J. Math. Soc. Japan
Vol. 45, No. 3, 1993

Unicity theorems for the Gauss maps of complete minimal surfaces

Dedicated to Professor Shoshichi Kobayashi on his 60th birthday

By Hirotaka FUJIMOTO

(Received June 19, 1992)

§1. Introduction.

Let M be a minimal surface in $\boldsymbol{R}^3$, or more precisely, a connected oriented minimal surface immersed in $\boldsymbol{R}^3$. By definition, the Gauss map G of M is the map which maps each point $p\in M$ to the unit normal vector $G(p)\in S^2$ of M at p. Instead of G, we study the map $g:=\pi\cdot G: M\rightarrow\bar{\boldsymbol{C}}:=\boldsymbol{C}\cup\{\infty\}$ for the stereographic projection π of S^2 onto $\bar{\boldsymbol{C}}$. The surface M is canonically considered as an open Riemann surface with a conformal metric and g is a meromorphic function on M. For a complete minimal surface in $\boldsymbol{R}^3$ g has many properties which have similarities to results in value distribution theory of meromorphic functions on $\boldsymbol{C}$. The author obtained some of them in the previous papers [**5**], [**6**] and [**7**]. The purpose of this paper is to give some unicity theorems for the Gauss map of minimal surfaces in $\boldsymbol{R}^3$ which are similar to the following theorem for meromorphic functions given by R. Nevanlinna ([**9**]):

THEOREM. *If two nonconstant meromorphic functions g and $\tilde{g}$ on $\boldsymbol{C}$ have the same inverse images for five distinct values, then $g\equiv\tilde{g}$.*

Let M and $\tilde{M}$ be two nonflat minimal surfaces in $\boldsymbol{R}^3$ and assume that there is a conformal diffeomorphism Φ of M onto $\tilde{M}$. Consider the maps $g:=\pi\cdot G$ and $\tilde{g}:=\pi\cdot\tilde{G}\cdot\Phi$, where G and $\tilde{G}$ are the Gauss maps of M and $\tilde{M}$ respectively. Suppose that there are q distinct points $\alpha_1, \alpha_2, \cdots, \alpha_q$ such that $g^{-1}(\alpha_j)=\tilde{g}^{-1}(\alpha_j)$ $(1\leqq j\leqq q)$. The main result in this paper is stated as follows:

THEOREM I. *If $q\geqq 7$ and either M or $\tilde{M}$ is complete, then $g\equiv\tilde{g}$.*

For a particular case, we can show the following:

THEOREM II. *If $q\geqq 6$ and both of M and $\tilde{M}$ are complete and have finite total curvature, then $g\equiv\tilde{g}$.*

In Theorem I, the number seven is the best-possible. In fact, we can construct two mutually isometric complete minimal surfaces whose Gauss maps are dis-

This research was partially supported by Grant-in-Aid for Scientific Research (No. 04640141), Ministry of Education, Science and Culture.

tinct and have the same inverse images for six distinct values. It is an interesting open problem to ask whether the number six in Theorem II is the best-possible or not.

§ 2. A pseudo-metric with strictly negative curvature.

As in the previous paper [8], for each $\alpha, \beta \in \bar{C}$ we define

$$|\alpha, \beta| := \frac{|\alpha-\beta|}{\sqrt{1+|\alpha|^2}\sqrt{1+|\beta|^2}}$$

if $\alpha \neq \infty$ and $\beta \neq \infty$, and $|\alpha, \beta| = |\beta, \alpha| := 1/\sqrt{1+|\alpha|^2}$ if $\beta = \infty$.

For later use, we give the following:

PROPOSITION 2.1. *Let g and $\tilde{g}$ be mutually distinct nonconstant meromorphic functions on a Riemann surface M such that, for q distinct numbers $\alpha_1, \alpha_2, \cdots, \alpha_q$, $g^{-1}(\alpha_j) = \tilde{g}^{-1}(\alpha_j)$ $(1 \leqq j \leqq q)$, where we assume $q > 4$. For $a_0 > 0$ and ε with $q-4 > q\varepsilon > 0$ set*

$$\lambda := \left(\prod_{j=1}^{q} |g, \alpha_j| \log\left(\frac{a_0}{|g, \alpha_j|^2}\right)\right)^{-1+\varepsilon}, \quad \tilde{\lambda} := \left(\prod_{j=1}^{q} |\tilde{g}, \alpha_j| \log\left(\frac{a_0}{|\tilde{g}, \alpha_j|^2}\right)\right)^{-1+\varepsilon},$$

and define

$$d\tau^2 := |g, \tilde{g}|^2 \lambda \tilde{\lambda} \frac{|g'|}{1+|g|^2} \frac{|\tilde{g}'|}{1+|\tilde{g}|^2} |dz|^2 \tag{2.2}$$

outside the set $E := \bigcup_{j=1}^{q} g^{-1}(\alpha_j)$ and $d\tau^2 := 0$ on E. Then, for a suitably chosen a_0, $d\tau^2$ is continuous on M and has strictly negative curvature on the set $\{d\tau^2 \neq 0\}$.

For the proof, we use the following result in the previous paper ([8, Proposition 2]).

PROPOSITION 2.3. *Let g be a nonconstant meromorphic function on $\Delta_R := \{z\,;\, |z| < R\}$. Take q distinct values $\alpha_1, \cdots, \alpha_q$, where $q > 1$. Then, for each $\rho > 0$ and ε with $q-1 > q\varepsilon > 0$, there exists some positive constants a_0 and C such that*

$$\Delta \log \frac{(1+|g|^2)^\rho}{\prod_{j=1}^{q} \log(a_0/|g, \alpha_j|^2)}$$
$$\geqq C \frac{|g'|^2}{(1+|g|^2)^2} \prod_{j=1}^{q} \left(|g, \alpha_j|^2 \log^2 \frac{a_0}{|g, \alpha_j|^2}\right)^{-1+\varepsilon}.$$

PROOF OF PROPOSITION 2.1. Take an arbitrary point $z_0 \in M$. The pseudometric $d\tau^2$ remains unaltered by Möbius transformations of $\bar{C}$ corresponding to rotations of S^2. To see the continuity of $d\tau^2$ at z_0, we may assume that $g(z_0) \neq \infty$ and $\tilde{g}(z_0) \neq \infty$. If $z_0 \notin E$, then $d\tau^2$ is obviously continuous at z_0. Assume that $z_0 \in E$, so that $g(z_0) = \alpha_j$ for some j. Then, $g'/(g-\alpha_j)$ and $\tilde{g}'/(\tilde{g}-\alpha_j)$ have poles of order one and $g - \tilde{g}$ $(=(g-\alpha_j)-(\tilde{g}-\alpha_j))$ has a zero at z_0, whence $d\tau^2$ is continuous at z_0.

Now, taking an arbitrary holomorphic local coordinate z on an open subset of $\{d\tau^2\neq 0\}$, consider the nonnegative function μ with $d\tau^2=\mu^2|dz|^2$. We can write

$$\mu^2 = u(1+|g|^2)^\rho(1+|\tilde{g}|^2)^\rho \Big/ \prod_{j=1}^{q}\left(\log\frac{a_0}{|g,\alpha_j|^2}\log\frac{a_0}{|\tilde{g},\alpha_j|^2}\right)^{1-\varepsilon}$$

on $\{d\tau^2\neq 0\}$, where u is a positive function with $\Delta\log u=0$ and $\rho:=q(1-\varepsilon)/2-2$ (>0). By the use of Proposition 2.3 we have

$$\begin{aligned}\Delta\log\mu^2 &= \Delta\log\frac{(1+|g|^2)^\rho}{\prod_{j=1}^{q}\log^{1-\varepsilon}(a_0/|g,\alpha_j|^2)}+\Delta\log\frac{(1+|\tilde{g}|^2)^\rho}{\prod_{j=1}^{q}\log^{1-\varepsilon}(a_0/|\tilde{g},\alpha_j|^2)}\\ &\geqq C_1\frac{\lambda^2|g'|^2}{(1+|g|^2)^2}+C_2\frac{\tilde{\lambda}^2|\tilde{g}'|^2}{(1+|\tilde{g}|^2)^2}\\ &\geqq C_3\frac{\lambda\tilde{\lambda}|g'||\tilde{g}'|}{(1+|g|^2)(1+|\tilde{g}|^2)}\end{aligned}$$

for some positive constants C_j's. Since $|g,\tilde{g}|\leqq 1$, we obtain the inequality

$$\Delta\log\mu^2 \geqq C_3\mu^2\,.$$

This shows that $d\tau^2$ has strictly negative curvature. The proof of Proposition 2.1 is completed.

COROLLARY 2.4. *Let g and $\tilde{g}$ be meromorphic functions on Δ_R satisfying the same assumption as in Proposition* 2.1. *Then, for the metric $d\tau^2$ defined by* (2.2), *there is a constant $C>0$ such that*

$$d\tau^2 \leqq C\frac{4R^2}{(R^2-|z|^2)^2}|dz|^2\,.$$

PROOF. This is an immediate consequence of Proposition 2.1 and the generalized Schwarz lemma given by L. V. Ahlfors in [1].

§ 3. The proof of Theorem I.

As is stated in § 1, we consider two nonflat minimal surfaces $x:=(x_1,x_2,x_3)$: $M\to\boldsymbol{R}^3$ and $\tilde{x}:=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3):\tilde{M}\to\boldsymbol{R}^3$ such that there is a conformal diffeomorphism Φ of M onto $\tilde{M}$, and assume that there are q distinct values $\alpha_1,\cdots,\alpha_q$ such that $g^{-1}(\alpha_j)=\tilde{g}^{-1}(\alpha_j)$ for the meromorphic functions $g:=\pi\cdot G$, $\tilde{g}:=\pi\cdot\tilde{G}\cdot\Phi$, where π is the stereographic projection and G and $\tilde{G}$ are the Gauss maps of M and $\tilde{M}$ respectively. Here, there is no harm in assuming that $\alpha_q:=\infty$. As in Theorem I, we assume that $q>6$ and either M or $\tilde{M}$, say M, is complete and, furthermore, $g\not\equiv\tilde{g}$. We may consider M and $\tilde{M}$ as open Riemann surfaces with conformal metrics ds^2 and $d\tilde{s}^2$ respectively. Then the given map Φ gives a biholomorphic isomorphism between M and $\tilde{M}$. As is well-known (e.g., [**11**]), setting

$$\omega := \partial x_1 - \sqrt{-1}\partial x_2 , \qquad \tilde{\omega} := \partial \tilde{x}_1 - \sqrt{-1}\partial \tilde{x}_2 ,$$

we can write

$$ds^2 = (1+|g|^2)^2 |\omega|^2 , \qquad d\tilde{s}^2 = (1+|\tilde{g}|^2)^2 |\tilde{\omega}|^2 .$$

Therefore, for each holomorphic local coordinate z defined on a simply connected open set U we can find a nowhere zero holomorphic function h_z such that

$$ds^2 = |h_z|^2 (1+|g|^2)(1+|\tilde{g}|^2)|dz|^2 . \tag{3.1}$$

Taking some η with $q-6>q\eta>0$, we set

$$\tau := \frac{2}{q-4-q\eta} \quad (<1) \tag{3.2}$$

and define the pseudo-metric $d\sigma^2$ by

$$d\sigma^2 := |h_z|^{2/(1-\tau)} \left(\frac{\prod_{j=1}^{q-1}(|g-\alpha_j||\tilde{g}-\alpha_j|)^{1-\eta}}{|g-\tilde{g}|^2 |g'||\tilde{g}'| \prod_{j=1}^{q-1}(1+|\alpha_j|^2)^{1-\eta}} \right)^{\tau/(1-\tau)} |dz|^2, \tag{3.3}$$

which does not depend on a choice of holomorphic local coordinate z and so well-defined on $M' := M - E'$, where

$$E' := \{z \in M ; g'(z) = 0,\ \tilde{g}'(z) = 0 \text{ or } g(z)(=\tilde{g}(z)) = \alpha_j \text{ for some } j\}.$$

On the other hand, setting $\varepsilon := \eta/2$, we can define another pseudo-metric $d\tau^2$ on M by (2.2), which has strictly negative curvature on M'.

Take an arbitrary point z in M'. Using the fact that $d\sigma^2$ is flat on M', we take the largest R ($\leqq +\infty$) such that there is a holomorphic map $\Psi : \Delta_R \to M'$ with $\Psi(0)=z$ which is a local isometry with respect to the standard metric on Δ_R and the metric $d\sigma^2$ on M'. Observe the pseudo-metric $\Psi^* d\tau^2$ on Δ_R, which has strictly negative curvature. Since there is no metric with strictly negative curvature on $\boldsymbol{C}$, we have necessarily $R<+\infty$. Moreover, by the same arguments as in the previous papers [**5**] and [**8**], we can choose a point w_0 with $|w_0|=R$ such that, for the line segment

$$\Gamma : w = t w_0 \quad (0 \leqq t < 1),$$

the image $\gamma := \Psi(\Gamma)$ tends to the boundary of M' as t tends to 1. Here, if we suitably choose the constant η in the definition (3.2) of τ, γ tends to the boundary of M.

Since Ψ is a local isometry, we may take the coordinate w as a holomorphic local coordinate on M' and we may write $d\sigma^2 = |dw|^2$. By (3.3) we obtain

$$|h_w|^2=\left(\frac{|g-\tilde{g}|^2|g'||\tilde{g}'|\prod_{j=1}^{q-1}(1+|\alpha_j|^2)^{1-\eta}}{\prod_{j=1}^{q-1}(|g-\alpha_j||\tilde{g}-\alpha_j|)^{1-\eta}}\right)^{\tau}.$$

According to (3.1), we have

$$\begin{aligned}ds^2 &= |h_w|^2(1+|g|^2)(1+|\tilde{g}|^2)|dw|^2\cdot\\ &=\left(\frac{|g-\tilde{g}|^2|g'||\tilde{g}'|(1+|g|^2)^{1/\tau}(1+|\tilde{g}|^2)^{1/\tau}\prod_{j=1}^{q-1}(1+|\alpha_j|^2)^{1-\eta}}{\prod_{j=1}^{q-1}(|g-\alpha_j||\tilde{g}-\alpha_j|)^{1-\eta}}\right)^{\tau}|dw|^2\\ &=\left(\mu^2\prod_{j=1}^{q}(|g,\alpha_j||\tilde{g},\alpha_j|)^{\varepsilon}\left(\log\frac{a_0}{|g,\alpha_j|^2}\log\frac{a_0}{|\tilde{g},\alpha_j|^2}\right)^{1-\varepsilon}\right)^{\tau}|dw|^2,\end{aligned}$$

where μ is the function with $d\tau^2=\mu^2|dw|^2$. On the other hand, since the function $x^{\varepsilon}\log^{1-\varepsilon}(a_0/x^2)$ $(0<x\leqq 1)$ is bounded, we have

$$ds^2\leqq C\left(\frac{|g,\tilde{g}|^2|g'||\tilde{g}'|\lambda\tilde{\lambda}}{(1+|g|^2)(1+|\tilde{g}|^2)}\right)^{\tau}|dw|^2$$

for some $C>0$. Therefore, by the use of Corollary 2.4 we have

$$ds\leqq C'\left(\frac{2R}{R^2-|w|^2}\right)^{\tau}|dw|$$

for some C'. This yields that

$$\int_{\gamma}ds\leqq C'\int_{\Gamma}\left(\frac{2R}{R^2-|w|^2}\right)^{\tau}|dw|<+\infty,$$

which contradicts the assumption of completeness of M. We have necessarily $g\equiv\tilde{g}$. The proof of Theorem I is completed.

§4. The proof of Theorem II.

To prove of Theorem II is given by reduction to absurdity. Under the assumption of Theorem II suppose that $g\not\equiv\tilde{g}$. According to Chern-Osserman's theorem ([2, Theorem 1]), M may be identified with $\overline{M}-\{a_1,\cdots,a_k\}$ for a compact Riemann surface $\overline{M}$. Moreover, the maps g, $\tilde{g}$ and the metric ds^2, $\Phi^*d\tilde{s}^2$ may be considered as meromorphic functions and pseudo-metrics on $\overline{M}$ with singularities like poles at $a_1,\cdots,a_k$ respectively. By the assumption, $g^{-1}(\alpha_j)\cap M=\tilde{g}^{-1}(\alpha_j)\cap M$ $(1\leqq j\leqq q)$. We denote by d_g and $d_{\tilde{g}}$ the degrees of g and $\tilde{g}$ respectively and by v_g and $v_{\tilde{g}}$ the total branching orders of g and $\tilde{g}$ on $\overline{M}$ respectively. Set

$$n_j:=\#(g^{-1}(\alpha_j)\cap M)=\#(\tilde{g}^{-1}(\alpha_j)\cap M)\quad(1\leqq j\leqq q).$$

We see easily

$$qd_g\leqq k+\sum_{j=1}^{q}n_j+v_g.$$

On the other hand, we have

$$2\gamma-2=v_g-2d_g$$

by Riemann-Hurwitz formula (e.g., [3, p. 140]) and

$$\frac{1}{2\pi}C(M)=-2d_g \leqq \chi(M)-k=2-2\gamma-2k$$

by Chern-Osserman's theorem ([11, Theorem 9.3]), where γ, $C(M)$ and $\chi(M)$ denote the genus of M, the total curvature of M and the Euler characteristic of M respectively. These imply that

$$(q-4)d_g \leqq \sum_{j=1}^{q} n_j-k .$$

Similarly,

$$(q-4)d_{\tilde{g}} \leqq \sum_{j=1}^{q} n_j-k .$$

Consider the function

$$\varphi:=\frac{1}{g-\tilde{g}} .$$

By the assumption, we have

$$\sum_{j=1}^{q} n_j \leqq \text{the number of poles of } \varphi \leqq d_g+d_{\tilde{g}} .$$

Therefore, we conclude

$$(q-4)(d_g+d_{\tilde{g}}) \leqq 2(d_g+d_{\tilde{g}})-2k$$

and so

$$2k \leqq (6-q)(d_g+d_{\tilde{g}}) .$$

Since $k>0$, we have necessarily $q \leqq 5$. This contradicts the assumption. The proof of Theorem II is completed.

§ 5. An example.

In this section, we shall give an example which shows that the number seven in Theorem I is the best-possible. To this end, taking a number α with $\alpha \neq 0, \pm 1$, we consider the meromorphic functions

$$h(z):=\frac{1}{z(z-\alpha)(\alpha z-1)}, \qquad g(z)=z$$

and the universal covering surface M of $\boldsymbol{C}-\{0, \alpha, 1/\alpha\}$. The functions h and g may be considered as holomorphic functions on M. As is well-known, by setting

$$x_1 := \mathrm{Re}\int_0^z h(1-g^2)dz, \quad x_2 := \mathrm{Re}\int_0^z \sqrt{-1}h(1+g^2)dz, \quad x_3 := 2\mathrm{Re}\int_0^z hgdz,$$

we can construct a minimal surface $x=(x_1, x_2, x_3): M \to \boldsymbol{R}^3$ in $\boldsymbol{R}^3$ whose Gauss map is essentially the same as g. It is easily seen that M is complete. On the other hand, if we construct another minimal surface $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3): \tilde{M} \to \boldsymbol{R}^3$ in the similar manner by the use of the meromorphic functions

$$h(z) := \frac{1}{z(z-\alpha)(\alpha z-1)}, \qquad \tilde{g}(z) = \frac{1}{z},$$

we can easily check that $\tilde{M}$ is isometric with M, so that the identity map $\Phi: z \in M \mapsto z \in \tilde{M}$ is a conformal diffeomorphism. For the maps g and $\tilde{g}$ we have $g \not\equiv \tilde{g}$ and $g^{-1}(\alpha_j) = \tilde{g}^{-1}(\alpha_j)$ for six values

$$\alpha_1 := 0, \quad \alpha_2 := \infty, \quad \alpha_3 := \alpha, \quad \alpha_4 := \frac{1}{\alpha}, \quad \alpha_4 := 1, \quad \alpha_5 := -1\,.$$

These show that the number seven in Theorem I cannot be replaced by six.

References

[1] L. V. Ahlfors, An extension of Schwarz's lemma, Trans. Amer. Math. Soc., **43** (1938), 359-364.

[2] S. S. Chern and R. Osserman, Complete minimal surfaces in euclidean n-space, J. Analyse Math., **19** (1967), 15-34.

[3] O. Forster, Lectures on Riemann surfaces, Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[4] H. Fujimoto, A unicity theorem for meromorphic maps of a complete Kähler manifold into $P^N(\boldsymbol{C})$, Tôhoku Math. J., **38** (1986), 327-341.

[5] H. Fujimoto, On the number of exceptional values of the Gauss map of minimal surfaces, J. Math. Soc. Japan, **40** (1988), 235-247.

[6] H. Fujimoto, Modified defect relations for the Gauss map of minimal surfaces, J. Differential Geom., **29** (1989), 245-262.

[7] H. Fujimoto, Modified defect relations for the Gauss map of minimal surfaces, II, J. Differential Geom., **31** (1990), 365-385.

[8] H. Fujimoto, On the Gauss curvature of minimal surfaces, J. Math. Soc. Japan., **44** (1992), 427-439.

[9] R. Nevanlinna, Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen, Acta Math., **48** (1926), 367-391.

[10] R. Osserman, Global properties of minimal surfaces in E^3 and E^n, Ann. of Math., **80** (1964), 340-364.

[11] R. Osserman, A survey of minimal surfaces, 2nd ed., Dover Publ. Inc., New York, 1986.

Hirotaka FUJIMOTO

Department of Mathematics
Faculty of Science
Kanazawa University
Kakuma-Machi
Kanazawa 920-11
Japan