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A generalization of Axiom A

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§1. Introduction.

In [1], J. Baumgartner introduced the class of partial orderings for Axiom A which includes c.c.c. p.o. sets, ω_1-closed p.o. sets and various notions of forcing which add new subsets of ω. If partial orderings which satisfy Axiom A are iterated under countable support, then the iteration, regardless of its length, satisfies the following covering property: If $\mathring{X}$ is a countable subset of the ordinals in the generic extension via the iteration, then there is $X \in V$ (the ground model) which is countable in V with $\mathring{X} \subseteq X$. This covering property implies that ω_1 is preserved. The main procedure involved in showing this is to produce what we call a fusion sequence which has a lower bound. It is not plausible, however, that the iteration itself satisfies Axiom A.

In this paper we generalize the class of partial orderings for Axiom A so that our generalization is iterable under countable support. The difference between these two classes is that: When we construct a nice descending sequence (fusion sequence) $\langle p_n \rangle_{n<\omega}$, the choice of p_{n+1} depends only on p_n for Axiom A and depends on $p_0, \cdots, p_n$ for our generalization.

Let us begin with a quick review of definitions.

§2. Preliminaries.

A binary relation $(P, \leqq)$ is a *preordering* if $(P, \leqq)$ is reflexive and transitive. A preordering $(P, \leqq)$ satisfies *Axiom A* if there is a sequence $\langle \leqq_n \rangle_{n<\omega}$ such that

(1) $(P, \leqq_n)$ is a preordering for all $n<\omega$,

(2) if $p \leqq_n q$, then $p \leqq q$,

(3) if $p \leqq_{n+1} q$, then $p \leqq_n q$,

(4) if $\langle p_n \rangle_{n<\omega}$ is a sequence of conditions from P with $p_{n+1} \leqq_n p_n$ for each $n<\omega$, then there is a condition p in P such that $p \leqq_n p_n$ for all $n<\omega$,

(5) for any n in ω, any p in P and any dense subset D of P below p (i.e.

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for any r in P, if $r \leqq p$, then there is a condition d in D with $d \leqq r$), there are q in P and a countable subset D' of D such that $q \leqq_n p$ and $q \leqq \vee D'$ (i. e. for any condition r with $r \leqq q$, there are d in D' and a in P with $a \leqq r$ and $a \leqq d$).

An *infinite game* G for a preordering P is played by two players I and II. I initiates a play by choosing a condition p_0 in P, then II follows by choosing a condition p_1 with $p_1 \leqq p_0$, then I picks a condition p_2 with $p_2 \leqq p_1$, and II picks p_3 with $p_3 \leqq p_2, \cdots$ etc. This way, they finish the play $\langle p_n \rangle_{n<\omega}$. If the play had a lower bound, then II wins, otherwise I wins the play. By a *winning strategy* σ for II, we mean a function from the collection of finite sequences of P into P such that any play of the form:

$$p_0 \geqq \sigma(p_0) \geqq p_2 \geqq \sigma(p_0, p_2) \geqq p_4 \geqq \cdots$$

has a lower bound.

For a regular uncountable cardinal θ, $H(\theta)$ denotes the collection of sets which are hereditarily of size less than θ. For each countable ordinal δ, let $S(\delta, H(\theta))$ be the collection of sequences $\langle a_i \rangle_{i \leq \beta}$ such that

(1) $\beta \leqq \delta$,

(2) for each $i \leqq \beta$ a_i is a countable subset of $H(\theta)$ and

(3) $\langle a_i \rangle_{i \leq \beta}$ is continuously increasing. (i. e. $j \leqq i \leqq \beta$ implies $a_j \subseteqq a_i$ and if $i \leqq \beta$ is a limit ordinal, then $a_i = \bigcup_{j<i} a_j$.)

A sequence $\langle N_i \rangle_{i \leq \delta}$ is *nice* if

(1) for each $i \leqq \delta$ $(N_i, \in)$ is a countable elementary substructure of $(H(\theta), \in)$, which we denote by $N_i \prec H(\theta)$,

(2) for each $i<\delta$, $\langle N_j \rangle_{j \leq i} \in N_{i+1}$ and

(3) $\langle N_i \rangle_{i \leq \delta}$ is continuous.

Note that since $N_i \in N_{i+1}$ and N_i is countable, $N_i \subseteqq N_{i+1}$ holds. For a condition q in P and a set N, q is (P, N)-*generic* if for each dense subset D of P in N, $q \leqq \vee (D \cap N)$ holds.

For any uncountable set A, let $[A]^\omega$ be the collection of subsets of A which has size ω. A subset D of $[A]^\omega$ is *closed unbounded* if D is closed (i. e. for $\langle X_n \rangle_{n<\omega}$ with $X_n \in D$ and $X_n \subseteqq X_{n+1}$ for all $n<\omega$, $\bigcup_{n<\omega} X_n \in D$) and cofinal (i. e. for any $X \in [A]^\omega$, there is $Y \in D$ s. t. $X \subseteqq Y$).

A preordering $(P, \leqq)$ is δ-*proper* if there are a regular uncountable cardinal θ with $P \in H(\theta)$ and a function C from $S(\delta, H(\theta)) \cup \{\emptyset\}$ into the collection of closed unbounded sets of $[H(\theta)]^\omega$ such that for any nice sequence $\langle N_i \rangle_{i \leq \delta}$ with $N_0 \in C(\emptyset)$ and $N_{i+1} \in C(\langle N_j \rangle_{j \leq i})$ for all $i<\delta$ and for any $p \in N_0 \cap P$, there is a condition q in P such that $q \leqq p$ and q is (P, N_i)-generic for all $i \leqq \delta$.

$((P_\beta, \leqq_\beta, 1_\beta)_{\beta \leq \alpha}, (\mathring{Q}_\beta, \mathring{\leqq}_\beta, \mathring{1}_\beta)_{\beta<\alpha})$ is a *countable support iteration* of length $\alpha+1$ if

(1) the elements of P_β are sequences of length β and $(P_\beta, \leqq_\beta)$ is a preordering with a greatest element 1_β,

(2) $\Vdash_{P_\beta}$ "$(\mathring{Q}_\beta, \mathring{\leqq}_\beta)$ is a preordering with a greatest element $\mathring{1}_\beta$",

(3) $P_{\beta+1} \subseteqq \{p^\frown\langle\tau\rangle: p \in P_\beta \text{ and } \Vdash_{P_\beta} \tau \in \mathring{Q}_\beta\}$,

(4) if $p \Vdash_{P_\beta} \tau \in \mathring{Q}_\beta$, then there is a condition q in $P_{\beta+1}$ such that $q \restriction \beta = p$ and $p \Vdash_{P_\beta} q(\beta) = \tau$,

(5) if $p \in P_\beta$ and $q \in P_{\beta+1}$, then $p^\frown\langle q(\beta)\rangle \in P_{\beta+1}$,

(6) $p \leqq_{\beta+1} q$ iff $p \restriction \beta \leqq_\beta q \restriction \beta$ and $p \restriction \beta \Vdash_{P_\beta} p(\beta) \mathring{\leqq}_\beta q(\beta)$,

(7) $1_{\beta+1} = 1_\beta{}^\frown\langle\mathring{1}_\beta\rangle$.

If β is a limit ordinal, then

(8) P_β is the collection of sequences of length β such that for each $\rho < \beta$ $p \restriction \rho \in P_\rho$ holds and $\mathrm{supp}(p) = \{\rho < \beta : p(\rho) \neq \mathring{1}_\rho\}$ is countable,

(9) $p \leqq_\beta q$ iff for all $\rho < \beta$ $p \restriction \rho \leqq_\rho q \restriction \rho$ and

(10) $1_\beta = \langle\mathring{1}_\rho\rangle_{\rho<\beta}$.

If G_β is a P_β-generic filter over V (the ground model) and τ is a P_β-name, then the object decided by τ and G_β is denoted by $\tau[G_\beta]$. We simply write $(P_\beta, \mathring{Q}_\beta)_{\beta \leq \alpha, \beta < \alpha}$ for an iteration.

§ 3. The Axiom C.

DEFINITION 1. A preordering $(P, \leqq)$ satisfies *Axiom C* if there is a subset R of the collection of finite sequences of P such that

(1) $R(p)$ for all p in P,

(2) $R(p_0, \cdots, p_n)$ implies $p_0 \geqq \cdots \geqq p_n$,

(3) for any $n < \omega$ and any sequence of conditions $\langle p_k \rangle_{k<\omega}$ from P if $R(p_0, \cdots, p_n, \cdots, p_{n+i})$ holds for all $i < \omega$, then there is a condition p in P such that $R(p_0, \cdots, p_i, p)$ holds for all $i \geqq n-1$,

(4) if $R(p_0, \cdots, p_n)$ and D is dense below p_n, then there are a condition $p_{n+1} \in P$ and a countable subset D' of D such that $R(p_0, \cdots, p_n, p_{n+1})$ and $p_{n+1} \leqq \vee D'$ hold.

We call the sequence $\langle p_k \rangle_{k<\omega}$ appeared in (3) a *fusion sequence* and the condition p, a *fusion* of the fusion sequence.

PROPOSITION 2.

(0) (4) *in Definition* 1 *is equivalent to: If* $R(p_0, \cdots, p_n)$ *and* $p_n \Vdash_P$"τ *is an ordinal", then there are a condition* p_{n+1} *in* P *and a countable collection of ordinals* X *such that* $R(p_0, \cdots, p_n, p_{n+1})$ *and* $p_{n+1} \Vdash_P \tau \in X$.

(1) *If* $(P, \leqq)$ *satisfies Axiom A, then* $(P, \leqq)$ *satisfies Axiom C.*

(2) *If* $(P, \leqq)$ *satisfies Axiom C, then* $(P, \leqq)$ *is* δ*-proper for all* $\delta < \omega_1$.

(3) *Countable support iterations of Axiom A of arbitrary length satisfy*

Axiom C.

(4) *Countable support products of perfect set forcing satisfy Axiom C.*

(5) *If the player* II *has a winning strategy in the game G for P, then the preordering P satisfies Axiom C.*

PROOF. (0) Suppose (4) in Definition 1. Suppose $R(p_0, \cdots, p_n)$ and $p_n \Vdash$"τ is an ordinal". Define $D=\{p\in P$: there is an ordinal α s.t. $p\Vdash\tau=\alpha\}$. D is dense below p_n. Thus by (4) in Definition 1, we have p_{n+1} in P and a countable subset D' of D such that $R(p_0, \cdots, p_n, p_{n+1})$ and $p_{n+1}\leqq\bigvee D'$ hold. Let $X=\{\alpha: \exists d\in D'\ d\Vdash\tau=\alpha\}$. X works.

Conversely, assume $R(p_0, \cdots, p_n)$ and D is dense below p_n. We want to show that the conclusion of (4) in Definition 1 holds. Let $\langle d_\xi\rangle_{\xi<\rho}$ be an enumeration of D. Since D is dense below p_n, we have a P-name $\mathring{\xi}$ s.t. $p_n\Vdash d_{\mathring{\xi}}\in\mathring{G}$, where $\mathring{G}$ is the canonical name for a P-generic filter. Therefore there is a condition p_{n+1} in P and a countable collection of ordinals X such that $R(p_0, \cdots, p_n, p_{n+1})$ and $p_{n+1}\Vdash\mathring{\xi}\in X$. Let $D'=\{d_\xi: \xi\in X\cap\rho\}$. D' works.

(1) Suppose $(P, \leqq)$ satisfies Axiom A with $\langle\leqq_n\rangle_{n<\omega}$. We define R as follows: $R(p)$ if and only if $p\in P$; and for $p_0, \cdots, p_{n+1}$ in P, $R(p_0, \cdots, p_{n+1})$ if and only if $p_{n+1}\leqq_n\cdots\leqq_1 p_1\leqq_0 p_0$. This R works.

(2) Suppose $(P, \leqq)$ satisfies Axiom C with R. Take a regular uncountable cardinal θ with $P\in H(\theta)$ and define the function C with the domain $S(\delta, H(\theta))\cup\{\emptyset\}$ such that $C(\emptyset)=\{N\prec H(\theta): N$ is countable and $P, \leqq, R\in N\}$ and for any $\sigma\in S(\delta, H(\theta))$, $C(\sigma)=\{N\prec H(\theta): N$ is countable$\}$. Let $\langle N_i\rangle_{i\leqq\delta}$ be a nice sequence with $N_0\in C(\emptyset)$ and $N_{i+1}\in C(\langle N_j\rangle_{j\leqq i})$ for all $i<\delta$. It suffices to show that

(a) For any p in $P\cap N_0$, there is a condition p_0 such that $p_0\leqq p$ and p_0 is (P, N_0)-generic.

(b) For all α, β with $\alpha<\beta\leqq\delta$, if $R(p_0, \cdots, p_n)$ with $p_0, \cdots, p_n\in N_{\alpha+1}$ and p_n is (P, N_i)-generic for all $i\leqq\alpha$, then there is a condition p_{n+1} such that $R(p_0, \cdots, p_n, p_{n+1})$ and p_{n+1} is (P, N_i)-generic for all $i\leqq\beta$.

To show (a), let p be a condition in $P\cap N_0$ and let $\langle D_n\rangle_{n<\omega}$ enumerate the dense subsets of P which belong to N_0. By (1) and (4) in Definition 1 we have a condition a_0 and a countable subset D_0' of D_0 such that $R(p, a_0)$ and $a_0\leqq\bigvee D_0'$ hold. Since p, R, $\leqq$, D_0 and P are all in N_0, there are a_0 and D_0' in N_0 as such. Since D_0' is countable, D_0' is a subset of N_0. Therefore, we have $a_0\leqq\bigvee(D_0\cap N_0)$. We repeat this argument for D_1, D_2, D_3 and so forth to get $\langle a_n\rangle_{n<\omega}$ and $\langle D_n'\rangle_{n<\omega}$ such that $R(p, a_0, \cdots, a_n)$ and $a_n\leqq\bigvee(D_n\cap N_0)$ for all $n<\omega$. By (3) in Definition 1, we have a condition p_0 such that $R(p, a_0, \cdots, a_{n-1}, p_0)$ for all $n<\omega$. By (2) in Definition 1 we have $a_n\geqq p_0$ for all $n<\omega$. Thus $p_0\leqq\bigvee(D_n\cap N_0)$ for all $n<\omega$, and so p_0 is (P, N_0)-generic. Note that we can retake such a p_0 in N_1, if $1\leqq\delta$.

To show (b), we proceed by induction on β (for all $\alpha<\beta$). Notice that if $x\leqq y$ and y is (P, M)-generic, then so is x. Also notice that if x is (P, M_j)-generic for all $j<i$, then x is $(P, \bigcup_{j<i}M_j)$-generic. We use these facts. The rests are similar to the above argument.

(3) We assume that the reader is familiar with Lemmas 7.2 and 7.3 in [1]. Let $(P_\alpha, \mathring{Q}_\alpha)_{\alpha\leq\nu, \alpha<\nu}$ be a countable support iteration of preorderings such that $\Vdash_{P_\alpha}$"$\mathring{Q}_\alpha$ satisfies Axiom A with $\langle\mathring{\leqq}^\alpha_n\rangle_{n<\omega}$" for all $\alpha<\nu$. For each $p, q\in P_\nu$, $n<\omega$ and a finite subset F of ν, we define as in [1] $q\geqq_{F,n}p$ if $q\geqq p$ and for any α in F $p\restriction\alpha\Vdash_{P_\alpha}$"$p(\alpha)\mathring{\leqq}^\alpha_n q(\alpha)$" holds. For each p in P_ν, let us fix a function f_p from ω such that $\nu\supseteqq f_p$ "$\omega\supseteqq\mathrm{supp}(p)$. (Here we assume $\nu\neq 0$.) We define $F_{p_0,\cdots,p_n}=f_{p_0}$"$(n+1)\cup\cdots\cup f_{p_n}$"$(n+1)$ for each $p_0, \cdots, p_n\in P_\nu$ and define inductively a subset R of the finite sequences of P_ν.

(a) $R(p)$ if $p\in P_\nu$.

(b) $R(p_0, \cdots, p_{n+1})$ if $R(p_0, \cdots, p_n)$ and $p_n\geqq_{F,n}p_{n+1}$ holds, where $F=F_{p_0,\cdots,p_n}$.

This R works.

(4) Similar to (3) using Lemma 1.6 and Corollary 1.10 in [2].

(5) Let σ be a winning strategy for the player II. We define R as follows: $R(p)$ if and only if p in P; and for $p_0, \cdots, p_{n+1}$ in P $R(p_0, \cdots, p_{n+1})$ if and only if $p_0\geqq\sigma(p_0)\geqq p_1\geqq\sigma(p_0, p_1)\geqq\cdots\geqq\sigma(p_0, \cdots, p_n)\geqq p_{n+1}$. This R works.

Q. E. D.

It is known concerning Axiom A:

THEOREM (J. Baumgartner, unpublished). *Let $(P_\alpha, \mathring{Q}_\alpha)_{\alpha\leq\nu,\alpha<\nu}$ be a countable support iteration such that $\Vdash_{P_\alpha}$"$\mathring{Q}_\alpha$ satisfies Axiom A" for all $\alpha<\nu$. If $\nu<\omega_1$, then we may show that P_ν satisfies Axiom A.*

PROOF. Since ν is countable, we may fix a sequence of finite sets $\langle F_n\rangle_{n<\omega}$ such that $F_n\subseteqq F_{n+1}\subseteqq\nu$ for all $n<\omega$ and $\bigcup_{n<\omega}F_n=\nu$. Suppose that $\Vdash_{P_\alpha}$"$\mathring{Q}_\alpha$ satisfies Axiom A with $\langle\mathring{\leqq}^\alpha_n\rangle_{n<\omega}$" for all $\alpha<\nu$. For $p, q\in P_\nu$ define $q\leqq_n p$ if and only if $q\leqq p$ and for any α in F_n $q\restriction\alpha\Vdash_{P_\alpha}$"$q(\alpha)\mathring{\leqq}^\alpha_n p(\alpha)$". This is the same as (3) in Proposition 2. The difference is that the choice of the F_n is this time independent of $p, q\in P_\nu$. This $\langle\leqq_n\rangle_{n<\omega}$ works. Q. E. D.

THEOREM 3. *Let $(P_\alpha, \mathring{Q}_\alpha)_{\alpha\leq\nu,\alpha<\nu}$ be a countable support iteration such that $\Vdash_{P_\alpha}$"$\mathring{Q}_\alpha$ satisfies Axiom C" for all $\alpha<\nu$. We can show that P_ν satisfies Axiom C.*

To show Theorem 3, we fix a countable support iteration $(P_\alpha, \mathring{Q}_\alpha)_{\alpha\leq\nu,\alpha<\nu}$ and a sequence of names $\langle\mathring{R}_\alpha\rangle_{\alpha<\nu}$ such that $\Vdash_{P_\alpha}$"$\mathring{Q}_\alpha$ satisfies Axiom C with $\mathring{R}_\alpha$" for each $\alpha<\nu$. We first observe the following fusion lemma:

LEMMA 4. *Given a sequence $\langle E_n\rangle_{n<\omega}$ of disjoint finite subsets of δ with $\delta\leqq\nu$*

and a sequence of conditions $\langle p_n \rangle_{n<\omega}$ *from* P_δ. *If we assume that:*

(1) $i < \omega$,

(2) $\forall n<\omega\ p_n \geqq p_{n+1}$,

(3) $\bigcup_{n<\omega} \operatorname{supp}(p_n) \subseteqq \bigcup_{n<\omega} E_n$,

(4) $\forall n \geqq i\ \forall k<n\ \forall \alpha \in E_k\quad p_n \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_n(\alpha))$, *then there is a condition* p *in* P_δ *such that*

(5) $\forall n<\omega\quad p_n \geqq p$,

(6) $\forall n \geqq i\ \forall k<n\ \forall \alpha \in E_k\quad p \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_{n-1}(\alpha), p(\alpha))$.

PROOF. We construct the condition p by induction on $\alpha<\delta$. Suppose we have constructed $p \upharpoonright \alpha$ such that

(7) $\forall n<\omega\quad p_n \upharpoonright \alpha \geqq p \upharpoonright \alpha$,

(8) $\forall n \geqq i\ \forall k \leqq n\ \forall \beta \in E_k \cap \alpha\quad p \upharpoonright \beta \Vdash \mathring{R}_\beta(p_k(\beta), \cdots, p_{n-1}(\beta), p(\beta))$.

It suffices to get $p(\alpha)$: If α is not in any of E_n, then we put $p(\alpha) = \mathring{1}_\alpha$. If α is in some E_k, let us fix such a unique k. If $k<i$ holds, then for all $n \geqq i$ and for all β in E_k $p_n \upharpoonright \beta \Vdash \mathring{R}_\beta(p_k(\beta), \cdots, p_n(\beta))$ holds. But by (7) we have $p \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_n(\alpha))$ for all $n \geqq i$. Applying fusion inside the forcing relation, we have $p(\alpha)$ such that $p \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_{n-1}(\alpha), p(\alpha))$ for all $n \geqq i$. And so it is easy to see $p \upharpoonright \alpha \Vdash p_n(\alpha) \geqq p(\alpha)$ for all $n<\omega$. If $i \leqq k$ holds, then this time for all $n>k$ and for all β in E_k $p_n \upharpoonright \beta \Vdash \mathring{R}_\beta(p_k(\beta), \cdots, p_n(\beta))$ holds. But by (7) again, we have $p \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_n(\alpha))$ for all $n>k$. Therefore as in the previous case there is $p(\alpha)$ such that $p \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_{n-1}(\alpha), p(\alpha))$ holds for all $n>k$ and so $p \upharpoonright \alpha \Vdash p_n(\alpha) \geqq p(\alpha)$ for all $n<\omega$. Note that $\operatorname{supp}(p) \subseteqq \bigcup_{n<\omega} E_n$. Q.E.D.

LEMMA 5. *Suppose* $\langle E_i \rangle_{i<n}$ *is a sequence of disjoint finite subsets of* ρ *with* $\rho \leqq \nu$ *and* $\langle p_i \rangle_{i \leqq n}$ *is a sequence of conditions from* P_ρ *such that*

(1) $p_0 \geqq \cdots \geqq p_n$,

(2) $\forall k<n\ \forall \alpha \in E_k\quad p_n \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_n(\alpha))$.

If E_n *is a finite subset of* ρ *disjoint from* E_0 *through* E_{n-1} *and* D *is a dense subset of* P_ρ *below* p_n, *then there are a condition* p_{n+1} *in* P_ρ *and a countable subset* D' *of* D *such that*

(3) $p_n \geqq p_{n+1}$,

(4) $\forall k<n+1\ \forall \alpha \in E_k\quad p_{n+1} \upharpoonright \alpha \Vdash \mathring{R}_\alpha(p_k(\alpha), \cdots, p_{n+1}(\alpha))$,

(5) $p_{n+1} \leqq \bigvee D'$.

PROOF. We show by induction on $\rho \leqq \nu$. There are two cases:

Case 1. $\bigcup_{i \leqq n} E_i$ is bounded below ρ, say by $\delta<\rho$:

Since D is dense below p_n, we may fix a subset E of D such that

(6) $\forall e \in E\qquad e \leqq p_n$,

(7) $\forall e, e' \in E$ (if $e \neq e'$, then $e \upharpoonright \delta$ and $e' \upharpoonright \delta$ are incompatible in P_δ),

(8) $p_n \upharpoonright \delta \leqq \bigvee \{e \upharpoonright \delta : e \in E\}$.

Applying the induction hypothesis to $\langle E_i\rangle_{i\leqq n}$, $\langle p_i\restriction\delta\rangle_{i\leqq n}$ and $\{b\in P_\delta : \exists e\in E\ b\leqq e\restriction\delta\}$, we have a condition $p_{n+1}\restriction\delta$ (a notational abuse) and a countable subset D' of E such that (4) and $p_{n+1}\restriction\delta\leqq\bigvee\{e\restriction\delta : e\in D'\}$. Since D' is countable we have a condition p_{n+1} in P_ρ such that $p_n\geqq p_{n+1}$ and $e\leqq e\restriction\delta^\frown p_{n+1}\restriction[\delta,\rho)$ and $e\geqq e\restriction\delta^\frown p_{n+1}\restriction[\delta,\rho)$ for each e in D'. Now it is easy to check p_{n+1} and D' work.

Case 2. $\rho=\delta+1$ and $\delta\in E_k$ for some δ and some k with $0\leqq k\leqq n$:

Since D is dense below p_n, we have $p_n\restriction\delta\Vdash$"$\bar{D}=\{d(\delta)[G_\delta] : d\in D$ and $d\restriction\delta\in G_\delta\}$ is dense below $p_n(\delta)$". Thus we may fix P_δ-names $p_{n+1}(\delta)$ and $\bar{D}'$ such that $p_n\restriction\delta$ forces:

(9) $p_{n+1}(\delta)\leqq\bigvee\bar{D}'$,

(10) $p_{n+1}(\delta)\leqq p_n(\delta)$,

(11) $\bar{D}'$ is a countable subset of $\bar{D}$,

(12) $\mathring{R}_\delta(p_k(\delta),\cdots,p_n(\delta),p_{n+1}(\delta))$.

Since $\bar{D}'$ is countable, we may fix a sequence of P_δ-names $\langle\mathring{d}_m\rangle_{m<\omega}$ such that for each P_δ-generic filter G_δ over V with $p_n\restriction\delta\in G_\delta$:

(13) $\forall m<\omega$ ($\mathring{d}_m[G_\delta]\in D$ and $\mathring{d}_m[G_\delta]\restriction\delta\in G_\delta$) and

(14) $\bar{D}'[G_\delta]=\{\mathring{d}_m[G_\delta](\delta)[G_\delta] : m<\omega\}$ hold.

And so for each $m<\omega$ $D_m=\{q\in P_\delta : \exists d\in D\ q\Vdash\mathring{d}_m=d\}$ is dense below $p_n\restriction\delta$. By applying the induction hypothesis to $\langle E_i\cap\delta\rangle_{i<n}$, $\langle p_i\restriction\delta\rangle_{i\leqq n}$ and $\langle D_m\rangle_{m<\omega}$ repeatedly and by Lemma 4, we take a sequence $\langle D'_m\rangle_{m<\omega}$ and a condition $p_{n+1}\restriction\delta$ in P_δ such that

(15) $\forall m<\omega$ (D'_m is a countable subset of D_m),

(16) $\forall m<\omega$ $p_{n+1}\restriction\delta\leqq\bigvee D'_m$,

(17) $p_n\restriction\delta\geqq p_{n+1}\restriction\delta$,

(18) $\forall l<n+1\ \forall\alpha\in E_l\cap\delta$ $p_{n+1}\restriction\alpha\Vdash\mathring{R}_\alpha(p_l(\alpha),\cdots,p_{n+1}(\alpha))$.

Let $p_{n+1}=p_{n+1}\restriction\delta^\frown\langle p_{n+1}(\delta)\rangle$, then it is easy to check that $p_{n+1}\leqq\bigvee\{d\in D : \exists m<\omega\ \exists q\in D'_m\ q\Vdash\mathring{d}_m=d\}$ and $p_{n+1}\restriction\alpha\Vdash\mathring{R}_\alpha(p_l(\alpha),\cdots,p_{n+1}(\alpha))$ holds for all $l<n+1$ and for all $\alpha\in E_l$. Q.E.D.

Proof of Theorem 3. For each p in P_ν, let us fix a function f_p from ω with $\nu\supseteqq f_p``\omega\supseteqq\mathrm{supp}(p)$. For each $p$ in $P_\nu$, let $E_p=f_p``1$ and for each $p_0,\cdots,p_{n+1}\in P_\nu$, let $E_{p_0,\dots,p_{n+1}}=[f_{p_0}``(n+2)\cup\cdots\cup f_{p_{n+1}}``(n+2)]-[f_{p_0}``(n+1)\cup\cdots\cup f_{p_n}``(n+1)]$. We define a subset R of the collection of finite sequences of P_ν such that

(a) $R(p_0)$ for all $p_0\in P_\nu$ and

(b) $R(p_0,\cdots,p_{n+1})$ if $p_0\geqq\cdots\geqq p_{n+1}$ and $p_{n+1}\restriction\alpha\Vdash\mathring{R}_\alpha(p_k(\alpha),\cdots,p_{n+1}(\alpha))$ holds for all $k<n+1$ and for all $\alpha\in E_{p_0,\dots,p_k}$.

It is easy to check that this R works using Lemmas 4 and 5. Q.E.D.

Question. I do not know any example which is Axiom C but not Axiom A.

References

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