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Hausdorff dimension of Markov invariant sets

By Paweł WALCZAK

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Introduction.

One of the old questions about exceptional minimal sets of codimension-one C^2-foliations of compact manifolds reads (compare [**La**]): Is the Lebesgue measure $|\mathcal{M}|$ of any exceptional minimal set $\mathcal{M}$ equal to 0? The answer in general is still unknown. The class of Markov minimal sets was introduced by John Cantwell and Lawrence Conlon [**CC**] in the context of this question. Among the other results, they proved that $|\mathcal{M}|=0$ if $\mathcal{M}$ is a Markov exceptional minimal set. The same result in the particular case of a Markov exceptional minimal set with holonomy generated by two maps defined on a common interval was obtained in [**Mat**].

In [**LaW**], while studying relations between different invariants describing the dynamics of foliations, the authors observed that the question about the Hausdorff dimension $\dim_H$ of exceptional minimal sets is also of some interest. Since the inequality

$$\dim_H(\mathcal{M}) < \dim M\,, \tag{1}$$

M being the foliated manifold, implies that $|\mathcal{M}|=0$, Markov exceptional minimal sets seem to be good candidates to satisfy (1). In fact, this is our result here.

THEOREM. *If $\mathcal{M}$ is a Markov exceptional minimal set of a codimension-one C^2-foliation $\mathcal{F}$ of a compact manifold M, then $\mathcal{M}$ satisfies inequality* (1).

The Theorem follows immediately from the description of Markov exceptional minimal sets given in [**CC**] and the following.

PROPOSITION. *If Γ is a finitely generated Markov pseudogroup of local C^2-diffeomorphisms of the real line $\boldsymbol{R}$ and Z_0 is its Markov invariant set, then*

$$\dim_H(Z_0) < 1\,. \tag{1}$$

The idea of the proof of the Proposition is very similar to that of Theorem 3 in [**CC**]. We use several preparatory Lemmas of [**CC**] as well as some subtle estimates of [**Mat**]. However, we believe that the result itself as well as several details which differ the case of the Hausdorff dimension from that of Lebesgue measure justify writing down a short article.

1. Hausdorff dimension.

Let us recall (see [**HuW**], [**Ed**], etc.) that the *Hausdorff dimension* $\dim_H X$ of a metric space X is defined by

$$\dim_H X = \inf\{s>0\,;\ \mathcal{H}^s(X)=0\}, \tag{3}$$

where

$$\mathcal{H}^s(X) = \lim_{\varepsilon\to 0} \mathcal{H}^s_\varepsilon(X) = \sup_\varepsilon \mathcal{H}^s_\varepsilon(X), \tag{4}$$

$$\mathcal{H}^s_\varepsilon(X) = \inf_{\mathcal{A}\in C(\varepsilon)} H^s(\mathcal{A}), \tag{5}$$

$C(\varepsilon)$ is the family of all countable coverings of X by sets of diameter less than ε and

$$H^s(\mathcal{A}) = \sum_{A\in\mathcal{A}} (\operatorname{diam} A)^s. \tag{6}$$

The following facts are known quite well and follow immediately from the definition.

LEMMA 1. (i) *If X admits a sequence $(\mathcal{A}_n)$ of countable coverings such that*

$$\operatorname{diam}\mathcal{A}_n \longrightarrow 0 \quad \textit{and} \quad H^s(\mathcal{A}_n) \longrightarrow 0 \qquad \textit{as } n\to\infty, \tag{7}$$

then $\dim_H X \leqq s$.

(ii) *If $X\subset Y$, then* $\dim_H X \leqq \dim_H Y$.

(iii) *If $X=X_1\cup X_2\cup\cdots$, then*

$$\dim_H X = \sup_m \dim_H X_m. \tag{8}$$

(iv) *If X is countable, then* $\dim_H X=0$.

(v) *If metric spaces X and Y are quasi-isometric, then*

$$\dim_H X = \dim_H Y. \qquad \square \tag{9}$$

REMARK. In the sequel, we consider subspaces of the real line $\boldsymbol{R}$ only. In this case $\operatorname{diam}(A)$ in (6) can be replaced by the Lebesgue measure $|A|$ whenever $\mathcal{A}$ consists of intervals.

2. Markov pseudogroups.

Following [**CC**], we define *Markov pseudogroups* as pseudogroups Γ of local C^2-diffeomorphisms of the real line $\boldsymbol{R}$ generated by finite sets $\Gamma_0=\{h_1, \cdots, h_m\}$,

$m>1$, of maps between closed bounded intervals I_α and K_α, $h_\alpha: I_\alpha \to K_\alpha$, satisfying the following conditions:

(i) $K_\alpha \cap K_\beta = \emptyset$ when $\alpha \neq \beta$,

(ii) either $K_\alpha \subset I_\beta$ or $K_\alpha \cap I_\beta = \emptyset$.

(Since the maps h_α are differentiable of class C^2, they can be extended to C^2-maps $\tilde{h}_\alpha: \tilde{I}_\alpha \to \tilde{K}_\alpha$ between larger open intervals $\tilde{I}_\alpha \supset I_\alpha$ and $\tilde{K}_\alpha \supset K_\alpha$ satisfying analogous conditions.)

Let us say that the Markov property involved here is that the result of the "experiment" of building composed maps of the form $h_{\gamma_1} \circ \cdots \circ h_{\gamma_n}$ depends only on the existence of the compositions $h_{\gamma_i} \circ h_{\gamma_{i+1}}$ of consecutive maps.

For any sequence $\gamma = (\gamma_1, \cdots, \gamma_n)$, $\gamma_i \in \{1, \cdots, m\}$, set $|\gamma| = n$, $h_\gamma = h_{\gamma_1} \circ \cdots \circ h_{\gamma_n}$ and $K_\gamma = h_\gamma(I_{\gamma_n})$ whenever the composition exists. Let $\Gamma_+ = \{h_\gamma\}$, $\Gamma_n = \{h_\gamma;\ |\gamma| = n\}$,

$$Z = \bigcap_{n=1}^{\infty} \bigcup_{|\gamma|=n} K_\gamma \quad \text{and} \quad Z_0 = Z \setminus \inf(Z). \tag{10}$$

We shall say that Z_0 is the *Markov invariant set* of Γ. In typical situations (but not always) it is a minimal Γ-invariant Cantor set. On Z_0, the local homeomorphism $\tau: Z_0 \to Z_0$ is well defined by

$$\tau(x) = h_\alpha^{-1}(x) \qquad \text{whenever } x \in K_\alpha. \tag{11}$$

The map τ is said to be the *subshift* of Γ.

It is known ([**CC**], [**In**]) that any Markov pseudogroup can be realized as the holonomy pseudogroup of a codimension-one C^2-foliation of a compact manifold in a neighbourhood of an exceptional minimal set. Such an exceptional minimal set is said to be *Markov* as well.

3. Good and bad points.

Let Z_0 be the Markov invariant set of a Markov pseudogroup Γ. For any set $A \subset Z_0$ let A_∞ denote the set of all the points $x \in Z_0$ such that $\tau^n(x) \in A$ for infinitely many exponents n. A point $x \in Z_0$ is said to be *good* if there exists its neighbourhood V open in Z_0 and such that $\dim_H(V_\infty) < 1$. Let G be the set of all the good points and $B = Z_0 \setminus G$. The points of B are said to be *bad*. Obviously, G is open while B compact.

LEMMA 2. *If $A \subset G$ is compact, then* $\dim_H A_\infty < 1$.

PROOF. There are points $x_1, \cdots, x_n$ of A and their neighbourhoods $V_1, \cdots, V_n$ such that $A \subset V_1 \cup \cdots \cup V_n$ and $\dim_H(V_i)_\infty < 1$ for $i = 1, \cdots, n$. The statement of the Lemma follows immediately from Lemma 1, (ii) and (iii), and the relation

$$A_\infty \subset (V_1 \cup \cdots \cup V_n)_\infty = (V_1)_\infty \cup \cdots \cup (V_n)_\infty . \qquad \square \tag{12}$$

LEMMA 3. *$\dim_H Z_0 < 1$ if and only if $B = \emptyset$.*

PROOF. The implication "$\Rightarrow$" is obvious. Since Z_0 is compact, the converse follows immediately from Lemma 2. $\square$

4. Sacksteder estimates.

Let Γ_0 be a finite set of local C^2-diffeomorphisms of the real line $\boldsymbol{R}$ generating a pseudogroup Γ. If $g_k = h_k \circ h_{k-1} \circ \cdots \circ h_1 \in \Gamma$ for $k = 1, \cdots, l$, $h_1, \cdots, h_l \in \Gamma_0$, x and $y \in \boldsymbol{R}$, and the interval $[x, y]$ is contained in the domain of g_l, then

$$|g_l'(x)| \leqq |g_l'(y)| \cdot \exp\Big(\theta \cdot \sum_{j=0}^{l-1} |g_j(x) - g_j(y)|\Big), \tag{13}$$

where

$$\theta = \max\{\|(h^{\pm 1})''\| ; h \in \Gamma_0\} \cdot \max\{\|(h^{\pm 1})'\| ; h \in \Gamma_0\} \tag{14}$$

and $\|\cdot\|$ is the supremum norm ([**Sa**], p. 81). Hereafter, g_0 is the identity map.

If C is a closed Γ-invariant set, J is a gap of C (i.e., J is a connected component of $\boldsymbol{R} \setminus C$) and $x_0 \in \partial J$, then there exist constants $\sigma = \sigma(J)$ and $\delta = \delta(J)$ such that

$$\sum_{j=0}^{l} |g_j'(y_j)| \leqq \sigma \tag{15}$$

whenever $|y_j - x_0| < \delta$, $g_l \in \Gamma_+$ is defined on the interval $(x_0 - \delta, x_0 + \delta) \cup J$ and the intervals $J, g_1(J), \cdots, g_l(J)$ are mutually disjoint ([**Sa**], p. 82; compare [**Mat**], p. 85, for the particular case).

We shall denote by Γ_+^J the set of all $g = g_l \in \Gamma_+$ which satisfy the conditions above.

If Γ is a Markov pseudogroup, $a > 1$ is any fixed real number, $\mu = \sum |I_\alpha|$ and $C = Z_0$, then the constants

$$\delta(J) = \frac{|J| \cdot \log a}{a \theta \mu e^{\theta \mu}} \quad \text{and} \quad \sigma(J) = \frac{a \mu e^{\theta \mu}}{|J|} \tag{16}$$

do work.

The following generalizes Proposition 6 of [**Mat**].

LEMMA 4. *If $s \leqq 1$, $K \subset (x_0 - \delta, x_0 + \delta)$ is a closed nondegenerate interval and $\mathcal{L} = \{L_1, \cdots, L_p\}$, $p > 1$, is a finite family of pairwise disjoint closed intervals contained in K, then there exists $\lambda_s = \lambda_s(K, \mathcal{L}) > 0$ such that for any $g \in \Gamma_+^J$ the inequality*

(17) $$\sum_i |g(L_i^K)|^s \leqq \lambda_s \cdot |g(K)|^s,$$

holds. If $s > s_0$, *where*

(18) $$s_0 = s_0(K, \mathcal{L}) = \frac{\log p}{\log(p\kappa+p-1)-\log\kappa},$$

then we can choose λ_s *to satisfy* $\lambda_s < 1$.

Observe that $s_0 < 1$ for any $p > 1$ and $\kappa > 0$.

PROOF. For any $i=1, \cdots, p$ choose a gap J_i of $K \setminus \cup \mathcal{L}$ adjacent to L_i. Let $\kappa_i = |L_i| \cdot |J_i|^{-1} \cdot \exp(\theta\sigma|K|)$ and $\kappa = \max \kappa_i$. Inequalities (13) and (15) together with the mean value theorem imply that

(19) $$\begin{aligned}\frac{|g(L_i)|}{|g(J_i)|} &= \frac{|L_i|}{|J_i|} \cdot \frac{|g'(x_i)|}{|g'(y_i)|} \\ &\leqq \frac{|L_i|}{|J_i|} \exp\Big(\theta \sum_{j=0}^{l-1} |g_j(x_i)-g_j(y_i)|\Big) \leqq \frac{|L_i|}{|J_i|} \exp\Big(\theta \sum_j |g_j(K)|\Big) \\ &= \frac{|L_i|}{|J_i|} \exp\Big(\theta |K| \sum_j |g_j'(z_j)|\Big) \leqq \frac{|L_i|}{|J_i|} \exp(\theta\sigma|K|) = \kappa_i \leqq \kappa\end{aligned}$$

for any $g = g_l \in \Gamma_+^l$ and some points x_i, y_i and z_i of K. Since the number q of gaps of $K \setminus \mathcal{L}$ satisfies the inequality $p-1 \leqq q \leqq p+1$, it follows that

(20) $$\sum_i |g(L_i)| \leqq |g(K)| \cdot \frac{p\kappa}{p(\kappa+1)-1}.$$

This implies (17) with

(21) $$\lambda_s = p \cdot \Big(\frac{\kappa}{p(\kappa+1)-1}\Big)^s. \qquad \square$$

5. One-sided points.

Let Z_0 be a Markov invariant set, $x_0 \in Z_0$ and Z_0 accumulates on x_0 from at most one side. If x_0 is isolated in Z_0, then obviously $x_0 \in G$. Also, if $x_0 \in I_\alpha$ and $x_0 \in \mathrm{Domain}(g)$ only for finitely many $g \in \Gamma_+$, then $(I_\alpha \cap Z_0)_\infty = \emptyset$ and $x_0 \in G$. Otherwise, $x_0 \in \partial J$ for a gap J of Z_0 and we can find numbers δ and σ satisfying the conditions of Section 4. To fix ideas, assume that $x_0 = \inf J$. Take $\delta' < \delta(J)$ such that $x_0 - \delta' \notin Z$ and let $V = (x_0-\delta', x_0] \cap Z_0$.

Now, if the point x_0 is non-cyclic (i.e., $\tau^k(x_0) \neq x_0$ for $k > 0$), choose n_0 big enough and cover the set V by a finite family $\mathcal{K}$ of closed intervals K of the form either K_γ or $K_\gamma \cap (x_0-\delta', x_0]$, $|\gamma| = n_0$, contained in $(x_0-\delta', x_0]$. For any $K \in \mathcal{K}$, let $\mathcal{L}_K$ be the family of all the nondegenerate intervals of the form $K_{\tilde{\gamma}} \cap K$, where $|\tilde{\gamma}| = n_0+1$. The families $\mathcal{A}_n = \{g(K);\, g \in \Gamma_l,\ l \geqq n$ and $K \in \mathcal{K}\}$

and $\mathcal{C}_n=\{g(L); g\in\Gamma_l,\ l\geqq n,\ L\in\mathcal{L}_K$ and $K\in\mathcal{K}\}$, $n=1, 2, \cdots$, cover V_∞. Moreover, $V_\infty=\bigcap_n \bigcup\mathcal{A}_n=\bigcap_n \bigcup\mathcal{C}_n$. Inequality (17) yields

$$\mathcal{H}^s(V_\infty)=\lim_{n\to\infty}\sum_{A\in\mathcal{A}_n}|A|^s\leqq\lambda\cdot\lim_{n\to\infty}\sum_{C\in\mathcal{C}_n}|C|^s=\lambda\cdot\mathcal{H}^s(V_\infty), \tag{22}$$

where $\lambda=\max\{\lambda_s(K, \mathcal{L}_K); K\in\mathcal{K}\}$. For $s>\tilde{s}_0=\max\{s_0(K, \mathcal{L}_K); K\in\mathcal{K}\}$, $\lambda<1$ and we obtain that $\mathcal{H}^s(V_\infty)=0$. Therefore, $\dim_H(V_\infty)\leqq\tilde{s}_0<1$ and $x_0\in G$.

If x_0 is cyclic, then Lemma 5.8 of [CC] implies that

$$V_\infty=A\cup\bigcup_{i=1}^{p}(f_i(V))_\infty, \tag{23}$$

where A is a countable set, $f_i\in\Gamma_+$ and the points $f_i(x_0)$ are not cyclic. Note that the elements f_i of Γ_+ depend only on x_0, not on V. Equality (23) together with the argument for non-cyclic points and Lemma 1, (iii) and (iv), implies that $\dim_H V_\infty<1$ when δ' is small enough to provide the estimates $\dim_H(f_iV)_\infty<1$ for $i=1, \cdots, p$. Therefore, $x_0\in G$ again.

In this way, we proved the following.

LEMMA 5. *Any point $x_0\in Z_0$ such that Z_0 accumulates on x_0 from at most one side is good.* □

6. Final arguments.

To prove the Proposition it is enough to show that $B=\emptyset$ (Lemma 3). Assume not and take any $y\in B$. From Lemma 5 it follows that Z_0 accumulates on y from both sides. Therefore, $y\in I_\alpha$ for some α and there exists a gap $J=(a, b)\subset I_\alpha$ and a multiindex $\gamma=(\gamma_1, \cdots, \gamma_n)$ such that $K_\gamma=[c, d]\subset I_\alpha$ and $a<b<c<y_0<d$, where $y_0=\min(B\cap[b, \infty))$. We may assume that $|K_\gamma|$ is as small as needed, for example that $K_\gamma\subset(b^*-\delta(J), b^*+\delta(J))$.

Let $n_0=|\gamma|$ and $\Gamma_{n_0}=\{f_1, \cdots, f_{p+r}\}$, where the enumeration is such that the range of f_i is disjoint from (a, c) if and only if $i\leqq p$. Write $f_i=h_{\alpha_i}\circ h_{\gamma_i}$, where $|\gamma_i|=n_0-1$. Let $f_i^*=h_{\alpha_i}|K_{\gamma_i}$ and denote by Γ^* the pseudogroup generated by $f_1^*, \cdots, f_p^*$. Γ^* is a Markov pseudogroup again, so we may consider the corresponding Markov invariant set Z_0^*. Let b^* be the right end point of this gap J^* of Z_0^* which contains J. Clearly, $b^*\in K_\gamma$.

Lemma 5.15 of [CC] shows that for any $V\subset Z_0$ one has

$$V_\infty\subset\bigcup_{i=1}^{r}(A_i)_\infty\cup\bigcup_{k\geq 0}\tau^{-k}((V\cap Z_0^*)_{\infty*}), \tag{24}$$

where A_i is the range of f_{p+i} and the sets of the form $B_{\infty*}$ are defined analogously to B_∞ while replacing τ by τ^*, the Markov subshift determined by Γ^*. Note that, by the construction, $A_i\subset[b, c)\subset G$ for any i.

As in [**CC**] we have to consider three cases:

(i) If $b^*=d$ then $(a, d)\cap Z_0^*=\emptyset$, so for $V=(a, d)\cap Z_0$ one has $V_\infty\subset\bigcup_{i=1}^{r}(A_i)_\infty$ and therefore $\dim_H V_\infty<1$ by Lemmas 1, (ii)-(iii), and 2. Since $y_0\in V$, $y_0\in G$, a contradiction.

(ii) If $b^*<d$ but $\tau^n(b^*)$ is not defined for some $n\geqq 1$, then any neighbourhood V of y_0 contained in $(c, d)\cap Z_0$ satisfies $(V\cap Z_0^*)_{\infty*}=\emptyset$ and, as in case (i), the inequality $\dim_H V_\infty<1$ follows from (24) and Lemmas 1 and 2. Again, $y_0\in G$ contradicting the choice of y_0.

(iii) Finally, assume that $b^*<d$ and $\tau^n(b^*)$ is defined for all n. In this case, we have to apply the argument of Section 5 to the pseudogroup Γ^* and the gap J^* of Z_0^*. Since $K_\gamma\subset(b^*-\delta(J), b^*+\delta(J))$, $|J^*|\geqq|J|$ and the corresponding constants μ^* and θ^* for Γ^* satisfy the inequalities $\mu^*\leqq\mu$ and $\theta^*\leqq\theta$, it follows from (16) that $\delta^*=\delta(J^*)\geqq\delta(J)$. Therefore, the estimates analogous to (17) hold for segments $K\subset K_\gamma$ and the canonical extensions of maps of $\Gamma_\ddagger^*$. (Recall after Section 4 of [**CC**] that the canonical extension of a map $h_{j_1}^*\circ\cdots\circ h_{j_k}^*\in\Gamma^*$ is defined as $h_{\alpha_{j_1}}\circ\cdots\circ h_{\alpha_{j_k}}$.) Consequently,

$$\dim_H (V\cap Z_0^*)_{\infty*} < 1 \tag{25}$$

for $V=(c, d)\cap Z_0$. Inequality (25) together with (24) and Lemma 1, (iii) and (v), implies again that $\dim_H V_\infty<1$ and $y_0\in G$ providing us with a contradiction as before. $\square$

7. Some remarks.

The classical Denjoy example ([**De**], compare [**Ta**]) shows that the assumption of C^2-differentiability is essential. In the C^1 case, the equality $\dim_H Z_0=1$ as well as the inequality $|Z_0|>0$ may hold.

More subtle estimates of $\dim_H Z_0$ from above (as well as from below) should be possible to obtain in terms of the maps $h_1, \cdots, h_m$ (i.e., their domains, ranges and derivatives) generating Γ. The calculation could be, however, very hard. (Compare [**Bo**] to have an idea.) In general, $\dim_H Z_0$ can be arbitrarily close to 0 as well as to 1. In some "degenerate" cases (Z_0 finite, for example), the equality $\dim_H Z_0=0$ can hold. However, for a typical Markov pseudogroup Γ, Z_0 is of positive Hausdorff dimension and (as well as the corresponding exceptional minimal set in a suitable foliated manifold) becomes a fractal in the sense of Mandelbrot [**Man**] (compare [**Ed**], p. 151).

It should be possible to obtain similar results while replacing the Hausdorff dimension by other fractal dimensions like, for example, the packing dimension $\dim_P$ or the lower (resp., upper) entropy dimension $\dim_E^l$ (resp., $\dim_E^u$) (see [**Ba**], [**TT**] and [**Ed**], pp. 181-185). This should be still of some interest because of the inequalities

$$\dim_H \leqq \dim_E^l \leqq \dim_E^u \leqq \dim_P . \tag{26}$$

Another related question is whether the equality

$$\dim_H Z_0 = \dim_P Z_0 \tag{27}$$

holds for Markov invariant sets. In other words, is—typically—Z_0 a fractal in the sense of Taylor?

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Paweł WALCZAK
Instytut Matematyki
Uniwersytet Łódzki
Banacha 22, 90238 Łódź
Poland
(E-mail : pawelwal@krysia.uni.lod2.pl)