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Sharp characters with irrational values

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§ 1. Introduction.

Let G be a finite group, and let χ be a character of G of degree n. Let L be the set of values of χ on the nonidentity elements of G. We call L the *type* of the pair (G, χ). Put $f_L(x)=\prod_{\alpha\in L}(x-\alpha)$. It is known that $f_L(n)$ is a rational integer and $|G|$ is a divisor of $f_L(n)$ (see [**3**], [**4**]). The pair (G, χ) is said to be *sharp* if $f_L(n)=|G|$.

The notion of sharpness was first introduced for permutation characters by Ito and Kiyota in [**5**] as a generalization of sharply t-transitive permutation representations. Indeed, if G is a sharply t-transitive permutation group and χ is the associated permutation character, then (G, χ) is sharp of type $\{0, 1, \cdots, t-1\}$. In [**4**], Cameron and Kiyota extended the definition of sharp pair to that given above and posed the problem of classifying, for a given set L of algebraic integers, the sharp pairs (G, χ) of type L.

The pair (G, χ) is said to be *normalized* if χ does not contain the principal character of G as a constituent. If χ does contain the principal character 1_G as a constituent with multiplicity m, and if we put $\chi'=\chi-m\cdot 1_G$, $L'=\{\alpha-m \mid \alpha\in L\}$, then (G, χ) is sharp of type L if and only if (G, χ') is sharp of type L'. Thus it is no loss of generality to consider only normalized pairs (G, χ) when classifying sharp pairs.

In this paper we give a complete classification of sharp pairs of type L when L contains an irrational number. Several special cases have appeared in the literature. For example, we have the following results. (See also [**1**], [**9**], [**6**], [**7**], [**8**].)

THEOREM 1.1 (Cameron-Kiyota, [**4**]). *Suppose (G, χ) is sharp and normalized of type L, where L consists of a single algebraic conjugacy class of irrational values. Then G is cyclic of odd prime order, and χ is either linear or the sum of two complex conjugate linear characters of G.*

THEOREM 1.2 (Alvis-Kiyota-Lenstra-Nozawa, [**2**]). *Suppose (G, χ) is sharp and normalized of type L, where $L\cap \boldsymbol{Z}=\emptyset$. Then G is a cyclic group of odd order, and χ is either a linear character or the sum of two complex conjugate* *linear characters of* G.

Our main result is the following.

THEOREM 1.3. *Suppose* (G, χ) *is sharp and normalized and* χ *assumes an irrational value. Then one of the following holds.*

(i) G *is cyclic of order* m, *and either* $m\geqq 3$ *and* χ *is linear, or else* $m\geqq 5$ *and* χ *is the sum of two complex conjugate linear characters of* G.

(ii) G *is dihedral of order* $2m$, *where* $m\geqq 5$ *is odd, and* χ *is irreducible of degree* 2.

(iii) G *is dihedral or generalized quaternion of order* $2m$, *where* $m\geqq 8$ *is even, and* $\chi=\psi$ *or* $\chi=\psi+\varepsilon$, *where* ψ *is irreducible of degree* 2 *and* ε *is linear with cyclic kernel of order* m.

(iv) G *is isomorphic to the binary octahedral group and* χ *is irreducible of degree* 2.

(v) G *is isomorphic to* $SL(2, 5)$ *and* χ *is irreducible of degree* 2.

(vi) G *is isomorphic to* A_5 *and* χ *is irreducible of degree* 3.

§2. The proof of Theorem 1.3.

We require the following two results, the second of which is proved in the next section.

THEOREM 2.1 (Blichfeldt [3], Cameron-Kiyota [4]). *Let* G *be a finite group, and let* χ *be a character of* G. *Let* L *be the set of values of* χ *on* $G\setminus\{1\}$. *Then* $f_L(n)$ *is a rational integer, and* $|G|$ *divides* $f_L(n)$.

THEOREM 2.2. *Suppose* (G, χ) *is sharp and normalized,* $g\in G$, *and* $\chi(g)$ *is irrational. Let* ρ *be a representation of* G *affording* χ, *and let* $L(g)$ *be the set of values of* χ *on* $\langle g\rangle\setminus\{1\}$. *Then the following hold.*

(i) $C_G(g)$ *is cyclic, and* χ *assumes irrational values on the generators of* $C_G(g)$.

(ii) *Any nonidentity eigenvalue of* $\rho(g)$ *occurs with multiplicity* 1. *Moreover, if* ω *and* ω' *are distinct nonidentity eigenvalues of* $\rho(g)$, *then* $\omega'=\bar{\omega}$.

(iii) *If* $\chi\neq\bar{\chi}$ *or* $o(g)$ *is odd, then* $f_{L(g)}(n)=o(g)$. *If* $\chi=\bar{\chi}$ *and* $o(g)$ *is even, then* $f_{L(g)}(n)=2o(g)$.

PROOF OF THEOREM 1.3. Throughout this proof we suppose (G, χ) is sharp and normalized of type L, where L contains an irrational number. Also, we denote by ρ a representation of G affording χ. By Theorem 2.2, there is some $g\in G$ such that $\chi(g)$ is irrational and $C_G(g)=\langle g\rangle$. *We fix such an element* g *for the remainder of this proof.* Put $H=C_G(g)$, $N=N_G(H)$.

STEP 1. *If* $\chi\neq\bar{\chi}$, *then* G *is cyclic and* χ *is linear.*

PROOF. Suppose $\chi \neq \bar{\chi}$. Then $\rho(g)$ has a single nonidentity eigenvalue with multiplicity 1 by Theorem 2.2. Thus $N=C_G(g)=H$. Also, $C_G(h)=H$ for $h\in H\setminus\{1\}$, so and ${}^xH\cap H=\{1\}$ for $x\in G\setminus H$. Thus G is a Frobenius group with Frobenius complement H. Let K be the Frobenius kernel, so $G\setminus K$ is the disjoint union of the sets ${}^xH\setminus\{1\}$ as x ranges over K. Since (G, χ) is normalized, we have

$$0=|G:K|\cdot(\chi, 1)_G=(\chi, 1)_K+|H|\cdot(\chi, 1)_H-n,$$

so $(\chi, 1)_K=1-(|H|-1)(n-1)\geqq 0$, and hence $n=1$ since $|H|\geqq 3$. Therefore χ is linear, and so G is abelian since χ is faithful. Thus $G=C_G(g)=\langle g\rangle$ as required. □

We suppose for the rest of this proof that $\chi=\bar{\chi}$. Let ω be a nonidentity eigenvalue of $\rho(g)$, and let λ be the linear character of H such that $\lambda(g)=\omega$. Let ψ be the unique irreducible character of G such that ψ is a constituent of both χ and the induced character λ^G. Put $m=|H|$ and $d=\psi(1)$. Since $\rho(g)$ has only two nonidentity eigenvalues by Theorem 2.2, $|N:H|\leqq 2$. Also, $C_G(h)=H$ for any $h\in H$ such that $h^2\neq 1$.

STEP 2. *Suppose $\psi\neq\bar{\psi}$. Then G is cyclic and χ is the sum of two complex conjugate linear characters.*

PROOF. In this case g is not conjugate to g^{-1}, so $N=H$. Let $K=\ker(\chi-\psi-\bar{\psi})$. Then K contains H, and hence $|K|>|G:H|\cdot(|H|-2)>|G|/2$ since $|H|\geqq 5$. Therefore $K=G$ and $\chi=\psi+\bar{\psi}$ since (G, χ) is normalized. It follows that ψ is faithful.

Note $\psi_H=(d-1)\cdot 1_H+\lambda$. Since $(\psi, \psi)_G=1$, we have

$$|G:H|\sum_{x\in H,\, x^2\neq 1}|d-1+\lambda(x)|^2<|G|.$$

If m is even, then $m((d-1)^2+1)-d^2-(d-2)^2<m$, so $d=1$ since $m\geqq 8$. If m is odd, then $m((d-1)^2+1)-d^2<m$, so $d=1$ since $m\geqq 5$. In either case ψ is linear, so G is abelian since ψ is faithful. Therefore $G=C_G(g)=\langle g\rangle$, as required. □

For the remainder of this proof we suppose $\psi=\bar{\psi}$, so $d\geqq 2$ and G is non-abelian. Put $K=\ker(\chi-\psi)$, so $H\subseteqq K$. If m is even, then $|K|>|G:N|\cdot(|H|-2)>|G|/3$ since $m\geqq 8$. If m is odd, then $|K|>|G:N|\cdot(|H|-1)>|G|/3$ since $m\geqq 5$. In either case we have $|G:K|\leqq 2$. Observe that $\chi=(n-d)\varepsilon+\psi$, where ε is the linear character of G with kernel K. (If $K=G$, then $n=d$ and $\chi=\psi$.)

If m is divisible by an odd prime p and P is the Sylow p-subgroup of H, then $C_G(P)=H$, so $N_G(P)=N$, and thus $P\in \mathrm{Syl}_p(G)$ since $|N:H|\leqq 2$.

Suppose m is a power of 2. If $x\in N\setminus H$, then $\rho(x)$ exchanges the nonidentity eigenspaces of $\rho(g)$, so $\rho(x)$ has a pair of eigenvalues ζ, $-\zeta$, and thus

$\chi(x)$ is rational by Theorem 2.2. In particular, if X is the conjugacy class of g in G, then $N\cap X\subseteqq\{g, g^{-1}\}$. Therefore $N\in\mathrm{Syl}_2(G)$ since $N_G(N)=N$.

STEP 3. *Suppose H is normal in K. If m is odd, then G is dihedral of order $2m$ and $\chi=\psi$ is irreducible of degree 2. If m is even, then G is either dihedral or generalized quaternion of order $2m$, and $\chi=\psi$ or $\chi=\psi+\varepsilon$ where ψ is irreducible of degree 2.*

PROOF. Suppose that m is divisible by an odd prime p. Let P be the Sylow p-subgroup of H. Then P is the unique Sylow p-subgroup of G since H is normal in K, and thus $H=C_G(P)$ is normal in G. Now suppose m is a power of 2, so $N\in\mathrm{Syl}_2(G)$. Then H is normal in G since $G=N\cdot K$.

Therefore H is normal in G in either case, so $G=N$ and $|G:H|=2$. Thus G is dihedral if m is odd, G is dihedral or generalized quaternion if m is even, and $d=2$ in either case. Note that if $x\in G\setminus H$, then $\chi(x)=-(n-2)$. Let $L(g)$ be as in Theorem 2.2. If m is odd, then $f_{L(g)}(n)=m=|G|/2$, and hence $-(n-2)=n-2$ since (G, χ) is sharp, so $n=2$ and $\chi=\psi$. If m is even, then $f_{L(g)}(n)=2m=|G|$, so $-(n-2)\in\chi(H)\cap\boldsymbol{Z}\subseteqq\{n-4, n-3, n-2, n-1, n\}$ since (G, χ) is sharp, so $n=2$ or $n=3$, and therefore $\chi=\psi$ or $\chi=\psi+\varepsilon$. □

For the remainder of this proof we suppose H is not normal in K. Let $M=\psi(K\setminus\{1\})$, $M_0=M\cap\boldsymbol{Z}$, $M^*=M\setminus\boldsymbol{Z}$. Note ψ is faithful on K since χ is faithful, so $d\notin M_0$. Since $\rho(x)$ cannot have -1 as an eigenvalue if $\chi(x)$ is irrational by Theorem 2.2, χ assumes only rational values on $G\setminus K$.

STEP 4. *$M^*\subseteqq\psi(H)$.*

PROOF. Suppose not, so there is some $x\in K$ such that $\psi(x)$ is irrational and $\psi(x)\notin\psi(H)$. By Theorem 2.2 we have $C_G(x)=\langle g'\rangle$, where $\chi(g')$ is irrational and $C_G(g')=\langle g'\rangle$. Put $H'=\langle g'\rangle$, $N'=N_G(H')$, $m'=|H'|$. Let ω' be a nonidentity eigenvalue of $\rho(g')$, and let λ' be the linear character of H' defined by $\lambda'(g')=\omega'$. Note $\psi_{H'}=(d-2)\cdot 1_{H'}+\lambda'+\bar{\lambda}'$. Also, if $z\in G$, $z^2\neq 1$, and z is contained in a conjugate of H and a conjugate of H', then both H and H' are conjugate to $C_G(z)$, which is impossible since $\psi(x)\notin\psi(H)$. Therefore we have

$$\frac{1}{|N|}\sum_{x\in H,\, x^2\neq 1}|d-2+\lambda(x)+\overline{\lambda(x)}|^2+\frac{1}{|N'|}\sum_{y\in H',\, y^2\neq 1}|d-2+\lambda'(y)+\overline{\lambda'(y)}|^2<1.$$

If m and m' are both odd, then

$$\frac{1}{2m}[m((d-2)^2+2)-d^2]+\frac{1}{2m'}[m'((d-2)^2+2)-d^2]<1,$$

which is impossible since $d\geqq 2$ and $m, m'\geqq 5$. If m is even and m' is odd, then

$$\frac{1}{2m}[m((d-2)^2+2)-d^2-(d-4)^2]+\frac{1}{2m'}[m'((d-2)^2+2)-d^2]<1,$$

which is a contradiction since $d\geqq 2$, $m\geqq 8$ and $m'\geqq 5$. A similar contradiction occurs of m is odd and m' is even. Finally, if m and m' are both even, then

$$\frac{1}{2m}[m((d-2)^2+2)-d^2-(d-4)^2]+\frac{1}{2m'}[m'((d-2)^2+2)-d^2-(d-4)^2]<1,$$

which is impossible since $d\geqq 2$ and m, $m'\geqq 8$. Therefore $M^*\subseteqq\psi(H)$. □

It follows from Step 4 that all irrational values of χ occur on conjugates of H.

STEP 5. *Suppose $d=2$. Then G is isomorphic to the binary octahedral group or $SL(2, 5)$, and $\chi=\psi$ is irreducible of degree 2.*

PROOF. We have $M_0\subseteqq\{-2, -1, 0, 1\}$ since ψ is faithful on K. Moreover, $-2\in M_0$ if and only if $-2\in\psi(H)$ since $Z(K)\subseteqq H$. Thus $M_0\setminus\psi(H)\subseteqq\{-1, 0, 1\}$.

Case 1. Suppose $gcd(6, m)=1$. Then $|K|$ divides $6f_{\psi(H)\setminus\{d\}}(n)=6m$ by Theorem 2.1 and part (iii) of Theorem 2.2. Let p be a prime divisor of m, and let P be the Sylow p-subgroup of H. Since $C_G(P)=H$ and H is not normal in K, we must have $p=5$, $H=P$, and $|K|=6m$. The action of K on $\mathrm{Syl}_5(G)$ induces a homomorphism $K\to S_6$ whose kernel is a 2-subgroup of $N_K(H)$, so $m=5$ and $|K|=30$. This is impossible since $C_G(H)=H$.

Case 2. Suppose $gcd(6, m)=2$. Then $M_0\setminus\psi(H)\subseteqq\{-1, 0, 1\}$, and $M_0\setminus\psi(H)\subseteqq\{-1, 1\}$ if $m/2$ is even. Therefore $|K|$ divides $12m$ if $m/2$ is odd, and $|K|$ divides $6m$ if $m/2$ is even. Also,

$$\frac{1}{m}\sum_{x\in H,\, x^2\neq 1}|\lambda(x)+\overline{\lambda(x)}|^2=\frac{2(m-4)}{m}<|N:H|,$$

and therefore $|N:H|=2$ since $m\geqq 8$.

Suppose m has an odd prime divisor p. Let P be the Sylow p-subgroup of H, so $P\in\mathrm{Syl}_p(G)$. Then $p=5$ or $p=11$ since $|G:N|$ divides 12 and $N\neq G$.

Suppose $p=5$. Then $|G:N|=6$, and $|G|=12m$. The action of G on $\mathrm{Syl}_5(G)$ induces a homomorphism $\sigma: G\to S_6$ whose kernel is a 2-subgroup of N. Thus $|P|=5$. Let Q be the Sylow 2-subgroup of H. Then $Q\subseteqq\ker(\sigma)$ since S_6 has no elements of order 10. If $m/2$ is even, then $C_G(Q)=H$, so Q is not characteristic in $\ker(\sigma)$ since H is not normal in G, and thus $|H|=20$ and $\ker(\sigma)$ is isomorphic to the quaternion group. Then the image of σ has order 30, which is impossible. Thus $m/2$ is odd, so $|H|=10$ and $|G|=120$. Note $\ker(\sigma)=\langle z\rangle$, where z is a central involution in G, and $\sigma(G)\approx A_5$. Also, $G\not\approx A_5\times \boldsymbol{Z}_2$ since $d=2$. It follows that G is isomorphic to the double cover of A_5, that is, $SL(2, 5)$, and $\chi=\psi$ is irreducible of degree 2.

Now suppose $p=11$. Then $|G:N|=12$, so $|G|=24m$, $|K|=12m$, and hence $K\cap N=H$. Note $m/2$ is a power of 11. The action of K on $\mathrm{Syl}_{11}(G)$ induces a homomorphism $\sigma: K\to S_{12}$ whose kernel is a 2-subgroup of H. Therefore $|P|=11$, $|H|=22$ and $|K|=264$. Moreover, $\ker(\sigma)=\langle z\rangle$ for some involution $z\in Z(K)$ since S_{12} contains no elements of order 22. Then $\bar{H}=H/\langle z\rangle$ is a TI-set in $\bar{K}=K/\langle z\rangle$, so $\bar{K}$ has a normal subgroup of order 12, and hence K has a normal subgroup K_0 of order 24. However, P must act on $K_0\setminus\langle z\rangle$ without fixed points, so we have a contradiction.

Finally, assume m is a power of 2, say $m=2^t$. Then $N\in\mathrm{Syl}_2(G)$, and thus $|G|=6m=3\cdot 2^{t+1}$. The action of G on $\mathrm{Syl}_2(G)$ induces a homomorphism $\sigma: G\to S_3$ whose kernel contains $H_0=\langle g^2\rangle$. If $t>3$, then H_0 is characteristic in $\ker(\sigma)$, so H_0 is normal in G, and hence $H=C_G(H_0)$ is normal in G, a contradiction. Therefore $t=3$ and $|G|=48$. Let $x\in\ker(\sigma)\setminus H$. Since $x\in N$ and $x^2\in H$, we have $o(x)=2$ or $o(x)=4$. If $o(x)=2$, then $\ker(\sigma)$ is dihedral of order 8, so H_0 is characteristic in $\ker(\sigma)$ and we have a contradiction. Therefore $o(x)=4$. It follows that G is isomorphic to the binary octahedral group and $\chi=\psi$ is irreducible of degree 2.

Case 3. Suppose $gcd(6, m)=3$. Then $M_0\setminus\psi(H)\subseteqq\{0, 1\}$, so $|K|$ divides $2m$ and H is normal in K, a contradiction.

Case 4. Suppose $gcd(6, m)=6$. If $m/2$ is even, then $M_0\subseteqq\psi(H)$, and therefore $|K|$ divides $2m$ and H is normal in K, a contradiction. Therefore $m/2$ is odd, so $M_0\setminus\psi(H)\subseteqq\{0\}$, and thus $|K|$ divides $4m$. Let P be the Sylow 3-subgroup of H. Then $P\in\mathrm{Syl}_3(G)$ and $N_G(P)=N$, so $|G:N|=4$, $|G|=8m$ and $m/2$ is a power of 3. The action of G on $\mathrm{Syl}_3(G)$ induces a homomorphism $G\to S_4$ whose kernel is a 2-subgroup of N. Therefore $|P|=3$ and $|H|=6$, which is impossible since χ assumes irrational values on H. □

STEP 6. *Suppose $d\geqq 3$. Then G is isomorphic to A_5, and χ is irreducible of degree* 3.

PROOF. Since $(\psi, \psi)_G=1$, we have

$$|G:N|\sum_{x\in H,\, x^2\neq 1}|d-2+\lambda(x)+\overline{\lambda(x)}|^2<|G|.$$

Suppose m is even. Then $m((d-2)^2+2)-d^2-(d-4)^2<|N|$, and it follows that $d=3$, $m=8$, and $|N:H|=2$. Note $-3\notin M_0$ since $Z(K)\subseteqq H$ and ψ does not assume the value -3 on H. Also, $-2\notin M_0$ since $d=3<\varphi(5)$ and $\psi(K)\subseteqq \boldsymbol{Q}(\omega)$. Therefore $M_0\subseteqq\{-1, 0, 1, 2\}$, so $|K|$ divides $24\cdot 16$ by Theorem 2.1 and part (iii) of Theorem 2.2. Since $N\in\mathrm{Syl}_2(G)$, we must have $|G|=48$ and $|\mathrm{Syl}_2(G)|=3$. Arguing as in Case 2, we see G is isomorphic to the binary octahedral group. Then G has no irreducible characters of degree 3 with irrational values, so a contradiction is reached.

Therefore m is odd, and hence $m((d-2)^2+2)-d^2<|N|$, so $d=3$, $m\in\{5, 7\}$ and $|N:H|=2$. We have $-3\notin M_0$ since $Z(K)\subseteqq H$ and $-3\notin\psi(H)$. Suppose $m=7$. Then $-2\notin M_0$ since $d=3<\varphi(5)$ and $\psi(K)\subseteqq \boldsymbol{Q}(\omega)$. Therefore $M_0\subseteqq\{-1, 0, 1, 2\}$, so $|K|$ divides $24\cdot 7$. Since $H\in \mathrm{Syl}_7(G)$, it follows that $|\mathrm{Syl}_7(G)|=8$, which is impossible since $|N:H|=2$. Therefore $p=5$. Put $L_0=L\cap\boldsymbol{Z}$, $L^*=L\setminus\boldsymbol{Z}$. Since $H\in\mathrm{Syl}_5(G)$, we have $|G|_5=5$, and hence $gcd(5, f_{L_0}(n))=1$ since $f_{L^*}(n)=5$. Thus $-2\notin M_0$ since otherwise $n-5\in L_0$. Therefore $M_0\subseteqq\{-1, 0, 1, 2\}$, and hence $|K|$ divides $24\cdot 5=120$. Since $|N:H|=2$, we must have $|G|=60$, so G has 6 Sylow 5-subgroups. Therefore G is isomorphic to A_5, $K=G$, and $\chi=\psi$ is irreducible of degree 3. □

We have shown that Theorem 1.3 will follow once Theorem 2.2 is established.

§ 3. The proof of Theorem 2.2.

In this section we shall prove Theorem 2.2, thus completing the proof of Theorem 1.3. We require several lemmas. We suppose that (G, χ) is sharp and normalized of type L. Let $n=\chi(1)$, and let ρ be a representation of G affording χ.

Let ζ be a primitive complex k-th root of unity. We extend the definitions of the Euler function φ and Möbius function μ to complex roots of unity by setting $\varphi(\zeta)=\varphi(k)$ and $\mu(\zeta)=\mu(k)$.

LEMMA 3.1 (Alvis [**1**]). *Suppose $g\in G$ and $\chi(g)$ is irrational. Let L' be the Galois orbit containing $\chi(g)$. Let $\nu(\zeta)$ denote the multiplicity of ζ as an eigenvalue of $\rho(g)$, and put $c=\nu(1)$, $t=|L'|$. Then*

$$f_{L'}(n)\geqq\left(\frac{n-c}{2}\right)^t\prod_{\zeta\neq 1}N(1-\zeta)^{2t\nu(\zeta)/(n-c)\varphi(\zeta)},$$

where $N(\alpha)$ denotes the product of the distinct algebraic conjugates of α. In particular,

$$f_{L'}(n)\geqq\left(\frac{n-c}{2}\right)^t p^{2t/\varphi(p^l)}$$

if g has prime power order p^l. Moreover, if equality holds in either case, then $L'\subset\boldsymbol{R}$ and $t<\varphi(o(g))$.

LEMMA 3.2. *Suppose $g\in G$, $\chi(g)$ is irrational, and $\chi(g)\neq\chi(h)$ for all elements h of G of prime power order. Let L' be the Galois orbit of $\chi(g)$, and let c be the multiplicity of 1 as an eigenvalue of $\rho(g)$. Then $f_{L'}(n)=1$ and $n-c\leqq 2$. Moreover, if $n-c=2$ then $\chi(g)$ is real.*

PROOF. Let $P \in \mathrm{Syl}_p(G)$, where p is a prime divisor of $|G|$. Since (P, χ_P) has type contained in $L \setminus L'$ by assumption, $|P|$ divides $f_L(n)/f_{L'}(n)$ by Theorem 2.1. In particular, $|G|$ divides $f_L(n)/f_{L'}(n)$, and therefore $f_{L'}(n)=1$ since (G, χ) is sharp of type L. Also, it follows from Lemma 3.1 that

$$\left(\frac{n-c}{2}\right)^t \leqq f_{L'}(n) = 1$$

where $t=|L'|$, and thus we have $n-c=1$ or $n-c=2$. If $n-c=2$, then equality holds in the first inequality of Lemma 3.1, and therefore $\chi(g)$ is real. □

LEMMA 3.3. *Suppose $g \in G$, $\chi(g)$ is irrational, and g is a p-element of G for some prime p. Let L' be the Galois orbit of $\chi(g)$. Then $f_{L'}(n)$ is a power of p, and $f_{L'}(n)>1$.*

PROOF. Let ω be a primitive $o(g)$-th root of unity. Since $L' \subseteqq \boldsymbol{Q}(\omega) \setminus \boldsymbol{Q}$, p is a unique prime such that $L' \subseteqq \boldsymbol{Q}(\omega)$. Thus if $Q \in \mathrm{Syl}_q(G)$ for some prime q, $q \neq p$, then (Q, χ_Q) has type contained in $L \setminus L'$, so $|Q|$ divides $f_L(n)/f_{L'}(n)$. In particular, the p'-part of $|G|$ divides $f_L(n)/f_{L'}(n)$, and thus $f_{L'}(n)$ is a power of p since (G, χ) is sharp of type L. Also since g is a p-element of G, we have $n-\chi(g) \in (1-\omega)\boldsymbol{Z}[\omega]$, and it follows that p divides $f_{L'}(n)$. □

LEMMA 3.4 (Alvis-Kiyota-Lenstra-Nozawa [2]). *Assume $\chi(g)$ is irrational, where $g \in G$ has order p^l for some prime p. Let L' be the Galois orbit of $\chi(g)$, and assume $f_{L'}(n)$ is a power of p. Then there is a primitive complex p^l-th root ω of unity and a nonnegative integer s such that the eigenvalues of $\rho(g)$ are* 1 *and ω (with multiplicities $n-p^s$ and p^s, respectively) or* 1, *ω and $\bar{\omega}$ (with multiplicities $n-2p^s$, p^s and p^s, respectively).*

We say an element g of G is χ-*singular* if $\chi(g)$ is irrational and the product of the algebraic conjugates of $n-\chi(g)$ is equal to 1. If g is χ-singular, then $\chi(g) \neq \chi(h)$ for any p-element h of G whenever p is prime by Lemma 3.3.

LEMMA 3.5. *Suppose $g \in G$. Then one of the following holds.*

(i) *$\chi(g)$ is rational.*

(ii) *g is χ-singular. Moreover, if ξ is a nonidentity eigenvalue of $\rho(g)$, then the eigenvalues of $\rho(g)$ are either* 1 *and ξ, with multiplicities $n-1$ and* 1, *respectively, or else* 1, *ξ and $\bar{\xi}$, with multiplicities $n-2$,* 1 *and* 1, *respectively.*

(iii) *$\chi(g)$ is irrational and $\chi(g)=\chi(y)$ for some p-element y of G, where p is prime. Moreover, if $o(g) \neq o(y)$, then $o(y)=3$ if $\chi(g)$ is not real, and $o(y)=5$ if $\chi(g)$ is real.*

PROOF. Assume $\chi(g)$ is irrational and $\chi(g) \neq \chi(y)$ for all elements y of G of prime power order. Let c be the multiplicity of 1 as an eigenvalue of $\rho(g)$.

Then g is χ-singular and $n-c\leqq 2$, and $\chi(g)$ is real if $n-c=2$, by Lemma 3.2. Therefore (ii) holds.

Suppose now that $\chi(g)$ is irrational and $\chi(g)=\chi(y)$ for some p-element y of G. Let $p^l=o(y)$, and let ω be a nonidentity eigenvalue of $\rho(y)$. By Lemma 3.4 we have $\chi(y)=n-p^s+p^s\omega$ or $\chi(y)=n-2p^s+p^s\omega+p^s\bar{\omega}$ for some nonnegative integer s.

Let ζ be a primitive $|G|$-th root of unity, and let ζ_p, $\zeta_{p'}$ be the p-part and p'-part of ζ, respectively. For $\xi\in\langle\zeta\rangle$, denote by $\nu(\xi)$ the multiplicity of ξ as an eigenvalue of $\rho(g)$. Let $\mathrm{Tr}: \boldsymbol{Q}(\zeta)\to\boldsymbol{Q}(\zeta_p)$ be the trace mapping. Recall that we have extended the Euler function φ and the Möbius function μ to complex roots of unity. Define

$$X^+ = \{\xi\in\langle\zeta_{p'}\rangle \mid \mu(\xi)=1\}, \quad X^- = \{\xi\in\langle\zeta_{p'}\rangle \mid \mu(\xi)=-1\}.$$

Put

$$\nu^+(\alpha)=\sum_{\xi\in X^+}\nu(\xi\alpha)/\varphi(\xi), \quad \nu^-(\alpha)=\sum_{\xi\in X^-}\nu(\xi\alpha)/\varphi(\xi),$$

and define $\nu_0(\alpha)=\nu^+(\alpha)-\nu^-(\alpha)$. Then we have

$$\begin{aligned}\mathrm{Tr}(\chi(g)) &= \sum_{\alpha\in\langle\zeta_p\rangle}\Big(\sum_{\xi\in\langle\zeta_{p'}\rangle}\nu(\xi\alpha)\mathrm{Tr}(\xi)\Big)\alpha \\ &= \sum_{\alpha\in\langle\zeta_p\rangle}\Big(\sum_{\xi\in\langle\zeta_{p'}\rangle}\nu(\xi\alpha)\mu(\xi)\varphi(\zeta_{p'})/\varphi(\xi)\Big)\alpha \\ &= \varphi(\zeta_{p'})\sum_{\alpha\in\langle\zeta_p\rangle}\nu_0(\alpha)\alpha. \qquad (3.1)\end{aligned}$$

Let δ_p be a primitive complex p-th root of unity.

Case 1. Suppose $\chi(g)$ is not real. We have

$$\mathrm{Tr}(\chi(g))=\varphi(\zeta_{p'})(n-p^s)+\varphi(\zeta_{p'})p^s\omega. \qquad (3.2)$$

We claim that either $o(g)=o(y)=p^l$ or else $p^l=3$. First suppose $l=1$. Comparing (3.1) and (3.2), we see

$$n-p^s=\nu_0(1)-\nu_0(\delta), \quad p^s=\nu_0(\omega)-\nu_0(\delta)$$

for any $\delta\in\langle\delta_p\rangle\setminus\{1,\omega\}$. Therefore

$$n\leqq\nu^+(1)+\nu^+(\omega)+2\nu^-(\delta)\leqq\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega))+2\sum_{\xi\in X^-}\nu(\xi\delta).$$

Since $n=\sum_{\xi\in\langle\zeta\rangle}\nu(\xi)$, we have

$$(p-2)\sum_{\delta}\sum_{\xi\in X^-}\nu(\xi\delta)\leqq(p-2)\Big(n-\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega))\Big)\leqq 2\sum_{\delta}\sum_{\xi\in X^-}\nu(\xi\delta),$$

where δ in the sums ranges over $\langle\delta_p\rangle\setminus\{1,\omega\}$. Suppose $p>3$. Then $\nu(\xi\delta)=0$ for $\xi\in X^-$ and $\delta\in\langle\delta_p\rangle\setminus\{1,\omega\}$. Therefore

$$\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)) \leqq n \leqq \sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega))/\varphi(\xi),$$

and hence $n=\nu(1)+\nu(\omega)$ and $\nu(\xi)=0$ for $\xi\notin\{1, \omega\}$. Thus $o(g)=o(y)=p$, and the claim holds.

Now suppose $l>1$. Then

$$n-p^s = \nu_0(1)-\nu_0(\delta), \quad p^s = \nu_0(\omega)-\nu_0(\delta\omega)$$

for any $\delta\in\langle\delta_p\rangle\setminus\{1\}$. Therefore

$$n \leqq \nu^+(1)+\nu^+(\omega)+\nu^-(\delta)+\nu^-(\delta\omega) \leqq \sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega))+\sum_{\xi\in X^-}(\nu(\xi\delta)+\nu(\xi\delta\omega)),$$

and summing over $\delta\in\langle\delta_p\rangle\setminus\{1\}$ gives

$$\begin{aligned}(p-1)\sum_{\delta}\sum_{\xi\in X^-}(\nu(\xi\delta)+\nu(\xi\delta\omega)) &\leqq (p-1)\Big(n-\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega))\Big)\\ &\leqq \sum_{\delta}\sum_{\xi\in X^-}(\nu(\xi\delta)+\nu(\xi\delta\omega)).\end{aligned}$$

If $p>2$, then $\nu(\xi\delta)=\nu(\xi\delta\omega)=0$ for $\xi\in X^-$ and $\delta\in\langle\delta_p\rangle\setminus\{1\}$, so

$$n \leqq \nu^+(1)+\nu^+(\omega) \leqq \sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega))/\varphi(\xi). \tag{3.3}$$

On the other hand, if $p=2$ then

$$\begin{aligned} n &= \nu_0(1)-\nu_0(-1)+\nu_0(\omega)-\nu_0(-\omega)\\ &\leqq \nu^+(1)+\nu^+(\omega)+\nu^-(-1)+\nu^-(-\omega)\\ &\leqq \sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega))+\sum_{\xi\in X^-}(\nu(-\xi)+\nu(-\xi\omega))/2\end{aligned}$$

since $\varphi(\xi)\geqq 2$ for $\xi\in X^-$. Therefore $\nu(-\xi)=\nu(-\xi\omega)=0$ for $\xi\in X^-$ since $n=\sum_{\xi\in\langle\zeta\rangle}\nu(\xi)$, and hence (3.3) also holds if $p=2$. Thus $n=\nu(1)+\nu(\omega)$ and $\nu(\xi)=0$ for $\xi\notin\{1, \omega\}$, so $o(g)=o(y)=p^l$ as claimed.

Case 2. Suppose $\chi(g)$ is real. Note

$$\mathrm{Tr}(\chi(g)) = \varphi(\zeta_{p'})(n-2p^s)+\varphi(\zeta_{p'})p^s\omega+\varphi(\zeta_{p'})p^s\bar{\omega}. \tag{3.4}$$

We claim $o(g)=o(y)=p^l$ or else $p^l=5$. To see this, first suppose $l=1$. By (3.1) and (3.4) we have

$$n-2p^s = \nu_0(1)-\nu_0(\delta), \quad p^s = \nu_0(\omega)-\nu_0(\delta) = \nu_0(\bar{\omega})-\nu_0(\delta)$$

for any $\delta\in\langle\delta_p\rangle\setminus\{1, \omega, \bar{\omega}\}$. Therefore

$$\begin{aligned} n &= \nu_0(1)+\nu_0(\omega)+\nu_0(\bar{\omega})-3\nu_0(\delta)\\ &\leqq \nu^+(1)+\nu^+(\omega)+\nu^+(\bar{\omega})+3\nu^-(\delta)\\ &\leqq \sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)+\nu(\xi\bar{\omega}))+3\sum_{\xi\in X^-}\nu(\xi\delta).\end{aligned}$$

Summing over δ, we obtain

$$(p-3)\sum_{\delta}\sum_{\xi\in X^-}\nu(\xi\delta)\leqq(p-3)\Big(n-\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)+\nu(\xi\bar{\omega}))\Big)$$
$$\leqq 3\sum_{\delta}\sum_{\xi\in X^-}\nu(\xi\delta).$$

If $p>5$, then $\nu(\xi\delta)=0$ for $\delta\in\langle\delta_p\rangle\setminus\{1, \omega, \bar{\omega}\}$ and $\xi\in\langle\zeta_{p'}\rangle$, so

$$\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)+\nu(\xi\bar{\omega}))\leqq n\leqq\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)+\nu(\xi\bar{\omega}))/\varphi(\xi),$$

and hence $n=\nu(1)+\nu(\omega)+\nu(\bar{\omega})$ and $\nu(\xi)=0$ for $\xi\notin\{1, \omega, \bar{\omega}\}$. We conclude $o(g)=o(y)=p$ if $p>5$, as claimed.

Now suppose $l>1$. Then

$$n-2p^s=\nu_0(1)-\nu_0(\delta),\quad p^s=\nu_0(\omega)-\nu_0(\delta\omega)=\nu_0(\bar{\omega})-\nu_0(\delta\bar{\omega})$$

for any $\delta\in\langle\delta_p\rangle\setminus\{1\}$. Thus

$$\begin{aligned}n&=\nu_0(1)+\nu_0(\omega)+\nu_0(\bar{\omega})-\nu_0(\delta)-\nu_0(\delta\omega)-\nu_0(\delta\bar{\omega})\\&\leqq\nu^+(1)+\nu^+(\omega)+\nu^+(\bar{\omega})+\nu^-(\delta)+\nu^-(\delta\omega)+\nu^-(\delta\bar{\omega})\\&\leqq\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)+\nu(\xi\bar{\omega}))+\sum_{\xi\in X^-}(\nu(\xi\delta)+\nu(\xi\delta\omega)+\nu(\xi\delta\bar{\omega})).\end{aligned}$$

Summing over δ gives

$$\begin{aligned}&(p-1)\sum_{\delta}\sum_{\xi\in X^-}(\nu(\xi\delta)+\nu(\xi\delta\omega)+\nu(\xi\delta\bar{\omega}))\\&\leqq(p-1)\Big(n-\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)+\nu(\xi\bar{\omega}))\Big)\\&\leqq\sum_{\delta}\sum_{\xi\in X^-}(\nu(\xi\delta)+\nu(\xi\delta\omega)+\nu(\xi\delta\bar{\omega})).\end{aligned}$$

If $p>2$, then $\nu(\xi\delta)=\nu(\xi\delta\omega)=\nu(\xi\delta\bar{\omega})=0$ for $\delta\in\langle\delta_p\rangle\setminus\{1\}$ and $\xi\in X^-$, so

$$n\leqq\nu^+(1)+\nu^+(\omega)+\nu^+(\bar{\omega})=\sum_{\xi\in X^+}(\nu(\xi)+\nu(\xi\omega)+\nu(\xi\bar{\omega}))/\varphi(\xi). \tag{3.5}$$

Suppose $p=2$. Then

$$\begin{aligned}n&=\nu_0(1)+\nu_0(\omega)+\nu_0(\bar{\omega})-\nu_0(-1)-\nu_0(-\omega)-\nu_0(-\bar{\omega})\\&\leqq\nu^+(1)+\nu^+(\omega)+\nu^+(\bar{\omega})+\nu^-(-1)+\nu^-(-\omega)+\nu^-(-\bar{\omega})\\&\leqq\nu^+(1)+\nu^+(\omega)+\nu^+(\bar{\omega})+\sum_{\xi\in X^-}(\nu(-\xi)+\nu(-\xi\omega)+\nu(-\xi\bar{\omega}))/2\end{aligned}$$

since $\varphi(\xi)\geqq 2$ for $\xi\in X^-$. It follows that $\nu(-\xi)=\nu(-\xi\omega)=\nu(-\xi\bar{\omega})=0$ for $\xi\in X^-$, so (3.5) also holds if $p=2$. Therefore $n=\nu(1)+\nu(\omega)+\nu(\bar{\omega})$ and $\nu(\xi)=0$ for $\xi\notin\{1, \omega, \bar{\omega}\}$, so $o(g)=o(y)=p^l$, as required. This completes the proof of the lemma. □

LEMMA 3.6. *Suppose g is a χ-singular element of G and $\langle g\rangle=\langle x\rangle$ whenever x is χ-singular and $\langle g\rangle\subseteq\langle x\rangle$. If $h\in C_G(g)$ is a p-element for some prime p and $h\notin\langle g\rangle$, then gh is not χ-singular.*

PROOF. Suppose not, so gh is χ-singular. Let g_p be the p-part of g. Let ω be a nonidentity eigenvalue of $\rho(g)$, and let λ be the eigenvalue of $\rho(h)$ on the ω-eigenspace of $\rho(g)$.

Suppose $\chi(g)=n-1+\omega$. Since all nonidentity eigenvalues of $\rho(gh)$ have the same order, the only nonidentity eigenvalue of $\rho(gh)$ is $\lambda\omega$, and the only nonidentity eigenvalue of $\rho(h)$ is λ. If $o(h)\leqq o(g_p)$, then $h\in\langle g_p\rangle$, a contradiction. On the other hand, if $o(h)>o(g_p)$, then $g_p\in\langle h\rangle$, so $\langle g\rangle\subset\langle gh\rangle$, another contradiction.

Suppose $\chi(g)=n-2+\omega+\bar{\omega}$. Let λ' be the eigenvalue of $\rho(h)$ on the $\bar{\omega}$-eigenspace of $\rho(g)$. Then $\lambda\omega$ and $\lambda'\bar{\omega}$ are the only nonidentity eigenvalues of $\rho(gh)$, and therefore $\overline{\lambda\omega}=\lambda'\bar{\omega}$, so $\lambda'=\bar{\lambda}$. As above, we have $h\in\langle g_p\rangle$ if $o(h)\leqq o(g_p)$, and $\langle g\rangle\subset\langle gh\rangle$ if $o(h)>o(g_p)$, both of which are contradictions. This completes the proof of the lemma. □

LEMMA 3.7. *Suppose g is χ-singular. If $h\in C_G(g)$ is a p-element for some prime p, then $\chi(gh)$ is not equal to $\chi(y)$ for any p-element y of G.*

PROOF. Suppose to the contrary that $\chi(gh)=\chi(y)$ for some p-element y of G. Let ω be a primitive $|G|$-th root of unity, and let ω_0 be the p-part of ω. Then $n-\chi(g)=(\chi(1)-\chi(y))+(\chi(gh)-\chi(g))\in(1-\omega_0)\boldsymbol{Z}[\omega]$, and hence p divides the product of the algebraic conjugates of $n-\chi(g)$, a contradiction. □

Suppose now that α and β are complex roots of unity whose orders are relatively prime. Assume $S\subseteq\langle\alpha\rangle$ and $T\subseteq\langle\beta\rangle$ are $\boldsymbol{Q}$-bases for $\boldsymbol{Q}(\alpha)$ and $\boldsymbol{Q}(\beta)$, respectively, and $1\in S\cap T$. Then $ST=\{\sigma\tau\mid\sigma\in S,\ \tau\in T\}$ is a $\boldsymbol{Q}$-basis for $\boldsymbol{Q}(\alpha\beta)$, and it follows that no nonzero linear combination of elements of $ST\setminus(S\cup T)$ is an element of the $\boldsymbol{Q}$-subspace $\boldsymbol{Q}(\alpha)+\boldsymbol{Q}(\beta)$ of $\boldsymbol{Q}(\alpha\beta)$. This observation is used in the proofs of the following two results.

LEMMA 3.8. *Suppose α and β are complex roots of unity whose orders are relatively prime. Assume $\alpha_0\in\langle\alpha\rangle$, $\beta_0\in\langle\beta\rangle$, and $\alpha_0\beta_0\in\boldsymbol{Q}(\alpha)+\boldsymbol{Q}(\beta)$. Then $\alpha_0\in\{1,-1\}$ or $\beta_0\in\{1,-1\}$.*

PROOF. If $\alpha_0\neq\pm1$ and $\beta_0\neq\pm1$, then there are $\boldsymbol{Q}$-bases S and T for $\boldsymbol{Q}(\alpha)$ and $\boldsymbol{Q}(\beta)$, respectively, such that $\{1,\alpha_0\}\subseteq S$ and $\{1,\beta_0\}\subseteq T$. Then $\alpha_0\beta_0\in ST\setminus(S\cup T)$, and a contradiction is reached. □

LEMMA 3.9. *Suppose α and β are complex roots of unity whose orders are relatively prime. Assume α_0, $\alpha_1\in\langle\alpha\rangle$, β_0, $\beta_1\in\langle\beta\rangle$, and $\alpha_0\beta_0+\alpha_1\beta_1\in\boldsymbol{Q}(\alpha)+\boldsymbol{Q}(\beta)$. Suppose further that 1, β_0, β_1 are distinct and $\{1,\beta_0,\beta_1\}$ is linearly independent* *over* $\boldsymbol{Q}$. *Then* $\{\alpha_0, \alpha_1\} \subseteqq \{1, -1\}$.

PROOF. First suppose $\{\alpha_0, \alpha_1\} \cap \{1, -1\} = \emptyset$. Let T be a $\boldsymbol{Q}$-basis for $\boldsymbol{Q}(\beta)$ such that $\{1, \beta_0, \beta_1\} \subseteqq T$. Assume $\{1, \alpha_0, \alpha_1\}$ is linearly independent over $\boldsymbol{Q}$. Then there is a $\boldsymbol{Q}$-basis S for $\boldsymbol{Q}(\alpha)$ such that $\{1, \alpha_0, \alpha_1\} \subseteqq S$. Then $\alpha_0\beta_0, \alpha_1\beta_1 \in ST \setminus (S \cup T)$ and $\alpha_0\beta_0 + \alpha_1\beta_1 \in \boldsymbol{Q}(\alpha) + \boldsymbol{Q}(\beta)$, and a contradiction is reached. Next, suppose $\{1, \alpha_0, \alpha_1\}$ is linearly dependent over $\boldsymbol{Q}$. Renumbering if necessary, we may suppose $\alpha_1 = a_0 + a_1\alpha_0$ with $a_0, a_1 \in \boldsymbol{Q}$. Let S be a $\boldsymbol{Q}$-basis for $\boldsymbol{Q}(\alpha)$ such that $\{1, \alpha_0\} \subseteqq S$. Then $\alpha_0\beta_0, \alpha_0\beta_1 \in ST \setminus (S \cup T)$, and $\alpha_0\beta_0 + a_1\alpha_0\beta_1 = \alpha_0\beta_0 + \alpha_1\beta_1 - a_0\beta_1 \in \boldsymbol{Q}(\alpha) + \boldsymbol{Q}(\beta)$, and another contradiction is reached.

Therefore $\{\alpha_0, \alpha_1\} \cap \{1, -1\} \neq \emptyset$. If $\alpha_0 = \pm 1$, then $\alpha_1\beta_1 \in \boldsymbol{Q}(\alpha) + \boldsymbol{Q}(\beta)$, so $\{\alpha_0, \alpha_1\} \subseteqq \{1, -1\}$ by Lemma 3.8 since $\beta_1 \neq \pm 1$. Similarly, $\{\alpha_0, \alpha_1\} \subseteqq \{1, -1\}$ if $\alpha_1 = \pm 1$. $\square$

For k a positive integer, ω_k will denote a primitive complex k-th root of unity.

LEMMA 3.10. *Suppose g is χ-singular and $h \neq 1$ is a p-element in $C_G(g)$ for some prime p. Assume $g'h$ is not χ-singular, where g' is the p'-part of g. Then $\chi(g'h)$ is rational.*

PROOF. Suppose to the contrary that $\chi(g'h)$ is irrational. By Lemma 3.5, $\chi(g'h) = \chi(y)$ for some element y of G of order q, where $q=3$ or $q=5$. Lemma 3.7 applies to show $q \neq p$. Thus q divides the order of g'.

Let ω be a nonidentity eigenvalue of $\rho(g)$, and let λ be the eigenvalue of $\rho(h)$ on the ω-eigenspace of $\rho(g)$. Let ω' be the p'-part of ω, so $\omega' \neq \pm 1$.

Case 1. Suppose $\chi(g)$ is not real. In this case $\chi(g'h) = \chi(h) - \lambda + \lambda\omega' = \chi(y)$, so $\lambda\omega' \in \boldsymbol{Q}(\omega_{o(h)}) + \boldsymbol{Q}(\omega_q) \subseteqq \boldsymbol{Q}(\omega_{o(h)}) + \boldsymbol{Q}(\omega')$. Thus $\lambda = \pm 1$ by Lemma 3.8, so $\omega' \in (\boldsymbol{Q}(\omega_{o(h)}) + \boldsymbol{Q}(\omega_q)) \cap \boldsymbol{Q}(\omega') = \boldsymbol{Q}(\omega_q)$, and hence we may choose ω_q so that $\omega' = \pm\omega_q$. Then $\chi(h) \in \boldsymbol{Q}(\omega_{o(h)}) \cap \boldsymbol{Q}(\omega_q) = \boldsymbol{Q}$. Thus $\chi(g'h) \notin \boldsymbol{R}$, and hence $q=3$.

Now, $\chi(h) - \lambda \pm \lambda\omega_3 = \chi(g'h) = \chi(y) = n - 3^s + 3^s\omega_3^j$ for some s and j by Lemma 3.4. Comparing imaginary parts, we find $3^s = 1$. Then $\chi(h) = n - 1 + \lambda + \omega_3^j - \lambda\omega' \in \{n-1, n-2, n-3\}$ since χ is faithful. Since $n - \chi(h)$ must be divisible by p, we have $p=2$ and $\chi(h) = n-2$, so $\lambda = -1$ and $\omega' = -\omega_3^j$, and hence p divides $o(g')$, a contradiction.

Case 2. Suppose $\chi(g)$ is real. Let λ' be the eigenvalue of $\rho(h)$ on the $\bar{\omega}$-eigenspace of $\rho(g)$. In this case $\chi(g'h) = \chi(h) - \lambda - \lambda' + \lambda\omega' + \lambda'\bar{\omega}' = \chi(y)$, so $\lambda\omega' + \lambda'\bar{\omega}' \in \boldsymbol{Q}(\omega_{o(h)}) + \boldsymbol{Q}(\omega_q) \subseteqq \boldsymbol{Q}(\omega_{o(h)}) + \boldsymbol{Q}(\omega')$, and hence $\{\lambda, \lambda'\} \subseteqq \{1, -1\}$ or else $\{1, \omega', \bar{\omega}'\}$ is linearly dependent over $\boldsymbol{Q}$ by Lemma 3.9.

Suppose $\{\lambda, \lambda'\} \not\subseteqq \{1, -1\}$, so $\{1, \omega', \bar{\omega}'\}$ is linearly dependent over $\boldsymbol{Q}$. Then $q=3$ and ω_3 can be chosen so that $\omega' = \pm\omega_3$. Note that if $\lambda = \pm 1$, then $\lambda'\bar{\omega}_3 \in \boldsymbol{Q}(\omega_{o(h)}) + \boldsymbol{Q}(\omega_3)$, so $\lambda' = \pm 1$ by Lemma 3.8, a contradiction. Thus $\lambda \neq \pm 1$, and similarly $\lambda'\neq\pm1$. Also, $(\lambda-\lambda')\omega_3=\lambda\omega_3+\lambda'\bar{\omega}_3-\lambda'\in\boldsymbol{Q}(\omega_{o(h)})+\boldsymbol{Q}(\omega_3)$. Therefore $\lambda-\lambda'\in\boldsymbol{Q}$. Since $p\neq3$ and $\{\lambda,\lambda'\}\cap\{1,-1\}=\emptyset$, we must have $\lambda=\lambda'$. Then $\lambda\omega_3+\lambda'\bar{\omega}_3=-\lambda$, so $\chi(g'h)=\chi(y)\in\boldsymbol{Q}(\omega_{o(h)})\cap\boldsymbol{Q}(\omega_q)=\boldsymbol{Q}$, a contradiction.

Therefore $\{\lambda,\lambda'\}\subseteqq\{1,-1\}$. Since $\omega'\pm\bar{\omega}'\in(\boldsymbol{Q}(\omega_{o(h)})+\boldsymbol{Q}(\omega_q))\cap\boldsymbol{Q}(\omega')=\boldsymbol{Q}(\omega_q)$ and q divides the order of g', it follows that ω_q can be chosen so that $\omega'=\pm\omega_q$. Thus $\chi(h)\in\boldsymbol{Q}(\omega_{o(h)})\cap\boldsymbol{Q}(\omega_q)=\boldsymbol{Q}$.

If $\lambda=\lambda'$, then $\chi(y)$ is real, so $q=5$. Then $\chi(h)-2\lambda\pm\lambda(\omega_5+\bar{\omega}_5)=\chi(g'h)=\chi(y)=n-2\cdot5^s+5^s\omega_5^j+5^s\bar{\omega}_5^j$ for some $s\geqq0$ and some j by Lemma 3.4. Since $\chi(h)$ is rational, we must have $5^s=1$. Then $\chi(h)=n-2+2\lambda+\omega_5^j+\bar{\omega}_5^j-\lambda(\omega'+\bar{\omega}')$, and thus $\chi(h)\in\{n-1, n-4, n-5\}$ since χ is faithful. Since p must divide $n-\chi(h)$, we have $p=2$ and $\chi(h)=n-4$, so $\lambda=-1$ and $\omega'=-\omega_5^j$ or $\omega'=-\bar{\omega}_5^j$, which is impossible since $o(g')$ is odd.

Thus $\lambda\neq\lambda'$, so $\chi(y)$ is not real, and hence $q=3$. Then $\chi(h)\pm(\omega_3-\bar{\omega}_3)=\chi(g'h)=\chi(y)=n-3^s+3^s\omega_3^j$ for some j. Comparing imaginary parts, we see $\sqrt{3}=3^s\sqrt{3}/2$, which is impossible. This completes the proof of the lemma. □

LEMMA 3.11. *Suppose g is a χ-singular element of G and $\langle g\rangle=\langle h\rangle$ whenever h is χ-singular and $\langle g\rangle\subseteqq\langle h\rangle$. Then $C_G(g)=\langle g\rangle$.*

PROOF. Suppose not. Let h be a p-element of $C_G(g)$ for some prime p such that $h\notin\langle g\rangle$. Then gh is not χ-singular by Lemma 3.6. Let g_p be the p-part of g. Then $g_ph\neq1$, so Lemma 3.10 applies to show $\chi(gh)$ is rational. Similarly, $\chi(g^{-1}h)$ is rational.

Let ω be a nonidentity eigenvalue of $\rho(g)$, and let λ denote the eigenvalue of $\rho(h)$ on the ω-eigenspace of $\rho(g)$.

Case 1. Suppose $\chi(g)$ is not real. Then $\chi(gh)-\chi(g^{-1}h)=\lambda(\omega-\bar{\omega})\in\boldsymbol{Q}$, and hence $o(\omega)=12$ and $\lambda^2=-1$. This is impossible since $\chi(gh)=\chi(h)-\lambda+\lambda\omega\in\boldsymbol{Q}$.

Case 2. Suppose $\chi(g)$ is real. Let λ' be the eigenvalue of $\rho(h)$ on the $\bar{\omega}$-eigenspace of $\rho(g)$. Then $\chi(gh)-\chi(g^{-1}h)=(\lambda-\lambda')(\omega-\bar{\omega})\in\boldsymbol{Q}$, so $\lambda'=\lambda$ or $\lambda'=\bar{\lambda}$.

Suppose p does not divide $o(g)$. Since $\lambda\omega+\lambda'\bar{\omega}=\chi(gh)-\chi(h)+\lambda+\lambda'\in\boldsymbol{Q}(\omega_{o(h)})$ and $\{1,\omega,\bar{\omega}\}$ is linearly independent over $\boldsymbol{Q}$, Lemma 3.9 applies to show $\{\lambda,\lambda'\}\subseteqq\{1,-1\}$, and hence $\lambda=\lambda'=\pm1$ by the above. Then $\omega+\bar{\omega}\in\boldsymbol{Q}(\omega_{o(h)})\cap\boldsymbol{Q}(\omega)=\boldsymbol{Q}$, so $\chi(g)$ is rational, a contradiction.

Therefore p divides $o(g)$. Next assume $o(h)=p$. Replacing h by an appropriate element of $\langle g_p, h\rangle\setminus\langle g_p\rangle$ if necessary, we may suppose $\lambda=\lambda'=1$. Then $\chi(h)=\chi(gh)-\omega-\bar{\omega}+2$ is irrational and real, and hence $p>3$. We may suppose ω_p is an eigenvalue of $\rho(h)$. Then the nonidentity eigenvalues of $\rho(h)$ are ω_p and $\bar{\omega}_p$ with multiplicity p^s for some $s\geqq0$ by Lemma 3.4. There is some j such that $\omega^j=\omega_p$. Then the nonidentity eigenvalues of $\rho(g^jh)$ are ω_p and $\bar{\omega}_p$, both with multiplicity p^s+1, contradicting Lemma 3.4.

Now we return to the general case. By the above, $\langle g_p, h\rangle$ has a unique subgroup of order p, so $\langle g_p, h\rangle$ is cyclic. Therefore $o(h)>o(g_p)$ and $g_p\in\langle h\rangle$ since $h\notin\langle g\rangle$. We may as well suppose $h^p=g_p$. Then λ and λ' are the only eigenvalues of $\rho(h)$ of order greater than p.

Suppose $\chi(h)$ is irrational. Then λ and λ' are the only nonidentity eigenvalues of $\rho(h)$ by Lemma 3.4. If $\lambda'=\bar{\lambda}$, then gh is χ-singular and $\langle g\rangle\subset\langle gh\rangle$, so we have a contradiction. Therefore $p=2$ and $\lambda'=\lambda$ by Lemma 3.4. Since $h^2=g_2$, we have $(\lambda')^2=\bar{\lambda}^2$, and therefore $\lambda^2=-1$. Then $\chi(gh)=n-2+\lambda\omega+\lambda\bar{\omega}\notin\boldsymbol{R}$, which is a contradiction.

Therefore $\chi(h)$ is rational. Then $p=2$ and $\lambda'=-\lambda$. Note $\lambda'=\bar{\lambda}$, so $\lambda^2=-1$ and $o(h)=4$. Also, $(\lambda-\lambda')(\omega-\bar{\omega})=2\lambda(\omega-\bar{\omega})\in\boldsymbol{Q}$, and therefore $o(g)=12$. This is a contradiction since $o(h)>o(g_2)$. This completes the proof of the lemma. □

LEMMA 3.12. *Suppose g is an element of G of prime power order p^l such that $\chi(g)$ is irrational. Assume that $C_G(g)$ contains no χ-singular elements. Then $C_G(g)$ is a p-group.*

PROOF. Suppose not. Let h be an element of $C_G(g)$ of prime order q, $q\neq p$. Let V_ξ denote the ξ-eigenspace of $\rho(g)$, and let θ_ξ be the character of $C_G(g)$ acting on V_ξ. Let ω be a nonidentity eigenvalue of $\rho(g)$. Note that $\dim(V_\omega)=p^s$ for some nonnegative integer s by Lemma 3.4. Put $\alpha=\theta_\omega(h)$, $\beta=\theta_{\bar{\omega}}(h)$, so α, $\beta\in\boldsymbol{Q}(\omega_q)$. We claim that it suffices to prove that $\alpha=0$. Indeed, if $\alpha=0$, then $\theta_\omega(h^j)=0$ for $1\leqq j\leqq q-1$ since $p\neq q$, and hence V_ω affords a multiple of the regular character of $\langle h\rangle$. Thus q divides $\dim(V_\omega)=p^s$, a contradiction.

Suppose $\chi(gh)$ is rational. Then $\chi(g^jh)$ is rational whenever p does not divide j. In particular $\chi(gh)-\chi(g^{-1}h)=(\alpha-\beta)(\omega-\bar{\omega})\in\boldsymbol{Q}$, so $\alpha-\beta\in\boldsymbol{Q}(\omega)\cap\boldsymbol{Q}(\omega_q)=\boldsymbol{Q}$, and thus $\alpha=\beta$. Thus $\alpha(\omega+\bar{\omega})=\chi(gh)-\theta_1(h)\in\boldsymbol{Q}(\omega_q)$. If $\alpha\neq0$, then $\omega+\bar{\omega}\in\boldsymbol{Q}$, and hence $p^l=3$ or $p^l=4$. In either case $\bar{\omega}$ is not an eigenvalue of $\rho(g)$, so $\beta=0$, and thus $\alpha=0$, a contradiction. Hence $\alpha=0$ and we are done.

For the remainder of the proof we suppose $\chi(gh)$ is irrational. By our assumptions, gh is not χ-singular. Therefore Lemma 3.5 gives $\chi(gh)=\chi(y)$ for some element y of order r, with $r=3$ if $\chi(gh)\notin\boldsymbol{R}$ and $r=5$ if $\chi(gh)\in\boldsymbol{R}$. Note that $r=p$ or $r=q$. We have $\chi(gh)\in\boldsymbol{Q}(\omega_r)$, and also $\chi(g^{-1}h)\in\boldsymbol{Q}(\omega_r)$ since $p\neq q$. Then

$$\chi(gh)-\chi(g^{-1}h)=(\alpha-\beta)(\omega-\bar{\omega})\in\boldsymbol{Q}(\omega_r)\subseteqq\boldsymbol{Q}(\omega)+\boldsymbol{Q}(\omega_q)$$

since $r\in\{p, q\}$. Since $\{1, \omega-\bar{\omega}\}$ extends to a $\boldsymbol{Q}$-basis for $\boldsymbol{Q}(\omega)$, it follows that $\alpha-\beta\in\boldsymbol{Q}$. Then $\alpha(\omega+\bar{\omega})=\chi(gh)-\theta_1(h)+(\alpha-\beta)\bar{\omega}\in\boldsymbol{Q}(\omega)+\boldsymbol{Q}(\omega_q)$. Therefore either $\omega+\bar{\omega}\in\boldsymbol{Q}$ or else $\alpha\in\boldsymbol{Q}$. If $\omega+\bar{\omega}\in\boldsymbol{Q}$, then $p^l=3$ or $p^l=4$ and $\bar{\omega}$ is not an eigenvalue of $\rho(g)$, so $\beta=0$ and $\alpha\in\boldsymbol{Q}$. Thus $\alpha\in\boldsymbol{Q}$ in all cases, and hence we also have $\beta\in\boldsymbol{Q}$.

Case 1. Suppose $r=q$. Then $\alpha\omega+\beta\bar{\omega}=\chi(gh)-\theta_1(h)\in\boldsymbol{Q}(\omega)\cap\boldsymbol{Q}(\omega_q)=\boldsymbol{Q}$. If

$\{1, \omega, \bar{\omega}\}$ is linear dependent over $\boldsymbol{Q}$, then $p^l=3$ or $p^l=4$ and $\beta=0$ since $\chi(g)$ is irrational, and hence $\alpha=0$. On the other hand, if $\{1, \omega, \bar{\omega}\}$ is linearly independent over $\boldsymbol{Q}$, then $\alpha=\beta=0$. Thus $\alpha=0$ in either case, and we are done.

Case 2. Suppose $r=p$. In this case $\theta_1(h)=\chi(gh)-\alpha\omega-\beta\bar{\omega}\in\boldsymbol{Q}(\omega)\cap\boldsymbol{Q}(\omega_q)=\boldsymbol{Q}$. Thus $\alpha\omega+\beta\bar{\omega}=\chi(y)-\theta_1(h)\in\boldsymbol{Q}(\omega_p)$. We may suppose $l=1$, for otherwise $\alpha=\beta=0$ and the proof is complete.

If $r=p=3$, then $\theta_1(h)+\alpha\omega_3=\chi(gh)=\chi(y)=n-3^t+3^t\omega_3^j$ for some t and j by Lemma 3.4, and thus $\alpha=\pm 3^t$ since $\theta_1(h)$ is rational. If $\alpha=3^t$, then $\theta_1(h)=n-3^t$ and $h=1$ since χ is faithful, a contradiction. Thus $\alpha=-3^t$, $\theta_1(h)=n-2\cdot 3^t$ and so $n-\chi(gh)=3^{t+1}+3^t\omega_3$. The product of the algebraic conjugates of $n-\chi(gh)$ is divisible by $(3+\omega_3)(3+\bar{\omega}_3)=7$. However, if M is the set of values of χ on $P\setminus\{1\}$, where $P\in\mathrm{Syl}_7(G)$, then $\chi(gh)\notin M$ and $f_M(n)$ is divisible by $|P|$ by Theorem 2.1, so $f_L(n)$ is divisible by $7|P|$, which is impossible since (G, χ) is sharp of type L.

Suppose $r=p=5$. Note $\alpha=\beta$ since $\chi(gh)$ is real. Then $\theta_1(h)+\alpha(\omega_5+\bar{\omega}_5)=n-2\cdot 5^t+5^t(\omega_5^j+\bar{\omega}_5^j)$ for some j and t by Lemma 3.4. Thus $\alpha=\pm 5^t$ since $\theta_1(h)$ is rational. If $\alpha=5^t$, then $\theta_1(h)=n-2\cdot 5^t$, so $h=1$, a contradiction. On the other hand, if $\alpha=-5^t$, then $\theta_1(h)=n-3\cdot 5^t$, so q divides $n-2\cdot 5^t-\theta_1(h)=5^t$, a contradiction. This completes the proof of the lemma. □

LEMMA 3.13. *Let p be prime. Suppose g and h are commuting p-elements of G of different orders such that $\chi(g)$ and $\chi(h)$ are irrational. Then $\langle g, h\rangle$ is cyclic.*

PROOF. We suppose $o(g)<o(h)$. Let ω be a nonidentity eigenvalue of $\rho(g)$, and let ξ be a nonidentity eigenvalue of $\rho(h)$. Let V_λ and W_λ denote the λ-eigenspaces of $\rho(g)$ and $\rho(h)$, respectively.

First suppose $\chi(gh)$ and $\chi(gh^{-1})$ are rational. Let α_1 be the trace of $\rho(g)$ on W_1, and let α be the trace of $\rho(g)$ on $W_\xi\oplus W_{\bar{\xi}}$. Note $\alpha(\xi-\bar{\xi})=\chi(gh)-\chi(gh^{-1})\in\boldsymbol{Q}$. If $\alpha\neq 0$, then $\xi-\bar{\xi}\in\boldsymbol{Q}(\xi^p)$, so $\xi^2=1$ and $\chi(h)$ is rational, a contradiction. Thus $\alpha=0$. Then $2\alpha_1=\chi(gh)+\chi(gh^{-1})\in\boldsymbol{Q}$, and therefore $\chi(g)=\alpha_1\in\boldsymbol{Q}$, a contradiction.

Replacing h by h^{-1} if necessary, we may suppose $\chi(gh)$ is irrational. Then all nonidentity eigenvalues of $\rho(gh)$ have order $o(h)$ by Lemma 3.4, so $V_\omega\oplus V_{\bar{\omega}}\subseteq W_\xi\oplus W_{\bar{\xi}}$. Since $\omega\xi$ and $\omega\bar{\xi}$ cannot both be eigenvalues of $\rho(gh)$ by Lemma 3.4, we have $V_\omega\subseteq W_\xi$ or $V_\omega\subseteq W_{\bar{\xi}}$. We may suppose ξ has been chosen so that $V_\omega\subseteq W_\xi$. Since $\omega\xi$ and $\bar{\omega}\xi$ cannot both be eigenvalues of $\rho(gh)$, we have $V_{\bar{\omega}}\subseteq W_{\bar{\xi}}$. Suppose $V_1\cap W_\xi\neq\{0\}$. Then ξ is an eigenvalue of $\rho(gh)$, and hence $\xi=\overline{\omega\xi}$ by Lemma 3.4, so $p=2$, $\xi^2=\bar{\omega}$, and $\chi(g)$ and $\chi(h)$ are real. Note $o(h)>o(g)\geqq 8$. Thus $\chi(g^{-1}h)$ is irrational and $\rho(g^{-1}h)$ has nonidentity eigenvalues ξ, $\bar{\xi}$, ξ^3 and $\bar{\xi}^3$, contradicting Lemma 3.4. Therefore $V_1\cap W_\xi=\{0\}$, so $V_\omega=W_\xi$. A similar argument shows $V_1\cap W_{\bar{\xi}}=\{0\}$ if $V_{\bar{\omega}}\neq\{0\}$. If $V_1\cap W_{\bar{\xi}}\neq\{0\}$ and $V_{\bar{\omega}}=\{0\}$, then $p=2$ and $W_{\bar{\xi}}\subseteq V_1$, and the only nonidentity eigenvalues of $\rho(g^{o(g)/2}h)$ are

$-\xi$ and $\bar{\xi}$. Hence $\bar{\xi}=-\xi$, so $\xi^2=-1$ and $\chi(g)$ is rational, a contradiction. Therefore $V_1\cap W_{\bar{\xi}}=\{0\}$, so $V_{\bar{\omega}}=W_{\bar{\xi}}$. Thus $g\in\langle h\rangle$, as required. □

LEMMA 3.14. *Let p be prime. Suppose g is a p-element of G such that $\chi(g)$ is irrational and $C_G(g)$ is a p-group. Assume $h\in C_G(g)$, $o(h)=o(g)$ and $\chi(h)$ is irrational. Then $\langle g\rangle=\langle h\rangle$.*

PROOF. Let ω be a nonidentity eigenvalue of $\rho(g)$. Replacing h by a power if necessary, we may suppose ω is an eigenvalue of $\rho(h)$. Denote by V_ξ and W_ξ the ξ-eigenspaces of $\rho(g)$ and $\rho(h)$, respectively. Let $p^s=\dim(V_\omega)$, $p^t=\dim(W_\omega)$. Put $X=\{x\in G\,|\,\chi(x)=\chi(g)\}$.

Case 1. Suppose $o(g)\notin\{3, 4, 5, 8\}$. In this case the eigenvalues of $\rho(gh)$ are elements of the set $\{1, \omega, \bar{\omega}, \omega^2, \bar{\omega}^2\}$, which is linearly independent over $\boldsymbol{Q}$.

We claim that $\chi(gh)$ or $\chi(gh^{-1})$ is rational. Suppose to the contrary that both $\chi(gh)$ and $\chi(gh^{-1})$ are irrational. Assume $(V_\omega\oplus V_{\bar{\omega}})\cap W_1\neq\{0\}$. Then $\rho(gh)$ has an eigenvalue ω or $\bar{\omega}$, and therefore $\rho(gh)$ has no eigenvalue ω^2 or $\bar{\omega}^2$ by Lemma 3.4. Thus $V_\omega\cap W_\omega=V_{\bar{\omega}}\cap W_{\bar{\omega}}=\{0\}$. The same argument applied to gh^{-1} shows $V_\omega\cap W_{\bar{\omega}}=V_{\bar{\omega}}\cap W_\omega=\{0\}$. Thus $V_\omega\oplus V_{\bar{\omega}}\subseteqq W_1$ and $W_\omega\oplus W_{\bar{\omega}}\subseteqq V_1$. Then $\rho(gh^2)$ has both ω and ω^2 as eigenvalues, which is impossible.

Therefore $(V_\omega\oplus V_{\bar{\omega}})\cap W_1=\{0\}$. Similarly, $V_1\cap W_\omega\oplus W_{\bar{\omega}}=\{0\}$. Hence $V_1=W_1$ and $V_\omega\oplus V_{\bar{\omega}}=W_\omega\oplus W_{\bar{\omega}}$. If $\chi(g)$ and $\chi(h)$ are not real, then $V_\omega=W_\omega$, so $g=h$ and we are done. If only one of $\chi(g)$ and $\chi(h)$ is real, then $p=2$ and $W_\omega=V_\omega\oplus V_{\bar{\omega}}$ or $V_\omega=W_\omega\oplus W_{\bar{\omega}}$. Then the nonidentity eigenvalues of $\rho(g^2h)$ or $\rho(gh^2)$ are ω and ω^3, contradicting Lemma 3.4. Hence we may suppose $\chi(g)$ and $\chi(h)$ are both real, so $s=t$.

Let $d=\dim(V_\omega\cap W_\omega)$. The multiplicity of ω^2 as an eigenvalue of $\rho(gh)$ is equal to d, while the multiplicity of ω^2 as an eigenvalue of $\rho(gh^{-1})$ is equal to p^s-d. These multiplicities are equal to 0 or a power of p by Lemma 3.4. Hence $d=p^s$, $d=0$, or else $p=2$ and $d=2^{s-1}$. However, if $p=2$ and $d=2^{s-1}$, then $\rho(g^2h)$ has nonidentity eigenvalues ω, ω^3, $\bar{\omega}$ and $\bar{\omega}^3$, so $\chi(g^h)$ is irrational since $o(g)>8$, and we have a contradiction to Lemma 3.4. Hence $d=p^s$ or $d=0$, so $g=h$ or $g=h^{-1}$ and we are done.

Therefore $\chi(gh)$ or $\chi(gh^{-1})$ is rational, and thus $g=h^{-1}$ or $g=h$, so $\langle g\rangle=\langle h\rangle$.

Case 2. Suppose $o(g)=5$ or $o(g)=8$ and $\chi(g)$ is not real. In this case the eigenvalues of $\rho(gh)$ are elements of the set $\{1, \omega, \omega^2, \bar{\omega}\}$, which is linearly independent over $\boldsymbol{Q}$. The argument given in Case 1 shows that $\chi(gh)$ or $\chi(gh^{-1})$ is rational. Hence $g=h^{-1}$ or $g=h$, so $\langle g\rangle=\langle h\rangle$.

Case 3. Suppose $o(g)=3$. In this case ω is the only nonidentity eigenvalue of $\rho(g)$ and $\rho(h)$. We suppose $\langle g\rangle\neq\langle h\rangle$ and arrive at a contradiction.

CLAIM 1. *We have $s=t\geqq 1$. Moreover,* $\dim(V_1\cap W_\omega)=\dim(V_\omega\cap W_1)=3^{s-1}$ *and* $\dim(V_\omega\cap W_\omega)=2\cdot 3^{s-1}$.

PROOF OF CLAIM 1. Let $a=\dim(V_{\omega}\cap W_1)$, $b=\dim(V_1\cap W_{\omega})$, $c=\dim(V_{\omega}\cap W_{\omega})$, so $a+c=\dim(V_{\omega})=3^s$ and $b+c=\dim(W_{\omega})=3^t$. If $a=0$ or $b=0$, then $c=3^s=3^t$, so $g=h$, a contradiction. Suppose $a>0$ and $b>0$. Then $\rho(gh^{-1})$ has both ω and $\bar{\omega}$ as eigenvalues, so $\chi(gh^{-1})$ is rational by Lemma 3.4, and thus $a=b$ and $s=t$. Also, $\rho(gh)$ has ω as an eigenvalue with multiplicity $2a$, so $\chi(gh)$ is rational by Lemma 3.4. Therefore $a=b=3^{s-1}$ and $c=2\cdot 3^{s-1}$. $\square$

CLAIM 2. $X\cap C_G(\langle g, h\rangle)\subseteqq\langle g, h\rangle$.

PROOF OF CLAIM 2. Suppose not. Let $x\in X\cap C_G(\langle g, h\rangle)\setminus\langle g, h\rangle$. Let U_{ξ} be the ξ-eigenspace of $\rho(x)$. Put $d=\dim(V_1\cap W_1\cap U_{\omega})$. Applying Claim 1 to the pairs $\{g, h\}$, $\{g, x\}$ and $\{h, x\}$, we see

$$\dim(V_{\omega})=\dim(W_{\omega})=\dim(U_{\omega})=3^s,$$

$$\dim(V_1\cap W_1\cap U_{\omega})=\dim(V_1\cap W_{\omega}\cap U_1)=\dim(V_{\omega}\cap W_1\cap U_1)=d,$$

$$\dim(V_1\cap W_{\omega}\cap U_{\omega})=\dim(V_{\omega}\cap W_1\cap U_{\omega})=\dim(V_{\omega}\cap W_{\omega}\cap U_1)=3^{s-1}-d,$$

and

$$\dim(V_{\omega}\cap W_{\omega}\cap U_{\omega})=3^{s-1}+d.$$

Therefore the multiplicities of ω and $\bar{\omega}$ as eigenvalues of $\rho(ghx^{-1})$ are $3^{s-1}+3d$ and 3^{s-1}, respectively. Hence $d=0$ by Lemma 3.4. Thus $\rho(x)$ acts as ω on $V_1\cap W_{\omega}$ and $V_{\omega}\cap W_1$, and the remaining 3^{s-1} nonidentity eigenvalues of $\rho(x)$ occur on $V_{\omega}\cap W_{\omega}$.

Suppose $y\in X\cap C_G(\langle g, h, x\rangle)\setminus\langle g, h\rangle$. Then $\rho(x)$ and $\rho(y)$ agree on V_1+W_1 by the above. Let a be the multiplicity of ω as an eigenvalue of $\rho(y)$ on $V_{\omega}\cap W_{\omega}\cap U_{\omega}$. Then $\rho(xy)$ has ω as an eigenvalue with multiplicity $2(3^{s-1}-a)$ and $\bar{\omega}$ with multiplicity $2\cdot 3^{s-1}+a$. It follows that $a=0$ or $a=3^{s-1}$ by Lemma 3.4. If $a=0$ then $y=(ghx)^{-1}$, while if $a=3^{s-1}$ then $y=x$.

Put $X_0=\{g, h, x, (ghx)^{-1}\}$. Let $H=C_G(\langle g, h, x\rangle)$, so H is a 3-group. We have shown $X\cap H=X_0$. Let $N=N_G(H)$, and let $N_3\in\mathrm{Syl}_3(N)$. If $z\in N_3\setminus H$, then $\rho(z)$ has an orbit of size 3 on the eigenspaces of $\rho(H)$, so $\rho(z)$ has eigenvalues ξ, $\omega\xi$ and $\bar{\omega}\xi$ for some ξ, and thus $\chi(z)$ is rational by Lemma 3.4. Therefore $N_3\cap X=X_0$, so $N_G(N_3)\subseteqq N$, and thus $N_3\in\mathrm{Syl}_3(G)$. Note $|N_3:H|\leqq 3$.

We next show χ assumes only rational values on $H\setminus\langle g, h, x\rangle$. Suppose to the contrary that $z\in H\setminus\langle g, h, x\rangle$ and $\chi(z)$ is irrational. If $o(z)>3$, then $\langle g\rangle=\langle h\rangle\subseteqq\langle z\rangle$ by Lemma 3.13, a contradiction. Then $o(z)=3$ and $z\in X$ or $z^{-1}\in X$ by Claim 1. Hence $z\in X_0$ or $z^{-1}\in X_0$, so $z\in\langle g, h, x\rangle$, a contradiction.

Now let θ be the character of H afforded by $V_{\omega}\cap W_{\omega}\cap U_{\omega}$. Then θ vanishes on $H\setminus\langle g, h, x\rangle$, and hence $|H|$ divides $|H|\cdot(\theta, \theta)_H=3^3(3^{s-1})^2=3^{2s+1}$. Also, we have $\chi(gh)=n-2\cdot 3^s$, $\chi(gh^{-1})=n-3^s$, and therefore $|G|_3$ is divisible by $3^s(1-\omega)3^s(1-\bar{\omega})3^{2s}=3^{4s+1}$ since (G, χ) is sharp. Therefore $3^{4s+1}\leqq 3^{2s+2}$, which is impossible since $s\geqq 1$. This contradiction completes the proof of Claim 2. $\square$

Now put $H=C_G(\langle g, h\rangle)$, so H is a 3-group. Also, $H\cap X=\{g, h\}$, and therefore $H\in\mathrm{Syl}_3(G)$. As in the proof of Claim 2, χ assumes only rational values on $H\smallsetminus\langle g, h\rangle$. Let θ be the character of H afforded by $V_\omega\cap W_\omega$. Then θ vanishes on $H\smallsetminus\langle g, h\rangle$. Therefore $|H|$ divides $|H|\cdot(\theta, \theta)_H=3^2(2\cdot 3^{s-1})^2=4\cdot 3^{2s}$. As in the proof of Claim 2, $|H|=|G|_3$ is divisible by $3^s(1-\omega)3^s(1-\bar{\omega})3^{2s}=3^{4s+1}$ since (G, χ) is sharp. Therefore $3^{4s+1}\leqq 3^{2s}$, and we have a contradiction since $s\geqq 1$. Hence $\langle g\rangle=\langle h\rangle$, as required.

Case 4. Suppose $o(g)=4$. In this case ω is the only nonidentity eigenvalue of $\rho(g)$ and $\rho(h)$. Let $d=\dim(V_\omega\cap W_\omega)$. If $d=0$, then $\chi(g^2h)$ is irrational and $\rho(g^2h)$ has an eigenvalue -1, contradicting Lemma 3.4. Therefore $d>0$ and $\rho(gh)$ has an eigenvalue -1. Then $\chi(gh)$ is rational, so $V_\omega=W_\omega$ and $g=h$.

Case 5. Suppose $o(g)=5$. The strategy used in this case is similar to that used in Case 3. By Case 2 we may suppose χ assumes only real values on $\langle g, h\rangle$. Thus $\dim(V_\xi\cap W_\lambda)=\dim(V_{\bar{\xi}}\cap W_{\bar{\lambda}})$ for any ξ, λ. We suppose $\langle g\rangle\neq\langle h\rangle$ and obtain a contradiction.

CLAIM 1. *We have* $s=t\geqq 1$. *Moreover*, $\dim(V_1\cap W_\omega)=\dim(V_\omega\cap W_1)=5^{s-1}$ *and* $\dim(V_\omega\cap W_\omega)=\dim(V_\omega\cap W_{\bar{\omega}})=2\cdot 5^{s-1}$.

PROOF OF CLAIM 1. Let $a=\dim(V_\omega\cap W_\omega)$, $b=\dim(V_\omega\cap W_{\bar{\omega}})$.

Suppose both $\chi(gh)$ and $\chi(gh^{-1})$ are irrational. Then by Lemma 3.4, $\rho(gh)$ has no eigenvalue ω^2 since $g\neq h$, and $\rho(gh^{-1})$ has no eigenvalue ω^2 since $g\neq h^{-1}$. Therefore $V_\omega\oplus V_{\bar{\omega}}\subseteqq W_1$ and $W_\omega\oplus W_{\bar{\omega}}\subseteqq V_1$. Hence $\chi(gh)$ is irrational and $\rho(gh)$ has ω as an eigenvalue with multiplicity 5^s+5^t, contradicting Lemma 3.4. Therefore $\chi(gh)$ or $\chi(gh^{-1})$ is rational.

Suppose $\chi(gh)$ is rational and $\chi(gh^{-1})$ is irrational. We have $b=0$ since $g\neq h^{-1}$. Then the multiplicity of ω and ω^2 as eigenvalues of $\rho(gh)$ are 5^s+5^t-2a and a, respectively, so $5^s+5^t=3a$. However, $a\leqq\dim(V_\omega)=5^s$ and $a\leqq\dim(W_\omega)=5^t$, so we have a contradiction. Similarly, a contradiction is obtained if $\chi(gh)$ is irrational and $\chi(gh^{-1})$ is rational.

Therefore $\chi(gh)$ and $\chi(gh^{-1})$ are both rational. Comparing multiplicities of ω and ω^2 as eigenvalues of $\rho(gh)$ and $\rho(gh^{-1})$, we see $5^s+5^t-2(a+b)=a=b$, and hence $s=t$ and $a=b=2\cdot 5^{s-1}$. It follows that $\dim(V_1\cap W_\omega)=\dim(V_\omega\cap W_1)=5^{s-1}$, and the claim is proved. $\square$

Note $\langle g, h\rangle\cap X=\{g^{\pm 1}, h^{\pm 1}\}$ by Claim 1. Suppose now that $x\in C_G(\langle g, h\rangle)$ and $\chi(x)$ is irrational. If $o(x)>5$, then $\langle g\rangle=\langle h\rangle\subseteqq\langle x\rangle$ by Lemma 3.13, a contradiction. If $o(x)=5$ and $\chi(x)$ is not real, then $\langle g\rangle=\langle x\rangle=\langle h\rangle$ by Case 2, another contradiction. Therefore $\chi(x)$ is real, and hence $\chi(x)$ is an algebraic conjugate of $\chi(g)$.

CLAIM 2. $X\cap C_G(\langle g, h\rangle)\subseteqq\langle g, h\rangle$.

PROOF OF CLAIM 2. Suppose to the contrary that $x\in X\cap C_G(\langle g, h\rangle)$ and $x\notin\langle g, h\rangle$. Let U_ξ be the ξ-eigenspace of $\rho(x)$, and put $d_{\alpha,\beta,\gamma}=\dim(V_\alpha\cap W_\beta\cap U_\gamma)$. Then $d_{\alpha,\beta,\gamma}=d_{\bar\alpha,\bar\beta,\bar\gamma}$ since χ assumes only real values on $C_G(\langle g, h\rangle)$. Applying Claim 1 to the pairs $\{g, h\}$, $\{g, x\}$ and $\{h, x\}$, we find

$$\sum_\xi d_{\alpha,\beta,\xi}=\sum_\xi d_{\alpha,\xi,\beta}=\sum_\xi d_{\xi,\alpha,\beta}=5^{s-1}\quad\text{if } \{\alpha, \beta\}=\{1, \omega\} \text{ or } \{1, \bar\omega\}, \tag{3.6}$$

and

$$\sum_\xi d_{\alpha,\beta,\xi}=\sum_\xi d_{\alpha,\xi,\beta}=\sum_\xi d_{\xi,\alpha,\beta}=2\cdot5^{s-1}\quad\text{if } \{\alpha, \beta\}=\{\omega\}, \{\bar\omega\}, \text{or } \{\omega, \bar\omega\}, \tag{3.7}$$

where ξ in the sums ranges over $\{1, \omega, \bar\omega\}$. It follows that $d_{1,1,\omega}=d_{1,\omega,1}$ since $d_{1,1,\omega}+d_{1,\omega,\omega}+d_{1,\bar\omega,\omega}=d_{1,\omega,1}+d_{1,\omega,\omega}+d_{1,\omega,\bar\omega}=5^{s-1}$ and $d_{1,\bar\omega,\omega}=d_{1,\omega,\bar\omega}$. Similarly, $d_{1,1,\omega}=d_{\omega,1,1}$.

Suppose that $\chi(ghx)$ is irrational. If ω is an eigenvalue of $\rho(ghx)$, then $d_{\omega,1,\omega}=d_{\omega,\omega,\omega}=0$ by Lemma 3.4, so $d_{\omega,\omega,\bar\omega}=2\cdot5^{s-1}$ by (3.7). Similarly $d_{\omega,\bar\omega,\omega}=d_{\bar\omega,\omega,\bar\omega}=2\cdot5^{s-1}$. Then $d_{1,\omega,\bar\omega}+d_{\omega,\omega,\bar\omega}+d_{\bar\omega,\omega,\bar\omega}\geqq4\cdot5^{s-1}$, contradicting (3.7). On the other hand, if ω^2 is an eigenvalue of $\rho(ghx)$, then $d_{1,\omega,\bar\omega}=2\cdot5^{s-1}$ since $d_{1,\omega,\bar\omega}+d_{\omega,\omega,\bar\omega}+d_{\bar\omega,\omega,\bar\omega}=2\cdot5^{s-1}$ and $d_{\omega,\omega,\bar\omega}=d_{\bar\omega,\omega,\bar\omega}=0$. However, $d_{1,\omega,1}+d_{1,\omega,\omega}+d_{1,\omega,\bar\omega}=5^{s-1}$, so we have a contradiction.

Therefore $\chi(ghx)$ is rational, so

$$3d_{1,1,\omega}+d_{\omega,\omega,\bar\omega}+d_{\omega,\bar\omega,\omega}+d_{\bar\omega,\omega,\omega}=d_{1,\omega,\omega}+d_{\omega,1,\omega}+d_{\omega,\omega,1}+d_{\bar\omega,\bar\omega,\bar\omega}.$$

From this together with (3.6) and (3.7) it follows that $d_{1,1,\omega}=0$.

Now suppose $\chi(ghx^2)$ is irrational. If $\rho(ghx^2)$ has an eigenvalue ω, then a contradiction arises. Therefore ω^2 is an eigenvalue of $\rho(ghx^2)$. It follows that $d_{\alpha,\beta,\gamma}=0$ or $d_{\alpha,\beta,\gamma}=5^{s-1}$ for all α, β, γ. Moreover, if I is the set of all triples (α, β, γ) for which $d_{\alpha,\beta,\gamma}=5^{s-1}$, then

$$\begin{aligned} I=\{&(1, \omega, \omega), (\omega, 1, \omega), (\omega, \omega, 1), (\omega, \omega, \bar\omega), (\omega, \bar\omega, \omega), (\omega, \bar\omega, \bar\omega),\\ &(1, \bar\omega, \bar\omega), (\bar\omega, 1, \bar\omega), (\bar\omega, \bar\omega, 1), (\bar\omega, \bar\omega, \omega), (\bar\omega, \omega, \bar\omega), (\bar\omega, \omega, \omega)\}.\end{aligned}$$

Using an argument similar to that used in the proof of Claim 2 of Case 3, it can be shown that if $y\in X\cap C_G(\langle g, h, x\rangle)\setminus\langle g, h\rangle$ and $\chi(ghy^2)$ is irrational, then either $y=x$ or $y=g^2h^2x^4$.

Now assume $\chi(ghx^2)$ is rational. Then $d_{\alpha,\beta,\gamma}=0$ or $d_{\alpha,\beta,\gamma}=5^{s-1}$ for all α, β, γ. Moreover, if I is defined as above, then one of the following holds.

$$\begin{aligned} I=\{&(1, \omega, \bar\omega), (\omega, 1, \omega), (\omega, \omega, \omega), (\omega, \omega, \bar\omega), (\omega, \bar\omega, 1), (\omega, \bar\omega, \bar\omega),\\ &(1, \bar\omega, \omega), (\bar\omega, 1, \bar\omega), (\bar\omega, \bar\omega, \bar\omega), (\bar\omega, \bar\omega, \omega), (\bar\omega, \omega, 1), (\bar\omega, \omega, \omega)\}\end{aligned} \tag{3.8}$$

$$I = \{(1, \omega, \bar{\omega}), (\omega, 1, \bar{\omega}), (\omega, \omega, 1), (\omega, \omega, \omega), (\omega, \bar{\omega}, \omega), (\omega, \bar{\omega}, \bar{\omega}),$$
$$(1, \bar{\omega}, \omega), (\bar{\omega}, 1, \omega), (\bar{\omega}, \bar{\omega}, 1), (\bar{\omega}, \bar{\omega}, \bar{\omega}), (\bar{\omega}, \omega, \bar{\omega}), (\bar{\omega}, \omega, \omega)\} \tag{3.9}$$

$$I = \{(1, \omega, \omega), (\omega, 1, \bar{\omega}), (\omega, \omega, \omega), (\omega, \omega, \bar{\omega}), (\omega, \bar{\omega}, 1), (\omega, \bar{\omega}, \omega),$$
$$(1, \bar{\omega}, \bar{\omega}), (\bar{\omega}, 1, \omega), (\bar{\omega}, \bar{\omega}, \bar{\omega}), (\bar{\omega}, \bar{\omega}, \omega), (\bar{\omega}, \omega, 1), (\bar{\omega}, \omega, \bar{\omega})\}. \tag{3.10}$$

Define $x_0=g^3h^4x^3$ if (3.8) holds, $x_0=x^{-1}$ if (3.9) holds, and $x_0=g^3h^4x^2$ if (3.10) holds. Then $x_0\in C_G(\langle g, h\rangle)\cap X\setminus\langle g, h\rangle$ and $\chi(ghx_0^2)$ is irrational.

Replacing x by x_0 if necessary, we may suppose $\chi(ghx^2)$ is irrational. Let $H=C_G(\langle g, h, x\rangle)$, and put $X_0=X\cap H$. Then

$$X_0 = \{g^{\pm 1}, h^{\pm 1}, x^{\pm 1}, (g^4h^2x^2)^{\pm 1}, (g^2h^4x^2)^{\pm 1}, (g^2h^2x^4)^{\pm 1}\}.$$

Let $N=N_G(H)$, and let N_5 be a Sylow 5-subgroup of N. If $y\in N_5\setminus H$, then y commutes with an even number of elements of X_0, and thus y commutes with exactly 2 elements of X_0. Therefore $|N_5 : H|\leqq 5$. Arguing as in the proof of Claim 2 of Case 3, we see $N_5\cap X=X_0$, so $N_5\in\mathrm{Syl}_5(G)$ and $|G|_5=|N_5|\leqq 5|H|$. Also, $|H|$ divides 5^{2s+1} since χ assumes only rational values on $H\setminus\langle g, h, x\rangle$. Since (G, χ) is sharp and χ assumes the values $\chi(gh)=n-2\cdot 5^s$, $\chi(gh^2)=n-3\cdot 5^s$, $\chi(g)=n-2\cdot 5^s+5^s(\omega+\bar{\omega})$, and $\chi(g^2)=n-2\cdot 5^s+5^s(\omega^2+\bar{\omega}^2)$, $|G|_5$ is divisible by 5^{4s+1}. Therefore $4s+1\leqq 2s+2$, which is impossible since $s\geqq 1$. This completes the proof of Claim 2. □

Put $H=C_G(\langle g, h\rangle)$, $X_0=H\cap X$. By Claim 2 we have $X_0=\{g^{\pm 1}, h^{\pm 1}\}$. Thus $H\in\mathrm{Syl}_5(G)$. Arguing as in Case 3, we see $|H|\leqq 5^{2s}$ since χ assumes rational values on $H\setminus\langle g, h\rangle$. As in the proof of Claim 2, we have $|G|_5\geqq 5^{4s+1}$. Therefore $4s+1\leqq 2s$, a contradiction. Thus $\langle g\rangle=\langle h\rangle$, as claimed.

Case 6. Suppose $o(g)=8$. The strategy used in this case is similar to that used in Case 3 and Case 5. We suppose $h\in C_G(g)\setminus\langle g\rangle$ and arrive at a contradiction. By Case 2 we may assume χ assumes only real values on $\langle g, h\rangle$. We may also assume ω has been chosen so that $\omega+\bar{\omega}=\sqrt{2}$.

CLAIM 1. *We have* $s=t\geqq 1$ *and* $\dim(V_\omega\cap W_\omega)=\dim(V_{\bar{\omega}}\cap W_\omega)=\dim(V_\omega\cap W_{\bar{\omega}})=\dim(V_{\bar{\omega}}\cap W_{\bar{\omega}})=2^{s-1}$.

PROOF OF CLAIM 1. Put $a=\dim(V_\omega\cap W_\omega)$, $b=\dim(V_\omega\cap W_{\bar{\omega}})$. If $a=b=0$, then $\chi(gh^4)$ is irrational and $\rho(gh^4)$ has an eigenvalue -1, contradicting Lemma 3.4. Thus $a>0$ or $b>0$. If $a>0$, then $\rho(gh^3)$ has -1 as an eigenvalue, so $\chi(gh^3)=n-2^{s+1}-2^{t+1}+2b+(2^s-2^t)\sqrt{2}$ is rational, and hence $s=t$. Similarly, $s=t$ if $b>0$. Therefore $s=t$ in both cases. In particular, $\chi(g)=\chi(h)=n-2^{s+1}+2^s\sqrt{2}$.

Now, if $a>0$, then $\rho(gh)$ has ω^2 and $-\omega^2$ as eigenvalues, so

$$\chi(gh) = n-2^{s+2}+2(a+b)+2(2^s-a-b)\sqrt{2}+a(\omega^2+\bar{\omega}^2)+2b$$

is rational, and hence $a+b=2^s$. Similarly, $a+b=2^s$ if $b>0$. Then

$$\chi(g^2h) = n-2^{s+1}+(b-a)\sqrt{2}.$$

If $\chi(g^2h)$ is irrational, then $b-a=2^s$ or $b-a=-2^s$, so $h=g^{-1}$ or $h=g$, contrary to our assumptions. Therefore $\chi(g^2h)$ is rational, so $a=b=2^{s-1}$. $\square$

Suppose now that $x\in C_G(\langle g, h\rangle)$ and $\chi(x)$ is irrational. If $o(x)<o(g)$, then $x\in\langle g\rangle$ by Lemma 3.13, and we have a contradiction since χ is rational on $\langle g\rangle\setminus\langle g^2\rangle$. If $o\langle x\rangle>o(g)$, then $\langle g\rangle=\langle h\rangle\subseteqq\langle x\rangle$ by Lemma 3.13 and we are done. Finally, if $o(x)=o(g)=8$, then $\chi(x)$ is an algebraic conjugate of $\chi(g)$ by Claim 1. Hence χ has only one algebraic conjugacy class of irrational values on $C_G(\langle g, h\rangle)$.

CLAIM 2. $C_G(\langle g, h\rangle)\cap X \subseteqq \langle g, h\rangle$.

PROOF OF CLAIM 2. Suppose to the contrary that $x\in C_G(\langle g, h\rangle)\cap X\setminus\langle g, h\rangle$. By Claim 1, the eigenvalues of $\rho(ghx)$ are elements of $\{1, \omega^3, \omega, \bar{\omega}, \bar{\omega}^3\}$. Let d be the multiplicity of ω^3 as an eigenvalue of $\rho(ghx)$. Applying Claim 1 to all pairs in $\{g, h, x\}$, we see that the multiplicities of $\omega, \bar{\omega}$, and $\bar{\omega}^3$ as eigenvalues of $\rho(ghx)$ are $3(2^{s-1}-d)$, $3d$, and $2^{s-1}-d$, respectively. Hence $\chi(ghx)=n-2^{s+1}+2d\bar{\omega}+2(2^{s-1}-d)\omega$. It follows that $s\geqq 2$ and $d=2^{s-2}$ since $\chi(ghx)\in\boldsymbol{R}$. Therefore $\chi(ghx)=n-2^{s+1}+2^{s-1}\sqrt{2}$, which is impossible since χ has a single algebraic conjugacy class of irrational values on $C_G(\langle g, h\rangle)$. $\square$

Now, it follows easily from Claim 1 that $\langle g\rangle\cap\langle h\rangle=\langle g^4\rangle$ and $\langle g, h\rangle$ has order 32. Put $K=C_G(\langle g, h\rangle)$, so $K\cap X\subseteqq\langle g, h\rangle$ by Claim 2. Considering eigenvalues, we see that $K\cap X=\{g, g^{-1}, h, h^{-1}\}$. Let $N=N_G(K)$. Then N acts on $K\cap X$ by conjugation, and N/K is isomorphic to a 2-subgroup of S_4. Hence $|N|$ divides $8|K|$. Now, let θ be the character of K afforded by V_ω, let W_ω be the ω-eigenspace of $\rho(h)$, and let ψ be the character of K afforded by $V_\omega\cap W_\omega$. Suppose $x\in K\setminus\langle g, h\rangle$. Then χ assumes only rational values on $x\langle g, h\rangle$ by Claim 2, so $\theta_\omega(x)=0$ and $\theta_\omega(ghx)=0$. Thus

$$0 = \theta_\omega(ghx) = \psi(x)\omega^2+(\theta_\omega(x)-\psi(x))\cdot 1 = \psi(x)(\omega^2-1),$$

and therefore $\psi(x)=0$. Since ψ vanishes on $K\setminus\langle g, h\rangle$, we have

$$|K|(\psi, \psi)_K = 32(2^{s-1})^2 = 2^{2s+3},$$

and therefore $|K|$ is a divisor of 2^{2s+3}. Hence $|N|$ is a divisor of $8\cdot 2^{2s+3}=2^{2s+6}$. If $x\in N\setminus K$, then $\rho(x)$ has a nontrivial orbit on the set of eigenspaces of $\langle g, h\rangle$ on V, so $\rho(x)$ has at least one pair of eigenvalues ζ, $-\zeta$, and thus $\chi(x)$ is rational by Lemma 3.4. Hence $N\cap X=K\cap X$, so $N_G(N)=N$ and $N\in\mathrm{Syl}_2(G)$.

Now, χ assumes the values $\chi(g^4)=n-2^{s+2}$, $\chi(g^2)=n-2^{s+1}$, $\chi(g)=n-2^{s+1}+2^s\sqrt{2}$, $\chi(g^3)=n-2^{s+1}-2^s\sqrt{2}$, $\chi(gh)=n-2^s$, and $\chi(g^5h)=n-3\cdot 2^s$ on K. Since

(G, χ) is sharp and $N \in \mathrm{Syl}_2(G)$, $|N|$ is divisible by 2^{6s+4}. Therefore $6s+4 \leqq 2s+6$, and hence $s=0$, contradicting Claim 1. Therefore $\langle g \rangle = \langle h \rangle$, and the proof of the lemma is complete. □

LEMMA 3.15. *Let p be prime. Assume g is a p-element of G, $\chi(g)$ is irrational, and $C_G(g)$ is a p-group. Suppose further that $\langle g \rangle = \langle h \rangle$ whenever $\langle g \rangle \subseteqq \langle h \rangle$ and $\chi(h)$ is irrational. Then $C_G(g) = \langle g \rangle$. Moreover, any nonidentity eigenvalue of $\rho(g)$ occurs with multiplicity* 1.

PROOF. Put $H = C_G(g)$, $N = N_G(H)$, $X = \{x \in G \mid \chi(x) = \chi(g)\}$. If $\chi(g)$ is not real, then $H \cap X = \{g\}$ by Lemma 3.14, and hence $N = H \in \mathrm{Syl}_p(G)$. On the other hand, if $\chi(g)$ is real, then $H \cap X \subseteqq \{g, g^{-1}\}$, so $|N : H| \leqq 2$, and thus $N \in \mathrm{Syl}_p(G)$ if $p=2$ and $H \in \mathrm{Syl}_p(G)$ if $p>2$. Let ω be a nonidentity eigenvalue of $\rho(g)$, and let p^s be the multiplicity of ω. Put $p^l = o(g)$.

Suppose $h \in H$ and $\chi(h)$ is irrational. If $o(h) > o(g)$, then $\langle g \rangle \subset \langle h \rangle$ by Lemma 3.13, and we have a contradiction. Therefore $o(h) \leqq o(g)$, and so $h \in \langle g \rangle$ by Lemmas 3.13 and 3.14. Hence χ assumes rational values on $H \setminus \langle g \rangle$. Let θ be the character of H afforded by the ω-eigenspace of $\rho(g)$. Then θ vanishes on $H \setminus \langle g \rangle$, so $|H|$ divides $|H| \cdot (\theta, \theta)_H = p^{2s+l}$. Therefore if $s=0$, then $H = \langle g \rangle$ and we are done.

Let M be the set of values of χ on $\langle g \rangle \setminus \{1\}$.

Case 1. Suppose $\chi(g)$ is not real. In this case we have

$$f_M(n) = \prod_{j=1}^{l} p^{s\varphi(p^j)+1} = p^{s(p^l-1)+l}.$$

Since (G, χ) is sharp, $|H| = |G|_p$ is divisible by $f_M(n)$, and therefore $s(p^l-1)+l \leqq 2s+l$. Thus either $s=0$ and we are done, or else $p^l=3$. Suppose $s>0$ and $p^l=3$. If $H \neq \langle g \rangle$, then χ assumes a rational value on $H \setminus \langle g \rangle$, and therefore $|G|_3$ is divisible by $3f_M(n)$, a contradiction. Therefore $H = \langle g \rangle$. But then $|G|_3 = 3$, and hence $s=0$.

Case 2. Suppose $\chi(g)$ is real. If $p=2$, then

$$f_M(n) = 2^{s+2} \cdot 2^{s+1} \cdot \prod_{j=3}^{l} 2^{s\varphi(2^j)/2+1} = 2^{2^{l-1}s+l+1}.$$

Since $|G|_2$ is divisible by $f_M(n)$ and $|G|_2 \leqq 2|H|$, we have $2^{l-1}s+l+1 \leqq 2s+l+1$, so $s=0$ since $l \geqq 3$ and we are done. Suppose $p>2$. Then

$$f_M(n) = \prod_{j=1}^{l} p^{s\varphi(p^j)/2+1} = p^{s(p^l-1)/2+l}.$$

Since $|G|_p = |H|$, we have $s(p^l-1)/2+l \leqq 2s+l$, so either $s=0$ and we are done, or else $p^l=5$. Suppose $s>0$ and $p^l=5$. If $H \neq \langle g \rangle$, then χ assumes a rational value on $H\setminus\langle g\rangle$, so $|G|_5\geqq 5f_M(n)$, and we have a contradiction. Thus $H=\langle g\rangle$, so $s=0$. This completes the proof of the lemma. □

LEMMA 3.16. *Suppose $\chi(x)$ is irrational and either x is χ-singular or else x has prime power order. If $g\in\langle x\rangle\setminus\{1\}$ and either $\chi(x)$ is not real or $o(g)>2$, then $C_G(g)=C_G(x)$.*

PROOF. It suffices to show that the eigenspaces of $\rho(x)$ and $\rho(g)$ coincide. If $\chi(x)$ is not real, then $\rho(x)$ has only one nonidentity eigenvalue, and hence we are done. If $\chi(x)$ is real, then $\rho(x)$ has two complex conjugate nonidentity eigenvalues, so $\rho(g)$ has two distinct nonidentity eigenvalues since $o(g)>2$, as required. □

PROOF OF THEOREM 2.2. Assume $g\in G$ and $\chi(g)$ is irrational. We claim that it suffices to prove $g\in C_G(x)=\langle x\rangle$ for some x such that $\chi(x)$ is irrational and either x is χ-singular or x has prime power order. Indeed, if this is the case then $C_G(g)=C_G(x)=\langle x\rangle$ by Lemma 3.16, so (i) holds. Also, (ii) holds by Lemma 3.5 if x is χ-singular, and (ii) holds by Lemma 3.15 if x has prime power order. Since (iii) is a consequence of (ii), the proof is complete.

Now, suppose g commutes with some χ-singular element x_0. Let $\langle x\rangle$ be maximal such that x is χ-singular and $\langle x_0\rangle\subseteqq\langle x\rangle$. Then $C_G(x)=\langle x\rangle$ by Lemma 3.11. Since $x_0\in\langle x\rangle$ and x_0 is χ-singular, we have $g\in C_G(x_0)=C_G(x)=\langle x\rangle$ by Lemma 3.16, and we are done by the claim.

Next, suppose $C_G(g)$ contains no χ-singular element of G. Suppose g is a p-element of G, where p is prime. Then $C_G(g)$ is a p-group by Lemma 3.12. Let $\langle x\rangle$ be maximal such that $\chi(x)$ is irrational and $\langle g\rangle\subseteqq\langle x\rangle$. Then x is a p-element, and $g\in C_G(x)=\langle x\rangle$ by Lemma 3.15, so again we are done by the claim.

Finally, suppose $C_G(g)$ contains no χ-singular element of G and g does not have prime power order. Then $\chi(g)=\chi(y)$ for some element y of G of order q, where $q=3$ if $\chi(g)$ is not real and $q=5$ if $\chi(g)$ is real by Lemma 3.5. Since y has prime order, we may apply the argument above to conclude $y\in C_G(x)=\langle x\rangle$, where $\chi(x)$ is irrational and either x is χ-singular or else x has prime power order. If X is the conjugacy class of x in G, then $X\cap\langle x\rangle\subseteqq\{x, x^{-1}\}$. Therefore $\langle x\rangle$ contains a Sylow q-subgroup of G, so we can assume the q-part g_q of g is an element of $\langle x\rangle$. Note $y\in\langle g_q\rangle$ since $o(y)=q$, and hence $g\in C_G(g_q)\subseteqq C_G(y)=\langle x\rangle$ by Lemma 3.16, and the proof of Theorem 2.2 is complete. □

References

[1] D. Alvis, On finite groups admitting certain sharp characters with irrational values, Comm. Algebra, 21 (1993), 535-554.

[2] D. Alvis, M. Kiyota, H. Lenstra and S. Nozawa, Sharp characters with only one rational value, Comm. Algebra, **22** (1994), 95–115.
[3] H. F. Blichfeldt, A theorem concerning the invariants of linear homogeneous groups with some applications to substitution-groups, Trans. Amer. Math. Soc., **5** (1904), 461-466.
[4] P. J. Cameron and M. Kiyota, Sharp characters of finite groups, J. Algebra, **115** (1988), 125–143.
[5] T. Ito and M. Kiyota, Sharp permutation groups, J. Math. Soc. Japan, **33** (1981), 435–444.
[6] T. Kataoka, Torsion part of $(R(G)/\boldsymbol{Z}\rho_G)^*$, preprint.
[7] T. Kataoka, Virtual characters of a finite group having prescribed values, preprint.
[8] T. Kataoka, On sharp characters of a finite group in the sense of Kiyota-Cameron, preprint.
[9] M. Kiyota and S. Nozawa, Sharp characters whose values at non-identity elements are 0 and a family of algebraic conjugates, J. Algebra, **161** (1993), 216–229.

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