)$, $(1\leqq i\leqq N)$, $\mathfrak{N}_n(R)$ are images of $\mathfrak{N}(P_i)$, $(1\leqq i\leqq N)$, $\mathfrak{N}(R)$ under f_n, respectively. Then, we can easily prove the following:

LEMMA 7. *f_n is a strong n-homomorphism of $\mathfrak{N}$ to $\mathfrak{N}_n$.*

Combining these results we have:

THEOREM 8. *Suppose that $\mathfrak{N}$ is an L-structure and n is a non-negative integer. Then;*

(xii) *The cardinality of the universe of $\mathfrak{N}_n$ and* $\mathrm{tr}(\mathfrak{N}_n)$ *are no more than K_n.*

(xiii) $\mathrm{tr}(\mathfrak{N}_n)$ *is a model of* Tr.

(xiv) *If $\mathfrak{N}$ is a model of* Sym, *then $\mathfrak{N}_n$ and* $\mathrm{tr}(\mathfrak{N}_n)$ *are models of it.*

(xv) *For any R-positive sentence A, if $\mathfrak{N}$ is a model of A, then $\mathfrak{N}_n$ and* $\mathrm{tr}(\mathfrak{N}_n)$ *are models of it.*

(xvi) *For each R-sentence A of R-degree$\leqq n$, if $\mathfrak{N}$ is a model of A, then $\mathfrak{N}_n$ is a model of it.*

(xvii) *For each R-sentence A of R-degree$\leqq n$, if $\mathfrak{N}$ is a model of A and* Tr, *then* $\mathrm{tr}(\mathfrak{N}_n)$ *is a model of A.*

From Theorem 8 above we can easily prove our main theorem as follows: Suppose that A is a sentence in F. If A has a model $\mathfrak{N}$, then by Theorem 8 above, at least one of $\mathrm{tr}(\mathfrak{N}_n)$ and $\mathfrak{N}_n$ is a model of A, whose cardinality is no more than K_n, where $n=R$-deg(A). This completes a proof of our main theorem.

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