J. Math. Soc. Japan
Vol. 44, No. 3, 1992

Modified analytic trivialization via weighted blowing up

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(Received July 31, 1991)

We consider the classification of real function-germs. It is well-known that there are modulus near non-simple germ for the differentiable equivalence. For topological equivalence it does not cause modulus, but seems to be *too* weak to provide a workable theory. T.-C. Kuo introduced the notion of the modified analytic trivialization (MAT) for a family of function-germs in [**4**], and generalized it naturally [**5**, **6**]. (We give the definition of it in more general form in § 1, and also call it MAT.) The associated equivalence relation preserves computability, but is slightly weaker than bianalyticity and much stronger than homeomorphism. He showed a finite classification theorem for isolated singularities in [**5**, **6**]. The next problem to be considered would be to describe MAT constant strata explicitly or what kind of singularities form a modified analytic equivalence class? Several authors have studied this problem, see e.g. [**4**, **2**, **7**]. In this paper we show a generalization of Kuo's theorem in [**4**], establishing MAT for a class of singularities in $\boldsymbol{R}^n$. As a consequence, we obtain that the Briançon-Speder family, for example, admits a MAT. In [**3**], S. Koike showed that the Briançon-Speder family does not preserve "tangency of arcs." Thus MAT does not preserve "tangency of arcs," as S. Koike conjectured before.

The author would like to thank Professor S. Koike for helpful advices.

§ 1. Definition.

Let $\pi : X \to \boldsymbol{R}^n$ be a real-analytic proper modification from a real space X to $\boldsymbol{R}^n$. Assume that there is a complexification of π, that is a complex-analytic proper modification. Let I be an open cube in $\boldsymbol{R}^m$, containing the origin 0, and $f_t(x) = F(x; t)$ a real analytic family of real analytic functions, defined in a neighborhood of $\{0\} \times I$ in $\boldsymbol{R}^n \times I$, with parameters $t \in I$. We say that F admits a *modified analytic trivialization* (*MAT*) *along* I *via* π if there is an analytic family of analytic isomorphisms H_t of neighborhoods of $\pi^{-1}(0)$ in X, which

This work was partly supported by Grand-in-Aid for encouragement of young scientists No. 02740060.

induces a family of homeomorphisms h_t of neighborhoods of 0 in $\boldsymbol{R}^n$ such that

$$f_t \circ h_t(x) = f_0(x), \qquad \text{for } t \in I.$$

This is a natural generalization of Kuo's original MAT in [**4**].

§ 2. Theorem.

Let $a=(a_1, \cdots, a_n)$ be an n-tuple of positive integers. Assume that the greatest common divisor of a_i's is 1. We can write

$$F(x;\, t) = H_k(x;\, t) + H_{k+1}(x;\, t) + \cdots, \qquad H_k \not\equiv 0,$$

where $H_j(x;\, t)=\sum a_\alpha(t)x^\alpha$, $a_1\alpha_1+\cdots+a_n\alpha_n=j$, $\alpha=(\alpha_1, \cdots, \alpha_n)$.

Theorem. *Suppose that the weighted initial form $H_k(x;\, t)$ of $F(x;\, t)$ satisfies the following condition (#) for $t \in I$.*

$$(\#) \qquad \left\{x \in \boldsymbol{R}^n :\ x_i^{a_i-1}\frac{\partial H_k}{\partial x_i}(x;\, t) = 0 \quad \text{for } i=1, \cdots, n\right\} = \{0\}.$$

Then F admits a MAT along I via the real weighted blowing up with weight a.

We give the definition of the real weighted blowing up in the next section. When $a=(1, \cdots, 1)$, this statement is basically the same as the theorem in § 3 in [**4**], although our X need not be smooth in general. Remember that, in [**2, 7**], it is required that the following supposition that the Newton polygon $\Gamma_+(f_t)$ is independent on t. But in our theorem the Newton polygon $\Gamma_+(f_t)$ may depend on t.

Example 1. $F(x;\, t)=x_3^5+tx_2^6x_3+x_1x_2^7+x_1^{15}$ (Briançon-Speder [**1**]). This is a family of weighted homogeneous polynomials with weight (1, 2, 3). Since $\{\partial F/\partial x_1=x_2\partial F/\partial x_2=x_3^2\partial F/\partial x_3=0\}=\{0\}$ for $t \in I$, F admits a MAT along I via the real weighted blowing up with weight (1, 2, 3). Here I is an open interval in $\boldsymbol{R}$, not containing $-15^{\frac{1}{7}}\cdot\left(\frac{7}{2}\right)^{\frac{4}{5}}/3$.

Example 2. $F(x;\, t)=x_3^3+tx_2^\alpha x_3+x_1x_2^\beta+x_1^{3\alpha}$ (Briançon-Speder [**1**]), where α is an odd number with $\alpha \geqq 3$, and $2\beta+1=3\alpha$. Similarly we obtain that F admits a MAT via the real weighted blowing up with weight $(1, 2, \alpha)$. Here I is an open interval in $\boldsymbol{R}$, such that $F(-, t)$ defines an isolated singularities at 0 for any $t \in I$.

Example 3. $F(x;\, t)=x_3(x_1^4+x_2^6+x_3^{12})+tx_2^7$. Since $\partial H_k/\partial x_3=x_1^4+x_2^6+13x_3^{12}$, F satisfies our condition (#) for $a=(3, 2, 1)$. Thus F admit a MAT via the real weighted blowing up.

EXAMPLE 4. $F(x; t)=x_1^4+2tx_1^2x_2^4+x_2^8$ admit a MAT along I via the real weighted blowing up with weight (2, 1), where I is an open interval in $\boldsymbol{R}$, not containing -1.

§3. The real weighted blowing up.

For an n-tuple $a=(a_1, \cdots, a_n)$ of positive integers, define a map φ of $\boldsymbol{C}^n$ to $\boldsymbol{C}^n$ by

$$\varphi: \boldsymbol{C}^n \ni (z_1, \cdots, z_n) \mapsto (x_1, \cdots, x_n) = (z_1^{a_1}, \cdots, z_n^{a_n}) \in \boldsymbol{C}^n.$$

Let $\tilde{\omega}: M \to \boldsymbol{C}^n$ be the blowing up at $\{0\}$, i.e. $M=\{(z_1, \cdots, z_n)\times[\zeta_1 : \cdots : \zeta_n] \in \boldsymbol{C}^n \times \boldsymbol{CP}^{n-1} : z_i\zeta_j=z_j\zeta_i$ for $1 \leqq i<j \leqq n\}$, and π is the restriction of the projection $\boldsymbol{C}^n \times \boldsymbol{CP}^{n-1} \to \boldsymbol{C}^n$. Let G be the direct product of the groups of a_i-th roots of the unity for $i=1, \cdots, n$. The group G acts on M by $g\cdot((z_1, \cdots, z_n) \times[\zeta_1 : \cdots : \zeta_n])=(g_1z_1, \cdots, g_nz_n)\times[g_1\zeta_1 : \cdots : g_n\zeta_n]$ for $g=(g_1, \cdots, g_n) \in G$. Then there is a natural map of $Y:=M/G$ to $\boldsymbol{C}^n$. This is an isomorphism over $\boldsymbol{C}^n-\{0\}$, and the inverse image of 0 is isomorphic to the weighted projective space with weight a. We call it the *(complex) weighted blowing up with weight a.*

Next we take the "real part" of the weighted blowing up $Y \to \boldsymbol{C}^n$. Let S be the subset of $\boldsymbol{C}^n$ such that the image of a point of S by φ is real, and $\tilde{S}$ the proper transform of S by $\tilde{\omega}$. Since G acts on $\tilde{S}$ invariantly, we obtain a real analytic map of the quotient space $X:=\tilde{S}/G$ to $\boldsymbol{R}^n$. We call $X \to \boldsymbol{R}^n$ the *real weighted blowing up with weight a.* Since the greatest common divisor of $a_1, \cdots, a_n$ is 1, $\boldsymbol{R}^n-\{0\}$ is dense in X. It is easy to see that X is a real analytic variety.

§4. Proof.

Let $z_1, \cdots, z_n$ be a complex coordinate system of $\boldsymbol{C}^n$. Let u_i, v_i be real coordinate functions with $z_i=u_i+\sqrt{-1}\, v_i$. We identify the real tangent space of $\boldsymbol{C}^n$ with the holomorphic tangent space of $\boldsymbol{C}^n$ using the map defined by

$$\frac{\partial}{\partial u_i} \mapsto \frac{\partial}{\partial z_i} \quad \text{and} \quad \frac{\partial}{\partial v_i} \mapsto \sqrt{-1}\frac{\partial}{\partial z_i}.$$

Then the usual euclid metric is given by

$$\left\langle \sum_i \alpha_i \frac{\partial}{\partial z_i}, \sum_i \beta_i \frac{\partial}{\partial z_i} \right\rangle = \operatorname{Re} \sum_i \alpha_i \bar{\beta}_i .$$

It is easy to see that

$$grad\ \mathrm{Re}\, f = \sum_i \overline{\frac{\partial f}{\partial z_i}} \cdot \frac{\partial}{\partial z_i}, \quad \text{and} \quad grad\ \mathrm{Im}\, f = \sqrt{-1}\, grad\ \mathrm{Re}\, f,$$

for a holomorphic function $f=f(z)$. Thus $\sum_i \alpha_i \frac{\partial}{\partial z_i}$ is tangent to each level surface of f if $\sum_i \alpha_i \frac{\partial f}{\partial z_i}=0$.

We return to our situation: $F(x, t): \boldsymbol{R}^n \times I \to \boldsymbol{R}$. First we consider the case for $m=1$. Using the coordinate x, $t=t_1$, we consider the complexification $F^C: \boldsymbol{C}^n \times \tilde{I} \to \boldsymbol{C}$ of $F: \boldsymbol{R}^n \times I \to \boldsymbol{R}$, where $\tilde{I} \subset \boldsymbol{C}$ is a small domain with $\tilde{I} \cap \boldsymbol{R} = I$. Put $\tilde{F} := F^C \circ (\varphi \times id_I)$, where id_I is the identity map of I. Define a real vector field V by

$$V = -\frac{\sum_{i=1}^{n} \frac{\partial \tilde{F}}{\partial t} \cdot \overline{\frac{\partial \tilde{F}}{\partial z_i}} \cdot \frac{\partial}{\partial z_i}}{\left|\frac{\partial \tilde{F}}{\partial z_1}\right|^2 + \cdots + \left|\frac{\partial \tilde{F}}{\partial z_n}\right|^2} + \frac{\partial}{\partial t},$$

where $\overline{}$ is the complex conjugation. This is tangent to each level surface of $\tilde{F}$, whenever V is defined. For $s=1, \cdots, n$, the functions $w_s = z_s$ and $w_j = \zeta_j/\zeta_s$, $(j \neq s)$ form a coordinate system on $\zeta_s \neq 0$ in M, and $\tilde{\omega}$ is expressed as $z_s = w_s$, $z_i = w_s w_i$, $i \neq s$. Then $\partial/\partial z_s = \partial/\partial w_s - \sum_{j \neq s} (w_j/w_s) \partial/\partial w_j$, $\partial/\partial z_i = (1/w_s) \partial/\partial w_i$, $i \neq s$. Note that $\partial \tilde{F}/\partial t$ and $\partial \tilde{F}/\partial z_i$ $(1 \leq i \leq n)$ are of order k and $k-1$ in z respectively; their lifts are therefore divisible by w_s^k and w_s^{k-1} respectively. Now observe that the denominator of z_i-component of V is equal to

$$\sum_{j=1}^{n} a_j^2 |z_j|^{2a_j-2} |\partial F/\partial x_j(z_1^{a_1}, \cdots, z_n^{a_n}; t)|^2.$$

Note that it is G-invariant. Its lift is equal to $w_s^{2k-2} U(w; t)$, where U is defined and analytic in a neighborhood of $w_s=0$; and U is positive in some neighborhood of $(\tilde{\omega}^{-1}(0) \cap \tilde{S}) \times I$ in $M \times \tilde{I}$ because of (#). Thus V has a real analytic lift $\tilde{V}$ there expressed in the form

$$\tilde{V} = w_s V_s(w; t) \frac{\partial}{\partial w_s} + \sum_{j \neq s} V_j(w; t) \frac{\partial}{\partial w_j} + \frac{\partial}{\partial t},$$

where $V_1, \cdots, V_n$ are real analytic near $(\tilde{\omega}^{-1}(0) \cap \tilde{S}) \times I$. The coefficient of $\partial/\partial w_s$ vanishes on $\{w_s=0\} = (\tilde{\omega}^{-1}(0) \times I) \cap \{\zeta_s \neq 0\}$.

Note that the numerator of the first term of V equals

$$\sum_{i=1}^{n} \frac{\partial F}{\partial t}(z_1^{a_1}, \cdots, z_n^{a_n}; t) \cdot a_i \overline{\frac{\partial F}{\partial x_i}(z_1^{a_1}, \cdots, z_n^{a_n}; t)} \left(\bar{z}_i^{a_i-1} \frac{\partial}{\partial z_i}\right).$$

It is easy to verify that

$$\bar{z}_i^{a_i-1}\frac{\partial}{\partial z_i}=r_i^{a_i-1}\cdot\cos a_i\theta_i\cdot\frac{\partial}{\partial r_i}+r^{a_i-2}\cdot\sin a_i\theta_i\cdot\frac{\partial}{\partial\theta_i},$$

where $z_i=r_i e^{\theta_i\sqrt{-1}}$; thus this is tangent to S, and G-equivaliant. Therefore $\hat{V}$ is tangent to $\tilde{S}$, and a G-equivaliant vector field. Then the trajectory of $\hat{V}$ gives analytic isomorphisms of $X=\tilde{S}/G$. By our construction these are the desired ones.

In the case for $m\geqq 2$, an argument similar to that in § 3 in [**4**] works; and we omit the details.

§ 5. Problems.

1. Can we replace the condition (#) in Theorem with the following condition (♮)?
(♮): H_k defines an isolated singularity at $\{0\}$.
2. Find a modified analytic invariant. The Milnor number $\mu(f)$ $(:=\dim_{\boldsymbol{R}}\boldsymbol{R}\{x\}/(\partial f/\partial x_1,\cdots,\partial f/\partial x_n))$ is not such an invariant. For example, $f_t=(x^2+y^2)^2+tx^{10}+x^{11}$. How about the Lojasiewicz exponent $(:=\min\{\alpha:\exists C,\ |\mathrm{grad}\,f|\geqq C|x|^{\alpha}$ near $0\})$?

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