J. Math. Soc. Japan
Vol. 43, No. 2, 1991

New properties of special varieties arising from adjunction theory

By Mauro C. BELTRAMETTI and Andrew J. SOMMESE

(Received Dec. 5, 1989)
(Revised June 25, 1990)

Introduction.

Let $X^{\wedge}$ be a connected projective submanifold of $\boldsymbol{P}_C$ and let $L^{\wedge}=\mathcal{O}_{\boldsymbol{P}_C}(1)_{X^{\wedge}}$. Studying the pair $(X^{\wedge}, L^{\wedge})$ adjunction theoretically leads to various classes of varieties with special fibre structures, e.g. scrolls and quadric fibrations. In this article we study these special fibre structures and show that they are in many cases even better behaved that might be expected. It is a blanket assumption in this paper that $\dim X^{\wedge}\geqq 3$.

The adjoint bundle, $K_{X^{\wedge}}+(n-1)L^{\wedge}$, is nef and big except for the following very special pairs (see [**S2**], [**S7**], [**SV**]):

a) $(X^{\wedge}, L^{\wedge})$ is either $(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1))$, a scroll over a curve, or a quadric Q in $\boldsymbol{P}^{n+1}$ with $L^{\wedge}{}_Q=\mathcal{O}_{\boldsymbol{P}^{n+1}}(1)_Q$,

b) $(X^{\wedge}, L^{\wedge})$ is a Del Pezzo variety, i.e. $K_{X^{\wedge}}\approx L^{\wedge -(n-1)}$,

c) $(X^{\wedge}, L^{\wedge})$ is a quadric bundle over a smooth curve,

d) $(X^{\wedge}, L^{\wedge})$ is a scroll over a surface.

The definitions of scrolls and quadric bundles are given in (0.6).

Given such a pair with $K_{X^{\wedge}}+(n-1)L^{\wedge}$ nef and big, there exists a new pair (X, L), *the reduction* of $(X^{\wedge}, L^{\wedge})$, where X is smooth and L is ample, and

◆ there exists a morphism $\pi: X^{\wedge}\to X$ expressing $X^{\wedge}$ as X with a finite set B blown up, $L=(\pi_* L^{\wedge})^{**}$,

◆ $L^{\wedge}\approx\pi^*L-[\pi^{-1}(B)]$ (equivalently $K_{X^{\wedge}}+(n-1)L^{\wedge}=\pi^*(K_X+(n-1)L)$),

◆ $K_X+(n-1)L$ is ample (and in fact very ample by the main result of [**SV**]).

Throughout this introduction $(X^{\wedge}, L^{\wedge})$ will always be a pair as above with a reduction (X, L). It further follows that $K_X+(n-2)L$ is nef and big except for a small list of exceptional pairs (see [**S7**], [**Fj**]):

a) $(X, L)=(\boldsymbol{P}^4, \mathcal{O}_{\boldsymbol{P}^4}(2))$ or $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(3))$,

b) $(X, L)=(Q, \mathcal{O}_Q(2))$ where $(Q, \mathcal{O}_Q(1))$ is a quadric in $\boldsymbol{P}^4$,

c) there is a holomorphic surjection $\phi: X\to C$ onto a smooth curve, C, where $K_X^2\otimes L^3\approx\phi^*H$ for an ample line bundle H on C; in particular the general fibre of ϕ is $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$,

d) (X, L) is a Mukai variety, i.e. $K_X = L^{-(n-2)}$,

e) (X, L) is a Del Pezzo fibration over a curve,

f) (X, L) is a quadric bundle over a surface,

g) (X, L) is a scroll of dimension $n \geqq 4$ over a threefold.

A Del Pezzo fibration over a curve is a pair (X, L) for which there exists a surjective map $\phi: X \to Y$ with connected fibres onto a smooth curve Y, with $K_X \otimes L^{n-2} \approx \phi^*(H)$ for an ample line bundle H on Y.

In this paper we are interested in a number of results about these classes.

The first special class of varieties that we are interested in are Del Pezzo fibrations over curves. In proposition (1.1) we show that the finite set B in the definition of the reduction meets a given fibre of ϕ in at most $f-3$ distinct points where f is the degree of a fibre of ϕ with respect to L. We further show that if $h^0(L^\wedge) \leqq 6$ and $d^\wedge \geqq 9$ then $f \geqq 5$; this is used in [**BSS**] to classify the smooth threefolds of degree 9 and 10 in $\boldsymbol{P}^5$.

A second special class of varieties are those pairs (X, L) with $n=3$ for which there exists a surjective map $\phi: X \to C$ with connected fibres onto a smooth curve, C, with $K_X^2 \otimes L^3 \approx \phi^*(H)$ for an ample line bundle H on C. In this case it is not hard to see that $(F, L_F) = (\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$ where F is the general fibre of ϕ. Our main result about this class is proposition (1.2) which states that ϕ is an algebraic fibre bundle. In (1.2.2) we state that for these smooth threefolds $h^0(L^\wedge) \geqq 7$ if $d^\wedge \neq 5$.

Proposition (3.1) states that $(X^\wedge, L^\wedge) \cong (X, L)$ if $(X^\wedge, L^\wedge)$ is a scroll or if $(X^\wedge, L^\wedge)$ is a quadric bundle over a normal variety, Y, with $\dim X^\wedge \geqq \dim Y+2$. A weaker, but sharp result, is given for $\dim X^\wedge = \dim Y+1$.

The main special class that we consider in this paper are the quadric bundles over surfaces. Here $K_X \otimes L^{n-2} \approx \phi^*(H)$ for an ample line bundle H on a normal surface, Y. The general fibre of ϕ is a smooth quadric. Theorem (2.3) states that Y has at worst rational double points of type A_1 as singularities. Further all fibres are of dimension $n-2$ if $n \geqq 4$. If $n=3$ then a fibre, F, of ϕ is either one dimensional, or it is isomorphic to either $\boldsymbol{F}_0$ with $L_F \approx E+2f$ where E and f are fibres of the two different rulings of $\boldsymbol{F}_0$, or to $\boldsymbol{F}_0 \cup \boldsymbol{F}_1$ with $L_{F_0} \sim \mathcal{O}_{F_0}(1, 1)$, and $L_{F_1} \sim E+2f$. We conjecture in (3.2) that $K_X \otimes L^{n-2} \approx \phi^*(K_Y \otimes \mathcal{H})$ for an ample line bundle $\mathcal{H}$. As evidence we prove proposition (3.3) which states that if the conjecture is true, then $(K_X \otimes L^{n-2})^2$ is spanned by global sections. It is known that if $K_X \otimes L^{n-2}$ is nef and (X, L) is not a quadric bundle, then $(K_X \otimes L^{n-2})^2$ is spanned by global sections.

Building on proposition (3.1) it is shown in §4 that if $(X^\wedge, L^\wedge)$ is a threefold in $\boldsymbol{P}^5$ of degree $d \geqq 9$, and $(X^\wedge, L^\wedge)$ is a scroll over a smooth surface, then $d \leqq 24$ and in fact there are only 6 possible choices of d.

We are very grateful to the referee for a number of valuable suggestions.

In particular the referee pointed out that we had left out the case $\boldsymbol{F}_0 \cup \boldsymbol{F}_1$ as a fibre in Theorem (2.3).

Both authors would like to thank the University of Genova and the University of Notre Dame for making their collaboration possible. The second author would also like to thank the National Science Foundation (DMS 87-22330 and DMS 89-21702).

§0. Background material.

We work over the complex field $\boldsymbol{C}$. By *variety* (*n-fold*) we mean an irreducible and reduced projective scheme V of dimension n. We denote its structure sheaf by $\mathcal{O}_V$. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$.

If V is normal, the *dualizing sheaf*, K_V, is defined to be $j_* K_{\mathrm{Reg}(V)}$ where $j: \mathrm{Reg}(V) \to V$ is the inclusion of the smooth points of V and $K_{\mathrm{Reg}(V)}$ is the canonical sheaf of holomorphic n-forms. Note that K_V is a line bundle if V is Gorenstein.

Let $\mathcal{L}$ be a line bundle on a normal variety V. $\mathcal{L}$ is said to be *numerically effective* (*nef*, for short) if $\mathcal{L} \cdot C \geqq 0$ for all effective curves C on V, and in this case $\mathcal{L}$ is said to be *big* if $c_1(\mathcal{L})^n > 0$ where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. We shall denote by $|\mathcal{L}|$ the complete linear system associated to $\mathcal{L}$ and by $\Gamma(\mathcal{L})$ the space of the global sections. We say that $\mathcal{L}$ *is spanned* if it is spanned by $\Gamma(\mathcal{L})$.

(0.1) We fix some more notation.

$\sim$(respectively $\approx$), the numerical (respectively linear) equivalence of line bundles;
$\chi(\mathcal{L}) = \sum(-1)^i h^i(\mathcal{L})$ the Euler characteristic of a line bundle $\mathcal{L}$;
$p_a(V) = (-1)^n(\chi(\mathcal{O}_V) - 1)$, the arithmetic genus of V; $q(V) = h^1(\mathcal{O}_V)$, the irregularity of V and $p_g(V) = h^0(K_V)$, the geometric genus, for V smooth;
$\kappa(V)$, the Kodaira dimension of V;
$e(V) = c_n(V)$, the topological Euler characteristic of V, for V smooth, where $c_n(V)$ is the n^{th} Chern class of the tangent bundle of V.

Abuses. Line bundles and divisors are used with little (or no) distinction. Hence we shall freely switch from the multiplicative to the additive notation and vice versa. Sometimes symbol "$\cdot$" of intersection of cycles is understood.

(0.2) For a line bundle $\mathcal{L}$ on a normal variety V, the *sectional genus* $g(\mathcal{L})$ of $(V, \mathcal{L})$ is defined by

(0.2.1) $$2g(\mathcal{L}) - 2 = (K_V + (n-1)\mathcal{L}) \cdot \mathcal{L}^{n-1}.$$

If there exist $n-1$ elements of $|\mathcal{L}|$ whose intersection is a reduced, irreducible curve C, $g(\mathcal{L})$ is simply the arithmetic genus $p_a(C)$ of C.

(0.3) Assumption. Throughout this paper it will be assumed that $X^\wedge$ is a smooth connected variety of dimension $n \geqq 2$ and $L^\wedge$ is a very ample line bundle on $X^\wedge$. Further if $n \geqq 3$, we denote by $S^\wedge$ a smooth surface obtained as transverse intersection of $n-2$ general elements of $|L^\wedge|$.

(0.4) Reduction. ([**S7**], (0.5)). Let $(X^\wedge, L^\wedge)$ be as in (0.3). We say that a pair (X, L) with X smooth is a *reduction* of $(X^\wedge, L^\wedge)$ if L is ample and

(0.4.1) there exists a morphism $\pi: X^\wedge \rightarrow X$ expressing $X^\wedge$ as X with a finite set B blown up, $L=(\pi_* L^\wedge)^{**}$;

(0.4.2) $L^\wedge = \pi^* L - [\pi^{-1}(B)]$ (equivalently, $K_X{}^\wedge + (n-1)L^\wedge \approx \pi^*(K_X + (n-1)L)$).

(0.4.3) REMARK. Note that the positive dimensional fibres of π are precisely the linear $\boldsymbol{P}^{n-1} \subset X^\wedge$ with normal bundle $\mathcal{O}_{\boldsymbol{P}^{n-1}}(-1)$. Furthermore by sending each element of $|L|$ to its proper transform, we obtain a $1-1$ correspondence between the smooth divisors in $|L|$ that contain B and the smooth elements of $|L^\wedge|$.

Recall also that if $K_{X^\wedge} + (n-1)L^\wedge$ is nef and big, then there exists a reduction π, (X, L) of $(X^\wedge, L^\wedge)$ and $K_X + (n-1)L$ is ample [**S7**], (4.5). Note that in this case such a reduction, (X, L), is unique up to isomorphism. In this paper we will refer to this reduction, (X, L), as *the reduction* of $(X^\wedge, L^\wedge)$. Indeed, except for the explicit list (see the introduction) of well understood pairs $(X^\wedge, L^\wedge)$ where $K_{X^\wedge} + (n-1)L^\wedge$ is not nef and big, we can assume that the reduction (X, L) of $(X^\wedge, L^\wedge)$ exists. Besides [**S1**], [**S2**], [**S7**] and [**SV**] we also refer to [**BSS**], § 0 where a number of general results on adjunction theory we use are collected with the appropriate references.

(0.5) The adjunction map. The following theorem is an easy consequence of [**S1**] and [**V**] (see also [**SV**], (0.1)).

(0.5.1) THEOREM. *Let* $(X^\wedge, L^\wedge)$ *be as in* (0.3). *Then* $K_{X^\wedge} + (n-1)L^\wedge$ *is spanned by global sections unless either*

i) $(X^\wedge, L^\wedge) \cong (\boldsymbol{P}^n, \mathcal{O}(1))$ *or* $(\boldsymbol{P}^2, \mathcal{O}(2))$;

ii) $(X^\wedge, L^\wedge) \cong (Q, \mathcal{O}_Q(1))$ *where* Q *is a smooth quadric in* $\boldsymbol{P}^{n+1}$;

iii) $X^\wedge$ *is a* $\boldsymbol{P}^{n-1}$ *bundle over a smooth curve and the restriction of* $L^\wedge$ *to a fibre is* $\mathcal{O}_{\boldsymbol{P}^{n-1}}(1)$. ■

Now suppose $K_{X^\wedge} + (n-1)L^\wedge$ to be spanned. Then we shall call the map $\Phi: X^\wedge \rightarrow \boldsymbol{P}^m$ determined by $K_{X^\wedge} + (n-1)L^\wedge$ *the adjunction map*. We shall write $\Phi = s \circ r$ for the Remmert-Stein factorization of Φ, so $r: X^\wedge \rightarrow Y$ is a morphism with connected fibres onto a normal variety $Y=r(X^\wedge)$ and s is a finite map.

Note that if $\dim \Phi(X^\wedge)=n$, then $r: X^\wedge \rightarrow r(X^\wedge)$, $L=(r_* L^\wedge)^{**}$ is the reduction of $(X^\wedge, L^\wedge)$ (see [**SV**], (0.3)). It should be also pointed out that if $K_{X^\wedge}+(n-2)L^\wedge$ is nef, $n \geqq 3$, then $(X^\wedge, L^\wedge)$ coincides with its reduction (X, L). Indeed in this case $K_{X^\wedge}+(n-1)L^\wedge$ is ample and hence spanned by (0.5.1). Therefore the pair $X=r(X^\wedge)$, $L=(r_* L^\wedge)^{**}$ is the reduction of $(X^\wedge, L^\wedge)$. Thus, either r is an isomorphism or $K_{X^\wedge}+(n-1)L^\wedge$ would be trivial on the positive dimensional fibres of r, a contradiction.

Note also that, if $n \geqq 3$, for a smooth $A \in |L^\wedge|$, the restriction of r to A is the adjunction map of the pair $(A, L^\wedge{}_A)$ given by $\Gamma(K_A+(n-2)L^\wedge{}_A)$. To see this note that from the Kodaira vanishing theorem it follows that $h^1(K_{X^\wedge}+(n-2)L^\wedge)=0$, and therefore that the restriction $\Gamma(K_{X^\wedge}+(n-1)L^\wedge) \rightarrow \Gamma(K_A+(n-2)L^\wedge{}_A)$ is onto.

(0.6) Some soecial varieties. Let V be a n-dimensional smooth connected variety, L a *very ample* line bundle on V. We say that (V, L) is a *scroll* (respectively a *quadric bundle*; respectively a *Del Pezzo fibration*) over a normal variety Y of dimension m if there exists a surjective morphism with connected fibres $p: V \rightarrow Y$, and an *ample* line bundle $\mathcal{L}$ on Y, such that $K_V+(n-m+1)L \approx p^* \mathcal{L}$ (respectively $K_V+(n-m)L \approx p^* \mathcal{L}$; respectively $K_V+(n-m-1)L \approx p^* \mathcal{L}$).

Note that if (V, L) is a scroll over either a curve or a surface Y then Y is smooth and (V, L) is a true $\boldsymbol{P}^k$ bundle, $k=n-\dim Y$. That is all fibres F of p are $\boldsymbol{P}^k$ and $L_F \cong \mathcal{O}_{\boldsymbol{P}^k}(1)$, or equivalently $(V, L) \cong (\boldsymbol{P}(\mathcal{E}), \mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1))$ for some ample and spanned (since L is very ample) locally free sheaf, $\mathcal{E}$, of rank $k+1$ on Y. This follows from a general result due to Sommese [**S7**], (3.3).

(0.6.1) LEMMA. *Let (V, L) be either a scroll over Y with* $\dim Y>1$, *a quadric bundle, or a Del Pezzo fibration. Then $K_V+(n-1)L$ is spanned.*

PROOF. In all the cases the adjoint bundle $K_V+(n-1)L \approx p^* \mathcal{L}$ is nef since $\mathcal{L}$ is ample. Then either $K_V+(n-1)L$ is spanned or (V, L) is as in (0.5.1), this contradicting the nefness of $K_V+(n-1)L$. Q.E.D.

(0.6.2) PROPOSITION. *Let (V, L) be a scroll over a smooth surface Y and S a smooth surface in $|L|$. Denote by L_S, $p_S: S \rightarrow Y$ the restriction to S of L, $p: V \rightarrow Y$ respectively and let $L_Y=(p_{S*} L_S)^{**}$. Then*

i) *(Y, L_Y) is the reduction of (S, L_S);*

ii) *L_Y is very ample;*

iii) *K_Y+L_Y is very ample.*

PROOF. First, note that K_S+L_S is nef and big since $K_S+L_S \approx p^* \mathcal{L}$ for some ample line bundle $\mathcal{L}$ on Y. Furthermore K_S+L_S is also spanned in view of (0.6.1). Now general results on adjunction theory (see (0.4.3), (0.5)) say that there exists a reduction (S', L'), $\pi: S \to S'$, of (S, L_S) and $K_{S'}+L'$ is ample since L_S is very ample. Since K_S+L_S is spanned and $p_S{}^*\mathcal{L} \approx K_S+L_S \approx \pi^*(K_{S'}+L')$ with $\mathcal{L}$ and $K_{S'}+L'$ both ample, it thus follows that p_S and π coincide, up to isomorphisms, with the Remmert-Stein factorization of the morphism associated to $\Gamma(K_S+L_S)$. So we conclude that $(Y, L_Y) \cong (S', L')$ and i) is proved.

To show ii), note that L_Y is spanned except at a finite set of points, say $y_1, \cdots, y_t$, where the fibres $f_i = p_S{}^{-1}(y_i)$, $i=1, \cdots, t$, are positive dimensional, since L_S is very ample. Now by choosing a smooth $A \in |L|$ transverse to the f_i's, we can assume that the restriction $p_A: A \to Y$ has positive dimensional fibres at points $a_1, \cdots, a_s$ with $\{a_j | j=1, \cdots, s\} \cap \{y_i | i=1, \cdots, t\} = \emptyset$. Therefore L_Y is spanned at $y_1, \cdots, y_t$ too and hence it is spanned. Furthermore the same argument above shows that L_Y is very ample on a Zariski open set U containing the points $y_1, \cdots, y_t$. Hence in particular L_Y separates any two points, so it is very ample.

Finally, iii) is a direct consequence of [**SV**] and the general fact (0.6.3) below. Indeed, in view of i) and ii), [**SV**] applies to say that K_Y+L_Y is very ample unless either:

a) (Y, L_Y) is a Del Pezzo surface with $L_Y \approx -3K_Y$, $K_Y \cdot K_Y = 1$,
b) (Y, L_Y) is a Del Pezzo surface with $L_Y \approx -2K_Y$, $K_Y \cdot K_Y = 2$, or
c) Y is a $\boldsymbol{P}^1$ bundle over an elliptic curve of invariant $e=-1$, $L_Y \approx 3\zeta$, ζ the tautological line bundle.

All the cases above are ruled out by Lemma (0.6.3) below. Let $V = \boldsymbol{P}(\mathcal{E})$, $\mathcal{E}$ locally free rank 2 vector bundle on Y. Note that $L_Y = \det \mathcal{E}$. In case a), since K_Y^{-1} is ample with $K_Y \cdot K_Y = 1$, there exists a smooth elliptic curve $C \in |K_Y^{-1}|$ and $\det \mathcal{E} \cdot C = L_Y \cdot C = 3\ K_Y \cdot K_Y = 3$. In case b), K_Y^{-1} is spanned since K_Y^{-1} is ample and $K_Y \cdot K_Y > 1$. Then we can choose again a smooth elliptic curve $C \in |K_Y^{-1}|$ and $\det \mathcal{E} \cdot C = L_Y \cdot C = 2\ K_Y \cdot K_Y = 4$. In case c), let C be a section of the $\boldsymbol{P}^1$ bundle Y with $C \cdot C = -e = 1$. Hence $\det \mathcal{E} \cdot C = L_Y \cdot C = 3\zeta \cdot C = 3$. Q. E. D.

(0.6.3) LEMMA. *Let Y be a smooth connected projective surface, $\mathcal{E}$ a locally free rank 2 coherent sheaf on Y. Let C be a smooth elliptic curve on Y. Assume that the tautological line bundle $\zeta = \mathcal{O}_{\boldsymbol{P}(\mathcal{E}_C)}(1)$ is very ample on $\boldsymbol{P}(\mathcal{E}_C)$. Then* $\deg \mathcal{E}_C = \det \mathcal{E} \cdot C \geqq 5$ *with equality only if $\boldsymbol{P}(\mathcal{E}_C)$ is embedded by ζ as a degree 5 surface in $\boldsymbol{P}^4$.*

PROOF. First of all, $\mathcal{E}_C$ is not the direct sum of line bundles. Assume otherwise that $\mathcal{E}_C = \mathcal{L} \oplus \mathcal{M}$ is a direct sum of two line bundles $\mathcal{L}$, $\mathcal{M}$ of degrees $l = \deg \mathcal{L}$, $m = \deg \mathcal{M}$. Then l, m are both positive and $l = h^0(\mathcal{L})$, $m = h^0(\mathcal{M})$ by the Riemann-Roch theorem. Therefore

$$h^0(\mathcal{E}_C) = h^0(\mathcal{L}) + h^0(\mathcal{M}) = l + m$$

If $\deg \mathcal{E} = l+m \leqq 4$, $\Gamma(\mathcal{E}_C) \cong \Gamma(\zeta)$ embeds the $\boldsymbol{P}^1$ bundle $\boldsymbol{P}(\mathcal{E}_C)$ over the elliptic curve C in $\boldsymbol{P}^3$, which is not possible. This shows that $\mathcal{E}_C$ is given by an extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}_C \longrightarrow \mathcal{M} \longrightarrow 0$$

where $\mathcal{L}$, $\mathcal{M}$ are line bundles of degrees l, m with either $l=m$ if $\deg \mathcal{E}_C$ is even or $1+l=m$ if $\deg \mathcal{E}_C$ is odd. Therefore a straightforward computation, by using the Riemann-Roch theorem shows that $h^0(\mathcal{E}_C) \leqq h^0(\mathcal{L}) + h^0(\mathcal{M}) \leqq 4$ whenever $\deg \mathcal{E}_C = l+m \leqq 4$ and the same argument as above gives the result. Q.E.D.

(0.6.4) The Hirzebruch surfaces. By $\boldsymbol{F}_r$ with $r \geqq 0$ we denote the r^{th} Hirzebruch surface. $\boldsymbol{F}_r$ is the unique holomorphic $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$ with a section E satisfying $E \cdot E = -r$. Let $p: \boldsymbol{F}_r \to \boldsymbol{P}^1$ denote the bundle projection. In the case $r=0$, $\boldsymbol{F}_0$ is simply $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. In the case $r \geqq 1$, E is the unique irreducible curve on $\boldsymbol{F}_r$ with negative self-intersection. By $\tilde{\boldsymbol{F}}_r$, with $r \geqq 1$, we denote the normal surface obtained from $\boldsymbol{F}_r$ by contracting E. In the case $r=1$, $\tilde{\boldsymbol{F}}_1$ is $\boldsymbol{P}^2$. We shall denote by $\boldsymbol{E}$, f a basis for the second integral cohomology of $\boldsymbol{F}_r$, f a fibre of p. A line bundle $aE+bf$ is ample if and only if it is very ample and it is very ample if and only if $a>0$ and $b>ar$.

(0.7) Castelnuovo's bound. Let $(X^\wedge, L^\wedge)$, $S^\wedge$ be as in (0.3). Assume that $|L^\wedge|$ embeds $X^\wedge$ in a projective space $\boldsymbol{P}^N$, $N \geqq 4$, and let $d^\wedge = L^{\wedge n}$. Then $g(C) = g(L^\wedge)$, where C is a smooth curve obtained as transverse intersection of $n-1$ general members of $|L^\wedge|$, and Castelnuovo's Lemma (see e.g. [**ACGH**] or [**BSS**], (0.11)) says that

$$g(L^\wedge) \leqq \left[\frac{d^\wedge-2}{N-n}\right]\left(d^\wedge - N + n - 1 - \left(\left[\frac{d^\wedge-2}{N-n}\right]-1\right)\frac{N-n}{2}\right) \tag{0.7.1}$$

where $N = h^0(L^\wedge)-1$ and $[x]$ means the greatest integer $\leqq x$.

Note that if $g(L^\wedge)$ does not reach the maximum with respect to (0.7.1) then the stronger bound

$$g(L^\wedge) \leqq d^\wedge(d^\wedge-3)/6+1 \tag{0.7.2}$$

holds true (see [**GP**]). ∎

We need some results on codimension two varieties.

(0.8) Lemma ([**H**], p. 434). *Let $(X^\wedge, L^\wedge)$, $S^\wedge$ be as in* (0.3). *Assume that $|L^\wedge|$ embeds $X^\wedge$ in a projective space $\boldsymbol{P}^N$ with $N=n+2$ and let $d^\wedge = L^{\wedge n}$. Then*

$$d^{\wedge 2} - 5d^\wedge - 10(g(L^\wedge)-1) + 12\chi(\mathcal{O}_{S^\wedge}) = 2K_{S^\wedge} \cdot K_{S^\wedge}.$$

(0.9) Lemma. *Let $(X^\wedge, L^\wedge)$, $S^\wedge$ be as in* (0.3). *Assume that $|L^\wedge|$ embeds*

$X^\wedge$ in a projective space $\boldsymbol{P}^N$ with $N=n+2$ and let $d^\wedge=L^{\wedge n}\geqq 4$. Then $K_{X^\wedge}+(n-1)L^\wedge$ is spanned.

PROOF. Assume that $K_{X^\wedge}+(n-1)L^\wedge$ is not spanned. By (0.5.1) it thus follows that $X^\wedge$ is a $\boldsymbol{P}^1$ bundle $p: X\to B$ over a smooth curve B and $L^\wedge_f\cong \mathcal{O}_{P^1}(1)$ for any fibre f. Now, a standard consequence of the Barth-Lefschetz Theorem implies that $q(X^\wedge)=0$, therefore $B\cong \boldsymbol{P}^1$ and hence $g(L^\wedge)=g(B)=0$. Furthermore the restriction $p: S^\wedge\to B$ is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$ so that $K_{S^\wedge}\cdot K_{S^\wedge}=8$ and $\chi(\mathcal{O}_{S^\wedge})=1$. Thus Lemma (0.8) gives $d^{\wedge 2}-5d^\wedge+6=0$, whence the contradiction $d^\wedge=2, 3$. Q.E.D.

(0.10) Congruences for 3-folds in $\boldsymbol{P}^5$. Let $(X^\wedge, L^\wedge)$ be as in (0.3) with $n=3$. Assume that $|L^\wedge|$embeds $X^\wedge$ in $\boldsymbol{P}^5$ and let $d^\wedge=L^{\wedge 3}$. Define

$$d_j^\wedge=(K_{X^\wedge}+L^\wedge)^j\cdot L^{\wedge 3-j}, \qquad j=0, 1, 2, 3,\ d_0^\wedge=d^\wedge .$$

Simply as a consequence of the Riemann-Roch theorem one has for any given value of $d^\wedge$ the following congruence (see [**BSS**], (0.17.5) and (3.7)).

$$\text{(0.10.1)} \qquad 11d^{\wedge 2}-2d^\wedge-d^\wedge d_1^\wedge+16d_1^\wedge+7d_2^\wedge+d_3^\wedge\equiv 0\ (24).$$

Note that $d_1^\wedge=K_{S^\wedge}\cdot L_{S^\wedge}$ and $d_2^\wedge=K_{S^\wedge}\cdot K_{S^\wedge}$. In particular if there exists a reduction (X, L) of $(X^\wedge, L^\wedge)$, let γ be the number of points blown up and define $d_j=(K_X+L)^j\cdot L^{3-j}$, $j=0, 1, 2, 3$. Then $d_j^\wedge=d_j-(-1)^j\gamma$ and the congruence above becomes

$$\text{(0.10.2)} \qquad 11d^{\wedge 2}-\gamma d^\wedge-2d^\wedge-d_1d^\wedge+16d_1+7d_2+d_3+10\gamma\equiv 0\ (24). \qquad \blacksquare$$

Let us recall a standard general fact we use in the sequel, as well as the Nakano's contractibility criterion in the form we use over and over in § 2.

(0.11) LEMMA. *Let $\varphi: X\to Y$ be a surjective morphism of complex irreducible projective varieties. Let* $\dim X=n$ *and assume* $\mathrm{Pic}(X)\cong \boldsymbol{Z}[L]$, *for some line bundle L on X. Then either* $\dim\varphi(X)=0$ *or n. Furthermore φ is finite to one if* $\dim\varphi(X)=n$.

PROOF. Let $\dim Y=s$, $1\leqq s<n$. We can assume that X is smooth. Let H be an ample line bundle on Y and let $\mathcal{L}=\varphi^*H$. Since mH is very ample for $m\gg 0$, the line bundle $\mathcal{L}$ is non trivial. Therefore $\mathcal{L}=\alpha L$ for some integer α, $\alpha\neq 0$. Thus $\alpha L\approx\varphi^*H$ would be trivial on the fibres of φ, a contradiction. The same argument shows that φ is finite to one if $\dim\varphi(X)=n$. Q.E.D.

(0.12) THEOREM (Nakano's contractibility criterion [**N**]). *Let X be a smooth projective variety and let $Z\subset X$ be a subvariety of X which is a $\boldsymbol{P}^k$ bundle $p: Z\to Z'$ over a variety Z'. Let $F\cong\boldsymbol{P}^k$ be a fiber of p. If $\mathcal{N}^X_{Z|F}\cong\mathcal{O}_{\boldsymbol{P}^k}(a)$ with*

$a<0$ there exists a holomorphic map $\tilde{p}$, an analytic variety X' and a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ Z' & \longrightarrow & X' \end{array}$$

such that $\tilde{p}$ induces a biholomorphism $X\setminus Z\cong X'\setminus Z'$. Furthermore X' is smooth if $a=-1$. ∎

For any further background material we refer to [**S5**], [**S7**] and [**BSS**].

§ 1. The Del Pezzo fibrations and the "special" P^2 bundle cases.

Let $X^{\wedge}$ be a smooth connected 3-fold and let $L^{\wedge}$ be a very ample line bundle on $X^{\wedge}$ such that $\Gamma(L^{\wedge})$ embeds $X^{\wedge}$ in a $\boldsymbol{P}^N$. In this section we deal with two special cases when $K_{X^{\wedge}}+2L^{\wedge}$ is nef and big and $(X^{\wedge}, L^{\wedge})$ admits as the reduction (X, L) either a Del Pezzo fibration or a "special" $\boldsymbol{P}^2$ bundle as in Proposition (1.2) below (compare with [**S7**], § 5). We prove here some new properties of such special varieties. We also find a lower bound for $h^0(L^{\wedge})$ which essentially shows that such a reduction, (X, L), cannot lie in $\boldsymbol{P}^5$, except for a few possible cases. Some of the following results are needed in [**BSS**], § 4.

(1.1) PROPOSITION. *Let $X^{\wedge}$ be a smooth connected threefold, $L^{\wedge}$ a very ample line bundle on $X^{\wedge}$ such that $\Gamma(L^{\wedge})$ embeds $X^{\wedge}$ in $\boldsymbol{P}^N$. Let (X, L) be the reduction of $(X^{\wedge}, L^{\wedge})$. Assume that (X, L) is a Del Pezzo fibration $\varphi: X\to B$ over a smooth curve B.*

(1.1.1) *Let F be the general fiber of φ and let $\deg F=K_F\cdot K_F=f$. Then there are no fibers of φ containing more than $f-3$ distinct points blown up under $r: X^{\wedge}\to X$. (Hence in particular $X^{\wedge}\cong X$ if $f=3$.)*

(1.1.2) *One has $N\geqq 6$ if $f<5$ and $d^{\wedge}=L^{\wedge 3}\geqq 9$.*

PROOF. Assume that there exists a fiber F of φ containing $x_1, \cdots, x_{f-2}$ distinct points of X blown up under r. Write $b=\varphi(F)$, $\boldsymbol{P}_i^2=r^{-1}(x_i)$, $i=1, \cdots, f-2$ and let $p=\varphi\circ r$. Then

$$p^{-1}(b)=R\cup \boldsymbol{P}_1^2\cup \cdots \cup \boldsymbol{P}_{f-2}^2$$

where R is a surface in $X^{\wedge}$. Since $\deg F=f$ and r is an isomorphism outside of a finite number of points, R is a (possibly singular, reducible or not reduced) quadric. If R is a smooth quadric, then the intersection $\boldsymbol{P}_i^2\cap R$ is a curve on

R which contracts to a point x_i, $i=1, \cdots, f-2$ and this is not possible. In all the other cases either R, a component $\boldsymbol{P}^2$ of R or R_{red} meets one the $\boldsymbol{P}_i^2$'s in a curve which again contracts to a point x_i. On the other hand, since $\text{Pic}(R) \cong \text{Pic}(\boldsymbol{P}^2) \cong \text{Pic}(R_{\text{red}}) \cong \boldsymbol{Z}$, the restriction of r to either R, $\boldsymbol{P}^2$, or R_{red} is finite to one by Lemma (0.11), this leading to a contradiction.

To prove (1.1.2), first recall that $3 \leqq f \leqq 9$ by general results on classification. Furthermore note that $N \geqq 5$, since otherwise $X^\wedge$ would have Kodaira dimension $\kappa(X^\wedge)=3$. Assume $N=5$ and $f \leqq 4$. Then $q(X^\wedge)=0$ since $X^\wedge$ is simply connected by the Barth-Lefschetz Theorem and hence $B \cong \boldsymbol{P}^1$. Let us denote by F a general fibre of the composition $p=\varphi \circ r$ (note that general fibers of p and φ are isomorphic) and look at the restriction $L^\wedge_F$ of $L^\wedge$ to F. The Riemann-Roch theorem gives us $h^0(L^\wedge_F)=f+1 \leqq 5$. Therefore the standard exact sequence

$$0 \longrightarrow L^\wedge \otimes \mathcal{O}_{X^\wedge}(-F) \longrightarrow L^\wedge \dashrightarrow L^\wedge_F \longrightarrow 0$$

shows that $h^0(L^\wedge \otimes \mathcal{O}_{X^\wedge}(-F)) \geqq 1$. Now the following Claim leads to a contradiction which gives the result.

CLAIM. If $N=5$, $h^0(L^\wedge \otimes \mathcal{O}_{X^\wedge}(-F))=0$ for any $f \geqq 3$.

PROOF. Let D be an effective divisor in $|L^\wedge - F|$. Since $L^\wedge - D \approx F = p^*\mathcal{O}_{\boldsymbol{P}^1}(1)$ we have $h^0(L^\wedge - D)=2$ which means that D is contained in a $\boldsymbol{P}^3$ of $\boldsymbol{P}^5$ given by the transverse intersection of two elements of $|L^\wedge|$. Hence in particular $\deg D = D \cdot L^\wedge \cdot L^\wedge$ in $\boldsymbol{P}^3 = L^\wedge \cdot L^\wedge$ so that

$$K_D \approx \mathcal{O}_D(L^\wedge \cdot L^\wedge \cdot D - 4).$$

We also have

$$K_D \approx (K_{X^\wedge} + D)_{|D} \approx (K_{X^\wedge} + L^\wedge - F)_{|D}$$

which shows that K_D is trivial on the fibres of the restriction $D \to \boldsymbol{P}^1$. Thus

$$L^\wedge \cdot L^\wedge \cdot D - 4 = L^\wedge \cdot L^\wedge \cdot (L^\wedge - F) - 4 = 0$$

that is $d^\wedge = f+4 \leqq 8$, a contradiction. Q.E.D.

(1.2) PROPOSITION. *Let $X^\wedge$ be a smooth connected threefold, $L^\wedge$ a very ample line bundle on $X^\wedge$ such that $\Gamma(L^\wedge)$ embeds $X^\wedge$ in $\boldsymbol{P}^N$. Let (X, L) be the reduction of $(X^\wedge, L^\wedge)$. Assume that there is a holomorphic surjection with connected fibres $\varphi: X \to B$ onto a nonsingular curve B, whose general fibre is $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$ and $K_X^2 \otimes L^3 \approx \varphi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on B. Then we have*

(1.2.1) *$(F, L_F) \cong (\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$ for any fibre F of φ;*

(1.2.2) *$N \geqq 6$ if $d^\wedge = L^{\wedge 3} \neq 5$.*

PROOF. By (0.5.1), $K_{X^\wedge} + 2L^\wedge$ is spanned, and hence so is $H := K_X + 2L$. H

is ample since $2H\approx(2K_X+3L)+L\approx\varphi^*\mathcal{L}+L$. The restriction of H to any general fibre ($\cong \boldsymbol{P}^2$) is $\mathcal{O}_{\boldsymbol{P}^2}(1)$. So $H^2\cdot F=1$ for any fibre F. Hence $(F, H_F)\cong(\boldsymbol{P}^2, H_F)$ since H is ample and spanned. This implies (1.2.1).

To prove (1.2.2), assume $N=5$. Then the results of [**BSS**], §4 (see proof of (4.3), (4.3.2), (4.3.8)) apply to say that the sectional genus $g(L)$ of (X, L) is odd and

$$(1.2.3)\qquad 12(g(L)-1)=2d\hat{}(d\hat{}-3)+24-7d\hat{}+\gamma\,,$$

$$(1.2.4)\qquad 54(g(L)-1)=4d\hat{}^2-43d\hat{}+17\gamma+168\,,$$

where γ denotes the points of X blown up under $r: X\hat{}\to X$. Thus from (1.2.3), (1.2.4) we find

$$(1.2.5)\qquad \gamma=(10d\hat{}^2-31d\hat{}-120)/25\,.$$

Now either $g(L)=(d\hat{}^2+3)/4-d\hat{}$ or $g(L)\leqq d\hat{}(d\hat{}-3)/6+1$ by (0.7). In the first case, by using (1.2.3) and (1.2.5) we find an equation in $d\hat{}$ with no integer solutions. In the second case we easily see from (1.2.3) that $7d\hat{}\geqq 24+\gamma$ and hence we get from (1.2.5)

$$5d\hat{}^2-103d\hat{}+240\leqq 0$$

which implies that $d\hat{}\leqq 17$. A straightforward check shows that only $d\hat{}=5$ satisfies the congruence $10d^{2}\hat{}-31d\hat{}-120\equiv 0\ (25)$ which comes out from (1.2.5). This completes the proof. Q.E.D.

§2. The quadric bundle case.

Let $X\hat{}$ be a smooth connected n-fold and let $L\hat{}$ be a very ample line bundle on $X\hat{}$. Through this section we consider the case when $K_{X\hat{}}+(n-1)L\hat{}$ is nef and big and the reduction (X, L) of $(X\hat{}, L\hat{})$ exists. Let $r: X\hat{}\to X$ be the reduction map. Assume further that for $t\gg 0$, $\Gamma(t(K_X+(n-2)L))$ gives a morphism φ down to a surface. Note that, if $s\circ\phi$ is the Remmert-Stein factorization of φ, one has $K_X+(n-2)L\approx\phi^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$, which means that (X, L) is a *quadric bundle* over a normal surface via ϕ.

Let us recall the following standard facts we use over and over. The main reference in this section is [**S5**] (see also [**F**]).

(2.1) LEMMA. *With the notation as above, let D be an irreducible reduced divisor such that* $\dim\varphi(D)=0$. *Then there are no locally complete intersection, irreducible, reduced curves C with* $\dim\varphi(C)=0$ *such that $\mathcal{N}_C^X$ is spanned by global sections, $h^1(\mathcal{N}_C^X)=0$ and $D\cdot C>0$. In particular there exist no locally complete intersection, irreducible, reduced curves C on D such that $\mathcal{N}_C^X$ is spanned by global* *sections,* $h^1(\mathcal{N}_C^X)=0$ *and* $\deg(\mathcal{N}_{D|C}^X)>0$.

PROOF. Since $\varphi(C)$ is a point, by the rigidity property of proper maps there exists a tubular neighborhood U_C of C such that $\varphi(U_C)$ is contained in an affine neighborhood V_C of $\varphi(C)$. Now, since $\mathcal{N}_C^X$ is spanned and $h^1(\mathcal{N}_C^X)=0$, Kodaira-Spencer deformation theory applies to say that deformations of C exist to fill up a neighborhood $\mathcal{T}_C$ of C (see also [**Bl**]). Therefore, since V_C contains no compact subvarieties, $\varphi(C')=0$ for all $C'\in U_C\cap\mathcal{T}_C$. But $D\cdot C>0$ and hence $\varphi(C')=\varphi(C)$ for all $C'\in U_C\cap\mathcal{T}_C$. It thus follows that $\dim\varphi(U_C)$ $(=\dim\varphi(X))=0$, a contradiction. Q.E.D.

(2.2) LEMMA ([**S5**], (0.5.3)). *With the notation as above, let B be the finite set of points of X blown up to get $X^\wedge$ and let C be an effective curve on X such that $L\cdot C=1$. Then C is a smooth rational curve and $C\cap B=\emptyset$.*

PROOF. Since L is ample, $L\cdot C=1$ implies that C is irreducible and reduced. Let C' be the proper transform of C under r. If C meets B then C' would meet $r^{-1}(B)$. This would imply that

$$L^\wedge\cdot C' = (r^*L - r^{-1}(B))\cdot C' = L\cdot C - r^{-1}(B)\cdot C' \leqq 1-1 = 0\,.$$

Since $L^\wedge$ is ample we conclude that $C\cap B=\emptyset$. Therefore $C'\cong C$ and $L^\wedge\cdot C'=1$. Since $L^\wedge$ is spanned by global sections, this implies that C' and hence C is a smooth rational curve. Q.E.D.

The following is the main result of this section. The proof of the first part of the statement consists of a sequence of several Lemmas which take up the rest of the section.

(2.3) THEOREM. *Let (X, L) be the reduction of an n-dimensional polarized pair $(X^\wedge, L^\wedge)$ with $X^\wedge$ smooth and $L^\wedge$ very ample. Assume that, for all $t\gg 0$, $\Gamma(t(K_X+(n-2)L))$ gives a morphism φ onto a normal surface Y.*

(2.3.1) *Either φ has equal dimensional fibres or $n=3$ and the only divisorial fibres are*
- a) *isomorphic to $\boldsymbol{F}_0$ with $L_{F_0}\sim\mathcal{O}_{F_0}(1, 2)$, or*
- b) *isomorphic to $\boldsymbol{F}_0\cup\boldsymbol{F}_1$ with $L_{F_0}\sim\mathcal{O}_{F_0}(1, 1)$, $L_{F_1}\sim E+2f$. In this case φ is described in Lemma (2.6).*

(2.3.2) *The surface Y has at worst Gorenstein rational singular points, y, of type A_1, such that $\varphi^{-1}(y)$ is a 1-dimensional, non-reduced fibre.*

PROOF. By taking t large enough, we can assume that Y is normal, φ does not depend on t, and φ has connected fibres.

First, assume $n=3$. If a fibre F has a two dimensional irreducible component then all irreducible components of F are two dimensional. To see this assume otherwise. Then there is an irreducible one dimensional component C on F and an irreducible two dimensional component D such that $D\cdot C>0$. Let C' be the proper transform of C under the reduction map $r: X^{\wedge}\to X$. Since the general fibre of the composition of $r: X^{\wedge}\to X$ and φ is a curve of degree 2 relative to $L^{\wedge}$, it follows that the intersection of an element A of $|L^{\wedge}|$ and $r^{-1}(F)$ has at most two connected components. From this and the fact that $L^{\wedge}$ is very ample it follows that $L^{\wedge}\cdot C'\leqq 2$. Therefore C' is smooth rational. Note that for small deformations r' of the restriction map, $r_{C'}$, $r'(C')$ is in the same homology class as C and therefore $r'(C')\cdot D>0$ and $\varphi(r'(C'))=\varphi(F)$. Noting that $\deg(r_{C'}{}^*T_X)=\deg(r_{C'}{}^*K_X^{-1})=L\cdot C\geqq 1$, it follows (e.g. from Proposition 3 of [**M**]) that there is at least a 4-dimensional connected family of deformations r' of the map $r_{C'}$. Since $\dim \mathrm{Aut}(\boldsymbol{P}^1)=3$ it follows that there is at least a one dimensional non-trivial family of curves near C that go to $\varphi(C)$. This contradicts the assertion that C is an irreducible one dimensional component of F.

Let F' be a 2-dimensional fibre of φ. Let $F=F'_{\mathrm{red}}=\bigcup_{i=1}^{s} D_i$ where the D_i's denote the irreducible components of F. By [**S5**], (0.5.2) a general smooth surface $S\in|L|$ meets each D_i along an irreducible reduced curve C_i, $i=1,\cdots,s$. One has

$$K_S\cdot C_i=(K_{X|S}+L_S)\cdot C_i=(K_X+L)\cdot C_i=0$$

and hence $C_i^2<0$ by the Hodge index theorem. Therefore $C_i^2=-2$ and C_i is a -2 smooth rational curve, $i=1,\cdots,s$. The intersection, $S\cap D_i$, is of dimension 1 and is smooth. This shows that S meets D_i transversely along a smooth $\boldsymbol{P}^1$ contained in the smooth points of D_i. Then by [**S3**], (0.6.2), D_i is either $\tilde{\boldsymbol{F}}_2$, $\boldsymbol{P}^2$, or $\boldsymbol{F}_r$ for some $r\geqq 0$, $i=1,\cdots,s$.

Note that there are no 3 distinct components, D_1, D_2, D_3, of F with non-empty intersection $D_1\cap D_2\cap D_3$. Otherwise, let $x\in D_1\cap D_2\cap D_3$ and choose a smooth $S\in|L|$ that contains x. Slicing with S we see that S contains a configuration of at least three -2 rational curves meeting at x. This is not possible (see [**S5**], (0.7)).

After Lemma (2.6) it will be shown that there are at most 2 distinct irreducible components of a two dimensional fibre of φ.

The proof of (2.3.1) runs now parallel to that of Theorem (1.0.1) of [**S5**]. It will be a consequence of the following lemmas where a case by case analysis of all possible configurations of the D_i's is carried out. Recall that we are assuming $\dim X=3$. The notation is as above.

(2.4) LEMMA. *Let D_1 and D_2 be two irreducible reduced surfaces on X such that $D_1\cap D_2$ is non-empty and* $\dim\varphi(D_i)=0$ *for $i=1, 2$. Then neither D_1 nor D_2*

are isomorphic to $\tilde{\boldsymbol{F}}_2$. Also D_1 and D_2 meet transversely in a smooth rational curve C satisfying $L\cdot C=1$.

PROOF. Let C be the curve intersection of D_1 and D_2. As above, by [**S5**], (0.5.2), we can choose a general $S\in|L|$ which is smooth and meets D_1 and D_2 in irreducible reduced curves C_1, C_2 respectively; furthermore C_1, C_2 are -2 smooth rational curves and (compare with [**S5**], (0.7)) they meet transversely in a single point x. Therefore

$$C_1\cdot C_2=L_{D_1}\cdot D_{2|D_1}=L\cdot D_1\cdot D_2=1\,.$$

Since only even numbers arise as intersections of Cartier divisors on $\tilde{\boldsymbol{F}}_2$ we also conclude that neither D_1 nor D_2 are isomorphic to $\tilde{\boldsymbol{F}}_2$. Hence each D_i is isomorphic to either $\boldsymbol{F}_r$ with $r\geqq 0$ or $\boldsymbol{P}^2$. Since $L\cdot D_1\cdot D_2=L\cdot C=1$ we conclude from Lemma (2.2) that C is a smooth rational curve. Therefore the same argument as in the proof of (1.2) shows that the intersection of D_1 and D_2 is transverse. Q. E. D.

(2.5) LEMMA. *Let F' be a divisorial fibre of $\varphi: X\to Y$ and let $F=F'_{\mathrm{red}}$. Then there are no irreducible components D of F such that $D\cong\boldsymbol{P}^2$. (In particular F cannot be isomorphic to $\boldsymbol{P}^2$.)*

PROOF. Let D_1, D_2, $C=D_1\cap D_2$ be as in (2.4) and assume $D_1\cong\boldsymbol{P}^2$. The normal bundle $\mathcal{N}_C^X$ of C in X is of the form $\mathcal{N}_C^X\cong\mathcal{N}_1\oplus\mathcal{N}_2$ where $\mathcal{N}_i$ denotes the normal bundle of C in D_i, $i=1, 2$. Since $(K_X\otimes L)_{|C}\sim\mathcal{O}_C$ and $L\cdot C=1$ one has $K_X\cdot C=-1$. Therefore from the adjunction formula $K_C\approx K_{X|C}\otimes\det(\mathcal{N}_C^X)$, it follows that

$$\deg\mathcal{N}_C^X=\deg\mathcal{N}_1+\deg\mathcal{N}_2=-1\,. \tag{2.5.1}$$

Then both D_1 and D_2 cannot be isomorphic to $\boldsymbol{P}^2$. Otherwise, since $L\cdot C=1$ it would follow that they meet in a line and we would have $\deg\mathcal{N}_1=\deg\mathcal{N}_2=1$, contradicting (2.5.1).

Thus, recalling (2.4), D_2 is isomorphic to $\boldsymbol{F}_r$ for some $r\geqq 0$. Note that $L_{\boldsymbol{F}_r}\sim E+bf$, $b\geqq 1$. Indeed we know by the above that $L_{\boldsymbol{F}_r}$ is a -2 smooth rational curve. Therefore from the classification of polarized surfaces carrying an ample line bundle of sectional genus zero it turns out that $L_{\boldsymbol{F}_r}\cdot f=1$, and hence $L_{\boldsymbol{F}_r}\sim E+bf$ with $b\geqq r+1$ by ampleness.

(2.5.2) CLAIM. One has $r=2$, $C\sim E$ and $L_{\boldsymbol{F}_2}\sim E+3f$ on $\boldsymbol{F}_2$.

PROOF. Write $C\sim\alpha E+\beta f$ and suppose $C\neq E, f$. Then $C\cdot E\geqq 0$ and $C\cdot f>0$ lead to $\beta\geqq\alpha r$, $\alpha>0$. Hence from

$$L_{\boldsymbol{F}_r}\cdot C=(E+bf)\cdot(\alpha E+\beta f)=1\,,$$

and $b \geqq r+1$, we find $\alpha(r+1) \leqq 1$. Since $\alpha > 0$, one has $\alpha = 1$, $r=0$. Then by interchanging the role of E and f we get $\beta > 0$ whence the contradiction $L_{F_0} \cdot C = b + \beta = 1$.

Thus either $C \sim E$ or $C \sim f$ on $\boldsymbol{F}_r$. From $C \cong \boldsymbol{P}^1$ and $L \cdot C = L_{\boldsymbol{P}^2} \cdot C = 1$, it easily follows that C is a line on $D_1 \cong \boldsymbol{P}^2$. If $C \sim f$ one has $\mathcal{N}_C^X \approx \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}$, which contradicts (2.5.1). Therefore $C \sim E$ and $\mathcal{N}_C^X \approx \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-r)$ leads to $r=2$. Now from $L_{\boldsymbol{F}_2} \cdot C = (E+bf) \cdot E = 1$ we find $b=3$. ∎

Thus $D_1 \cong \boldsymbol{P}^2$, $D_2 \cong \boldsymbol{F}_2$. We claim that there are no more components of F. Indeed, suppose there exists a more component D_3. Then by the above we know that either $D_3 \cong \boldsymbol{P}^2$ or $D_3 \cong \boldsymbol{F}_r$, $r \geqq 0$; further if $D_3 \cong \boldsymbol{P}^2$ one has $D_3 \cap D_1 = \emptyset$ and D_3 meets $\boldsymbol{F}_2$ along E $(= \boldsymbol{P}^2 \cap \boldsymbol{F}_2)$, which is a contradiction. Hence $D_3 \cong \boldsymbol{F}_r$. Note that if D_3 meets $\boldsymbol{P}^2$ along a curve, say γ, then $\gamma \cap E \neq \emptyset$ and therefore D_3 meets $\boldsymbol{F}_2$ along a curve C' too. Hence claim (2.5.2) applies to say that either $C' \sim E$ or $C' \sim f$. In both the cases, by slicing with a generic smooth $S \in |L|$ we find three -2 smooth rational curves meeting in a point and this configuration is not possible (compare again with [**S5**], (0.7)).

Now we have on $D_2 \cong \boldsymbol{F}_2$:

$$K_{X|\boldsymbol{F}_2} \sim -L_{\boldsymbol{F}_2} \sim -E-3f\,;\ \mathcal{N}_{\boldsymbol{F}_2}^X \sim -E-f\,.$$

Then $\mathcal{N}_{\boldsymbol{F}_2|f}^X \cong \mathcal{O}_f(-1)$ so that the Nakano's contractibility criterion (0.12) applies to say that we can smoothly contract $\boldsymbol{F}_2$ along f to get an analytic manifold X' and a proper holomorphic modification $q: X \to X'$. By denoting $D_1' = q(D_1) \cong \boldsymbol{P}^2$ on X', the standard exact sequence

$$0 \longrightarrow \mathcal{N}_C^{\boldsymbol{P}^2} \cong \mathcal{O}_{\boldsymbol{P}^1}(1) \longrightarrow \mathcal{N}_C^X \cong \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-2) \longrightarrow \mathcal{N}_{\boldsymbol{P}^2|C}^X \longrightarrow 0$$

gives $\mathcal{N}_{\boldsymbol{P}^2|C}^X \cong \mathcal{O}_{\boldsymbol{P}^1}(-2)$, whence $\mathcal{N}_{\boldsymbol{P}^2}^X \cong \mathcal{O}_{\boldsymbol{P}^2}(-2)$. Since f meets $D_1 \cong \boldsymbol{P}^2$ on X in a point it thus follows that $\mathcal{N}_{D_1'}^{X'} \cong \mathcal{O}_{\boldsymbol{P}^2}(-1)$. Then $D_1' \cong \boldsymbol{P}^2$ smoothly contracts to a point p of a smooth birational analytic model X'' of X'. Let $q'' \circ q': X \to X''$ be the contraction. Since F is a fibre of φ, there exists a morphism $\sigma: X'' \to Y$ which makes the following diagram commute

$$\begin{tikzcd} X \arrow[rr, "q'' \circ q'"] \arrow[dr, "\varphi"'] & & X'' \arrow[dl, "\sigma"] \\ & Y & \end{tikzcd}\,.$$

Note that the fibre $\sigma^{-1}(\varphi(F)) = p$ is 0-dimensional while the general fibres of σ are 1-dimensional, which is a contradiction. This proves that the case $D_1 \cong \boldsymbol{P}^2$, $D_2 \cong \boldsymbol{F}_2$ does not occur.

In particular the arguments above show that the fibre F is irreducible. Let us consider the remaining case $F \cong \boldsymbol{P}^2$. Let $\mathcal{N}_F^X \cong \mathcal{O}_{\boldsymbol{P}^2}(a)$, $a \in \boldsymbol{Z}$ be the normal bundle of F in X.

If $a<0$ the general Grauert's contractibility criterion [**Gr**] applies to give an analytic variety and a proper holomorphic modification $q: X \rightarrow X'$ such that $q(F)$ is a possibly singular point on X'. Again, since F is a fibre of φ, there exists a morphism $\beta: X' \rightarrow Y$ such that $\varphi=\beta \circ q$ and one has the same contradiction as above.

If $a \geqq 0$, $\Gamma(\mathcal{N}_{\boldsymbol{P}^2}^X)$ spans $\mathcal{N}_{\boldsymbol{P}^2}^X$ and $h^1(\mathcal{N}_{\boldsymbol{P}^2}^X)=0$. Then the Kodaira-Spencer theory says that the union of smooth deformations of $\boldsymbol{P}^2$ fills up a neighborhood U of $\boldsymbol{P}^2$ in X such that $\varphi(U)$ is contained is some open affine neighborhood V of $\varphi(\boldsymbol{P}^2)$. Since V does not contain any compact subvariety one sees that all smooth deformations of $\boldsymbol{P}^2$ in U are fibres of φ. Then we have the contradiction that there exists a family of dimension $\geqq 1$ of 2-dimensional fibres of φ. This completes the proof of (2.5). Q.E.D.

(2.6) LEMMA. *Let F' be a divisorial fibre of $\varphi: X \rightarrow Y$ and let $F=F'_{\mathrm{red}}$. Let D_1, D_2 be irreducible components of F such that the intersection $D_1 \cap D_2$ is non-empty. Then after renaming, D_1 is isomorphic to $\boldsymbol{F}_1$ and D_2 is isomorphic to $\boldsymbol{F}_0$, D_1, D_2 meet along a section E of $\boldsymbol{F}_1$, and $L_{\boldsymbol{F}_0} \sim \mathcal{O}_{\boldsymbol{F}_0}(1,1)$, $L_{\boldsymbol{F}_1} \sim E+2f$. The fibre F' is reduced, i.e. $F=F'_{\mathrm{red}}$, and $\varphi(F)$ is a smooth point of Y. Furthermore φ factors, $\varphi=\rho \circ q'' \circ q'$, where $q': X \rightarrow W$ is the smooth contraction of $\boldsymbol{F}_0$ along the fibres $f'=E$, and $q'': W \rightarrow Z$ is the blowing up at a smooth point of Z. The map ρ is locally a product projection in an inverse image $\rho^{-1}(\mathcal{V})$ of a neighborhood $\mathcal{V}$ of $\varphi(F)$.*

PROOF. Assume that D_1 is isomorphic to $\boldsymbol{F}_a$ and D_2 is isomorphic to $\boldsymbol{F}_b$. Let E, f denote the basis for $H_2(\boldsymbol{F}_a, \boldsymbol{Z})$ and let E', f' denote the corresponding basis for $H_2(\boldsymbol{F}_b, \boldsymbol{Z})$. From Lemma (2.4) we know that D_1, D_2 meet along a smooth rational curve C and $L \cdot C=1$. Further we know that $\mathcal{N}_C^X \cong \mathcal{N}_a \oplus \mathcal{N}_b$ where $\mathcal{N}_a$, $\mathcal{N}_b$ denote the normal bundles of C in $\boldsymbol{F}_a$, $\boldsymbol{F}_b$ respectively and, as in the proof of (2.5), we have

$$\deg \mathcal{N}_C^X=\deg \mathcal{N}_a+\deg \mathcal{N}_b=-1 .$$

Now the same argument as in the proof of claim (2.5.2) shows that either $C \sim E$ or $C \sim f$ on $\boldsymbol{F}_a$ and either $C \sim E'$ or $C \sim f'$ on $\boldsymbol{F}_b$.

If $C \sim f$ on $\boldsymbol{F}_a$ and $C \sim f'$ on $\boldsymbol{F}_b$ one has $\mathcal{N}_C^X \cong \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}$, contradicting $\deg \mathcal{N}_C^X=-1$. In particular by noting that the role of E and f can be switched on $\boldsymbol{F}_0$, it follows that not both a and b can be zero. Therefore, after possibly renumbering, it can be assumed without loss of generality that $a>0$.

If $C \sim E$ on $\boldsymbol{F}_a$ and $C \sim E'$ on $\boldsymbol{F}_b$ one has $\mathcal{N}_C^X \cong \mathcal{O}_{\boldsymbol{P}^1}(-a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-b)$ and hence $a+b=1$. Since $a>0$, we get $a=1$, $b=0$. So, by switching the roles of E' and f' on $\boldsymbol{F}_b$ we fall in the remaining case when $C \sim E$ on $\boldsymbol{F}_a$ and $C \sim f'$ on $\boldsymbol{F}_b$.

In this case from $L\cdot C=L_{F_a}\cdot E=1$ we find $L_{F_a}\sim E+(a+1)f$. Since $-L_{F_a}\sim K_{X|F_a}$ the adjunction formula yields $\mathcal{N}^X_{F_a}\sim -E-f$.

(2.6.1) CLAIM. We have $L_{F_b}\sim E'+(b+1)f'$ and $\mathcal{N}^X_{F_b}=-E'-f'$.

PROOF. Since $L\cdot C=L_{F_b}\cdot f'=1$, $L_{F_b}\sim E'+kf'$ with $k\geqq b+1$ by ampleness. Then, as above,

$$\mathcal{N}^X_{F_b}\approx K_{F_b}+L_{F_b}\sim -E'-(2+b-k)f'\,.$$

If $k>b+1$ we have $\mathcal{N}^X_{F_b}\cdot(E'+bf')=k-b-2\geqq 0$. Note that $|E'+bf'|$ is spanned (see (0.6.4)) and take a smooth $\Gamma\in|E'+bf'|$. Note also that Γ is a rational curve. Therefore from the exact sequence

$$*)\qquad 0\longrightarrow \mathcal{N}^{F_b}_{\Gamma}\cong \mathcal{O}_{P^1}(b)\longrightarrow \mathcal{N}^X_{\Gamma}\longrightarrow \mathcal{N}^X_{F_b|\Gamma}\cong \mathcal{O}_{P^1}(x)\longrightarrow 0$$

where $x=k-b-2\geqq 0$ we infer that $H^0(\mathcal{N}^X_{\Gamma})$ spans $\mathcal{N}^X_{\Gamma}$ and $h^1(\mathcal{N}^X_{\Gamma})=0$. Lemma (2.1) applies to give a contradiction. To see this note that the curve Γ meets $\boldsymbol{F}_a$ since $\Gamma\cdot C=\Gamma\cdot f'=1$ on $\boldsymbol{F}_b$. ∎

Since $\deg \mathcal{N}^X_{F_a}\cdot f=-1$ we apply Nakano's criterion (0.12) to have a proper holomorphic smooth modification $q: X\to X'$ where $\boldsymbol{F}_a$ contracts along the fibres f. Note that, since $\dim\varphi(\boldsymbol{F}_a)=0$, there is a morphism $\sigma: X'\to Y$ such that $\varphi=\sigma\circ q$. From $\mathcal{N}^X_{F_b}\sim -E'-f'$ and since f meets $\boldsymbol{F}_b$ in a point we infer that $\mathcal{N}^{X'}_{F_b}\sim -E'$ on X'. Now the same argument as in the proof of (2.6.1) applies. Let $\Gamma'\in|E'+bf'|$ be a smooth rational curve on $\boldsymbol{F}_b$ in X'. One has $\mathcal{N}^{X'}_{F_b}\cdot(E'+bf')=-E'\cdot(E'+bf')=0$. Therefore the exact sequence $*)$ on X' leads to

$$**)\qquad 0\longrightarrow \mathcal{O}_{P^1}(b)\longrightarrow \mathcal{N}^{X'}_{\Gamma'}\longrightarrow \mathcal{O}_{P^1}\longrightarrow 0$$

so that $\mathcal{N}^{X'}_{\Gamma'}$ is spanned and $h^1(\mathcal{N}^{X'}_{\Gamma'})=0$. Hence we see that all smooth deformations of Γ' are general fibres of σ. This implies that Γ' has a 2-dimensional family of deformations. On the other hand the sequence $**)$ yields

$$h^0(\mathcal{N}^{X'}_{\Gamma'})=h^0(\mathcal{O}_{P^1}(b)\oplus\mathcal{O}_{P^1})=b+2$$

and hence $b+2=2$ that is $b=0$. Thus $D_2\cong\boldsymbol{F}_0$. Since

$$\mathcal{N}^X_{F_0}\cdot f'=(-E'-f')\cdot f'=-1\,,$$

then Nakano's criterion applies again to smoothly contract $\boldsymbol{F}_0$ along f' $(=E)$ to a smooth curve γ. Let $q': X\to W$ be the contraction. Since W is smooth and $q'(\boldsymbol{F}_a)$ has an isolated singularity, $q'(\boldsymbol{F}_a)$ is normal, i.e. it is isomorphic to $\tilde{\boldsymbol{F}}_a$. Thus either $a=2$ or $a=1$ (see again [**S3**], (0.6.2)). Note that $|E+af|$ is base-point free and take a smooth curve $\mathcal{D}\in|E+af|$ on $\boldsymbol{F}_a$. Since $\mathcal{D}\cdot E=0$, we have $\mathcal{D}\cap E=\emptyset$ and hence we can identify $\mathcal{D}$ with its image in W. Let $p=q'(E)$ on W. Assume $a=2$ and compute

$$\mathcal{N}^W_{\tilde{F}_2}\cdot\mathcal{D}=\mathcal{N}^{W-p}_{\tilde{F}_2-p}\cdot\mathcal{D}=\mathcal{N}^{X-E}_{F_2-E}\cdot\mathcal{D}=\mathcal{N}^X_{F_2}\cdot\mathcal{D}=-1\,.$$

This is absurd since $\mathcal{D}$ is a Cartier divisor on $\tilde{\boldsymbol{F}}_2$ and only even numbers arise as intersections of Cartier divisors on $\tilde{\boldsymbol{F}}_2$.

Thus we conclude that $D_1\cong\boldsymbol{F}_1$, $D_2\cong\boldsymbol{F}_0$ on X and $q'(D_1)\cong\boldsymbol{P}^2$ on W. Furthermore $L_{F_0}\sim E'+f'$, $L_{F_1}\sim E+2f$. Note that $\mathcal{N}^W_{P^2}\cong\mathcal{O}_{P_2}(-1)$ since $\mathcal{N}^W_{P^2}\cdot\mathcal{D}=\mathcal{N}^X_{F_1}\cdot\mathcal{D}=-1$. Then Grauert's contractibility criterion applies to give an analytic variety and a proper holomorphic modification $q'':W\to Z$ such that $q''(\boldsymbol{P}^2)$ is a smooth point on Z. Let $\rho:Z\to Y$ be the morphism such that $\varphi=\rho\circ q''\circ q'$. Let $\xi=q''(\gamma)$ on Z. Since

$$\mathcal{N}^W_\gamma\cong\mathcal{N}^X_{F_0|f'}\oplus\mathcal{N}^X_{F_0|f'}\cong\mathcal{O}_{P^1}(-1)\oplus\mathcal{O}_{P^1}(-1)$$

we infer that $\mathcal{N}^Z_\xi\cong\mathcal{O}_\xi\oplus\mathcal{O}_\xi$. Therefore general results from deformation theory imply that there exists a neighborhood of ξ of the form $\xi\times B$ where B is an open set in $\boldsymbol{C}^2$. From this it follows that there exists a neighborhood, B, of $y=\varphi(F)$ with $\rho^{-1}(B)\cong\xi\times B$. It thus follows that the germs of holomorphic functions defined in some complex neighborhood of F that vanish on F, define F. Since $\varphi_*(\mathcal{O}_X)\cong\mathcal{O}_Y$ this is enough to conclude that F' is reduced, i.e. $F'=F$. It also follows that the image of F in Y is a smooth point of Y. Q.E.D.

Note that there are at most two distinct irreducible components of a two dimensional fibre F' of φ. To see this assume otherwise. Let D_1, D_2, D_3 be three irreducible components of $F=F'_{\mathrm{red}}$. The case when they have non-empty intersection is dealt with before Lemma (2.4). Therefore we can assume without loss of generality that the intersection of $D_1\cap D_2\cap D_3$ is empty. Since F is connected it can be further assumed that $D_1\cap D_2$ is non-empty and $D_2\cap D_3$ is non-empty. $D_1\cap D_3$ is empty. If not we could intersect with a smooth $S\in|L|$ to obtain a configuration of three -2 rational curves C_1, C_2, C_3 on S with

$$(C_1+C_2+C_3)^2=C_1^2+C_2^2+C_3^2+2(C_1\cdot C_2+C_2\cdot C_3+C_3\cdot C_1)\geqq-6+2(1+1+1)=0$$

in contradiction to the Hodge index theorem. Note that by Lemma (2.6) either

$$D_1\cong\boldsymbol{F}_0,\quad D_2\cong\boldsymbol{F}_1\quad\text{and}\quad D_3\cong\boldsymbol{F}_0$$

or

$$D_1\cong\boldsymbol{F}_1,\quad D_2\cong\boldsymbol{F}_0\quad\text{and}\quad D_3\cong\boldsymbol{F}_1\,.$$

The first case is not possible. This follows by a further use of Lemma (2.6) to see that D_3 and D_1 both meet D_2 in the unique section, E, on $\boldsymbol{F}_1$ with $E\cdot E=-1$. This contradicts the fact that $D_1\cap D_2\cap D_3$ is empty.

Now consider the second case. Contract D_1 and D_3 along their rulings, $\alpha:X\to\Omega$. The map φ factors as $\beta\circ\alpha$. Note that the image, $\alpha(D_2)$, is isomorphic to $\boldsymbol{F}_0$ and the normal bundle in Ω of $\boldsymbol{F}_0$ is isomorphic to $-E'+f'$ where E'

and f' denote the rulings of $\alpha(D_2)$. By the usual deformation argument, we see that E' deforms to fill out a neighborhood of E' on Ω. Since $E'\cdot\alpha(D_2)=(-E'+f')\cdot E'>0$, we conclude that β is the constant map. This contradiction gives the desired conclusion.

By combining Lemmas (2.4), (2.5), (2.6) we see that, if $n=3$, all possible divisorial fibres F' of $\varphi: X\to Y$ are either

a) irreducible with $F\ (=F'_{\mathrm{red}})\cong \boldsymbol{F}_r$, $r\geqq 0$, or $F\cong \tilde{\boldsymbol{F}}_2$, or

b) reducible and reduced, i.e. $F'=F$, and $F=\boldsymbol{F}_0\cup \boldsymbol{F}_1$.

We can prove more.

(2.7) LEMMA. *Let (X, L), $\varphi: X\to Y$ be as in* (2.3) *with $n=3$ and let F' be an irreducible divisorial fibre of φ. Let $F=F'_{\mathrm{red}}$. Then the case $F\cong\tilde{\boldsymbol{F}}_2$ does not occur and the only possible cases are $F\cong \boldsymbol{F}_0$ with $L_F\sim E+2f$, or $F\cong \boldsymbol{F}_r$, $r>0$, and $L_{F_r}\sim E+(r+1)f$.*

PROOF. First, let us assume $F\cong\tilde{\boldsymbol{F}}_2$. Note that $\mathcal{N}_F^X\cong\mathcal{O}_F(a)$ for some integer a since $\mathrm{Pic}(F)\cong\boldsymbol{Z}$. If $a<0$, the Grauert contractibility criterion applies to give a contradiction as in the proof of (2.5). If $a\geqq 0$, since $\tilde{\boldsymbol{F}}_2$ is locally complete intersection we use [**Bl**] to conclude that deformations of $\tilde{\boldsymbol{F}}_2$ exist and fill up a neighborhood U of $\tilde{\boldsymbol{F}}_2$ in X such that $\varphi(U)$ is contained in some open affine neighborhood of $\varphi(\tilde{\boldsymbol{F}}_2)$. The same argument as at the end of the proof of (2.5) applies again to give the usual contradiction.

Thus we can assume that all divisorial fibres F of $\varphi: X\to Y$ are isomorphic to $\boldsymbol{F}_r$ with $r\geqq 0$.

(2.7.1.) CLAIM. With the notation as above, either $F\cong\boldsymbol{F}_0$ with $L_F\sim E+2f$ or $r>0$ with $L_{F_r}\sim E+(r+1)f$ and $\mathcal{N}_{F_r}^X\approx -E-f$.

PROOF. We know that $L_{F_r}\cdot f=1$, so $L_{F_r}\sim E+bf$ where $b\geqq r+1$ by ampleness. Write $b=r+1+x$ for some $x\geqq 0$. Then

$$\mathcal{N}_{F_r}^X \approx K_{F_r}+L_{F_r} \sim -E+(x-1)f\,.$$

Note that $|E+rf|$ is base point free and take a smooth curve $\Gamma\in|E+rf|$. Note also that Γ is a rational curve. Therefore if $x\geqq 1$, the exact sequence

$$0\longrightarrow \mathcal{N}_\Gamma^{F_r}\cong\mathcal{O}_{P^1}(r)\longrightarrow\mathcal{N}_\Gamma^X\longrightarrow\mathcal{N}_{F_r|\Gamma}^X\cong\mathcal{O}_{P^1}(x-1)\longrightarrow 0$$

says that $H^0(\mathcal{N}_\Gamma^X)$ spans $\mathcal{N}_\Gamma^X$ and $h^1(\mathcal{N}_\Gamma^X)=0$. If $x\geqq 2$, Lemma (2.1) leads to a contradiction. So let $x=1$. Note that $L\cdot\Gamma=2$. Indeed, from the proof of Lemma (2.1) it follows that all smooth deformations of Γ are general fibres of φ since F is isolated. Then $L\cdot\Gamma=2$ since (X, L) a conic bundle. Then $L\cdot\Gamma=(E+(r+2)f)\cdot(E+rf)=2$ leads to $r=0$. Thus either we are done or $x=0$. But in this case $\mathcal{N}_F^X$ is negative and F can be contracted to a point. This contradicts the upper semi-continuity of dimensions of fibres. This proves the claim and hence Lemma (2.7). Q. E. D.

Conclusion of the proof of (2.3.1.). Recall that we are assuming $n=3$. If F' is reducible, the $n=3$ statement follows from Lemma (2.6). So we can assume that F' is irreducible. First we show that the case F $(=F'_{\mathrm{red}})\cong \boldsymbol{F}_r$ with $r>0$ does not occur. After this it will follow by Lemma (2.7) that $F\cong \boldsymbol{F}_0$.

Assume that $F\cong \boldsymbol{F}_r$. Note that there exists a sequence of (smooth) fibres f_i's of φ such that the points $\varphi(f_i)$ are in the flat locus and the curve C defined as the limit of the f_i's in Hilbert scheme of the f_i's is contained in F. Let $C\sim aE+bf$ with a, b both non-negative. Clearly

$$L\cdot C=L_{\boldsymbol{F}_r}\cdot C=(E+(r+1)f)\cdot(aE+bf)=2$$

since (X, L) is a conic bundle. Then we find $a+b=2$ so that either $a=0$, $b=2$, $a=b=1$ or $a=2$, $b=0$.

Let $a=0$, $b=2$. We have $\mathcal{N}^X_{\boldsymbol{F}_r}\approx -E-f$ and $\mathcal{N}^X_{\boldsymbol{F}_r}\cdot f=-1$. Then by Nakano's contractibility criterion there exists a smooth analytic contraction $q: X\to X'$ such that $q(\boldsymbol{F}_r)=\boldsymbol{P}^1$ in X'. Now $C\sim 2f$ is a fibre of $q_{\boldsymbol{F}_r}: \boldsymbol{F}_r\to \boldsymbol{P}^1$. Thus there exists a neighborhood U of $q(C)$ with no compact subvarieties but points. Therefore C cannot be obtained as limit of the f_i's, otherwise we have the contradiction that infinitely many of the $q(f_i)$'s are one dimensional and fall in U.

Let $a=2$, $b=0$. Take a point $x\in \boldsymbol{F}_r$, $x\notin E$. One can choose a sequence of smooth one dimensional fibres f_j's of φ in such a way that the curve C', defined as the limit of the f_j's, contains x. Note that $C\sim C'$ on $\boldsymbol{F}_r$. Indeed, clearly C and C' are homologous on X and by the above either $C'\sim 2E$ or $C'\sim E+f$. Since $C\sim 2E$ we have to rule out the case $C'\sim E+f$. If $C'\sim E+f$, then E would be homologous to f on X and hence $\deg \mathcal{N}^X_{\boldsymbol{F}_r|E}=\deg \mathcal{N}^X_{\boldsymbol{F}_r|f}$ that is $(-E-f)\cdot E=(-E-f)\cdot f$ on $\boldsymbol{F}_r$, whence the contradiction $r=0$. Thus $C'(\sim C\sim 2E)$ does not move on $\boldsymbol{F}_r$ since $\mathcal{N}^{\boldsymbol{F}_r}_C\approx \mathcal{O}_C(-4r)$ has negative degree. Therefore the fact that $x\notin E$, $x\in C'$ leads again to a contradiction.

Let $a=b=1$. First, we need the following rather general argument to show that $r=2$. Since $\mathcal{N}^X_{\boldsymbol{F}_r|f}\approx \mathcal{O}_f(-1)$ we apply once again the Nakano's criterion to get a proper analytic modification X' of X such that $\boldsymbol{F}_r$ contracts along f to a smooth curve γ on X'. Let $q: X\to X'$ be the contraction. As usual there is a morphism $\sigma: X'\to Y$ such that $\varphi=\sigma\circ q$. Now

$$\mathcal{N}^{X'}_\gamma \cong \mathcal{N}^X_{\boldsymbol{F}_r|E}\oplus \mathcal{N}^X_{\boldsymbol{F}_r|\Gamma}\cong \mathcal{O}_{\boldsymbol{P}^1}(r-1)\oplus \mathcal{O}_{\boldsymbol{P}^1}(-1)$$

where $\Gamma\in|E+rf|$. Then $h^0(\mathcal{N}^{X'}_\gamma)=r$, $h^1(\mathcal{N}^{X'}_\gamma)=0$. It thus follows that there exists a r-dimensional family of deformations of γ, further, since $\dim \sigma(\gamma)=0$, the usual rigidity argument says that deformations of γ are contained in nearby fibres Λ of σ and by the semi-continuity of the dimension of fibres we have

$$1 = \dim \gamma \geqq \dim \Lambda \geqq 1$$

that is $\dim \Lambda = 1$. Therefore one has in X' a r-dimensional family τ_r of 1-dimensional fibres of σ and hence it has to be $r=2$. Now, let $U \subseteq \boldsymbol{C}^2$ be the neighborhood parameterizing τ_2. Then by the above we get a holomorphic map $\theta: U \to Y$. Since the curves in τ_2 go to different curves in X', we see that θ must be generically one to one. Since Y is normal, the Zariski Main Theorem applies to say that $\theta: U \xrightarrow{\sim} V$ is a biholomorphism onto some neighborhood V of $\sigma(\gamma)$ in Y. Thus $\tau_2 = \sigma^{-1}(V)$ and hence $\mathcal{N}_\gamma^{\tau_2} \cong \mathcal{N}_\gamma^{X'} \cong \mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(1)$. On the other hand, γ is a fibre on which $\tau_2 \to U$ is of maximal rank, and therefore $\mathcal{N}_\gamma^{\tau_2} \cong \mathcal{O}_\gamma \oplus \mathcal{O}_\gamma$, which leads to a contradiction.

Thus we conclude that $F \cong \boldsymbol{F}_0$. To see that F' is reduced, i.e. $F'=F$, note that since $\varphi_*(\mathcal{O}_X) \cong \mathcal{O}_Y$, it suffices to show that the germs of holomorphic functions defined in some complex neighborhood of F that vanish on F, define F. To see this note that from claim (2.7.1) it follows that the normal bundle of F is linearly equivalent to $-E$. From this it follows that X is the blow up, $\rho: X \to X'$, of a smooth manifold X' along a smooth rational curve, $\mathcal{C}$, with $\rho^{-1}(\mathcal{C}) = F$. Note that the normal bundle of $\mathcal{C}$ is $\mathcal{O}_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}$. From this it follows that there is a complex neighborhood of $\mathcal{C}$ of the form $\Delta \times \mathcal{C}$ where Δ is an open set in $\boldsymbol{C}^2$. From this it follows that the germs of holomorphic functions defined in some complex neighborhood of F that vanish on F, define F. It also follows that

●) the image of F in Y is a smooth point of Y.

To conclude the proof of (2.3.1) it remains to show that all the fibres of $\varphi: X \to Y$ are equal dimensional if X has dimension $n \geqq 4$. Let D be a $(n-1)$-dimensional fiber of $\varphi: X \to Y$. By slicing down with smooth elements of $|L|$ corresponding to smooth members of $|L^\wedge|$ we can assume $n=4$. Let A be a smooth element in $|L|$. Then A meets transversely D in a 2-dimensional fiber, say F, of the restriction $\varphi_A: A \to Y$. By the above we know that either $F \cong \boldsymbol{F}_0$ or $F = \boldsymbol{F}_0 \cup \boldsymbol{F}_1$.

Let $F = \boldsymbol{F}_0$. Since F is smooth D is irreducible, reduced with isolated Gorenstein singularities. From [**S7**], (0.22) we know that the number of isolated nonrational singular points of D is bounded by $h^0(K_D + L_D)$. Now, $K_D + L_D$ has no sections since the restriction $(K_D + L_D)_F \cong K_{F_0}$ has no sections. Hence D has only (isolated, Gorenstein) rational singularities. Then general results (see [**S6**], §0) apply to say that $K_D + 3L_D$ is spanned unless $D \cong \boldsymbol{P}^3$ and $L_D \cong \mathcal{O}_{P^3}(1)$. This exception is impossible since $L_{D|F_0} \approx L_{F_0} \sim E + 2f$ on $\boldsymbol{F}_0$. Thus we conclude that $K_D + 3L_D$ gives a morphism, say α. Since

$$(K_D + 3L_D)_{F_0} \approx K_{F_0} + 2L_{F_0} \sim 2f$$

we see that $\alpha(\boldsymbol{F}_0)$ is 1-dimensional and hence $\alpha(\boldsymbol{F}_0)=\boldsymbol{P}^1$. Therefore $\alpha(D)=\boldsymbol{P}^1$ in view of [S7], (0.3.2) and $\alpha: D\to\boldsymbol{P}^1$ is in fact a $\boldsymbol{P}^2$ bundle since $K_D+3L_D=\alpha^*\mathcal{O}(1)$ (see (0.6)). Let $\boldsymbol{P}^2$ be a fiber of α. Clearly, $\mathcal{N}^X_{D|\boldsymbol{P}^2}\approx\mathcal{O}_{\boldsymbol{P}^2}(b)$ for some integer b and hence $\mathcal{N}^X_{D|\boldsymbol{P}^2\cap A}\approx\mathcal{O}_C(b)$ where $\boldsymbol{P}^2\cap A=C\cong\boldsymbol{P}^1$ is a fiber of $\boldsymbol{F}_0$. Furthermore since all the intersections are transverse we have

$$(\mathcal{N}^X_{D|\boldsymbol{P}^2})_C \approx (\mathcal{N}^A_{D\cap A})_C \approx \mathcal{N}^A_{\boldsymbol{F}_0|C}\,.$$

Therefore we find $\mathcal{O}_C(b)\approx\mathcal{O}_C(-1)$ which gives $b=-1$. Thus by the Nakano contractibility criterion there exists an analytic variety and a proper holomorphic modification $q: X\to X'$ such that D smoothly contracts along the $\boldsymbol{P}^2$'s and $q(D)=\boldsymbol{P}^1$. Since D is a fibre of $\varphi: X\to Y$, there exists a morphism $\sigma: X'\to Y$ such that $\varphi=\sigma\circ q$, whence the usual contradiction that $\sigma^{-1}(\varphi(D))$ is 1-dimensional while the general fibres of σ are 2-dimensional.

Let $F=\boldsymbol{F}_0\cup\boldsymbol{F}_1$. Then $D=D_0\cup D_1$. Exactly the same argument as above shows that D_0, D_1 are irreducible, reduced with only isolated, rational singularities. Furthermore $K_{D_i}+3L_{D_i}$ is spanned unless $(D_i, L_{D_i})\cong(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(1))$, $i=0, 1$. But this exception is impossible since neither $\boldsymbol{F}_0$ nor $\boldsymbol{F}_1$ can be a member of $|L_{D_i}|=|\mathcal{O}_{\boldsymbol{P}^3}(1)|$. Therefore $K_{D_i}+3L_{D_i}$ is spanned for $i=0, 1$. Since $L_{F_0}\sim E+f$ we have

$$(K_{D_0}+3L_{D_0})_{F_0} \approx K_{F_0}+2L_{F_0} \approx \mathcal{O}_{F_0}\,.$$

It thus follows that $K_{D_0}+3L_{D_0}\approx\mathcal{O}_{D_0}$, i.e. (D_0, L_{D_0}) is a hyperquadric in $\boldsymbol{P}^4$. Therefore $\mathcal{N}^X_{D_0}\approx K_{D_0}-K_{X|D_0}\approx -L_{D_0}$, so that D_0 can be contracted to a point. Let α be the morphism given by $\Gamma(K_{D_1}+3L_{D_1})$. Since

$$(K_{D_1}+3L_{D_1})_{F_1} \approx K_{F_1}+2L_{F_1}\sim f\,,$$

we see that $\alpha(\boldsymbol{F}_1)$ is 1-dimensional and hence $\alpha(\boldsymbol{F}_1)=\boldsymbol{P}^1$. Therefore the same argument as above shows that $\alpha: D_1\to\boldsymbol{P}^1$ is a $\boldsymbol{P}^2$ bundle. The divisors, D_0 and D_1, meet since $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ meet along a section. Hence there is at least one fibre $\boldsymbol{P}^2$ of $\alpha: D_1\to\boldsymbol{P}^1$ which meets D_0 in a curve and doesn't belong to D_0. Then, since D_0 contracts to a point, $D_0\cap\boldsymbol{P}^2$ would be a curve on $\boldsymbol{P}^2$ which contracts to a point. This absurdity completes the proof of (2.3.1). Q.E.D.

Proof of (2.3.2). By taking general hyperplane sections we can assume $n=\dim X=3$. The fact that Y is smooth at the points corresponding to the divisorial fibres follows from Lemma (2.6) and the consequence ◉) above in the proof of (2.3.1). Let $S^\wedge$ be a general member of $|L^\wedge|$ and let $S\in|L|$ be the corresponding smooth surface on X. Let f_y be a 1-dmensional fibre over a point y of Y. Then S intersects f_y in two, possibly coincident, points. If $S\cap f_y$ consists of two distinct points, Y is smooth at y. Note that if $f=f_1\cup f_2$ is the disjoint union of two $\boldsymbol{P}^1$'s meeting in a point, say p, one easily sees that it cannot be $S\cap f_y=p$. Otherwise, denoting by f_1' the proper transform of f_1 under $r: X^\frown\to X$ we would have the contradiction $L^\frown\cdot f_1'=0$. If $S\cap f_y$ consists of a point x of multiplicity 2, let V be a complex neighborhood of $\varphi(x)$ such that $V-\varphi(x)$ is smooth and the restriction, φ_U, is finite where $U=\varphi^{-1}(V)\cap S$. Then there exists an involution $i: U\to U$ leaving x fixed with $V=U/\langle i\rangle$ where $\langle i\rangle=\mu_2$ is the cyclic group of the square roots of 1. This shows that V has 2-Gorenstein rational singularities at worst. General results on singularities for which we refer to [**D**], in particular §5 and Appendix II, apply to give the result. Q.E.D.

(2.8) EXAMPLE-REMARK. Let (X, L) and $\varphi: X\to Y$ be as in (2.3) with $\dim X=3$. The case of a divisorial fibre $F\cong \boldsymbol{F}_0$ of $\varphi: X\to Y$ really occurs. Indeed, let $X=\boldsymbol{P}^2\times\boldsymbol{P}^1$. Let $p: X\to\boldsymbol{P}^2$ be the projection and $\pi: \boldsymbol{P}^{2\frown}\to\boldsymbol{P}^2$ be the blowing up of $\boldsymbol{P}^2$ at a point y. Look at the base change diagram

$$\begin{array}{ccc} X^\frown & \longrightarrow & X=\boldsymbol{P}^2\times\boldsymbol{P}^1 \\ {\scriptstyle p^\frown}\downarrow & & \downarrow{\scriptstyle p} \\ \boldsymbol{P}^{2\frown} & \longrightarrow & \boldsymbol{P}^2 \end{array} .$$

Let $q: X^\frown\to\boldsymbol{P}^1$ be the canonical projection over $\boldsymbol{P}^1$. Then $K_{X^\frown}\approx p^{\frown *}K_{\boldsymbol{P}^{2\frown}}\otimes q^*K_{\boldsymbol{P}^1}$. Let $E=\pi^{-1}(y)$ and define $\mathcal{L}=p^{\frown *}(\pi^*\mathcal{O}_{\boldsymbol{P}^2}(4)-E)\otimes q^*\mathcal{O}_{\boldsymbol{P}^1}(2)$. Thus $\mathcal{L}$ is very ample and, since $K_{\boldsymbol{P}^{2\frown}}\approx\pi^*K_{\boldsymbol{P}^2}+E$,

$$K_{X^\frown}\otimes\mathcal{L}\approx p^{\frown *}(\pi^*(K_{\boldsymbol{P}^2}\otimes\mathcal{O}_{\boldsymbol{P}^2}(4))\approx p^{\frown *}\pi^*\mathcal{O}_{\boldsymbol{P}^2}(1)$$

which shows that $(X^\frown, \mathcal{L})$ is a quadric bundle over $\boldsymbol{P}^2$, with the only divisorial fibre $\boldsymbol{F}_0=(\pi\circ p^\frown)^{-1}(y)$ over y.

Let (X, L), $\varphi: X\to Y$ be as in (2.3) with $\dim X=3$ and assume that there exists a unique divisorial fibre $F\cong\boldsymbol{F}_0$ of φ. Let $\sigma: Y'\to Y$ be the blow up of Y at $\varphi(F)$. Then there is a morphism $\varphi': X\to Y'$ such that $\varphi=\sigma\circ\varphi'$ and all the fibres of φ' are 1-dimensional since $\varphi'^{-1}(s)\cong\boldsymbol{P}^1$ for any $s\in\sigma^{-1}(\varphi(F))$. Furthermore $K_X\otimes L\approx\varphi'^*\sigma^*\mathcal{L}$ where $\mathcal{L}$ is an ample line bundle on Y such that $K_X\otimes L\approx\varphi^*\mathcal{L}$. Therefore φ' expresses (X, L) as a quadric bundle over Y' except for the fact that $\sigma^*\mathcal{L}$ is merely nef.

§3. Some further results.

The first result we prove in this section is a general property of scrolls and quadric bundles.

(3.1) PROPOSITION. *Let $X^\frown$ be a smooth connected n-fold with $n\geqq 3$ and $L^\frown$ a very ample line bundle on $X^\frown$. Assume that $(X^\frown, L^\frown)$ admits a reduction (X, L), $r: X^\frown\to X$.*

(3.1.1) *If (X, L) is a scroll, then $(X^{\wedge}, L^{\wedge})\cong(X, L)$;*

(3.1.2) *Let (X, L) be a quadric bundle $\phi: X\to Y$ over a normal variety Y. If $n-\dim Y\geqq 2$, then $(X^{\wedge}, L^{\wedge})\cong(X, L)$. If $n-\dim Y=1$, the points blown up under r lie either on divisorial fibers of ϕ or on smooth 1-dimensional fibres. Furthermore any smooth 1-dimensional fibre contains at most one point blown up. In particular if $n=3$, there are no points blown up under r on divisorial fibres.*

PROOF. It is enough to prove (3.1.2). Indeed a slight and easier modification of the argument below gives the scroll statement.

Let $\phi: X\to Y$ be a quadric bundle and assume $n-\dim Y\geqq 2$. Let $x\in X$ be a point blown up under r. We can choose a sequence F_i of general fibres of ϕ such that the point x is contained in the limit of the F_i's. Let F be the limit of the F_i's in the Hilbert scheme of the F_i's. The fibres F_i are smooth quadrics of dimension $\geqq 2$ and hence F is a (possibly singular) quadric of dimension $\geqq 2$. Therefore there exists a line $\mathcal{C}$ on F passing through x and such that $L\cdot\mathcal{C}=1$. Denote by $\mathcal{C}'$ the proper transform of $\mathcal{C}$ under r. Then

$$L^{\wedge}\mathcal{C}'\leqq r^{*}L\cdot\mathcal{C}'-r^{-1}(x)\cdot\mathcal{C}'=L\cdot\mathcal{C}-r^{-1}(x)\cdot\mathcal{C}'=0$$

contradicting the ampleness of $L^{\wedge}$. It thus follows that $(X^{\wedge}, L^{\wedge})\cong(X, L)$.

Assume now $n-\dim Y=1$ and let $x\in X$ be a point blown up under r. The same argument as above shows that x lies either on a divisorial fibre or on a smooth 1-dimensional fibre f, otherwise we contradict again the ampleness of $L^{\wedge}$. If $x\in f$, let f' be the proper transform of f under r and compute

$$L^{\wedge}\cdot f'\leqq r^{*}L\cdot f'-r^{-1}(x)\cdot f'=L\cdot f-r^{-1}(x)\cdot f'=2-r^{-1}(x)\cdot f'\leqq 1\,.$$

This shows that at most one point blown up under r can lie on f.

If $n=3$ we know from Theorem (2.3) that all divisorial fibres F of ϕ are isomorphic to either $\boldsymbol{F}_0$ or $\boldsymbol{F}_0\cup\boldsymbol{F}_1$. If a point x blown up under r lies on F, by using Lemmas (2.6), (2.7), we can find a line $\mathcal{C}$ on F passing through x and such that $L\cdot\mathcal{C}=1$. Then the usual argument leads to a contradiction. Q.E.D.

Examples we have looked and the above structure theorem (2.3) suggest the following conjecture.

(3.2) CONJECTURE. *Let (X, L) be a quadric bundle of dimension n over a normal surface Y and let $\phi: X\to Y$ be the quadric bundle projection. Then $K_X+(n-2)L\approx\phi^{*}\mathcal{L}$ where $\mathcal{L}$ is of the form $\mathcal{L}\approx K_Y+H$ and both $\mathcal{L}$ and H are ample line bundles on Y.*

To make this conjecture plausible, first note that Y is Gorenstein by Theorem (2.3) and thus $H:=\mathcal{L}\otimes K_Y^{-1}$ is a line bundle. Furthermore, assume $n=3$ and let C be a smooth curve on Y with $\phi^{-1}(C)$ smooth. Then $H\cdot C>0$. This follows by noting that the same argument as in [**S1**], § 2 yields

$$(K_X \otimes L \otimes \phi^* \mathcal{O}_Y)_{\phi^{-1}(C)} \approx K_{\phi^{-1}(C)} \otimes L_{\phi^{-1}(C)} \approx \phi^*(K_C \otimes M)$$

for some ample line bundle M on C. Therefore we have

$$\phi^*(K_Y \otimes H \otimes \mathcal{O}_Y(C))_{\phi^{-1}(C)} \approx \phi^*(K_C \otimes M)$$

and hence $(K_Y \otimes H \otimes \mathcal{O}_Y(C))_C \approx K_C \otimes M$ which leads to $H_C \approx M$.

The above conjecture has the following nice consequence. Let $X^\wedge$ be a smooth connected 3-fold and let $L^\wedge$ be a very ample line bundle on $X^\wedge$. Assume that $K_{X^\wedge} + (n-1)L^\wedge$ is nef and big and the reduction (X, L) of $(X^\wedge, L^\wedge)$ exists. Define $\mathcal{K}_X = K_X + L$. Then assuming the conjecture true we can prove the general result that $2\mathcal{K}_X$ is always spanned by its global sections except for 4 very well understood cases where $\mathcal{K}_X$ is not nef (see [**S7**]). Note that the proof below shows that $2\mathcal{K}_X$ is always spanned if $\mathcal{K}_X$ is nef, unless (X, L) is a quadric bundle.

(3.3) PROPOSITION. *Let (X, L) be as above and let $\mathcal{K}_X = K_X + L$. Assume that $\mathcal{K}_X$ is nef and that conjecture* (3.2) *is true. Then $2\mathcal{K}_X$ is spanned by its global sections.*

PROOF. The result is already known when (X, L) is of log-general type, i.e. when $\mathcal{K}_X$ is nef and big. Indeed in this case it was previously proved in [**BBS**], (0.8.1), (2.1), by lifting to X Bombieri's results on a smooth $S \in |L|$, that $5\mathcal{K}_X$ is spanned as well as $3\mathcal{K}_X$ is spanned whenever $K_S \cdot K_S \geqq 3$. Note that S is a minimal surface of general type since $K_S = \mathcal{K}_{X|S}$. Then in the note added in proof to [**S8**], the stronger result, that $2\mathcal{K}_X$ is spanned by its global sections, is shown.

Thus we can assume that $\mathcal{K}_X$ is nef and not big. Then from [**S7**], we know that either

i) $K_X \approx -L$,

ii) (X, L) is a Del Pezzo fibration over a smooth curve, or

iii) (X, L) is a quadric bundle over a surface.

In case i), $2\mathcal{K}_X \approx \mathcal{O}_X$ is clearly spanned. So let us assume that $p: X \to C$ is a Del Pezzo fibration over a smooth curve C. From [**S8**], (0.5) we know that $\mathcal{K}_X \approx p^*\mathcal{L}$ for some ample line bundle $\mathcal{L}$ on C of degree $\deg \mathcal{L} = 2q(S) - 2 + \chi(\mathcal{O}_S)$. Define $H = \mathcal{L} - K_C$. Then $\deg H = \chi(\mathcal{O}_S) \geqq 0$ since $K_S \approx p^*\mathcal{L}$ is nef. If $\deg(K_C + 2H) \geqq 2$, one has

$$\deg 2\mathcal{L} = \deg(2K_C + 2H) \geqq 2q(S)$$

so that $2\mathcal{L}$ is spanned and hence $2\mathcal{K}_X$ is spanned too. Therefore we only have to rule out the case when $\deg(K_C + 2H) = 2q(S) - 2 + 2\chi(\mathcal{O}_S) \leqq 1$. This implies $q(S) + \chi(\mathcal{O}_S) \leqq 1$. Then either $q(S) = \chi(\mathcal{O}_S) = 0$, $q(S) = 1$, $\chi(\mathcal{O}_S) = 0$, or $q(S) = 0$, $\chi(\mathcal{O}_S)$

$=1$; each of these cases contradicts the ampleness condition $\deg \mathcal{L}>0$. This shows that $2\mathcal{K}_X$ is spanned in case ii) (compare with [S8], (0.5.1)).

Thus it remains to consider the quadric bundle case iii). Let $\phi: X \to Y$ be the quadric bundle projection. By the assumptions made we know that $\mathcal{K}_X \approx \phi^*\mathcal{L}$ where $\mathcal{L} \approx K_Y+H$ for some ample line bundle H on Y. Let $\sigma: \tilde{Y} \to Y$ be the minimal desingularization of Y. Now $\sigma^* K_Y \approx K_{\tilde{Y}}$ since Y has only rational singularities by Theorem (2.3). Write $\tilde{\mathcal{L}}=\sigma^*\mathcal{L}$, $\tilde{H}=\sigma^*H$ and assume $(\tilde{H}+\tilde{\mathcal{L}})^2 \geqq 5$. Then a well known result of Reider states that $K_{\tilde{Y}}+\tilde{H}+\tilde{\mathcal{L}}=2\tilde{\mathcal{L}}$ is spanned by its global sections unless there exists an effective divisor D such that

$$(\tilde{H}+\tilde{\mathcal{L}}) \cdot D-1 \leqq D \cdot D < (\tilde{H}+\tilde{\mathcal{L}}) \cdot D/2 < 1$$

which leads to either $D \cdot (\tilde{H}+\tilde{\mathcal{L}})=0$, $D \cdot D=1$ or $D \cdot (\tilde{H}+\tilde{\mathcal{L}})=1$, $D \cdot D=0$. In the former case, we have $\sigma_* D \cdot (H+\mathcal{L})=0$ so $\sigma_* D$ is a point and hence D is contained in the set of the -2 curves, that is $D=\sum \lambda_i E_i$, $E_i^2=-2$ which contradicts $D \cdot D=1$. In the latter case, $\sigma_* D$ is a curve and $\sigma_* D \cdot (H+\mathcal{L})=1$ which contradicts the ampleness of H and $\mathcal{L}$. Therefore $2\tilde{\mathcal{L}}$ is spanned, and hence $2\mathcal{K}_X$ is also, provided that $(\tilde{H}+\tilde{\mathcal{L}})^2 \geqq 5$. Now compute $(\tilde{H}+\tilde{\mathcal{L}})^2=(H+\mathcal{L})^2=H^2+2H \cdot \mathcal{L}+\mathcal{L}^2 \geqq 4$. Then either we are done or $(H+\mathcal{L})^2=4$ with $H^2=\mathcal{L}^2=H \cdot \mathcal{L}=1$. But in this case $H \sim \mathcal{L}$ by the Hodge index theorem so that K_Y is numerically trivial and H^2 is even by the genus formula, a contradiction. Q.E.D.

§4. $\boldsymbol{P}^1$ bundles in $\boldsymbol{P}^5$.

Let X be a smooth threefold and let L be a very ample line bundle on X. Assume that (X, L) is a $\boldsymbol{P}^1$ bundle over a smooth surface S', i.e. there exists a morphism $p: X \to S'$ such that any fibre f is a linear $\boldsymbol{P}^1$ and $L \cdot f=1$. Note that a scroll over a surface in the sense of (0.6) is a $\boldsymbol{P}^1$ bundle in this sense. Furthermore assume that $\Gamma(L)$ embeds X in $\boldsymbol{P}^5$. Let $d=L^3$ be the *degree* of (X, L). The complete classification of smooth threefolds in $\boldsymbol{P}^5$ of degree $\leqq 8$ was previously worked out by Ionescu [**I1**], [**I2**], [**I3**] and Okonek [**O1**], [**O2**]; therefore we shall assume that $d \geqq 9$. Indeed also the degree $d=9$, 10 cases are covered in the more recent paper [**BSS**]. Then in particular the adjoint bundle K_X+2L is spanned in view of (0.9). Let $g(L)$ be the sectional genus of (X, L) and let $\#=e(S)-e(S')$ denote the number of positive dimensional fibres of the restriction $p: S \to S'$, where S is a general smooth element of $|L|$. Recall that (S', L'), where $L'=(p_{S*}L_S)^{**}$, is the reduction of (S, L). Also recall that L' and $K_{S'}+L'$ are very ample (see (0.6.2)). In this section we show that only six possible 3-tuples $(d, g(L), \#)$ can occur for $d \geqq 9$. We also refer to [**BSS**], §1. First note that

$$\# > 0. \tag{4.0}$$

This was previously proved by Sommese in [**S4**], (1.3). Indeed the condition $L \cdot f=1$ implies that S is a meromorphic section of X over S'. The fact that S cannot be a holomorphic section follows from the general fact that there are no connected manifolds V of dimension $\geqq 3$ carrying an ample divisor A with a continuous map $\tau: V \to A$ and such that $\tau_A: A \to A$ is a homotopy equivalence (see also [**FS**], (0.11)). Furthermore, from [**BSS**], (1.9) we know that

$$\# = 11(g(L)-1)-2(d^2-5d). \tag{4.1}$$

Then (4.0), (4.1) lead to

$$g(L) > 1+2(d^2-5d)/11. \tag{4.2}$$

Note also that, since K_S+L_S is spanned, the same argument as in [**S6**], § 3 applies to give

$$(K_S+L_S)^2 \geqq p_g(S)+g(L)-2, \tag{4.3}$$

and, if $p_g(S)>0$,

$$(K_S+L_S)^2 \geqq 2(p_g(S)+g(L)-2). \tag{4.3$'$}$$

A strong numerical constraint is given by the following congruence.

(4.4) PROPOSITION. *Let (X, L) be a $\boldsymbol{P}^1$ bundle embedded by $\Gamma(L)$ in $\boldsymbol{P}^5$ as above. Then the invariants d, $g(L)$ satisfy the congruence*

$$7d^2-(g(L)-1)(d+21)-d \equiv 0 \ (12).$$

PROOF. Note that $(K_X+2L)^3=0$ since $(K_X+2L)\cdot f=0$ for any fibre f of $p: X \to S'$. Then we have $d_3=-3d_2-3d_1-d$ where the invariants d_i's are defined as in (0.10) and hence the general congruence (0.10.1) for threefolds in $\boldsymbol{P}^5$ becomes

$$11d^2-d_1(d+11)-3d+4d_2 \equiv 0 \ (24).$$

Therefore, since $d_1=2g(L)-2-d$, we find

$$12d^2-(g(L)-1)(2d+22)+8d+4d_2 \equiv 0 \ (24). \tag{4.4.1}$$

Now Lemma (0.8) yields

$$4d_2 = 2d^2-10d-20(g(L)-1)+24\chi(\mathcal{O}_S). \tag{4.4.2}$$

Thus, by combining (4.4.1), (4.4.2) we get the result. Q. E. D.

To go on, we first rule out the case when (X, L) is a Castelnuovo variety.

(4.5) PROPOSITION. *With the notations as above, let (X, L) be a $\boldsymbol{P}^1$ bundle on a smooth surface S' embedded by $\Gamma(L)$ in $\boldsymbol{P}^5$. Then the sectional genus $g(L)$ does not reach the maximum with respect to the Castelnuovo's bound* (0.7).

PROOF. Let $d'=L'^2$. Since $(K_S+L_S)=\pi^*(K_{S'}+L')$ by (0.6.2) one has $(K_{S'}+L')^2=(K_S+L_S)^2$ so that the inequality (4.3) holds on S' too. Then by using the genus formula we find

$$K_{S'}\cdot K_{S'}+3g(L)-2 \geqq d' . \tag{4.5.1}$$

Let d be even. Then $g(L)=(d-2)^2/4$ so that (4.1) yields $\#=3d^2/4-d$ and hence $d'=3d^2/4>3g(L)$. Therefore $K_{S'}\cdot L'<0$ by the genus formula, so that S' is a ruled surface and $K_{S'}\cdot K_{S'}\leqq 9$. Thus by (4.5.1), $d'\leqq 3g(L)+7$ which leads to the contradiction $3d^2\leqq 3(d-2)^2+28$. Let d be odd. Then $g(L)=(d^2-4d+3)/4$ so (4.1) yields $\#=(3d^2-11)/4-d$ and hence $d'=(3d^2-11)/4>2g(L)-2$. Therefore $K_{S'}\cdot L'<0$ and the same argument as above gives a contradiction. Q. E. D.

Thus, from now on, we can assume that (see (0.7))

$$g(L) \leqq d(d-3)/6+1 \tag{4.6}$$

which combined with (4.1) leads to $d^2-27d+1\leqq 0$, i.e. $d\leqq 26$.

Next step is to rule out the case when the surface S has geometric genus $p_g(S)=0$.

(4.7) PROPOSITION. *With the notation as above. Let (X, L) be a $\boldsymbol{P}^1$ bundle on a surface S' embedded by $\Gamma(L)$ in $\boldsymbol{P}^5$. Then $p_g(S)>0$ for a smooth $S\in|L|$.*

PROOF. A systematic use of all the previous relations (4.0), (4.1), (4.2), (4.4), (4.6), together with the Hodge inequality

$$d'(K_{S'}\cdot K_{S'}) = (d+\#)(K_S\cdot K_S+\#) \leqq (K_S\cdot L_S-\#)^2 = (K_{S'}\cdot L')^2$$

and Lemma (0.8) gives us, for $d\leqq 26$, the following list of numerical invariants.

d	$g(L)$	$\#$	$K_{S'}\cdot K_{S'}$	$(K_{S'}+L')^2$	$(2K_{S'}+L')^2$
11	14	11	-15	15	-22
12	17	8	-24	20	-28
13	22	23	-24	24	-36
15	29	8	-51	38	-49
	30	19	-45	37	-50
	31	30	-39	36	-51
21	64	21	-120	90	-102
23	77	8	-159	114	-121
24	85	12	-174	126	-132

We have carried these computations out using a simple Pascal program.

Now, we know that both L' and $K_{S'}+L'$ are very ample by (0.6.2). On the other hand, since $(2K_{S'}+L')^2<0$ by the table, we conclude that $2K_{S'}+L'$ is not spanned. Therefore by (0.5.1) we know that $(S', K_{S'}+L')$ is either $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(1))$, $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$, $(Q, \mathcal{O}_Q(1))$ where Q is a smooth quadric in $\boldsymbol{P}^3$, or a $\boldsymbol{P}^1$ bundle over a smooth curve. Thus the values of either $(K_{S'}+L')^2$ or $K_{S'}\cdot K_{S'}$ given in the table lead to numerical contradictions. Q.E.D.

From now on we can assume $p_g(S)>0$. Look at the map $\pi: S'\to Y$ onto the minimal model Y of S'. Then by the above, $\kappa(S')=\kappa(Y)\geqq 0$ and hence $K_Y\cdot K_Y\geqq 0$. Denote by $\nu=K_Y\cdot K_Y-K_{S'}\cdot K_{S'}$ the number of points blown up under π and by E the union of the exceptional curves on S'. Now, since $K_{S'}=\pi^*K_Y+E$ and $K_{S'}+L'$ is ample we find

$$\begin{aligned}(K_{S'}+L')\cdot K_{S'} &= (K_{S'}+L')\cdot E+(\pi^*K_Y+L'+E)\cdot\pi^*K_Y\\ &\geqq \nu+K_Y\cdot K_Y+\pi^*K_Y\cdot L' = 2K_Y\cdot K_Y-K_{S'}\cdot K_{S'}+\pi^*K_Y\cdot L'\,.\end{aligned}$$

Hence from the Hodge index relation $d'K_Y\cdot K_Y\leqq(\pi^*K_Y\cdot L')^2$ we infer that

$$2K_{S'}\cdot K_{S'}+K_{S'}\cdot L'\geqq 2K_Y\cdot K_Y+(d'K_Y\cdot K_Y)^{1/2}\,. \tag{4.8}$$

The above relations become stronger and more useful when $K_{S'}\cdot K_{S'}<0$ and $K_Y\cdot K_Y>0$.

If $K_Y\cdot K_Y=0$, the following special arguments give us further numerical conditions.

(4.9) LEMMA. *With the notation as above, let (X, L) be a $\boldsymbol{P}^1$ bundle on a surface S' embedded by $\Gamma(L)$ in $\boldsymbol{P}^5$. Let $\pi: S'\to Y$ be the map onto the minimal model Y of S'. Assume that Y is an elliptic surface of Kodaira dimension 1 admitting a surjective morphism $\alpha: Y\to\boldsymbol{P}^1$ whose general fibre is an elliptic curve. Let $f'=\pi^{-1}(f)$ be the pullback to S' of a general fibre f of α. If f contains no points blown up under π then $L'\cdot f'\geqq 5$.*

PROOF. We can choose a smooth $S\in|L|$ such that the restriction $p_S: S\to S'$ of $p: X\to S'$ has a positive dimensional fibre over a point $x\in f'$. Let B be the proper transform of f' under $p_S: S\to S'$. Since $B\cong f$ is an elliptic curve and L is very ample we have $L\cdot B\geqq 3$ and hence $L'\cdot f'=1+L_S\cdot B\geqq 4$.

Now, let $A=p^{-1}(\pi^{-1}(f))$. Then A is a scroll over f'. Note that $\chi(\mathcal{O}_A)=0$ since $q(A)=1$ and also $L\cdot L\cdot A=L_A\cdot L_A=L'\cdot f'$. Assume $L'\cdot f'=4$ and look at the exact sequence

$$0\longrightarrow\mathcal{O}_A\longrightarrow L_A\longrightarrow L_C\longrightarrow 0$$

for a smooth $C\in|L|$. Since $1=(L\cdot f)_X=(L_A\cdot f_A)=(C\cdot f)_A$ we see that C is a section of $p: A\to f'$. Therefore C is an elliptic curve of degree $L\cdot C=L'\cdot f'=4$.

It thus follows that $h^0(L_C)=4$ and hence $h^0(L_A)\leqq 5$, Clearly $h^0(L_A)>4$ otherwise A would be embedded by $\Gamma(L)$ in $\boldsymbol{P}^3$, which contradicts $q(A)=1$. Then $h^0(L_A)=5$ and Lemma (0.8) applies to give the numerical contradiction $4-10(g(L_A)-1)=0$. Therefore $L'\cdot f'\geqq 5$ and we are done. Q.E.D.

(4.10) PROPOSITION. *Let (X, L) be a $\boldsymbol{P}^1$ bundle on a surface S' embedded by $\Gamma(L)$ in $\boldsymbol{P}^5$ as in (4.9). Assume that $p_g(S)>0$ for a smooth $S\in|L|$ and let Y be the minimal model of S'. Furthermore, assume $K_Y\cdot K_Y=0$. Then either $\chi(\mathcal{O}_S)\leqq 2$ or $K_{S'}\cdot L'+2K_{S'}\cdot K_{S'}\geqq 5(\chi(\mathcal{O}_S)-2)$.*

PROOF. Recall that $q(S)=q(Y)=0$ and $\kappa(Y)\geqq 0$ since $p_g(S)=p_g(Y)>0$. The assumption $K_Y\cdot K_Y=0$ implies that either Y is a $K3$ surface and $\chi(\mathcal{O}_S)=\chi(\mathcal{O}_Y)\leqq 2$ or Y is an elliptic surface with $\kappa(Y)=1$.

Let $\chi(\mathcal{O}_S)>2$. Then there exists a surjective morphism $\alpha: Y\to B$ onto a smooth curve B whose general fibre f is an elliptic curve. Furthermore, for some integer $N>0$,

$$NK_Y \approx \sum m_i f_{b_i}, \qquad m_i\in \boldsymbol{N},\ b_i\in B$$

and $\chi(\mathcal{O}_Y)=2\chi(\mathcal{O}_B)$ (see e.g. [**Bv**], IX, [**BPV**], V, (12.3)). It thus follows that $B\cong \boldsymbol{P}^1$ and $NK_Y\approx\alpha^*\mathcal{O}_{P^1}(m)$ where $m=\sum m_i$. Now from $h^0(\mathcal{O}_{P^1}(m))=h^0(NK_Y)\geqq N(h^0(K_Y)-1)+1$, we infer that $m\geqq N(\chi(\mathcal{O}_S)-2)$. For a general fiber f of $\alpha: Y\to \boldsymbol{P}^1$, let f' be the pull back under $\pi: S'\to Y$ of f. Then $L'\cdot f'\geqq 5$ by lemma (4.9). We compute

$$\begin{aligned}(4.10.1)\qquad L'\cdot\pi^*K_Y &= (L'\cdot\pi^*NK_Y)/N=(L'\cdot\pi^*\alpha^*\mathcal{O}_{P^1}(m))/N\\ &=(L'\cdot mf')/N\geqq 5m/N\geqq 5(\chi(\mathcal{O}_S)-2)\,.\end{aligned}$$

Since (S', L') is the reduction of (S, L_S), for any irreducible -1 curve $C\subset E$ one has $L'\cdot C\geqq 2$ so that $L'\cdot E\geqq 2\nu=-K_{S'}\cdot K_{S'}$. Therefore, since $L'\cdot K_{S'}=\pi^*K_Y\cdot L'+L'\cdot E$, (4.10.1) gives the result. Q.E.D.

For simplicity, let us first consider low values of d. A revised version of the program used to prove Proposition (4.7), running now for $p_g(S)>0$, gives us the following list of possible invariants for $9\leqq d\leqq 12$. Here $d'=L'\cdot L'$, $d_1'=K_{S'}\cdot L'$, $d_2'=K_{S'}\cdot K_{S'}$ and the constant c is defined as the biggest value of $K_Y\cdot K_Y$ which still satisfies the inequality (4.8).

d	d'	$g(L)$	#	d_2'	$(K_{S'}+L')^2$	$\chi(\mathcal{O}_S)$	d_1'	c
9	14	8	5	0	14	2	0	0
12	20	17	8	-6	38	4	12	6
				0	44	5		0
				6	50	6		6

The first two cases with $d=12$ do not occur. Indeed, if $d_2'=-6$, use (4.8). If $d_2'=c=0$ one has $Y\cong S'$, $K_Y\cdot K_Y=0$. Then, since $\chi(\mathcal{O}_S)>2$, Propositon (4.10) applies to give a contradiction. Note also that the degree $d=9$ case really occurs (see [**BSS**]). To conclude, by taking also into account the numerical conditions (4.8) and (4.10) it is a purely mechanical procedure to get the following final list of all possible cases for scrolls over surfaces in $\boldsymbol{P}^5$ of degree $\geqq 9$.

d	d'	$g(L)$	#	number of possible cases according to the values of d_1', d_2', $(K_{S'}+L')^2$ and $\chi(\mathcal{O}_S)$
9	14	8	5	1
12	20	17	8	1
15	23	29	8	8
21	42	64	21	30
23	31	77	8	83
24	36	85	12	84

We have carried these computations out using a Pascal program, which we don't include here but which is available on request.

It should be noted that the inequality (4.8) reduces from 14 to 8 the number of possible cases with degree $d=15$. In particular it rules out three cases with either $g(L)=30$ or 31, which we had previously shown to be the only possible cases with $d=15$ and $g(L)\neq 29$.

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, Geometry of Algebraic Curves, Volume I, Grundlehren, **267**, Springer-Verlag, 1985.

[BPV] W. Barth, C. Peters and A. Van de Ven, Compact Complex Surfaces, Ergebnisse der Math., **4**, Springer-Verlag, 1984.

[Bv] A. Beauville, Surfaces algébriques complexes, Astérisque, **54**, 1978.

[BBS] M. Beltrametti, A. Biancofiore and A.J. Sommese, Projective N-folds of log-general type. I, Trans. Amer. Math. Soc., **314** (1989), 825-849.

[BSS] M. Beltrametti, M. Schneider and A.J. Sommese, Threefolds of degree 9 and 10 in $\boldsymbol{P}^5$, Math. Ann., **288** (1990), 413-444.

[Bl] S. Bloch, Semiregularity and de Rham cohomology, Invent, Math., **17** (1972), 51-66.

[D] A. Durfee, Fifteen characterizations of rational double points and simple critical points, Enseign. Math., **25** (1979), 131-163.

[F] M.L. Fania, Configurations of -2 rational curves on sectional surfaces of n-folds, Math. Ann., **275** (1986), 317-325.

[Fj] T. Fujita, On polarized manifolds whose adjoint bundles are not semipositive, Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math., **10** (1987), 167-178.

[FS] M.L. Fania and A.J. Sommese, On the minimality of hyperplane sections of

Gorenstein threefolds, Contributions to several complex variables, in Honour of W. Stoll, Aspects of Math., E9, Vieweg-Verlag, 1986, 89-113.

[Gr] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331-368.

[GP] L. Gruson and C. Peskine, Genre des courbes de l'espace projectif, Algebraic Geometry, Proceedings Tromsö, Norway 1977, Lecture Notes in Math., **687**, Springer-Verlag, 1978.

[H] R. Hartshorne, Algebraic Geometry, G.T.M., **52**, Springer-Verlag, 1977.

[I1] P. Ionescu, Embedded projective varieties of small invariants, Proceedings of the Week of Algebraic Geometry, Bucharest 1982, Lecture Notes in Math., **1056**, Springer-Verlag, 1984, 142-187.

[I2] P. Ionescu, Embedded projective varieties of small invariants, II, Rev. Roumaine Math. Pures Appl., **31** (1986), 539-544.

[I3] P. Ionescu, Embedded projective varieties of small invariants, III, Preprint Series in Mathematics, **59** (1988), Bucharest.

[M] S. Mori, Projective manifolds with ample tangent bundles, Ann. of Math., **110** (1979), 593-606.

[N] S. Nakano, On the inverse of monoidal transformation, Publ. R.I.M.S., **6** (1971), 483-502.

[O1] C. Okonek, 3-Mannigfaltigkeiten in $\boldsymbol{P}^5$ und ihre zugehörigen stabilen Garben, Manuscripta Math., **38** (1982), 175-199.

[O2] C. Okonek, Über 2-codimensionale Untermannigfaltigkeiten vom Grad 7 in $\boldsymbol{P}^4$ und $\boldsymbol{P}^5$, Math. Zeit., **187** (1984), 209-219.

[S1] A. J. Sommese, Hyperplane sections of projective surfaces, I-The adjunction mapping, Duke Math. J., **46** (1979), 377-401.

[S2] A. J. Sommese, Hyperplane sections, Proceedings of the Algebraic Geometry Conference, University of Illinois at Chicago Circle, 1980, Lecture Notes in Math., **862**, Springer-Verlag, 1981, 232-271.

[S3] A. J. Sommese, On the minimality of hyperplane sections of projective threefolds, J. Reine Angew. Math., **329** (1981), 16-41.

[S4] A. J. Sommese, On the birational theory of hyperplane sections of projective threefolds, Unpublished 1981 Manuscript.

[S5] A. J. Sommese, Configurations of -2 rational curves on hyperplane sections of projective threefolds, Classification of Algebraic and Analytic Manifolds (ed. K. Ueno), Progr. Math., **39**, Birkhäuser, 1983, 465-497.

[S6] A. J. Sommese, Ample divisors on Gorenstein varieties, Proceedings of Complex Geometry Conference, Nancy 1985, Revue de l'Institut E. Cartan, **10** (1986).

[S7] A. J. Sommese, On the adjunction theoretic structure of projective varieties, Complex Analysis and Algebraic Geometry, Proceedings Göttingen 1985, Lecture Notes in Math., **1194**, Springer-Verlag, 1986, 175-213.

[S8] A. J. Sommese, On the nonemptiness of the adjoint linear system of a hyperplane section of a threefold, J. Reine Angew. Math., **402** (1989), 211-220; erratum, J. Reine Angew. Math., **41** (1990), 122-123.

[SV] A. J. Sommese and A. Van de Ven, On the adjunction mapping, Math. Ann., **278** (1987), 593-603.

[V] A. Van de Ven, On the 2-connectedness of very ample divisors on a surface, Duke Math. J., **46** (1979), 403-407.

Mauro C. BELTRAMETTI
Dipartimento di Matematica
Via L. B. Alberti 4, I-16132 Genova
Italy

Andrew J. SOMMESE
Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556
U. S. A.