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Sector theory and automorphisms for factor-subfactor pairs

Dedicated to Professor Masamichi Takesaki on his sixtieth birthday

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1. Introduction.

The index theory ([**19**]) for II_1-factors was initiated by Jones about ten years ago. Since then tremendous progress has been made in the subject matter. Especially, classification of subfactors with small indices in the AFD (II_1) factors is of particular interest (see [**41, 42, 45, 46**] and also [**18, 21**]), and this makes it possible to study automorphisms for factor-subfactor pairs in details (see for example [**22, 33, 47**]).

On the other hand, the notion of an index has been generalized to wider classes of operator algebras (for example [**25, 37, 52**]). In Longo's approach on index theory ([**37, 38**]) for factors of type III, the notion of a sector plays a fundamental role. This notion originally occurred in Quantum Field Theory, and it has been proved extremely useful by recent works of Izumi and Longo ([**14, 15, 16, 39, 40**]).

In our previous papers [**1, 28**], we saw that sectors are also useful to analyze automorphisms for factor-subfactor pairs. Let $M \supseteqq N$ be a factor-subfactor pair (with finite index), and $\theta \in Aut(M, N)$ be an automorphism for the pair. Let $\{M_k\}_{k=0,1,2,\dots}$ be the Jones tower, and we assume that θ is already extended to the tower in the canonical way. Then, θ is called strongly outer ([**1**]) if, for $x \in M_k$, the commutation relation $yx = x\theta(y)$ for all $y \in N$ forces $x=0$, and in ([**28**]) we saw that the strong outerness is characterized by making use of relevant sectors. Namely, θ is strongly outer if and only if it does not appear (as an irreducible component) in $\sqcup_k(\rho\bar{\rho})^k$, where ρ is a sector (or an endomorphism) satisfying $N=\rho(M)$ (see § 3 for details). In terms of bimodules naturally attached to the inclusion $M \supseteqq N$ in the Ocneanu approach ([**41, 42**]), this condition means that the $M-M$ bimodule canonically determined

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by θ does not appear in $\bigsqcup_k {}_M L^2(M_k)_M$. Note that, when $M=N$, this notion reduces to the usual outerness. This concept was independently considered by Popa and played an important role in his analysis on group actions ([**47**]). It should be also mentioned that this concept is closely related to Kawahigashi's work on $\chi(M, N)$ ([**22**]).

The purpose of the present article is to obtain further applications of the sector technique to the study of automorphisms for pairs. The first two sections §2 and §3 are preliminaries on sectors and strongly outer automorphisms respectively. In §4 we will determine when extended modular automorphisms in the sense of Connes-Takesaki ([**2**]) appear in $\bigsqcup_k(\rho\bar{\rho})^k$. Note that for factors of type III_λ ($\lambda\neq 0$) an extended modular automorphism simply means a modular automorphism and that the containment of such an automorphism in $\bigsqcup_k(\rho\bar{\rho})^k$ is very important for analysis on subfactors ([**16**]). In §5 we consider strongly free automorphisms (for pairs) in the sense of Winsløw ([**54**]). The strong freeness means that a similar property to the strong outerness is required at the level of the von Neumann algebras of type II_∞ (appearing in the structure analysis for factors of type III). This algebraic property is important when one deals with automorphisms for pairs in the type III setting. In fact, as was shown in [**54**], this property corresponds to the non-central triviality ([**24, 51**]) or the non-pointwise innerness ([**9, 10, 11**]) in the analysis on automorphisms on a single factor. We show that the strong freeness is stronger than the above strong outerness. Therefore, it is plain to see that the composition of an extended modular automorphism and a non-strongly outer automorphism is non-strongly free. In §6 we prove the converse in type III_λ ($\lambda\neq 0$) case. Therefore, in this case, an automorphism is non-strongly free exactly when it is the composition of a modular automorphism and a non-strongly outer automorphism. This result may be considered as a "subfactor version" of [**10**].

Basic facts on index theory can be found in [**6, 19, 25, 37, 38, 43**] while our basic reference for the modular theory and structure analysis on factors of type III is [**50**]. Results in the present article were announced in [**28, 29**].

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2. Sectors.

In this section we briefly recall basic facts on sectors, and further details can be found for example in [**14, 15, 38, 40**].

Throughout the article, let M be a properly infinite factor with a subfactor

N, and $E: M\mapsto N$ be a normal conditional expectation with $Ind\, E<\infty$ ([**25, 37**]). In the rest of the article we will also assume that M and N are isomorphic factors so that one can find an endomorphism ρ ($\in End(M)$, the unital normal $*$-endomorphisms of M) satisfying $N=\rho(M)$. As was pointed out for example in Remark (iii) after Theorem 3, [**28**], the assumption that $M\supseteq N$ are isomorphic properly infinite factors can be removed by standard tricks. Consequently, our main results below of course remain valid without this assumption.

Let $Sect(M)=End(M)/Int(M)$, the sectors, and we denote the class of $\rho\in End(M)$ by $[\rho]$. However, in most cases below, no confusion occurs and we will simply write ρ instead of $[\rho]$. For a sector ρ we define its statistical dimension $d\rho$ by

$$d\rho=\sqrt{[M:\rho(M)]_0},$$

where $[\cdot:\cdot]_0$ means the minimal index ([**12, 13, 37**]). Notice that $d\rho=1$ if and only if ρ is an automorphism of M. Throughout the article we will deal with sectors with finite statistical dimension. In the usual way, one can define the sum and the product of sectors:

$$\rho_1\oplus\rho_2,\quad \rho_1\rho_2.$$

The latter is just (the class of) the composition of endomorphisms while the former is the composition of

$$x\in M\mapsto\begin{pmatrix}\rho_1(x) & 0\\ 0 & \rho_2(x)\end{pmatrix}\in M\otimes M_2(\boldsymbol{C})$$

followed by the usual isomorphism $M\otimes M_2(\boldsymbol{C})\cong M$. (Notice that the class of the composition does not depend upon the choice of isometries realizing the second isomorphism.) The additivity of the square root ([**37**]) and the multiplicativity ([**30, 39**]) of the minimal index mean:

$$d(\rho_1\oplus\rho_2)=d\rho_1+d\rho_2,\quad d(\rho_1\rho_2)=d\rho_1 d\rho_2.$$

When $M\cap\rho(M)'=\boldsymbol{C}1$, ρ is called irreducible. If ρ is not irreducible (but $d\rho<\infty$), by using minimal projections in the finite dimensional algebra $M\cap\rho(M)'$, the intertwiners, one can obtain the irreducible decomposition

$$\rho=\rho_1\oplus\rho_2\oplus\cdots\oplus\rho_n.$$

This is completely analogous to the situation in the representation theory of (finite) groups, and for example the Frobenius reciprocity remains valid for sectors (see for example [**3, 5, 8, 28, 40, 56, 57, 58**]).

The conjugate sector $\overline{[\rho]}=[\bar\rho]$ is defined as

$$\bar\rho=\rho^{-1}\circ\gamma,$$

where γ $(=Ad(J_N J_M))$ is the Longo canonical endomorphism attached to $M \supseteqq N$ ([36]). In the bimodule (or equivalently, correspondence, see [44]) picture, considering conjugate sectors corresponds to looking at contragredient bimodules while the product of sectors corresponds to the relative tensor product ([48]) of relevant bimodules. One of the reasons why the notion of a conjugate sector is important is that this is related to the Jones tower of $M \supseteqq N=\rho(M)$. Namely,

$$M \supseteqq \rho(M)\ (=N) \supseteqq \rho\bar{\rho}(M)\ (=\gamma(M)) \supseteqq \rho\bar{\rho}\rho(M) \supseteqq \cdots$$

is exactly a downward Jones tunnel. In particular, the irreducible components in $\sqcup_n(\rho\bar{\rho})^n$, $\sqcup_n(\rho\bar{\rho})^n\rho$, $\sqcup_n(\bar{\rho}\rho)^n$, and $\sqcup_n(\bar{\rho}\rho)^n\bar{\rho}$ correspond to $M-M$, $N-M$, $N-N$, and $M-N$ bimodules respectively in the Ocneanu picture ([41, 42]).

3. Strongly outer automorphisms.

In this section we consider strongly outer automorphisms for pairs introduced in [1], and results in this section were announced in [28].

Let $\theta \in Aut(M, N)=\{\alpha \in Aut(M)\,;\, \alpha(N)=N\}$ be an automorphism for the pair $M \supseteqq N=\rho(M)$, and $E: M \mapsto N$ be the minimal conditional expectation $((d\rho)^2= Ind\, E<\infty)$. The uniqueness of a minimal conditional expectation guarantees $\theta \circ E=E \circ \theta$. Hence, θ can be uniquely (subject to the condition $\theta(e_i)=e_i$, where e_i are the Jones projections) extended to the Jones tower:

$$N \subseteqq M=M_0 \subseteqq M_1 \subseteqq M_2 \subseteqq M_3 \subseteqq M_4 \subseteqq \cdots.$$

DEFINITION 1 ([1]). An automorphism $\theta \in Aut(M, N)$ is called *strongly outer* when the following condition is satisfied for each i:

If $x \in M_i$ satisfies $nx=x\theta(n)$ for all $n \in N$, then we must have $x=0$.

In [47] this property is called the proper outerness and plays an important role in Popa's analysis on actions for inclusions. (See also [22] for related topics.) In [23], a more "quantized" version than the notion of the above strong outerness is considered by Kawahigashi to analyze "paragroup actions".

We would like to have a handy criterion for the strong outerness, and for this purpose automorphisms appearing in $\sqcup_k(\rho\bar{\rho})^k$ will play important roles. Notice that, even if an irreducible sector θ with $d\theta=1$ (i.e., an automorphism of M) appears in $(\rho\bar{\rho})^{n+1}$, we may not be able to find a unitary $u \in \mathcal{U}(M)$ such that $Adu \circ \theta \in Aut(M, N)$. As was remarked in [28], the Haagerup subfactor with index $(5+\sqrt{13})/2$ is an example where this phenomenon occurs. It is also possible to construct an abundance of such examples by looking at pairs $M^H \supseteqq M^G$ of fixed point algebras for suitable (finite) group-subgroup pairs $G \supseteqq H$ ([32]). The next result was proved in [28]. Since [28] may not be widely circulated, we repeat the proof here for the reader's convenience.

PROPOSITION 2. *An automorphism $\theta\in Aut(M)$ can be adjusted to an automorphism in $Aut(M, N)$ as above if and only if there exists an automorphism $\alpha\in Aut(M)$ satisfying $\theta\circ\rho=\rho\circ\alpha$ (as sectors). Furthermore, if $\theta\prec(\rho\bar{\rho})^{n}$ $(n\geqq 1)$ and $M\supseteqq N$ is irreducible, then $\alpha\prec(\bar{\rho}\rho)^{n}$.*

PROOF. When θ belongs to $Aut(M, N)$, we obviously have

$$\theta\circ\rho(M)=\theta(N)=N.$$

Therefore, one can find an automorphism $\alpha\in Aut(M)$ satisfying $\theta\circ\rho=\rho\circ\alpha$. Conversely, when $\theta\circ\rho=\rho\circ\alpha$ as sectors, one can find a unitary $u\in M$ satisfying $Adu\circ\theta\circ\rho(m)=\rho\circ\alpha(m)$, $m\in M$. Hence, the adjusted $Adu\circ\theta$ obviously leaves N invariant.

We now assume that ρ satisfies $\theta\circ\rho=\rho\circ\alpha$. The assumption $\theta\prec(\rho\bar{\rho})^{n}$ implies $id\prec(\rho\bar{\rho})^{n}\circ\bar{\theta}=((\rho\bar{\rho})^{n-1}\rho)\overline{(\theta\circ\rho)}$. Notice that $\theta\circ\rho$ is irreducible because

$$(\theta\circ\rho(M))'\cap M=\theta(\rho(M)'\cap M)$$

is one dimensional. Therefore, the Frobenius reciprocity implies

$$\theta\circ\rho\prec(\rho\bar{\rho})^{n-1}\rho.$$

We thus conclude

$$\alpha\prec\bar{\rho}\rho\circ\alpha=\bar{\rho}\circ\theta\circ\rho\prec\bar{\rho}(\rho\bar{\rho})^{n-1}\rho=(\bar{\rho}\rho)^{n}. \qquad \square$$

In the second half of the above proof, the irreducibility of ρ was essential. But, without this assumption, $\theta\prec(\rho\bar{\rho})^{n}$ still implies $\alpha\prec(\bar{\rho}\rho)^{n+1}$. In fact, we have

$$\alpha\prec\bar{\rho}\rho\circ\alpha=\bar{\rho}\circ\theta\circ\rho\prec\bar{\rho}(\rho\bar{\rho})^{n}\rho=(\bar{\rho}\rho)^{n+1}.$$

For two properly infinite factors M, N (no inclusion is assumed for a moment), we can define the notion of an $M-N$ sector. Let $End(M, N)$ be the unital normal $*$-endomorphisms from M to N, and we say that two endomorphisms are equivalent if they are related by a unitary in N in the obvious way. The quotient space is denoted by $Sect(M, N)$ ($Sect(M, M)=Sect(M)$ in § 2), and each class in $Sect(M, N)$ is called an $M-N$ sector. This notion is sometimes more convenient than that of an $(M-M)$ sector in § 2 because one has to deal with four kinds of bimodules in the index theory, and everything in § 2 goes through for $M-N$ sectors with suitable modifications ([**15**]). Actually, this notion will be also used in the proof of the theorem below. For example, for $\eta\in Sect(M, N)$, its conjugate $\bar{\eta}\in Sect(N, M)$ is defined as

$$\bar{\eta}=\eta^{-1}\circ\gamma$$

with the canonical endomorphism γ attached to the inclusion $N \supseteq \eta(M)$, and for $\eta_1 \in Sect(M, N)$ and $\eta_2 \in Sect(P, M)$ we have

$$\overline{\eta_1 \eta_2} = \bar{\eta}_2 \bar{\eta}_1 \ (\in Sect(N, P)).$$

This is seen via a bijective correspondence between $M-N$ sectors and $M-N$ bimodules ([**15**, **38**]).

When $M \supseteq N$ and $N = \rho(M)$ ($\rho \in End(M)$ or $Sect(M)$) as usual, one can also regard ρ as a (trivial) $M-N$ sector. As an $M-N$ sector the conjugate of ρ is $\rho^{-1} \in Sect(N, M)$, the canonical endomorphism attached to $N \supseteq \rho(M) = N$ being the identity. The conjugate of $\rho^{-1} \in Sect(N, M)$ is of course $\rho \in Sect(M, N)$.

The next characterization for strongly outer automorphisms is useful.

Theorem 3. *For an automorphism θ in $Aut(M, N)$, the following two statements are equivalent*:

(i) *The strong outerness breaks at the n-th extension M_n (i.e., some $x \neq 0 \in M_n$ satisfies $nx = x\theta(n)$ for all $n \in N$).*

(ii) *The automorphism α in Proposition 2 appears in $(\bar{\rho}\rho)^{n+1}$.*

Furthermore, when $M \supseteq N$ is irreducible, the above two conditions are also equivalent to

(iii) *The automorphism θ appears in $(\rho\bar{\rho})^{n+1}$.*

Therefore, θ is strongly outer if and only if θ does not appear in $\bigsqcup_k (\rho\bar{\rho})^k$ (even without the irreducibility $\cdots$ see the paragraph after the proof of Proposition 2).

Proof. (Equivalence between (i) and (ii)) Let N_{-1} be a downward basic extension of $M \supseteq N$. Notice that M_n can be considered as the $(n+1)$-st basic extension of $N \supseteq N_{-1}$. Since $N_{-1} = \rho\bar{\rho}(M)$, we have $N_{-1} = \rho \circ \bar{\rho} \circ \rho^{-1}(N)$. Let us consider the first ρ (resp. the third ρ^{-1}) as an $M-N$ (resp. $N-M$) sector while the middle $\bar{\rho} \in Sect(M)$ should be understood in the usual way. We set $\eta = \rho \circ \bar{\rho} \circ \rho^{-1}$ ($\in Sect(N)$). Then, Proposition 4 below applied to $(N \supseteq \eta(N) = N_{-1}, \theta|_N)$ guarantees that (i) is equivalent to

$$\theta|_N \prec (\eta\bar{\eta})^{n+1}$$

as $N-N$ sectors. (As usual after an inner perturbation we may and do assume that θ leaves N_{-1} invariant.) Recall that the conjugate ($N-M$) sector of $\rho \in Sect(M, N)$ is simply ρ^{-1}. Therefore, we compute

$$\begin{aligned}
\bar{\eta} &= \overline{\rho^{-1}} \circ \bar{\bar{\rho}} \circ \bar{\rho} = \rho \circ \rho \circ \rho^{-1},\\
\eta\bar{\eta} &= \rho \circ (\bar{\rho}\rho) \circ \rho^{-1},\\
(\eta\bar{\eta})^{n+1} &= \rho \circ (\bar{\rho}\rho)^{n+1} \circ \rho^{-1}.
\end{aligned}$$

For $\rho^{-1}\circ\theta|_N\circ\rho\in Sect(M, N)\times Sect(N)\times Sect(M, N)=Sect(M)$, we compute

$$\rho^{-1}\circ\theta|_N\circ\rho=\rho^{-1}\circ\rho\circ\alpha=\alpha$$

(see Proposition 2). Therefore, the containment $\theta|_N\prec(\eta\bar{\eta})^{n+1}$ (as $N-N$ sectors) is obviously equivalent (ii).

(Equivalence between (ii) and (iii)) Proposition 2 says that (ii) implies (iii). Since $\bar{\alpha}\circ\bar{\rho}=\bar{\rho}\circ\bar{\theta}$, the converse follows from Proposition 2 again where $\{\rho, \theta, \alpha\}$ is replaced by $\{\bar{\rho}, \bar{\alpha}, \bar{\theta}\}$. □

As mentioned in [**28**] (without a proof) we also have:

PROPOSITION 4. *Let $\theta\in Aut(M)$ (not necessarily in $Aut(M, N)$) be an automorphism of M. Then there exists a non-zero x in the n-th extension M_n satisfying $mx=x\theta(m)$ for all $m\in M$ if and only if $\theta\prec(\rho\bar{\rho})^n$.*

PROOF. Let us assume that $n=2k$ and a non-zero $x\in M_n$ satisfies $mx=x\theta(m)$. By applying γ^k to the both sides, we get

$$(\rho\bar{\rho})^k(m)\gamma^k(x)=\gamma^k(x)(\rho\bar{\rho})^k\circ\theta(m).$$

This means that the two sectors $(\rho\bar{\rho})^k$ and $(\rho\bar{\rho})^k\circ\theta$ admit a non-zero intertwiner $\gamma^k(x)\in M$ and hence they are not disjoint (i.e., they contain a common irreducible component). We thus get

$$id\prec\overline{(\rho\bar{\rho})^k}(\rho\bar{\rho})^k\circ\theta=(\rho\bar{\rho})^{2k}\circ\theta.$$

Hence, we conclude

$$\bar{\theta}\prec(\rho\bar{\rho})^n \text{ and } \theta\prec(\rho\bar{\rho})^n.$$

On the other hand, when $n=2k+1$, by applying γ^{k+1} we get

$$(\rho\bar{\rho})^{k+1}(m)\gamma^{k+1}(x)=\gamma^{k+1}(x)(\rho\bar{\rho})^{k+1}\circ\theta(m).$$

Notice that $\gamma^{k+1}(x)$ is a non-zero element in N. Therefore, by setting $\tilde{x}=\rho^{-1}\circ\gamma^{k+1}(x)\neq0\in M$ and by applying ρ^{-1} to the above both sides, we get

$$(\bar{\rho}\rho)^k\bar{\rho}(m)\tilde{x}=\tilde{x}(\bar{\rho}\rho)^k\bar{\rho}\circ\theta(m).$$

Therefore, $(\bar{\rho}\rho)^k\bar{\rho}$ and $(\bar{\rho}\rho)^k\bar{\rho}\circ\theta$ are not disjoint and as above we get

$$id\prec\overline{(\bar{\rho}\rho)^k\bar{\rho}}(\bar{\rho}\rho)^k\bar{\rho}\circ\theta=(\rho\bar{\rho})^{2k+1}\circ\theta$$

and $\theta\prec(\rho\bar{\rho})^n$.

When $\theta\prec(\rho\bar{\rho})^n$, we can obviously reverse the arguments so far. □

It is also possible to prove the results in this section by making use of bimodules ([**41**, **42**, **44**, **48**, **57**]) instead of sectors (see [7]).

4. Extended modular automorphisms.

From now on we will assume that $M \supseteqq N=\rho(M)$ are factors of type III and that $M \supseteqq N$ is irreducible. The irreducibility assumption is actually superficial, and all the results below remain valid (sometimes with trivial modifications). We choose and fix a faithful normal semi-finite weight ϕ on N and we set

$$\tilde{M}=M \rtimes_{\sigma^{\phi \circ E}} \boldsymbol{R} \supseteqq \tilde{N}=N \rtimes_{\sigma^{\phi}} \boldsymbol{R}.$$

This is an inclusion of von Neumann algebras of type II_∞ (which does not depend upon the choice of ϕ thanks to Connes' Radon-Nikodym theorem). The algebras $\tilde{M} \supseteqq \tilde{N}$ may have different centers. However, as was clarified in [**26**], analysis on inclusions of type III factors is more or less reduced to that in the case that $\tilde{M}$ and $\tilde{N}$ have the identical center. Therefore, in the rest we will further assume that $\mathcal{Z}(\tilde{M})=\mathcal{Z}(\tilde{N})$. (Notice that this assumption is automatic in the type III_1 case.)

As was clarified in [**16**] modular automorphisms sometimes appear as irreducible components in $\bigsqcup_n (\rho\bar{\rho})^n$ in the type III_λ, $\lambda \neq 0$, case, and this information is essential for analysis on inclusions of type III factors. In this section we will determine when extended modular automorphisms ([**2**]) appear in $\bigsqcup_n (\rho\bar{\rho})^n$. Recall that in [**27, 31**] typical inclusions of factors of type III_0 with this property were constructed. Also our experience shows that (when one wants to include the type III_0 case) an extended modular automorphism is a more natural object to investigate. (See for example [**10, 24, 51**].) In the second half of the section we will also show that certain inclusions with different type II and type III graphs can be expressed as simultaneous crossed products (by $\boldsymbol{Z}_n$- or $\boldsymbol{T}$-actions) of pairs with the same type II and type III graphs.

Let $\{\theta_t\}_{t \in \boldsymbol{R}}$ be the dual action on $\tilde{M} \supseteqq \tilde{N}$ so that thanks to the Takesaki duality the original inclusion $M \supseteqq N$ can be identified with

$$\tilde{M} \rtimes_\theta \boldsymbol{R} \supseteqq \tilde{N} \rtimes_\theta \boldsymbol{R}.$$

Let tr be the canonical trace on $\tilde{M}$ scaled in the usual way under the dual action. ($tr|_{\tilde{N}}$ is the canonical trace on $\tilde{N}$.) After the above identification we may and do assume that the dual weights $\hat{tr}$ and $(tr|_{\tilde{N}})\hat{}$ are $\phi \circ E$ and ϕ respectively. The modular action $\sigma_t^{\phi \circ E}$ being dual to θ_t, we have $\tilde{M}=M_{\phi \circ E}$, the fixed-point algebra. (Similarly $\tilde{N}=N_\phi$.) Let $e=\{c_t\}_{t \in \boldsymbol{R}}$ be a θ-cocycle with values in the unitary group of the center $\mathcal{U}(\mathcal{Z}(\tilde{M}))=\mathcal{U}(\mathcal{Z}(\tilde{N}))$ (i.e., $c_t\theta_t(c_s)=c_{t+s}$) and $\sigma_e=\sigma_e^{\phi \circ E}$ be the associated extended modular automorphism (on M). From the definition of σ_e ([**2**]), we have

$$\sigma_e(\lambda(t))=c_t\lambda(t), \quad \sigma_e|_{\tilde{M}}=id,$$

where $\lambda(t)$ is the usual generator in $M=\tilde{M}\rtimes_{\theta}\boldsymbol{R}$ corresponding to $\boldsymbol{R}$. (In particular, $\sigma_e\in Aut(M, N)$.) To determine if σ_e appears in $\sqcup_n(\rho\bar{\rho})^n$ (see Theorem 3), we set

$$\mathcal{H}=\{x\in M_n\,;\ yx=x\sigma_e(y)\ \text{for all}\ y\in N\}.$$

Let $\tilde{M}_n$ be the n-th extension of $\tilde{M}\supseteq\tilde{N}$ (and hence $M_n=\tilde{M}_n\rtimes_{\theta}\boldsymbol{R}$). We have

$$\{x\in\tilde{M}_n\cap\tilde{N}'\,;\ \theta_t(x)=c_tx\}\subseteq\mathcal{H}.$$

In fact, let us assume that $x\in\tilde{M}_n\cap\tilde{N}'\subseteq\tilde{M}_n\subseteq M_n$ satisfies $\theta_t(x)=c_tx$ as above. For $y\in\tilde{N}$ we have $\sigma_e(y)=y$ and $yx=x\sigma_e(y)$ since x came from the relative commutant $\tilde{M}_n\cap\tilde{N}'$. On the other hand, for $y=\lambda(t)$, we have

$$\lambda(t)x=\theta_t(x)\lambda(t)=c_tx\lambda(t)=xc_t\lambda(t)=x\sigma_e(\lambda(t)),$$

where the first equality is just the covariance. Since $\tilde{N}$ and $\lambda(\boldsymbol{R})$ generate N, we conclude that x is in $\mathcal{H}$.

We now start proving the reverse inclusion. Notice that $\mathcal{H}$ is the space of intertwiners between two (finite index) sectors. In particular, $\mathcal{H}$ is a finite dimensional linear space. Notice also that the obvious commutativity $\sigma_e\circ\sigma_t^{\phi\circ E}=\sigma_t^{\phi\circ E}\circ\sigma_e$ implies the invariance $\sigma_t^{\phi\circ E}(\mathcal{H})=\mathcal{H}$, $t\in\boldsymbol{R}$.

LEMMA 5. *There exists a linear space basis* $\{x_j\}_{j=1,2,\dots,m}$ *for* $\mathcal{H}$ *such that* $\sigma_t^{\phi\circ E}(x_j)=\exp(is_jt)x_j$ $(t\in\boldsymbol{R})$ *with some* $s_j\in\boldsymbol{R}$.

PROOF. Let m be a translation invariant mean on $\boldsymbol{R}$, and we set

$$\langle x,\,y\rangle=\int_{-\infty}^{\infty}\omega(\sigma_t^{\phi\circ E}(y^*x))dm(t)=(m(\omega(\sigma_{\cdot}^{\phi\circ E}(y^*x))))$$

with a faithful normal state $\omega\in M_*^+$. Obviously $\langle\cdot,\cdot\rangle$ defines an inner product on the space $\mathcal{H}$, and we have the invariance $\langle\sigma_t^{\phi\circ E}(x),\,\sigma_t^{\phi\circ E}(y)\rangle=\langle x,\,y\rangle$ from the construction. Hence, $\{\sigma_t^{\phi\circ E}|_{\mathcal{H}}\}_{t\in\boldsymbol{R}}$ gives rise to a one-parameter family of unitaries (relative to the inner product $\langle\cdot,\cdot\rangle$) on $\mathcal{H}$. By diagonalizing the unitaries, we get a basis with the desired property. $\square$

In the rest we will show that each eigenvalue s_j is zero by looking at type III_1, III_0, and III_λ $(0<\lambda<1)$ cases separately. Since $\sigma_t^{\phi\circ E}(\lambda(-s_j))=\exp(-is_jt)\lambda(-s_j)$, $z_j=x_j\lambda(-s_j)$ satisfies $\sigma_t^{\phi\circ E}(z_j)=z_j$ and $z_j\in(M_n)_{\phi\circ E}=\tilde{M}_n$. Consequently we have

$$x_j=z_j\lambda(s_j)\ \text{with}\ z_j\in\tilde{M}_n.$$

Since $x_j\in\mathcal{H}$, for $y\in\tilde{N}\subseteq N$ we compute

$$yz_j\lambda(s_j)=z_j\lambda(s_j)\sigma_e(y)=z_j\lambda(s_j)y=z_j\theta_{s_j}(y)\lambda(s_j)$$

and hence

$$(1) \qquad y z_j = z_j \theta_{s_j}(y), \quad y \in \tilde{N}.$$

At first, when $M \supseteqq N$ are of type III_1, $\tilde{M} \supseteqq \tilde{N}$ are factors of type II_∞. We remark that the conditional expectation between these factors of type II_∞ (arising from the unique trace) is minimal as was shown in [**30**]. If s_j were non-zero, $\theta_{s_j} \in Aut(\tilde{M}, \tilde{N})$ would scale a unique (up to a scalar multiple) trace by $\exp(-s_j) \neq 1$ and hence we would conclude that θ_{s_j} is strongly outer (see 1.6 of [**47**]). Therefore, (1) would imply $z_j = 0$ and $x_j = 0$, a contradiction.

Secondly, let us assume that $M \supseteqq N$ are of type III_0 and $s_j \neq 0$. Let $(\{F_t\}_{t \in \boldsymbol{R}}, \Omega)$ be a point-map realization of the restriction of $\{\theta_t\}_{t \in \boldsymbol{R}}$ to the center (i.e., the flow of weights of M). We set

$$\Omega_0 = \{\omega \in \Omega \,;\, F_{s_j}(\omega) = \omega\}.$$

The set Ω_0 is obviously invariant under $\{F_t\}_{t \in \boldsymbol{R}}$. Hence, the ergodicity of the flow of weights shows that Ω_0 must have measure zero. Therefore, we have seen that θ_{s_j} is centrally free. Thus, we would conclude $z_j = 0$, a contradiction. In fact, if $z_j \neq 0$, then the central support $c(z_j)$ majorizes a non-zero central projection p such that $p \perp \theta_{s_j}(p)$. However, because of

$$p = c(p z_j) = c(z_j \theta_{s_j}(p)) \leqq \theta_{s_j}(p),$$

$p \perp \theta_{s_j}(p)$ is impossible.

Finally, we deal with the type III_λ $(0 < \lambda < 1)$ case. In this case $\tilde{M} \supseteqq \tilde{N}$ can be identified with $M_0 \otimes L^\infty([0, -\log \lambda)) \supseteqq N_0 \otimes L^\infty([0, -\log \lambda))$ in a very explicit way. Here, $M_0 \supseteqq N_0$ are factors of type II_∞ appearing in the discrete decomposition picture. Basic facts on this identification are proved in [**9**], and for the reader's convenience we summarize them in Appendix.

LEMMA 6. *If $M \supseteqq N$ are factors of type III_λ, then $s_j = 0$.*

PROOF. In this case the flow of weights is periodic with period $-\log \lambda$. Hence, if s_j is not in $(-\log \lambda)\boldsymbol{Z}$, then θ_{s_j} is centrally free, and the identical argument as in the type III_0 case shows $z_j = 0$.

Therefore, let us assume $s_j = n \times (-\log \lambda)$ $(n \neq 0 \in \boldsymbol{Z})$ (to show the result by contradiction). Recall that $\sigma_t^{\phi \circ E}$ is periodic with the period $t_0 = -2\pi/\log \lambda$ and hence one can construct the natural isomorphism

$$\Psi : \tilde{M} \longmapsto (M \rtimes_{\sigma^{\phi \circ E}} (\boldsymbol{R}/t_0 \boldsymbol{Z})) \otimes L^\infty([0, -\log \lambda))$$

(see Appendix). Everything here is compatible with the basic extension, etc., the isomorphism being constructed by just the Fourier analysis. Let θ_0 be the dual automorphism of the torus action $\{\sigma_t^{\phi \circ E}\}_{0 \leq t < t_0}$. The formula (8) in Appendix shows that θ_{s_j} corresponds to $\theta_0^n \otimes id_{L^\infty([0, -\log \lambda))}$ via Ψ. Via this isomorphism, z_j corresponds to an $(M_n \rtimes_{\sigma^{\phi \circ E}} (\boldsymbol{R}/t_0 \boldsymbol{Z}))$-valued function on $[0, -\log \lambda)$, and the commutation relation (1) simply means

$$y(\omega)z_j(\omega) = z_j(\omega)\theta_0^n(y(\omega)) \quad \text{a.e. } \omega$$

for each $y(\omega)\in N\rtimes_{\sigma^\phi}(\boldsymbol{R}/t_0\boldsymbol{Z})$. Since θ_0^n is trace-scaling (by λ^n) and the relevant conditional expectation between the factors of type II_∞ is minimal (see Proposition 3.1, [**33**]), θ_0^n is strongly outer. Therefore, once again we conclude $z_j(\omega)=0$ a.e. ω, and $z_j=0$, a contradiction. □

Summing up the arguments so far, we have seen $s_j=0$ (for each j) in all the cases. Therefore, $x_j=z_j$ and $\mathcal{H}\subseteq(M_n)_{\phi\circ E}=\tilde{M}_n$. Since $\sigma_e|_{\tilde{N}}=id$, we actually have $\mathcal{H}\subseteq\tilde{M}_n\cap\tilde{N}'$. Since $x\in\mathcal{H}$ satisfies

$$\theta_t(x)\lambda(t) = \lambda(t)x = x\sigma_e(\lambda(t)) = c_t x\lambda(t),$$

we have

$$\{x \in M_n\,;\ yx = x\sigma_e(y),\ y \in N\} = \{x \in \tilde{M}_n\cap\tilde{N}'\,;\ \theta_t(x) = c_t x\}.$$

From this fact and Theorem 3 we obtain:

THEOREM 7. *Assume that $M\supseteq N=\rho(M)$ is an inclusion of type III factors such that the associated inclusion $\tilde{M}\supseteq\tilde{N}$ (with the dual action $\{\theta_t\}_{t\in\boldsymbol{R}}$) of von Neumann algebras of type II_∞ has the identical center. Let $e=\{c_t\}_{t\in\boldsymbol{R}}$ $(\in\boldsymbol{Z}_\theta^1(\boldsymbol{R}, \mathcal{U}(\mathcal{Z}(\tilde{M}))))$ be a cocycle. The corresponding extended modular automorphism σ_e appears in $(\rho\bar{\rho})^{n+1}$ if and only if there exists a non-zero element x in $\tilde{M}_n\cap\tilde{N}'$ satisfying $\theta_t(x)=c_t x$, $t\in\boldsymbol{R}$.*

REMARK 8. (i) When involved factors are of type III_λ $(\lambda\neq 0)$, a cocycle $e=\{c_t\}_{t\in\boldsymbol{R}}$ is specified by just a real number s (due to the transitivity of the flow of weights): $c_t=\exp(ist)$ (a constant function) and $\sigma_e=\sigma_s^{\psi\circ E}$ in this case. Therefore, the above condition reduces to the "eigenvalue condition" $\theta_t(x)=\exp(ist)x$, $t\in\boldsymbol{R}$. (See also the arguments in [**16**].)

(ii) The condition $\theta_t(x)=c_t x$ is stable under the multiplication by a coboundary $c_t'=u^*\theta_t(u)$ $(u\in\mathcal{U}(\mathcal{Z}(\tilde{M})))$. In fact, one can replace x by $ux\neq 0$.

(iii) When $M=N$, $\theta_t(x)=c_t x$ means that e is a coboundary, and hence σ_e is inner (as expected).

In fact, the above (iii) can be seen as follows: In this case we have $\tilde{M}_k\cap\tilde{N}'=\mathcal{Z}(\tilde{M})$. The condition $\theta_t(x)=c_t x$ implies $\theta_t(|x|)=|x|$, and the central ergodicity of θ_t shows that $|x|$ is a constant $(\neq 0)$. By dividing the both sides by this constant, we may and do assume that x is a unitary in the center. Therefore, we have the desired expression $c_t=x^*\theta_t(x)$.

Let $M\supseteq N$ be an inclusion of type III_λ factors with finite-depth, and let

$$M = A\rtimes_{\theta_0}\boldsymbol{Z} \supseteq N = B\rtimes_{\theta_0}\boldsymbol{Z}$$

be the common discrete decomposition (see [**33**]). Here, $A \supseteqq B$ is an inclusion of factors of type II_∞, and we will denote the corresponding Jones tower by $\{A_n\}_{n=0,1,2,\dots}$. We also assume that the graph of $M \supseteqq N$ (the type III graph) and that of $A \supseteqq B$ (the type II graph) are different. (Cf. [**16**].) By repeating almost the same arguments as in the first half of this section, we easily get

$$\{x \in M_k\,;\ yx = x\sigma_t^{\phi\circ E}(y),\ y \in N\} = \{x \in A_k \cap B'\,;\ \theta_0(x) = \exp(2\pi it/t_0)x\}$$

with $t_0 = -2\pi/\log\lambda$. In fact, x_j in Lemma 5 satisfies $\sigma_t^{\phi\circ E}(x_j) = \exp(is_jt)x_j$ as before. Then, the periodicity shows $\sigma_{t_0}^{\phi\circ E}(x_j) = \exp(is_jt_0)x_j = x_j$, and hence $s_j = 2\pi n/t_0$, i.e., $\sigma_t^{\phi\circ E}(x_j) = \exp(2\pi int/t_0)x_j$. Therefore, we conclude $x_j = z_j l^n$ $(z_j \in A_k)$, where l is the generator in the crossed product $M_k = A_k \rtimes_{\theta_0} \boldsymbol{Z}$ corresponding to $\boldsymbol{Z}$.

LEMMA 9. *The following conditions are equivalent*:
(i) *The modular automorphisms in* $\bigsqcup_k(\rho\bar{\rho})^k$ *are* id, $\sigma_{t_0/n}^{\phi\circ E}$, $\sigma_{2t_0/n}^{\phi\circ E}$, $\cdots$, $\sigma_{(n-1)t_0/n}^{\phi\circ E}$.
(ii) *On the tower* $\bigcup_k(A_k \cap B')$, *we have* $\theta_0^n = id$ *and* $\theta_0^m \neq id$ $(m = 1, 2, \cdots, n-1)$.

PROOF. Assume (i). Since $\sigma_{t_0/n}^{\phi\circ E}$ appears in $(\rho\bar{\rho})^{k+1}$, by the above and Theorem 3 one can choose $x \neq 0 \in A_k \cap B'$ satisfying $\theta_0(x) = \exp(2\pi i/n)x$. Therefore, $\theta_0^m \neq id$ $(m < n)$ on the tower $\bigcup_k(A_k \cap B')$ of relative commutants. Since $M \supseteqq N$ has finite-depth, so does $A \supseteqq B$. (See Corollary 3.2, [**33**].) Thus, θ_0 restricted to $A_{k_0} \cap B'$ (k_0 large enough) completely determines θ_0 on the whole tower. It is easy to see that θ_0 on the tower is periodic. (Equip $A_{k_0} \cap B'$ with the inner product determined by the canonical trace on the tower. Since this trace comes from relevant conditional expectations, we can regard θ_0 as a unitary matrix. If this unitary had an eigenvalue $\exp(2\pi is)$ with an irrational s, then $\sigma_{st_0}^{\phi\circ E}$ would appear in $\bigsqcup_k(\rho\bar{\rho})^k$, which contradicts the assumption that $M \supseteqq N$ is of finite-depth.) Let n_0 be the minimal integer such that $\theta_0^{n_0} = id$ on the tower. We have already known $n_0 \geqq n$. If $n_0 > n$, then as above we find a non-zero $x \in A_k \cap B'$ (for some k) such that $\theta_0(x) = \exp(2\pi i/n_0)x$. Hence, $\sigma_{t_0/n_0}^{\phi\circ E} \prec \bigsqcup_k(\rho\bar{\rho})^k$, which contradicts (i).

The converse can be proved by almost the identical argument. □

Assume that the type III and type II graphs of $M \supseteqq N$ are different, and let

$$id,\ \sigma_{t_0/n}^{\phi\circ E},\ \sigma_{2t_0/n}^{\phi\circ E},\ \cdots,\ \rho_{(n-1)t_0/n}^{\phi\circ E}$$

be the modular automorphisms in $\bigsqcup_k(\rho\bar{\rho})^k$. We set $\beta = \sigma_{t_0/n}^{\phi\circ E}$. This is an automorphism in $Aut(M, N)$ with period n. As was clarified in [**16**], the above modular automorphisms in $\bigsqcup_k(\rho\bar{\rho})^k$ are responsible for the difference of the two graphs. Therefore, it is natural to investigate the inclusion

$$P = M \rtimes_\beta \boldsymbol{Z}_n \supseteqq Q = N \rtimes_\beta \boldsymbol{Z}_n .$$

Notice that this inclusion is conjugate to the pair of the fixed point algebras (under the $\boldsymbol{Z}_n$-action β):

$$M^{\boldsymbol{Z}_n} \supseteq N^{\boldsymbol{Z}_n}.$$

Since

$$\beta(x)=\sigma_{t_0/n}^{\phi_0\circ E}(x)=\sum a_m \exp(2\pi i m/n) l^m \quad \text{for } x=\sum a_m l^m \in M=A \rtimes_{\theta_0} \boldsymbol{Z},$$

the above pair of the fixed point algebras are actually

$$(2) \qquad A \rtimes_{\theta_0^n} \boldsymbol{Z} \supseteq B \rtimes_{\theta_0^n} \boldsymbol{Z}.$$

Hence, $P \supseteq Q$ are factors of type III_{λ^n} (since θ_0^n scales the unique trace by λ^n), and (2) shows that the type II graph is the one for the pair $A \supseteq B$ (i.e., the type II graph of $M \supseteq N$). The type III graph of $P \supseteq Q$ can be computed as the fixed point of $A_k \cap B'$ under the automorphism θ_0^n. However, the n-th power of θ_0 being trivial on the tower (Lemma 9), the type III graph of $P \supseteq Q$ is the same as the type II graph.

Let $\hat{\beta}$ be the dual action of β. By the Takesaki duality we have:

PROPOSITION 10. *The pair $M \supseteq N$ can be expressed as a pair of simultaneous crossed products $P \rtimes_{\hat{\beta}} \boldsymbol{Z}_n \supseteq Q \rtimes_{\hat{\beta}} \boldsymbol{Z}_n$. Here, $P \supseteq Q$ are factors of type III_{λ^n}. The inclusion $P \supseteq Q$ has the same type III and type II graphs (=the type II graph of $M \supseteq N$).*

A similar phenomenon with $n=2$ (in the type III_0 setting) was pointed out in § 4 of [**31**] (see also Remark 13).

It is also possible to show the above result from the following: Recall $P=M \rtimes_\beta \boldsymbol{Z}_n=(A \rtimes_{\theta_0} \boldsymbol{Z}) \rtimes_\beta \boldsymbol{Z}_n$. Since β acts trivially on A and $\beta(l)=\exp(2\pi i/n)l$, this double crossed product can be expressed as $(A \otimes \lambda(\boldsymbol{Z}_n)'') \rtimes_{\tilde{\tilde{\theta}}_0} \boldsymbol{Z}$, with $\tilde{\tilde{\theta}}_0(a \otimes l_0)=\theta_0(a) \otimes \exp(2\pi i/n) l_0$ $(a \in A)$ for the generator l_0 $(l_0^n=1)$ in the group ring $\lambda(\boldsymbol{Z}_n)$. Therefore, by performing the Fourier transform on $l^2(\boldsymbol{Z}_n)$, we end up with

$$P \cong (A \otimes l^\infty(\boldsymbol{Z}_n)) \rtimes_{\tilde{\theta}_0} \boldsymbol{Z}.$$

Here, $\tilde{\theta}_0$ is the tensor product $\theta_0 \otimes Ad\, S$, and S is the $n \times n$ shift matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Of course the same expression (with B instead of A) is valid for Q.

Actually this approach is more fitting especially when one deals with factors of type III_1 (see below). In fact, in this case $\boldsymbol{Z}$-actions are around, and a crossed product by a $\boldsymbol{Z}$-action is easier to handle than a fixed point algebra under a $\boldsymbol{T}$-action for our purpose.

We now assume that $M \supseteqq N$ are factors of type III_1 with different type III and type II graphs (so that (type III) graph has infinite-depth, [**16**]) and that the type II graph is of finite-depth. (A typical example is a locally trivial inclusion with index 4 coming from a modular automorphism ([**49**]).)

Set

$$\Theta(t) = \theta_t|_{\cup_k(\tilde{M}_k \cap \tilde{N}')}.$$

(This is a "flow of weights" for the purpose of subfactor analysis, which played an important role in [**22, 33, 34, 35, 47, 53, 54**].) Since the type II graph is of finite-depth, $\Theta(t)$ is completely determined by θ_t restricted to the finite-dimensional algebra $\tilde{M}_k \cap \tilde{N}'$ (for k large enough). We further assume that the kernel of the continuous homomorphism $\Theta : t \in \boldsymbol{R} \mapsto \Theta(t)$ is of the form $Ker\,\Theta = t\boldsymbol{Z}$ for some $t>0$ ([**35**]).

LEMMA 11. *The following conditions are equivalent*:

(i) *The modular automorphisms in* $\sqcup_k(\rho\bar{\rho})^k$ *are* $\{\sigma^{\phi\circ E}_{ns_0};\ n \in \boldsymbol{Z}\}$.

(ii) $Ker\,\Theta = (2\pi/s_0)\boldsymbol{Z}$.

PROOF. Assume $Ker\,\Theta = (2\pi/s_0)\boldsymbol{Z}$, and let k_0 be the depth of $\tilde{M} \supseteqq \tilde{N}$. Being an $\boldsymbol{R}$-action, θ_t cannot move each of the minimal central projections in $\tilde{M}_{k_0} \cap \tilde{N}'$. In each direct summand (i.e., a full matrix algebra) here, θ_t is inner. Since $Ker\,\Theta = (2\pi/s_0)\boldsymbol{Z}$, one can find a non-zero $x \in \tilde{M}_{k_0} \cap \tilde{N}'$ satisfying $\theta_t(x) = \exp(is_0t)x$. Therefore, $\sigma^{\phi\circ E}_{s_0}$ (and hence all of $\sigma^{\phi\circ E}_{ns_0}$ $(n \in \boldsymbol{Z})$) appears in $\sqcup_k(\rho\bar{\rho})^k$ (Theorem 7). On the other hand, if $\sigma^{\phi\circ E}_{s_1}$ $(0<s_1<s_0)$ appeared in $\sqcup_k(\rho\bar{\rho})^k$, we would have a non-zero x with $\theta_t(x) = \exp(is_1t)x$ and $\theta_{2\pi/s_0}(x) = \exp(2\pi is_1/s_0)x \neq x$, a contradiction. The converse can be proved similarly. □

Assume $Ker\,\Theta = (2\pi/s_0)\boldsymbol{Z}$. Let $\beta = \sigma^{\phi\circ E}_{s_0} \in Aut(M, N)$ and we set

$$P = M \rtimes_\beta \boldsymbol{Z} \supseteqq Q = N \rtimes_\beta \boldsymbol{Z}.$$

With the dual action $\hat{\beta}$ ($\boldsymbol{T}$-action), we get:

PROPOSITION 12. *The pair $M \supseteqq N$ can be expressed as a pair of simultaneous crossed products $P \rtimes_{\hat{\beta}} \boldsymbol{T} \supseteqq Q \rtimes_{\hat{\beta}} \boldsymbol{T}$. Here, $P \supseteqq Q$ are factors of type III_λ with $\lambda =$* $\exp(-2\pi/s_0)$. *The inclusion $P \supseteqq Q$ has the same type III and type II graphs (=the type II graph of $M \supseteqq N$).*

PROOF. Notice

$$P = M \rtimes_\beta \boldsymbol{Z} = (\tilde{M} \rtimes_\theta \boldsymbol{R}) \rtimes_\beta \boldsymbol{Z} = (\tilde{M} \otimes \lambda(\boldsymbol{Z})'') \rtimes_{\tilde{\theta}} \boldsymbol{R}$$

with $\tilde{\tilde{\theta}}_t(m\otimes l)=\theta_t(m)\otimes\exp(is_0t)l$ since $\beta(\lambda(t))=\exp(is_0t)\lambda(t)$ and $\beta|_{\tilde{M}}=id$. Therefore, after the Fourier transform, we have

$$P\cong(\tilde{M}\otimes L^{\infty}(\boldsymbol{T}))\rtimes_{\tilde{\theta}}\boldsymbol{R},$$

where $\tilde{\theta}_t$ is θ_t tensored with the translation by $s_0t/2\pi$ on the torus $\boldsymbol{T}=\boldsymbol{R}/\boldsymbol{Z}$ (hence the flow of weights has period $2\pi/s_0$). We also have the similar expression for Q (with $\tilde{N}$ instead).

The type II graph of $P\supseteq Q$ can be seen from the inclusion $\tilde{M}\otimes L^{\infty}(\boldsymbol{T})\supseteq\tilde{N}\otimes L^{\infty}(\boldsymbol{T})$. Since tensoring with $L^{\infty}(\boldsymbol{T})$ does not do anything, the type II graph is the one determined by $\tilde{M}\supseteq\tilde{N}$. On the other hand, the type III graph of $P\supseteq Q$ can be determined from

$$((\tilde{M}_k\otimes L^{\infty}(\boldsymbol{T}))\cap(\tilde{N}\otimes L^{\infty}(\boldsymbol{T}))')_{\tilde{\theta}}=((\tilde{M}_k\cap\tilde{N}')\otimes L^{\infty}(\boldsymbol{T}))_{\tilde{\theta}}.$$

Since $\theta_{2\pi/s_0}=id$ on $\bigcup_k(\tilde{M}_k\cap\tilde{N}')$ (Lemma 11) and the flow has period $2\pi/s_0$ on the center $L^{\infty}(\boldsymbol{T})$, it is plain to see that the type III graph is also the one determined by $\tilde{M}\supseteq\tilde{N}$. □

The assumption $Ker\,\Theta=t\boldsymbol{Z}$ $(t>0)$ is actually essential in the above argument, and the author thanks F. Hiai for pointing out this fact.

EXAMPLE. Let P be a factor of type III_1, and we consider the locally trivial inclusion

$$M=P\otimes M_3(\boldsymbol{C})\supseteq N=\left\{\begin{pmatrix}x&0&0\\0&\sigma_1(x)&0\\0&0&\sigma_r(x)\end{pmatrix};\ x\in P\right\},$$

where σ denotes the modular action as usual and r is irrational. It is elementary to see that the type II inclusion is given by

$$\tilde{M}=\tilde{P}\otimes M_3(\boldsymbol{C})\supseteq\tilde{N}=\left\{\begin{pmatrix}x&0&0\\0&\tilde{\sigma}_1(x)&0\\0&0&\tilde{\sigma}_r(x)\end{pmatrix};\ x\in\tilde{P}\right\}$$

with the dual action $\theta_t^{\tilde{M}}=\theta_t^{\tilde{P}}\otimes Id$. Here, $\tilde{\sigma}_t$ means the canonical extension (to the II_∞-factor $\tilde{P}$) of σ_t so that

$$\tilde{\sigma}_1=Ad\,l(1)\ \text{and}\ \tilde{\sigma}_r=Ad\,l(r).$$

Therefore, $\tilde{M}\supseteq\tilde{N}$ is conjugate to

$$\tilde{P}\otimes M_3(\boldsymbol{C})\supseteq\tilde{P}\otimes\boldsymbol{C}1$$

by the inner conjugation

$$Ad\begin{pmatrix} 1 & 0 & 0 \\ 0 & l(1)^* & 0 \\ 0 & 0 & l(r)^* \end{pmatrix}.$$

Under this conjugation, the dual action $\theta_t^{\hat{p}} \otimes Id$ is transformed to

$$\theta_t^{\hat{p}} \otimes Ad\begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(it) & 0 \\ 0 & 0 & \exp(irt) \end{pmatrix}.$$

Since r is irrational, we obviously have $Ker\,\Theta = \{0\}$. On the other hand, it is plain to see that (when $N = \rho(M)$)

$$\rho\bar{\rho} = 1 \oplus 1 \oplus \sigma_1 \oplus \sigma_r$$

as a sector. The group G of all one dimensional sectors in $\sqcup_k(\rho\bar{\rho})^k$ is

$$G = \{\sigma_{m+nr}\,;\ n,\ m \in \boldsymbol{Z}\}$$

(which is isomorphic to a dense subgroup in $\boldsymbol{R}$). As before, $M \supseteq N$ can be identified with

$$(M \rtimes G) \rtimes \hat{G} \supseteq (N \rtimes G) \rtimes \hat{G}.$$

However, $M \rtimes G$ (and $N \rtimes G$) is no longer a factor of type III_λ. (It is of type III_0 and its flow of weights has pure point spectrum.)

Propositions 10, 12 dealt with type III_λ and type III_1 cases respectively, but we have not touched the type III_0 case. In this case, we are probably forced to deal with "modular endomorphisms" in the sense of Izumi ([**17**]). In fact, a recent important result by Izumi says that (in the III_0 case) a graph change occurs if and only if a modular endomorphism appears in $\sqcup_k(\rho\bar{\rho})^k$.

However, when the index is less than 4, Theorem 7 enables us to deal with the type III_0 case.

REMARK 13. Let $M \supset N$ be a pair of type III_0 factors with the type III graph A_{4m-3} and the type II graph D_{2m}. By the presence of the two end-points in the D_{2m} graph, $(1-e_0 \vee e_1 \vee \cdots \vee e_{2m-4})\,(\tilde{M}_{2m-3} \cap \tilde{N}')$ can be identified with $L^\infty(\Omega \times \{0, 1\})$. Since the type III and type II graphs are different, the dual action $\{\theta_t\}_{t \in \boldsymbol{R}}$ restricted to $(1-e_0 \vee e_1 \vee \cdots \vee e_{2m-3})(\tilde{M}_{2m-3} \cap \tilde{N}')$ gives rise to the two-to-one ergodic extension $(F_t',\ \Omega \times \{0, 1\})$ of $(F_t,\ \Omega)$, the flow of weights of M (=that of N):

$$F_t'(\omega,\ i) = (F_t(\omega),\ \varphi_{\omega,t}(i)).$$

Here, $\varphi : (\omega,\ t) \in \Omega \times \boldsymbol{R} \to \varphi_{\omega,t} \in S_2 \cong \{\pm 1\}$ is an F_t-cocycle. We set

$$c_t(\omega) = \varphi_{\omega,t} \in \{\pm 1\},$$
$$x = \chi_{\Omega \times \{0\}} - \chi_{\Omega \times \{1\}} \in L^\infty(\Omega \times \{0, 1\})\ (\subseteq \tilde{M}_{2m-3} \cap \tilde{N}').$$

It is elementary to see $\theta_t(x)=c_t x$. Notice that $e=\{c_t\}_{t\in\boldsymbol{R}}$ is a $\mathcal{U}(\mathcal{Z}(\tilde{M}))$-valued θ_t-cocycle, and hence by Theorem 7 the associated extended modular automorphism $\beta=\sigma_e$ (of period 2) appears in $\sqcup_k(\rho\bar{\rho})^k$. We set

$$P=M\rtimes_\beta \boldsymbol{Z}_2 \supseteq Q=N\rtimes_\beta \boldsymbol{Z}_2.$$

Then, the type *III* and type *II* graphs of the new pair are both D_{2m}. In fact, this fact can be seen based on the arguments in § 4 and Appendix A of [**31**], and details are left to the reader. As above $M\supseteq N$ can be identified with $P\rtimes_{\hat{\beta}}\boldsymbol{Z}_2\supseteq Q\rtimes_{\hat{\beta}}\boldsymbol{Z}_2$ by the Takesaki duality.

5. Strong freeness and strong outerness.

In [**54**] Winsløw studied automorphisms for pairs of factors of type III_λ. To do so, he considered a certain algebraic property (corresponding to the (non-)central triviality ([**24**, **51**]) or the non-pointwise innerness ([**9**, **10**, **11**]) in the study of automorphisms on single factors) for such automorphisms or actions. This property was called the strong freeness, and he obtained some classification results for non-strongly free actions of groups with certain properties. In this section we will show that the strong freeness is stronger than the strong outerness.

As in the previous section, let $M\supseteq N=\rho(M)$ be an inclusion of factors of type *III* such that the associated inclusion $\tilde{M}\supseteq\tilde{N}$ of von Neumann algebras of type II_∞ has the identical center. Let α $(\in Aut(M, N))$ be an automorphism for the pair in question, and we assume that α is already extended to the tower as was explained in § 3 (i.e., all the Jones projections are fixed).

Let $\tilde{\alpha}$ be the canonical extension of α in the sense of Haagerup-Størmer ([**9**]), that is, $\tilde{\alpha}$ is the automorphism of $\tilde{M}=M\rtimes_{\sigma^{\phi\circ E}}\boldsymbol{R}$ defined by

$$\tilde{\alpha}(\pi_{\sigma^{\phi\circ E}}(m))=\pi_{\sigma^{\phi\circ E}}(\alpha(m)),\quad m\in M \text{ (i.e., } \tilde{\alpha}|_M=\alpha \text{ for } M\hookrightarrow\tilde{M})$$

$$\tilde{\alpha}(\lambda(t))=\pi_{\sigma^{\phi\circ E}}((D(\phi\circ E\circ\alpha^{-1}):D(\phi\circ E))_t)\lambda(t),\quad t\in\boldsymbol{R}$$

with the Connes Radon-Nikodym cocycle $(D(\cdot):D(\cdot))_t$. Since the Radon-Nikodym cocycle belongs to N in the present case, the automorphism $\tilde{\alpha}$ actually belongs to $Aut(\tilde{M}, \tilde{N})$. It is also straight-forward to see that this extension procedure of an automorphism (for a pair) is compatible with the basic extension, etc..

DEFINITION 14 ([**54**, **55**]). An automorphism $\alpha\in Aut(M, N)$ is called *strongly free* if the following condition is satisfied for each k:

If $x\in\tilde{M}_k$ satisfies $yx=x\tilde{\alpha}(y)$ for all $y\in\tilde{M}$, then we must have $x=0$.

Since the canonical extension $\tilde{\alpha}$ commutes with the dual action $\{\theta_t\}_{t\in \boldsymbol{R}}$, one can further extend $\tilde{\alpha}$ to an automorphism of the second crossed product

$$\tilde{\tilde{M}} = \tilde{M} \rtimes_\theta \boldsymbol{R} = (M \rtimes_{\sigma^{\phi\circ E}} \boldsymbol{R}) \rtimes_\theta \boldsymbol{R} = \langle \tilde{M} \cong \pi_\theta(\tilde{M}),\ \lambda'(\boldsymbol{R})\rangle''.$$

Namely, we define $\tilde{\tilde{\alpha}} \in Aut(\tilde{\tilde{M}})$ (actually $\tilde{\tilde{\alpha}} \in Aut(\tilde{\tilde{M}}, \tilde{\tilde{N}})$) by

$$\tilde{\tilde{\alpha}}|_{\tilde{M}} = \tilde{\alpha},$$

$$\tilde{\tilde{\alpha}}(\lambda'(s)) = \lambda'(s) \quad (s \in \boldsymbol{R}).$$

The Takesaki duality says that the pairs $\tilde{\tilde{M}} \supseteq \tilde{\tilde{N}}$ and $M \otimes B(L^2(\boldsymbol{R})) \supseteq N \otimes B(L^2(\boldsymbol{R}))$ are conjugate via $\Pi: \tilde{\tilde{M}} = (M \rtimes_{\sigma^{\phi\circ E}} \boldsymbol{R}) \rtimes_\theta \boldsymbol{R} \mapsto M \otimes B(L^2(\boldsymbol{R}))$ satisfying

$$(\Pi(\pi_\theta \circ \pi_{\sigma^{\phi\circ E}}(m))\xi)(r) = \sigma^{\phi\circ E}_{-r}(m)\xi(r),$$

$$(\Pi(\pi_\theta(\lambda(t)))\xi)(r) = \xi(r-t),$$

$$(\Pi(\lambda'(s))\xi)(r) = \exp(isr)\xi(r),$$

where ξ is a vector in $L^2(M) \otimes L^2(\boldsymbol{R}) \cong L^2(\boldsymbol{R}; L^2(M))$. We now figure out how $\tilde{\tilde{\alpha}}$ looks like under this isomorphism. From the definition of $\tilde{\tilde{\alpha}}$, we obviously have

$$(\Pi(\tilde{\tilde{\alpha}}(\pi_\theta \circ \pi_{\sigma^{\phi\circ E}}(m)))\xi)(r) = \sigma^{\phi\circ E}_{-r} \circ \alpha(m)\xi(r), \tag{3}$$

$$(\Pi(\tilde{\tilde{\alpha}}(\pi_\theta(\lambda(t))))\xi)(r) = \sigma^{\phi\circ E}_{-r}((D(\phi\circ E\circ\alpha^{-1}) : D(\phi\circ E))_t)\xi(r-t), \tag{4}$$

$$(\Pi(\tilde{\tilde{\alpha}}(\lambda'(s)))\xi)(r) = \exp(isr)\xi(r). \tag{5}$$

Define the unitary v (acting on $L^2(M) \otimes L^2(\boldsymbol{R})$) by

$$(v\xi)(r) = (D(\phi\circ E\circ\alpha^{-1}) : D(\phi\circ E))_{-r}\xi(r).$$

Notice that v commutes with an arbitrary element in $M' \otimes \boldsymbol{C}1_{L^2(\boldsymbol{R})}$. Hence, v is a unitary in $M \otimes B(L^2(\boldsymbol{R}))$ (actually in $N \otimes B(L^2(\boldsymbol{R}))$), and $Ad\, v$ gives us an inner automorphism. The operator given by (5) obviously commutes with v. It is also elementary to see that the adjoint v^* is given by

$$(v^*\xi)(r) = (D(\phi\circ E\circ\alpha^{-1}) : D(\phi\circ E))^*_{-r}\xi(r).$$

Then, having (3) in mind, we compute

$$\begin{aligned}&(D(\phi\circ E\circ\alpha^{-1}) : D(\phi\circ E))_{-r}\sigma^{\phi\circ E}_{-r}\circ\alpha(m)(D(\phi\circ E\circ\alpha^{-1}) : D(\phi\circ E))^*_{-r}\xi(r)\\ &= \sigma^{\phi\circ E\circ\alpha^{-1}}_{-r}\circ\alpha(m)\xi(r)\\ &= \alpha\circ\sigma^{\phi\circ E}_{-r}(m)\xi(r),\end{aligned}$$

where the intertwining property of the Connes cocycle and the fact $\sigma^{\phi\circ E}_t = \alpha^{-1}\circ \sigma^{\phi\circ E\circ\alpha^{-1}}_t\circ\alpha$ were used. Similarly, having (4) in mind, (thanks to the cocycle property) we compute

$$\begin{aligned}&(D(\psi\circ E\circ\alpha^{-1}): D(\psi\circ E))_{-r}\sigma_{-r}^{\psi\circ E}((D(\psi\circ E\circ\alpha^{-1}): D(\psi\circ E))_t)\\&\qquad\times(D(\psi\circ E\circ\alpha^{-1}): D(\psi\circ E))^*_{-(r-t)}\xi(r-s)\\&=(D(\psi\circ E\circ\alpha^{-1}): D(\psi\circ E))_{-r+t}(D(\psi\circ E\circ\alpha^{-1}): D(\psi\circ E))^*_{-r+t}\xi(r-s)\\&=\xi(r-t).\end{aligned}$$

Therefore, we have shown:

LEMMA 15. *Under the Takesaki duality, the second extension* $\tilde{\tilde{\alpha}}\in Aut(\tilde{\tilde{M}}, \tilde{\tilde{N}})$ *corresponds to* $Ad\, v^*\circ(\alpha\otimes id_{B(L^2(\boldsymbol{R}))})\in Aut(M\otimes B(L^2(\boldsymbol{R})), N\otimes B(L^2(\boldsymbol{R})))$.

In the rest of the section, we will show that the strong freeness implies the strong outerness (by repeating similar arguments as those in § 4). To do so, we assume that $\alpha\in Aut(M, N)$ is not strongly outer. By Lemma 15, $\tilde{\tilde{\alpha}}\in Aut(\tilde{\tilde{M}}, \tilde{\tilde{N}})$ is not strongly outer, and

$$\mathcal{H}=\{x\in\tilde{\tilde{M}}_k\,;\ yx=x\tilde{\tilde{\alpha}}(y)\ \text{for all}\ y\in\tilde{\tilde{M}}\}$$

is a non-zero linear space (for some k). Let $\{\sigma_t\}_{t\in\boldsymbol{R}}$ $(\in Aut(\tilde{\tilde{M}}, \tilde{\tilde{N}}))$ be the dual action of $\{\theta_t\}_{t\in\boldsymbol{R}}$. Since $\sigma_t|_{\tilde{M}_k}=id$ and $\sigma_t(\lambda'(s))=\exp(ist)\lambda'(s)$, we easily observe $\sigma_t\circ\tilde{\tilde{\alpha}}=\tilde{\tilde{\alpha}}\circ\sigma_t$ (by applying the both sides to generators). Hence, we have the invariance $\sigma_t(\mathcal{H})=\mathcal{H}$ $(t\in\boldsymbol{R})$. As was pointed out in § 4, $\mathcal{H}$ is a finite dimensional space and one can choose a basis $\{x_1, x_2, \cdots, x_n\}$ such that $\sigma_t(x_j)=\exp(is_jt)x_j$ for some $s_j\in\boldsymbol{R}$. Hence, we can express each x_j as

$$x_j=z_j\lambda'(s_j)\ \text{for some}\ z_j\in(\tilde{\tilde{M}}_k)^{\sigma}=\tilde{M}_k.$$

Hence, the intertwining property shows that for each $y\in\tilde{M}$ $(\subseteqq\tilde{\tilde{M}})$ we have

$$\begin{aligned}yz_j\lambda'(s_j)&=z_j\lambda'(s_j)\tilde{\tilde{\alpha}}(y)\\&=z_j\lambda'(s_j)\tilde{\alpha}(y)\\&=z_j\theta_{s_j}\circ\tilde{\alpha}(y)\lambda'(s_j),\end{aligned}$$

and we have

$$yz_j=z_j\theta_{s_j}\circ\tilde{\alpha}(y)\quad\text{for all}\ y\in\tilde{M}. \tag{6}$$

LEMMA 16. *Each* s_j *is zero.*

PROOF. Since α is non-strongly outer, α appears in $\bigsqcup_m(\rho\bar{\rho})^m$ (Theorem 3). Therefore, mod$(\alpha)=1$, i.e., $\tilde{\alpha}|_{\mathcal{Z}(\tilde{M})}=id$ ([**16**]). As in § 3, by assuming $s_j\neq 0$, we will get the contradiction $z_j=0$.

At first we consider the type III_0 case. As in § 4, (6) implies $z_j=0$ by the central freeness of $\theta_{s_j}\circ\tilde{\alpha}$.

Secondly we consider the type III_1 case. Since the canonical trace tr on $\tilde{M}$ satisfies $tr\circ\tilde{\alpha}=tr$ (Proposition 12.2, [**9**]), we have $tr\circ\theta_{s_j}\circ\tilde{\alpha}=\exp(-s_j)tr$ and (6) shows $z_j=0$ as in § 4.

Finally we consider the type III_λ case. If s_j is not in $(-\log\lambda)\boldsymbol{Z}$, we have $z_j=0$ as in the type III_0 case. So let us assume $s_j=(-\log\lambda)n$ $(n\neq 0\in\boldsymbol{Z})$. Since $\mathrm{mod}(\alpha)=1$ we can find a unitary $u\in N$ such that $\phi\circ E\circ\alpha=\phi\circ E\circ Ad\, u$ (ϕ=dominant weight). In fact, $\phi\circ\alpha|_N=\phi(u\cdot u^*)$ for some $u\in\mathcal{U}(N)$ (Proposition 2.6, [**10**]). Since $E\circ\alpha=\alpha\circ E$, we have $\phi\circ E\circ\alpha=\phi\circ\alpha\circ E=\phi(uE(\cdot)u^*)=\phi\circ E(u\cdot u)^*$. Thus, after an inner perturbation, we can assume $\phi\circ E\circ\alpha=\phi\circ E$, which implies that α and $\sigma_t^{\phi\circ E}$ commute. Recall the factor of type II_∞

$$M_0=M\rtimes_{\sigma^{\phi\circ E}}(\boldsymbol{R}/t_0\boldsymbol{Z})=\langle M,\lambda_0(t)\rangle''$$

and the isomorphism $\Psi:\tilde{M}\mapsto M_0\otimes L^\infty([0,-\log\lambda))$ in Appendix. We extend α to $\alpha_0\in Aut(M_0)$ by setting $\alpha_0(\lambda_0(t))=\lambda_0(t)$. We know (Lemma A.1, (iii) in Appendix) that, under this isomorphism, $\tilde{\alpha}=\alpha_0\otimes id_{L^\infty([0,-\log\lambda))}$. Since θ_{s_j} corresponds to $\theta_0^n\otimes id_{L^\infty([0,-\log\lambda))}$ and z_j is regarded as an operator-valued function on $[0,-\log\lambda)$, (6) means

$$y(\omega)z_j(\omega)=z_j(\omega)\theta_0^n\circ\alpha_0(y(\omega))\quad \text{a.e. } \omega$$

for $y(\omega)\in M_0$. Since a (unique) trace τ on M_0 satisfies $\tau(\theta_0^n\alpha_0(\cdot))=\lambda^n\tau$ (Lemma A.1, (ii)), the above implies $z_j(\omega)=0$ a.e. ω and $z_j=0$. □

This lemma shows $z_j=x_j$ and $\mathcal{H}\subseteq\tilde{M}_k$. Hence, (6) shows that $x_j\neq 0\in\tilde{M}_k$ satisfies

$$yx_j=x_j\tilde{\alpha}(y)\quad\text{for all } y\in\tilde{M},$$

that is, $\tilde{\alpha}$ is not strongly free. So far we have been assuming that factors are of type III. Obviously, the same proof works for semi-finite factors since in this case the flow of weights is simply the reals $\boldsymbol{R}$ together with the usual translation (with speed 1).

Therefore, we have shown:

THEOREM 17. *Let $M\supseteq N$ be an inclusion of factors such that the associated inclusion $\tilde{M}\supseteq\tilde{N}$ of von Neumann algebras of type II_∞ has the identical center. If $\alpha\in Aut(M, N)$ is strongly free, then it is strongly outer.*

COROLLARY 18. *We keep the same assumptions as in the above theorem. Let $\alpha\in Aut(M, N)$ be an automorphism appearing in $\sqcup_k(\rho\bar{\rho})^k$ and let σ_e be an extended modular automorphism. Then, the composition $\sigma_e\circ\alpha$ is non-strongly free.*

PROOF. Since α appears in $\sqcup_k(\rho\bar{\rho})^k$, by Theorem 3 and Theorem 17 there exists a non-zero $x\in\tilde{M}_k$ (for some k) such that $yx=x\tilde{\alpha}(y)$ for all $y\in\tilde{M}$. Recall that the canonical extension of an extended modular automorphism is inner

([**10**]): $\tilde{\sigma}_e = Ad\, v_e$ for some $v_e \in \mathcal{U}(\tilde{M})$. Then, $(\sigma_e \circ \alpha)^{\sim} = \tilde{\sigma}_e \circ \tilde{\alpha}$ ([**9**]) and the non-zero element $x v_e^* \in \tilde{M}_k$ satisfies:

$$x v_e^* (\sigma_e \circ \alpha)^{\sim}(y) = x \tilde{\alpha}(y) v_e^* = y x v_e^*,$$

that is, $\sigma_e \circ \alpha$ is non-strongly free. □

When $M = N$, Haagerup and Størmer showed that $\tilde{\alpha}$ is inner (i.e., α is point-wise inner) if and only if α is an extended modular automorphism (up to an inner perturbation) (see [**9, 10, 11**]). The above result and Theorem 19 (in the next section) suggest that for analysis of automorphisms for pairs one has to look at automorphisms coming from $\bigsqcup_k (\rho\bar{\rho})^k$ as well.

6. Non-strongly free automorphisms.

In this section, we will obtain the converse of Corollary 18 for factors of type III_λ ($\lambda \neq 0$). Hence, we will be able to describe all the non-strongly free automorphisms (up to an inner perturbation) in this case if the irreducible decomposition of $\bigsqcup_k (\rho\bar{\rho})^k$ (i.e., the fusion rule) is known.

THEOREM 19. *Let $M \supseteqq N = \rho(M)$ be an inclusion of factors of type III_λ ($\lambda \neq 0$) such that the associated inclusion $\tilde{M} \supseteqq \tilde{N}$ of von Neumann algebras of type II_∞ has the identical center. If $\alpha \in Aut(M, N)$ is non-strongly free, then $\alpha = \sigma_t^{\phi \circ E} \circ \beta$ for some automorphism β appearing in $\bigsqcup_k (\rho\bar{\rho})^k$ and $t \in \boldsymbol{R}$.*

PROOF. Let us assume the existence of $x \neq 0 \in \tilde{M}_n$ satisfying

$$yx = x\tilde{\alpha}(y) \quad \text{for all } y \in \tilde{M}. \tag{7}$$

At first we consider the type III_1 case. Since $\tilde{M} \supseteqq \tilde{N}$ are factors,

$$\mathcal{H} = \{x \in \tilde{M}_n\, ;\ yx = x\tilde{\alpha}(y),\ y \in \tilde{M}\}$$

is a (non-zero) finite dimensional linear space. Since the dual action θ_t satisfies $\theta_t \circ \tilde{\alpha} = \tilde{\alpha} \circ \theta_t$, we have $\theta_t(\mathcal{H}) = \mathcal{H}$ and as before one can choose $x_1, x_2, \cdots, x_m \in \mathcal{H}$ such that $\theta_t(x_j) = \exp(i s_j t) x_j$ for some $s_j \in \boldsymbol{R}$. Thus, $x_j = z_j \lambda(s_j)$ with $z_j \in (\tilde{M}_n)_\theta = M_n$, and for $y \in M$ ($\subseteqq \tilde{M}$) we compute

$$y z_j \lambda(s_j) = z_j \lambda(s_j) \tilde{\alpha}(y) = z_j \lambda(s_j) \alpha(y) = z_j \sigma_{s_j}^{\phi \circ E} \circ \alpha(y) \lambda(s_j)$$

thanks to (7). Hence, $z_j \neq 0 \in M_n$ satisfies $y z_j = z_j \sigma_{s_j}^{\phi \circ E} \circ \alpha(y)$ and, by Proposition 4, $\beta = \sigma_{s_j}^{\phi \circ E} \circ \alpha$ appears in $(\rho\bar{\rho})^n$ (as a sector) and $\alpha = \sigma_{-s_j}^{\phi \circ E} \circ \beta$ (up to an inner perturbation).

We now go to the type III_λ case. The existence of $x \neq 0$ satisfying (7) forces that $\mathrm{mod}(\alpha) = 1$. In fact, there exists a unique central projection p such that $\tilde{\alpha}$ is the identity on $p\mathcal{Z}(\tilde{M})$ and free on $(1-p)\mathcal{Z}(\tilde{M})$. Since $\tilde{\alpha}$ commutes with the dual action θ_t, we have $\theta_t(p)=p$, $t\in\boldsymbol{R}$. Therefore, the central ergodicity implies that either we have $\mathrm{mod}(\alpha)=1$ or $\tilde{\alpha}$ is centrally free. However, if α were centrally free, then (7) would be impossible. (Recall the argument in the type III_0 case before Lemma 6.)

Thus, as in the proof of Lemma 16, we may and do assume $\phi\circ E\circ\alpha=\phi\circ E$ (and hence the extension α_0 $(\alpha_0(\lambda_0(t))=\lambda_0(t))$ satisfies $\tilde{\alpha}=\alpha_0\otimes id_{L^\infty([0,-\log\lambda))}$ via the isomorphism Ψ). Therefore, (7) guarantees the existence of a non-zero $x_0\in(M_0)_n$, the n-th extension of $M_0\supseteqq N_0$, satisfying $y_0x_0=x_0\alpha_0(y_0)$, $y_0\in M_0$. We now consider the non-zero linear space

$$\mathcal{H}_0=\{x\in(M_0)_n\,;\ yx=x\alpha_0(y)\ \text{for all}\ y\in M_0\}.$$

Once again $\mathcal{H}_0$ is finite dimensional since $M_0\supseteqq N_0$ are factors, and $\theta_0\circ\alpha_0=\alpha_0\circ\theta_0$ (Lemma A.1, (i)) implies $\theta_0(\mathcal{H}_0)=\mathcal{H}_0$. Considering $\theta_0|_{\mathcal{H}_0}$ as a matrix, one chooses an "eigenvector" $x_0\neq 0$ satisfying $\theta_0(x_0)=\mu x_0$ $(\mu\in\boldsymbol{C})$. Since θ_0 is an automorphism $(\|\theta_0(x_0)\|=\|x_0\|)$ we actually have $\theta_0(x_0)=\exp(is)x_0$ for some $s\in\boldsymbol{R}$. Therefore, $x_0=z_0\lambda_0(s_0)$ with some $z_0\in((M_0)_n)_{\theta_0}=M_n$ and $s_0\in[0, t_0)$. The rest of the proof is exactly the same as in the type III_1 case, and we are done. $\square$

This result probably remains valid for the type III_0 case as well with an extended modular automorphism instead, however, so far the author has been unable to prove it.

COROLLARY 20. *Let $M\supseteqq N$ be as in the previous theorem. When the identity is the only irreducible sector with statistical dimension* 1 *appearing in $\sqcup_k(\rho\bar{\rho})^k$, or more generally, when all the* (*non-trivial*) *irreducible sectors with statistical dimension* 1 *appearing in $\sqcup_k(\rho\bar{\rho})^k$ are modular automorphisms, then a non-strongly free automorphism in $Aut(M, N)$ is just a modular automorphism* (*up to an inner perturbation*).

We now assume that $M\supseteqq N$ are AFD factors of type III_λ, $0<\lambda<1$, with index (strictly) less than 4, and we will describe all the non-strongly free (non-trivial) automorphisms (up to an inner perturbation). Recall that in this case inclusions $M\supseteqq N$ have already been classified (**[20, 33, 47]**): The Dynkin diagrams A_n $(n\geqq 3)$, D_{2m} $(m\geqq 2)$, E_6, and E_8 appear. Except when the graph is given by the Dynkin diagram A_{4m-3}, the classification is the same as that in the AFD II_1 case and an inclusion splits, i.e., $M\supseteqq N$ is conjugate to $\mathcal{R}_\lambda\otimes A\supseteqq\mathcal{R}_\lambda\otimes B$, where $\mathcal{R}_\lambda$ is the Powers factor and $A\supseteqq B$ is an inclusion of AFD II_1-factors with the Dynkin diagram in question. When the graph is given by the Dynkin diagram A_{4m-3}, there are exactly two inclusions: the non-splitting inclusion (type II graph is the Dynkin diagram D_{2m} in this case) and the splitting inclusion.

1. The graph is A_{4m-3} and non-splitting: One non-trivial sector with statistical dimension 1 appears in $\sqcup_k(\rho\bar{\rho})^k$, but this is a modular automorphism

([**16**]). Therefore, the non-strongly free automorphisms are exactly the modular automorphisms due to Corollary 20 (see below). Among all the modular automorphisms $\sigma_t^{\phi\circ E}$ $(0<t<t_0)$, only $\sigma_{t_0/2}^{\phi\circ E}$ is non-strongly outer (Theorem 3) since it appears in $\sqcup_k(\rho\bar{\rho})^k$ (see below).

2. The graph is one of A_{even}, D_{2m} $(m\geqq 3)$, E_8 (and hence splitting): The only irreducible sector with statistical dimension 1 in $\sqcup_k(\rho\bar{\rho})^k$ is the identity. Hence, once again the non-strongly free automorphisms are exactly the modular automorphisms due to Corollary 20. However, in this case, all of them are strongly outer.

3. The graph is one of A_{odd} (and splitting), D_4, E_6: As noted above, $M=\mathcal{R}_\lambda\otimes A\supseteqq N=\mathcal{R}_\lambda\otimes B$. The decomposition of $\sqcup_k(\rho\bar{\rho})^k$ is obviously determined by just the inclusion $A\supseteqq B$ of II_1-factors. On the other hand, modular automorphisms come from $\mathcal{R}_\lambda$. Therefore, due to Corollary 18 and Theorem 19, the non-strongly free automorphisms are $\sigma_t\otimes 1$ $(0<t<t_0)$, and $\sigma_t\otimes\alpha$ $(0\leqq t<t_0)$. Here, σ_t is a modular automorphism (of $\mathcal{R}_\lambda$) and α is an automorphism attached to a non-trivial $A-A$ bimodule of index 1 appearing in $\sqcup_k {}_AL^2(A_k)_A$. There are two such automorphisms in the D_4 case (since $B=A^{\boldsymbol{Z}_3}$) while there is only one such α in the other cases. (The latter is a period 2 automorphism which played an important role in [**14, 15**].) Among these, only $1\otimes\alpha$ is non-strongly outer.

In Case 2 modular automorphisms are obviously coming from $\mathcal{R}_\lambda$ while in Case 1 they appear as follows: We start from the usual discrete decomposition $\mathcal{R}_\lambda=\mathcal{R}_{01}\rtimes_{\theta_0}\boldsymbol{Z}$. Let $C\supseteqq D$ be a unique inclusion of AFD II_1-factors with the graph D_{2m}, and let $\beta\in Aut(C, D)$ be a unique (period 2) automorphism with non-trivial Loi invariant. (Cf. [**4, 21**].) Or equivalently, we set $C=A\rtimes_\alpha\boldsymbol{Z}_2\supseteqq D=B\rtimes_\alpha\boldsymbol{Z}_2$ (by making use of $(A\supseteqq B, \alpha)$ in Case 3) and let β be the dual action of α. We set $\tilde{\theta}_0=\theta_0\otimes\beta\in Aut(\mathcal{R}_{01}\otimes C, \mathcal{R}_{01}\otimes D)$. Then, $M\supseteqq N$ in Case 1 is actually $(\mathcal{R}_{01}\otimes C)\rtimes_{\tilde{\theta}_0}\boldsymbol{Z}\supseteqq(\mathcal{R}_{01}\otimes D)\rtimes_{\tilde{\theta}_0}\boldsymbol{Z}$. In fact, the type II graph is obviously determined by $C\supseteqq D$ (i.e., the Dynkin diagram D_{2m}). Due to the presence of β, the type III graph (which is determined by $(C_n\cap D')_\beta$) shrinks to the Dynkin diagram A_{4m-3}. From this description, it is clear that the modular automorphisms in Case 1 appear as the dual action ($\boldsymbol{T}$-action) of $\tilde{\theta}_0$. Recall that the (extended) β switches the two end-points of the Dynkin diagram D_{2m} (the Loi invariant). Let p, q be the projections (in the relative commutant $C_{2m-3}\cap D'$) corresponding to the two end-points. The above description means that the (extended) $\tilde{\theta}_0$ satisfies $\tilde{\theta}_0(1_{\mathcal{R}_{01}}\otimes(p-q))=1_{\mathcal{R}_{01}}\otimes(q-p)$. Therefore, -1 turns out to be an eigenvalue, and hence $\sigma_{t_0/2}^{\phi\circ E}$ appears in $(\rho\bar{\rho})^{2m-2}$ thanks to Theorem 7 (or more precisely, the variant mentioned before Lemma 9 together with Theorem 3).

Similar analysis can be made when the index is 4 (based on [**34, 47**]), and this is left to the reader as an amusing exercise.

Appendix A. Type III_λ case.

Let $M \supseteqq N$ be factors of type III_λ $(0<\lambda<1)$ with the assumptions at the beginning of § 4. Since modular automorphisms $\sigma_t^{\phi\circ E} \in Aut(M, N)$ have the period $t_0=-2\pi/\log\lambda$, (as was explained in [**9**]) the inclusion $\tilde{M} \supseteqq \tilde{N}$ (acting on $L^2(\boldsymbol{R}; L^2(M))$) can be explicitly identified with the tensor product of an inclusion of factors of type II_∞ and the common abelian von Neumann algebra $L^\infty([0, -\log\lambda))$. For the reader's convenience, we briefly recall the identification in [**9**] together with some remarks. Full details can be found in [**9**].

We consider the inclusion of factors of type II_∞:

$$M_0 = M \rtimes_{\sigma^{\phi\circ E}} (\boldsymbol{R}/t_0\boldsymbol{Z}) \supseteqq N_0 = N \rtimes_{\sigma^{\phi\circ E}} (\boldsymbol{R}/t_0\boldsymbol{Z}).$$

The usual generators for these crossed products will be denoted by $\pi_0(x)$ ($x \in M$ or $x \in N$), $\lambda_0(t)$ in what follows. We then consider the inclusion

$$M_0 \otimes L^\infty([0, -\log\lambda)) \supseteqq N_0 \otimes L^\infty([0, -\log\lambda)).$$

The underlying Hilbert space here is

$$\mathcal{K} = L^2(\boldsymbol{R}/t_0\boldsymbol{Z}; L^2(M)) \otimes L^2([0, -\log\lambda)) \cong L^2((\boldsymbol{R}/t_0\boldsymbol{Z}) \times [0, -\log\lambda); L^2(M)).$$

Notice that $[0, -\log\lambda)$ can be identified with the dual group of $t_0\boldsymbol{Z}$ (by $(\gamma, nt_0) \in [0, -\log\lambda) \times t_0\boldsymbol{Z} \mapsto e^{i\gamma nt_0} = e^{2\pi in\gamma/(-\log\lambda)}$). Let $\xi \in \mathcal{K}$ and $t \in \boldsymbol{R}$, and write $t=\dot{t}+mt_0$ with $\dot{t} \in [0, t_0)$. Consider the operator T from $\mathcal{K}$ to $L^2(\boldsymbol{R}; L^2(M))$ defined by

$$\begin{aligned}(T\xi)(t) &= \frac{1}{\sqrt{-\log\lambda}} \int_0^{(-\log\lambda)} \xi(\dot{t}, \gamma) e^{-i\gamma t} d\gamma \\ &= \frac{1}{\sqrt{-\log\lambda}} \int_0^{(-\log\lambda)} \xi'(\dot{t}, \gamma) e^{-imt_0\gamma} d\gamma\end{aligned}$$

with $\xi'(\dot{t}, \gamma)=\xi(\dot{t}, \gamma)e^{-i\gamma\dot{t}}$. Then, it can be shown that T is a surjective isometry from $\mathcal{K}$ onto $L^2(\boldsymbol{R}; L^2(M))$ and that $\Psi=Ad\,T^*$ gives rise to an isomorphism from $\tilde{M}=\langle \pi_{\sigma^{\phi\circ E}}(M), \lambda(\boldsymbol{R})\rangle''$ onto $M_0 \otimes L^\infty([0, -\log\lambda))$. Let $m(e^{it\cdot})$ be the multiplication operator (acting on $L^2([0, -\log\lambda))$) defined by $(m(e^{it\cdot})\xi)(\gamma)=e^{it\gamma}\xi(\gamma)$. Then Ψ satisfies

$$\Psi(\pi_{\sigma^{\phi\circ E}}(x)) = \pi_0(x) \otimes 1 \quad (x \in M),$$

$$\Psi(\lambda(t)) = \lambda_0(t) \otimes m(e^{it\cdot}) \quad (t \in \boldsymbol{R}).$$

This isomorphism of course sends $\tilde{N}$ onto $N_0 \otimes L^\infty([0, -\log\lambda))$. Let θ_0 be the dual automorphism on the crossed products $M_0 \supseteqq N_0$. Via the isomorphism Ψ, when $t=n\times(-\log\lambda)+r$ with $r \in [0, -\log\lambda)$ the dual action θ_t is expressed as

$$(8) \qquad \theta_t = (\theta_0^n \otimes \beta_r) \oplus (\theta_0^{n+1} \otimes \beta_{r+\log\lambda}).$$

Here, $L^\infty([0, -\log\lambda))$ is considered as the direct sum of $L^\infty([0, -\log\lambda-r))$ and $L^\infty([-\log\lambda-r, -\log\lambda))$, and β_r is the shift: $(\beta_r f)(\gamma)=f(\gamma-r)$. ($\beta_r: L^\infty([0, -\log\lambda-r))\to L^\infty([r, -\log\lambda))$ and $\beta_{r+\log\lambda}: L^\infty([-\log\lambda-r, -\log\lambda)\to L^\infty([0, r)))$.

Let tr be the canonical trace on $\tilde{M}$ ($tr\circ\theta_t=e^{-t}tr$), and τ be the one on M_0 ($\tau\circ\theta_0=\lambda\tau$). Via Ψ, we have

$$tr = \int_{[0,-\log\lambda)}^{\oplus} e^{-\gamma}\tau_\gamma d\gamma \text{ on } M_0\otimes L^\infty([0, -\log\lambda)) \cong \int_{[0,-\log\lambda)}^{\oplus}(M_0)_\gamma d\gamma \tag{9}$$

with $(M_0)_\gamma=M_0$ and $\tau_\gamma=\tau$.

Let α be an automorphism in $Aut(M, N)$, and we assume the invariance $\phi\circ E\circ\alpha=\phi\circ E$. Since α commutes with $\sigma_t^{\phi\circ E}$, one can extend α to $\alpha_0\in Aut(M_0, N_0)$ by setting

$$\alpha_0(\pi_0(x)) = \pi_0(\alpha(x)),$$

$$\alpha_0(\lambda_0(t)) = \lambda_0(t).$$

LEMMA A.1. *The extension α_0 satisfies* (i) $\alpha_0\circ\theta_0=\theta_0\circ\alpha_0$, (ii) $\tau\circ\alpha_0=\tau$, *and* (iii) *via Ψ, the canonical extension $\tilde{\alpha}$ in the sense of Haagerup-Størmer corresponds to $\alpha_0\otimes id_{L^\infty([0,-\log\lambda))}$.*

PROOF. (i) This can be directly checked by applying the both sides to $\pi_0(x)$ and $\lambda_0(t)$.

(iii) The invariance implies $\tilde{\alpha}(\lambda(t))=\lambda(t)$ (see the paragraph before Definition 14).

Hence, we compute

$$\Psi(\tilde{\alpha}(\lambda(t))) = \Psi(\lambda(t)) = \lambda_0(t)\otimes m(e^{it\cdot}) = (\alpha_0\otimes id)(\lambda_0(t)\otimes m(e^{it\cdot})) = (\alpha_0\otimes id)\Psi(\lambda(t)).$$

Similarly we have $\Psi(\tilde{\alpha}(\pi_{\sigma^{\phi\circ E}}(x)))=(\alpha_0\otimes id)\Psi(\pi_{\sigma^{\phi\circ E}}(x))$.

(ii) This follows from (iii) and (9) thanks to $tr\circ\tilde{\alpha}=tr$ (Proposition 12.2, [**9**]). □

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