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# Character sums and the series $L(1,\chi)$ with applications to real quadratic fields

Dedicated to Professor Takashi Ono on his seventieth birthday

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**Abstract.** In this article, let $k \equiv 0$ or 1 (mod 4) be a fundamental discriminant, and let $\chi(n)$ be the real even primitive character modulo $k$. The series

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

can be divided into groups of $k$ consecutive terms. Let $v$ be any nonnegative integer, $j$ an integer, $0 \le j \le k-1$, and let

$$T(v,j,\chi) = \sum_{n=j+1}^{j+k} \frac{\chi(vk+n)}{vk+n}$$

Then $L(1,\chi) = \sum_{v=0}^{\infty} T(v,0,\chi) = \sum_{n=1}^{j} \chi(n)/n + \sum_{v=0}^{\infty} T(v,j,\chi)$.

In section 2, Theorems 2.1 and 2.2 reveal a surprising relation between incomplete character sums and partial sums of Dirichlet series. For example, we will prove that $T(v,j,\chi) \cdot M < 0$ for integer $v \ge \max\{1, \sqrt{k}/|M|\}$ if $M = \sum_{m=1}^{j-1} \chi(m) + 1/2\chi(j) \ne 0$ and $|M| \ge 3/2$. In section 3, we will derive algorithm and formula for calculating the class number of a real quadratic field. In section 4, we will attempt to make a connection between two conjectures on real quadratic fields and the sign of $T(0,20,\chi)$.

## 1. Introduction.

In this article, let $k \equiv 0$ or 1 (mod 4) be a fundamental discriminant, and let $\chi(n)$ be the real even primitive character modulo $k$. The series

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

can be divided into groups of $k$ consecutive terms. Let $v$ be any nonnegative integer, $j$ an integer, $0 \le j \le k-1$, and let

$$T(v,j,\chi) = \sum_{n=j+1}^{j+k} \frac{\chi(vk+n)}{vk+n} = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk+n}.$$

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Then $L(1,\chi)=\sum_{v=0}^{\infty}T(v,0,\chi)=\sum_{n=1}^{j}(\chi(n)/n)+\sum_{v=0}^{\infty}T(v,j,\chi)$. The following are some of the known results related to $T(v,j,\chi)$:

- If $k>3$ is a prime integer, then $T(v,j,\chi)\neq 0$ for nonnegative integers $v$ and $j$ (cf. [**8**]).
- $T(v,0,\chi)>0$ for all nonnegative integers $v$ and $k$ (cf. [**4**]).
- $T(v,[k/2],\chi)<0$ for all nonnegative integers $v$ and $k$, where $[x]$ denotes the greatest integer $\leq x$ (cf. [**9**]).

Combining the results of [**4**] and [**9**], which are mentioned above, we have the following interesting and important inequalities:

$$\sum_{n=1}^{k}\frac{\chi(n)}{n}<L(1,\chi)<\sum_{n=1}^{[k/2]}\frac{\chi(n)}{n}. \tag{1.1}$$

In section 2, Theorems 2.1 and 2.2 reveal a surprising relation between incomplete character sums and partial sums of Dirichlet series. For example, we will prove that $T(v,j,\chi)\cdot M<0$ for integer $v\geq\max\{1,\sqrt{k}/|M|\}$ if $M=\sum_{m=1}^{j-1}\chi(m)+(1/2)\chi(j)\neq 0$ and $|M|\geq 3/2$. Roughly speaking, the sign of $T(v,j,\chi)$ is dependent on the value of $\sum_{m=1}^{j-1}\chi(m)+(1/2)\chi(j)$. This result tells us more information about $T(v,j,\chi)$ than Theorem 2 of [**9**] does. Sections 3 and 4 illustrate the importance of Theorems 2.1 and 2.2. In section 3, we will derive algorithm and formula for calculating the class number of a real quadratic field. In section 4, we will attempt to make a connection between two conjectures on real quadratic fields and the sign of $T(0,20,\chi)$.

## 2. $T(v,j,\chi)$.

In this section we show that the sign of $T(v,j,\chi)$ has close relation to the sign of $\sum_{n=1}^{j-1}\chi(n)+(1/2)\chi(j)$.

For integer $j$ in the closed interval $[1,k-1]$, write

$$T(v,j,\chi)=\sum_{n=j+1}^{j+k}\frac{\chi(n)}{vk+n}=\frac{1}{k}\sum_{l=1}^{k}\frac{\chi(j+l)}{w+l/k},$$

where $w=v+(j/k)$. For $w=v+(j/k)>0$, consider the function

$$f(x)=\frac{1}{w+x}\quad\text{defined for } 0\leq x\leq 1.$$

Over the interval $(0,1)$, it has Fourier expansion

$$f(x)=\frac{1}{2}a_0+\sum_{m=1}^{\infty}(a_m\cos 2\pi mx+b_m\sin 2\pi mx),$$

where

$$\frac{1}{2}a_0=\int_0^1\frac{1}{w+x}\,dx=\log\left(1+\frac{1}{w}\right),\quad a_m=2\int_0^1\frac{\cos 2\pi mx}{w+x}\,dx$$

and

$$b_m=2\int_0^1\frac{\sin 2\pi mx}{w+x}\,dx\quad(\text{cf. } [\mathbf{11}, \text{pp. } 189]).$$

Using integration by parts, we have, for $m \geq 1$,

$$a_m = \frac{2}{(2\pi m)^2}\left\{\frac{1}{w^2} - \frac{1}{(w+1)^2}\right\} - \frac{4}{(2\pi m)^2}\int_0^1 \frac{\cos 2\pi mx}{(w+x)^3}\,dx.$$

Let $I_m = 4/(2\pi m)^2 \int_0^1 \cos 2\pi mx/(w+x)^3 dx$ and $J_m = 2/(2\pi m)^2\{1/w^2 - 1/(w+1)^2\}$. It is easy to see that $I_m = 4/(2\pi m)^2 \int_0^1 \cos 2\pi mx/(w+x)^3\,dx = 12/(2\pi m)^3 \int_0^1 \sin 2\pi mx/(w+x)^4\,dx > 0$ by looking at the graph of $y = \sin 2\pi mx/(w+x)^4$ on the interval [0, 1]. Since $0 < I_m = |I_m| < J_m$, we have $a_m = J_m - I_m = J_m\theta_m$, where $\theta_m = (J_m - I_m)/J_m$ and $0 < \theta_m < 1$. Similarly, we have $b_m = 2/(2\pi m)\{1/w - 1/(w+1)\} - 4/(2\pi m)^3\{1/w^3 - 1/(w+1)^3\} + 12/(2\pi m)^3 \int_0^1 \cos 2\pi mx/(w+x)^4\,dx$. Let $X_m = 12/(2\pi m)^3 \int_0^1 \cos 2\pi mx/(w+x)^4\,dx$ and $Y_m = 4/(2\pi m)^3\{1/w^3 - 1/(w+1)^3\}$. Then $X_m = 12/(2\pi m)^3 \int_0^1 \cos 2\pi mx/(w+x)^4\,dx = 48/(2\pi m)^4 \int_0^1 \sin 2\pi mx/(w+x)^5\,dx > 0$ and $X_m < Y_m$. Hence, we have

$$b_m = \frac{2}{2\pi m}\left(\frac{1}{w} - \frac{1}{w+1}\right) - \frac{4}{(2\pi m)^3}\left\{\frac{1}{w^3} - \frac{1}{(w+1)^3}\right\}\eta_m,$$

where $\eta_m = (Y_m - X_m)/Y_m$ and $0 < \eta_m < 1$. Now

$$\begin{aligned}
T(v,j,\chi) &= \frac{1}{k}\sum_{l=1}^{k}\frac{\chi(j+l)}{w+l/k} \qquad \left(w = v + \frac{j}{k} \text{ and } j \geq 1\right)\\
&= \frac{1}{k}\sum_{l=1}^{k-1}\chi(j+l)f\left(\frac{l}{k}\right) + \frac{1}{k}\chi(j+k)\frac{1}{w+1}\\
&= \frac{1}{k}\sum_{l=1}^{k-1}\chi(j+l)\left\{\frac{1}{2}a_0 + \sum_{m=1}^{\infty}\left(a_m\cos 2\pi m\frac{l}{k} + b_m \sin 2\pi m\frac{l}{k}\right)\right\} + \frac{1}{k}\frac{\chi(j)}{w+1}\\
&= \frac{\chi(j)}{k}\left(\frac{1}{w+1} - \frac{a_0}{2} - \sum_{m=1}^{\infty}a_m\right)\\
&\quad + \frac{1}{k}\sum_{l=1}^{k}\sum_{m=1}^{\infty}\left(a_m\chi(j+l)\cos 2\pi m\frac{l}{k} + b_m\chi(j+l)\sin 2\pi m\frac{l}{k}\right)\\
&\quad \left(\text{since } \sum_{m=1}^{\infty}a_m = \sum_{m=1}^{\infty}J_m\theta_m \text{ converges and } \sum_{l=1}^{k}\chi(j+l) = 0\right)\\
&= E + \frac{1}{k}\sum_{m=1}^{\infty}\left(a_m\sum_{l=1}^{k}\chi(j+l)\cos 2\pi m\frac{l}{k} + b_m\sum_{l=1}^{k}\chi(j+l)\sin 2\pi m\frac{l}{k}\right)\\
&\quad \left(\text{where } E = \frac{\chi(j)}{k}\left(\frac{1}{w+1} - \log\left(1+\frac{1}{w}\right) - \sum_{m=1}^{\infty}a_m\right)\right)\\
&= E + \frac{1}{k}\sum_{m=1}^{\infty}\left(a_m\chi(m)\sqrt{k}\cos 2\pi m\frac{j}{k} - b_m\chi(m)\sqrt{k}\sin 2\pi m\frac{j}{k}\right)
\end{aligned}$$

(cf. Lemma 2.3).

Hence

$$\sqrt{k}T(v,j,\chi) = \sqrt{k}E + \sum_{m=1}^{\infty}\left(a_m\chi(m)\cos 2\pi m\frac{j}{k} - b_m\chi(m)\sin 2\pi m\frac{j}{k}\right)$$

$$= \sqrt{k}E + \left(\frac{1}{w} - \frac{1}{w+1}\right)\left\{\frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\theta_m\cos 2\pi m(j/k)}{m^2}\right.$$

$$-\frac{2}{2\pi}\sum_{m=1}^{\infty}\frac{\chi(m)\sin 2\pi m(j/k)}{m}$$

$$\left.+\frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\eta_m\sin 2\pi m(j/k)}{m^3}\right\}, \qquad (2.1)$$

where $0 < \theta_m,\ \eta_m < 1$. Let

$$S_{vj} = \frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\left|\sum_{m=1}^{\infty}\frac{\chi(m)\theta_m\cos 2\pi m(j/k)}{m^2}\right|$$

$$+\frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\left|\sum_{m=1}^{\infty}\frac{\chi(m)\eta_m\sin 2\pi m(j/k)}{m^3}\right|$$

and

$$P_j = \left|\frac{2}{2\pi}\sum_{m=1}^{\infty}\frac{\chi(m)\sin 2\pi m(j/k)}{m}\right|$$

$$= \frac{1}{\sqrt{k}}\left|\sum_{m=1}^{j-1}\chi(m) + \frac{1}{2}\chi(j)\right| \quad \text{(cf. Proposition 2.4).}$$

Then we have

$$S_{vj} \le \frac{2}{(2\pi)^2}\left(\frac{1}{w} + \frac{1}{w+1}\right)\zeta(2) + \frac{4}{(2\pi)^3}\left(\frac{1}{w^2} + \frac{1}{w(w+1)} + \frac{1}{(w+1)^2}\right)\zeta(3)$$

$$\le \frac{2}{(2\pi)^2}\frac{2}{w}\frac{\pi^2}{6} + \frac{4}{(2\pi)^3}\frac{3}{w^2}\frac{\pi^2}{6} \quad \left(\zeta(2) = \frac{\pi^2}{6} > \zeta(3)\right)$$

$$= \frac{1}{w}\left(\frac{1}{6} + \frac{1}{4\pi}\frac{1}{w}\right)$$

$$< \frac{1}{4}\frac{1}{w} \quad (\text{if } v \ge 1).$$

If $\chi(j) = 0$ and $M = \sum_{m=1}^{j-1}\chi(m) \ne 0$, then $E = 0$ and $P_j = |M|/\sqrt{k}$. We thus have

$$P_j = \frac{|M|}{\sqrt{k}} \ge \frac{1}{4}\frac{1}{w} > S_{vj}$$

for integer $v \ge \max\{1, \sqrt{k}/4|M|\}$.

To sum up, we have proved the following:

THEOREM 2.1. *Let* $k \equiv 0$ *or* 1 (mod 4) *be a fundamental discriminant and* $\chi$ *the real even primitive character modulo* $k$. *If* $\chi(j) = 0$ *and* $M = \sum_{m=1}^{j-1} \chi(m) \neq 0$, *then*

$$T(v, j, \chi) \cdot \left\{ \sum_{m=1}^{j-1} \chi(m) \right\} < 0$$

*for integer* $v \geq max\{1, \sqrt{k}/4|M|\}$.

For the case $\chi(j) \neq 0$, we can obtain similar result as follows.

By the similar argument used earlier, we have, for $m \geq 1$,

$$\frac{2}{(2\pi m)^2} \left\{ \frac{1}{w^2} - \frac{1}{(w+1)^2} \right\} - \frac{12}{(2\pi m)^4} \left\{ \frac{1}{w^4} - \frac{1}{(w+1)^4} \right\}$$

$$< a_m < \frac{2}{(2\pi m)^2} \left\{ \frac{1}{w^2} - \frac{1}{(w+1)^2} \right\}.$$

This gives

$$\frac{2}{(2\pi)^2} \left\{ \frac{1}{w^2} - \frac{1}{(w+1)^2} \right\} \sum_{m=1}^{\infty} \frac{1}{m^2} - \frac{12}{(2\pi)^4} \left\{ \frac{1}{w^4} - \frac{1}{(w+1)^4} \right\} \sum_{m=1}^{\infty} \frac{1}{m^4}$$

$$< \sum_{m=1}^{\infty} a_m < \frac{2}{(2\pi)^2} \left\{ \frac{1}{w^2} - \frac{1}{(w+1)^2} \right\} \sum_{m=1}^{\infty} \frac{1}{m^2}.$$

Since $\sum_{m=1}^{\infty} (1/m^2) = (\pi^2/6)$ and $\sum_{m=1}^{\infty} (1/m^4) = (\pi^4/90)$, we have

$$\frac{1}{12} \left\{ \frac{1}{w^2} - \frac{1}{(w+1)^2} \right\} > \sum_{m=1}^{\infty} a_m \tag{2.2}$$

$$> \frac{1}{12} \left\{ \frac{1}{w^2} - \frac{1}{(w+1)^2} \right\} \left\{ 1 - \frac{1}{10} \left( \frac{1}{w^2} + \frac{1}{(w+1)^2} \right) \right\}.$$

For $w > 1$ (that is $v \geq 1$), by (2.2) and Lemma 2.5, we have

$$0 < \log\left(1 + \frac{1}{w}\right) - \frac{1}{w+1} + \sum_{m=1}^{\infty} a_m$$

$$< \frac{1}{w} - \frac{1}{w+1} + \frac{1}{12} \left\{ \frac{1}{w^2} - \frac{1}{(w+1)^2} \right\}$$

$$< \left( \frac{1}{w} - \frac{1}{w+1} \right) \left\{ 1 + \frac{1}{12} \left( \frac{1}{w} + \frac{1}{w+1} \right) \right\}.$$

This implies that, for $w > 1$ and $\chi(j) \neq 0$,

$$|E| = \frac{1}{k}\left\{\log\left(1+\frac{1}{w}\right) - \frac{1}{w+1} + \sum_{m=1}^{\infty} a_m\right\}$$

$$< \frac{1}{k}\left(\frac{1}{w} - \frac{1}{w+1}\right)\left\{1 + \frac{1}{12}\left(\frac{1}{w} + \frac{1}{w+1}\right)\right\}.$$

Hence, for $k \geq 5$, $w > 1$ and $\chi(j) \neq 0$,

$$\frac{\sqrt{k}|E|}{1/w - 1/(w+1)} + S_{vj} < \frac{1}{\sqrt{k}} + \frac{1}{12\sqrt{k}}\left(\frac{1}{w} + \frac{1}{w+1}\right) + S_{vj}$$

$$< \frac{1}{\sqrt{k}} + \frac{1}{6\sqrt{k}}\frac{1}{w} + \frac{1}{4}\frac{1}{w}$$

$$< \frac{1}{\sqrt{k}} + \frac{1}{12}\frac{1}{w} + \frac{1}{4}\frac{1}{w}$$

$$= \frac{1}{\sqrt{k}} + \frac{1}{3}\frac{1}{w}.$$

If $|M| = |\sum_{m=1}^{j-1} \chi(m) + (1/2)\chi(j)| \geq (3/2)$, then

$$P_j = \frac{|M|}{\sqrt{k}} \geq \frac{3+|M|}{3\sqrt{k}} = \frac{1}{\sqrt{k}} + \frac{|M|}{3\sqrt{k}}$$

$$> \frac{1}{\sqrt{k}} + \frac{1}{3w}$$

$$> \frac{1}{\sqrt{k}} + \frac{1}{12\sqrt{k}}\left(\frac{1}{w} + \frac{1}{w+1}\right) + S_{vj}$$

for integer $v \geq \max\{1, \sqrt{k}/|M|\}$.

By (2.1), we have the following theorem.

Theorem 2.2. *Let* $k \geq 5$ *and* $k \equiv 0$ *or* 1 (mod 4) *be a fundamental discriminant and* $\chi$ *the real even primitive character modulo* $k$. *If* $\chi(j) \neq 0$ *and* $|M| = |\sum_{m=1}^{j-1} \chi(m) + (1/2)\chi(j)| \geq (3/2)$, *then*

$$T(v,j,\chi)\cdot\left\{\sum_{m=1}^{j-1}\chi(m) + \frac{1}{2}\chi(j)\right\} < 0$$

*for integer* $v \geq max\{1, \sqrt{k}/|M|\}$.

Remark 1. Theorems 2.1 and 2.2 tell us more information about $T(v,j,\chi)$ than Theorem 2 of [**9**] does.

Finally, to close this section, we need to supply the following lemmas and proposition.

Lemma 2.3. $\sum_{l=1}^{k} \chi(j+l)e^{2\pi iml/k} = \chi(m)\sqrt{k}e^{-2\pi imj/k}$.

PROOF. Multiplying $e^{-2\pi imj/k}$ on both sides of the Gauss sum for real even primitive character $\chi$ modulo $k$, we have

$$\begin{aligned}\chi(m)\sqrt{k}e^{-2\pi imj/k} &= \sum_{n=1}^{k}\chi(n)e^{2\pi imn/k}e^{-2\pi imj/k}\\ &= \sum_{n=j+1}^{j+k}\chi(n)e^{2\pi imn/k}e^{-2\pi imj/k}\\ &= \sum_{l=1}^{k}\chi(j+l)e^{2\pi im(j+l)/k}e^{-2\pi imj/k}\\ &= \sum_{l=1}^{k}\chi(j+l)e^{2\pi iml/k}.\end{aligned}$$ □

To obtain Proposition 2.4, we apply a method used in [**10**].

PROPOSITION 2.4. *Let $j$ be any integer in the closed interval $[1, k-1]$. Then*

$$\sum_{n=1}^{j-1}\chi(n) + \frac{1}{2}\chi(j) = \frac{\sqrt{k}}{\pi}\sum_{n=1}^{\infty}\frac{\chi(n)\sin 2\pi n(j/k)}{n}.$$

PROOF. For fixed integer $j$, we define the periodic function $\phi$ with period $2\pi$ as follows:

$$\phi(x) = \begin{cases} 1, & \text{if } 0 < x < \dfrac{2\pi j}{k};\\ \dfrac{1}{2}, & \text{if } x = 0 \text{ or } x = \dfrac{2\pi j}{k} \text{ or } x = 2\pi;\\ 0, & \text{if } \dfrac{2\pi j}{k} < x < 2\pi. \end{cases}$$

Then $\sum_{n=1}^{j-1}\chi(n) + (1/2)\chi(j) = \sum_{n=1}^{k}\phi(2\pi n/k)\chi(n)$.

By exercise 17(c) of [**11**, Chapter 8], we have

$$\lim_{N\to\infty} s_N(\phi;0) = \frac{1}{2}(\phi(0+) + \phi(0-)) = \frac{1}{2}(1+0) = \phi(0)$$

and

$$\lim_{N\to\infty} s_N\left(\phi;\frac{2\pi j}{k}\right) = \frac{1}{2}\left(\phi\left(\frac{2\pi j}{k}+\right) + \phi\left(\frac{2\pi j}{k}-\right)\right) = \frac{1}{2}(0+1) = \phi\left(\frac{2\pi j}{k}\right),$$

where $s_N(\phi;x) = j/k + \sum_{n=1}^{N}(a_n\cos nx + b_n\sin nx)$, $a_n = 1/\pi\int_0^{2\pi}\phi(x)\cos nx\,dx$ and $b_n = 1/\pi\int_0^{2\pi}\phi(x)\sin nx\,dx$. Hence, over the interval $[0,2\pi]$, $\phi$ has Fourier expansion

$$\phi(x) = \frac{j}{k} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx),$$

where $a_n = 1/\pi \int_0^{2\pi} \phi(x)\cos nx\,dx = (1/n\pi)\sin 2\pi n(j/k)$ and $b_n = 1/\pi \int_0^{2\pi} \phi(x)\sin nx\,dx = 1/n\pi - (1/n\pi)\cos 2\pi n(j/k)$. Now

$$\begin{aligned}
\sum_{n=1}^{j-1}\chi(n) + \frac{1}{2}\chi(j) &= \sum_{m=1}^{k}\phi\left(\frac{2\pi m}{k}\right)\chi(m) \\
&= \sum_{m=1}^{k}\left\{\frac{j}{k} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi mn}{k} + b_n\sin\frac{2\pi mn}{k}\right)\right\}\chi(m) \\
&= \sum_{m=1}^{k}\left\{\sum_{n=1}^{\infty}\left(\chi(m)a_n\cos\frac{2\pi mn}{k} + \chi(m)b_n\sin\frac{2\pi mn}{k}\right)\right\} \\
&\quad \left(\text{since } \sum_{m=1}^{k}\chi(m) = 0\right) \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{k}\chi(m)a_n\cos\frac{2\pi mn}{k} + \sum_{m=1}^{k}\chi(m)b_n\sin\frac{2\pi mn}{k}\right) \\
&= \sum_{n=1}^{\infty}\left(a_n\sum_{m=1}^{k}\chi(m)\cos\frac{2\pi mn}{k} + b_n\sum_{m=1}^{k}\chi(m)\sin\frac{2\pi mn}{k}\right) \\
&= \sum_{n=1}^{\infty}a_n\chi(n)\sqrt{k} \\
&= \frac{\sqrt{k}}{\pi}\sum_{n=1}^{\infty}\frac{\chi(n)\sin 2\pi n(j/k)}{n}.
\end{aligned}$$

Here we used the fact that Gauss sum $\sum_{m=1}^{k}\chi(m)\exp(2\pi imn/k) = \chi(n)\sqrt{k}$ since $\chi(-1) = 1$.

Lemma 2.5. *For* $x > 0$, $1/x > \log(1 + 1/x) > 1/(x + 1)$.

Proof. For $x > 0$, from the inequality $e^x > 1 + x$, it is easy to derive $1/x > \log(1 + 1/x)$. Next, for $x > 0$, consider $G(x) = \log(1 + 1/x) - 1/(x + 1)$. Then $G'(x) < 0$ for $x > 0$. Suppose there exists $x_0 > 0$ such that $G(x_0) = 0$, then

$$1 + \frac{1}{x_0} = e^{1/(x_0+1)} = 1 + \frac{1}{x_0 + 1} + \frac{1/(x_0 + 1)^2}{2!} + \cdots + \frac{1/(x_0 + 1)^n}{n!} + \cdots$$

which gives

$$\begin{aligned}
\frac{1}{x_0} &< \frac{1}{x_0 + 1} + \frac{1}{(x_0 + 1)^2} + \cdots + \frac{1}{(x_0 + 1)^n} + \cdots \\
&= \frac{1/(x_0 + 1)}{1 - 1/(x_0 + 1)} = \frac{1}{x_0},
\end{aligned}$$

a contradiction. Since $G(1/2) = \log 3 - 2/3 > 0$, we have $G(x) > 0$ for $x > 0$. The lemma is proved. □

### 3. Class numbers of real quadratic fields.

The main results of this section are derived from the inequalities (1.1):

$$\sum_{n=1}^{k} \frac{\chi(n)}{n} < L(1,\chi) < \sum_{n=1}^{[k/2]} \frac{\chi(n)}{n}.$$

For $t \geq 0$, let $A(t) = \sum_{n=1}^{[t]} \chi(n)$. Then, by (1.1) and Abel's identity (cf. Theorem 2 of [**8**]), we have

$$\sum_{n=1}^{[m_1]} \frac{\chi(n)}{n} - \frac{A(m_1)}{m_1} + \int_{m_1}^{k} \frac{A(t)}{t^2}\,dt < L(1,\chi) \tag{3.1}$$

$$< \sum_{n=1}^{[m_2]} \frac{\chi(n)}{n} - \frac{A(m_2)}{m_2} + \int_{m_2}^{[k/2]} \frac{A(t)}{t^2}\,dt,$$

where $1 \leq m_1 \leq k$ and $1 \leq m_2 \leq [k/2]$. Let $r \leq [k/2]$ be a positive number such that $A(r) = 0$. Then we have

$$\sum_{n=1}^{[r]} \frac{\chi(n)}{n} + \int_{r}^{k} \frac{A(t)}{t^2}\,dt < L(1,\chi) \tag{3.2}$$

$$< \sum_{n=1}^{[r]} \frac{\chi(n)}{n} + \int_{r}^{[k/2]} \frac{A(t)}{t^2}\,dt.$$

By (3.2) and Pólya's inequality $|A(t)| \leq \sqrt{k}\log k$ [**1**, pp. 173], we derive

$$\sum_{n=1}^{[r]} \frac{\chi(n)}{n} - \frac{(k-r)}{rk}\sqrt{k}\log k < L(1,\chi) \tag{3.3}$$

$$< \sum_{n=1}^{[r]} \frac{\chi(n)}{n} + \frac{(k-2r)}{rk}\sqrt{k}\log k.$$

Hence

$$\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]} \frac{\chi(n)}{n} - \frac{(k-r)}{2r}\frac{\log k}{\log\varepsilon} < h = \frac{\sqrt{k}}{2\log\varepsilon} L(1,\chi) \tag{3.4}$$

$$< \frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]} \frac{\chi(n)}{n} + \frac{(k-2r)}{2r}\frac{\log k}{\log\varepsilon},$$

where $h$ is the class number, and $\varepsilon\ (>1)$ is the fundamental unit of $\boldsymbol{Q}(\sqrt{k})$.

To sum up, we have proved the following:

THEOREM 3.1. *For a real quadratic field* $\boldsymbol{Q}(\sqrt{k})$ *with fundamental unit* $\varepsilon\ (>1)$, *if there exists positive number* $r \leq [k/2]$ *such that* $A(r) = 0$ *and* $((2k-3r)/2r)(\log k/\log\varepsilon) < 1$, *then, by* (3.4), *the class number* $h$ *of* $\boldsymbol{Q}(\sqrt{k})$ *is*

$$h = \left[\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]}\frac{\chi(n)}{n} + \frac{(k-2r)}{2r}\frac{\log k}{\log\varepsilon}\right], \tag{3.5}$$

*where* $[t]$ *denotes the greatest integer* $\leq t$.

EXAMPLE. Let $k = 521$, then, by [**13**], $\varepsilon = 138377240 + 5624309\sqrt{521}$. For $r = 178$, we have $\sum_{m=1}^{178}\chi(m) = 0$. Hence, by (3.5), $h = [1.1511] = 1$.

REMARK 2. For an effective method to calculate $\varepsilon$, the fundamental unit of a real quadratic number field, see, for example, Theorem 15 of [**2**]. In [**2**], one also can find explicit formulas for the fundamental unit of $\boldsymbol{Q}(\sqrt{C})$ for some particular types of natural numbers $C$.

To continue our investigation, we quote a well-known result (Lemma 3.2) and results of Johnson and Mitchell [**5**]:

LEMMA 3.2. *If the discriminant of a quadratic field contains only one prime factor, then the class number of the field is odd.*

The proof can be found in [**3**, pp. 187].

LEMMA 3.3.
(1) *If prime* $p \equiv 5 \pmod 8$, *then* $A(p/6) = \sum_{n=1}^{[p/6]}\chi(n) = 0$.
(2) *If prime* $p \equiv 5 \pmod{24}$, *then* $A(p/12) = \sum_{n=1}^{[p/12]}\chi(n) = 0$.
*In both cases,* $\chi$ *is the real even primitive character modulo* $p$.

PROOF. See Johnson and Mitchell [**5**]. □

By Lemma 3.3, inequalities (3.3) and (3.4), we have immediately the following theorems.

THEOREM 3.4.
(1) *If prime* $p \equiv 5 \pmod 8$, *then*

$$\sum_{n=1}^{[p/6]}\frac{\chi(n)}{n} - \frac{5\log p}{\sqrt{p}} < L(1,\chi) < \sum_{n=1}^{[p/6]}\frac{\chi(n)}{n} + \frac{4\log p}{\sqrt{p}}.$$

(2) *If prime* $p \equiv 5 \pmod{24}$, *then*

$$\sum_{n=1}^{[p/12]}\frac{\chi(n)}{n} - \frac{11\log p}{\sqrt{p}} < L(1,\chi) < \sum_{n=1}^{[p/12]}\frac{\chi(n)}{n} + \frac{10\log p}{\sqrt{p}}.$$

*In both cases,* $\chi$ *is the real even primitive character modulo* $p$.

THEOREM 3.5.
(1) *If prime* $p \equiv 5 \pmod 8$, *then*

$$\frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/6]}\frac{\chi(n)}{n} - \frac{5}{2}\frac{\log p}{\log\varepsilon} < h < \frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/6]}\frac{\chi(n)}{n} + \frac{4}{2}\frac{\log p}{\log\varepsilon}.$$

(2) *If prime* $p \equiv 5 \pmod{24}$, *then*

$$\frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/12]}\frac{\chi(n)}{n}-\frac{11}{2}\frac{\log p}{\log\varepsilon}<h<\frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/12]}\frac{\chi(n)}{n}+\frac{10}{2}\frac{\log p}{\log\varepsilon}.$$

*In both cases, h is the class number, and* $\varepsilon$ $(>1)$ *is the fundamental unit of* $\boldsymbol{Q}(\sqrt{p})$.

Remark 3. Since the class number $h$ of $\boldsymbol{Q}(\sqrt{p})$ is odd for prime $p \equiv 5 \pmod 8$ (cf. Lemma 3.2), as an illustration of Theorem 3.5, we know that the class number $h$ is equal to the only odd integer lying in the closed interval $[(\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n - (5/2)\log p/\log\varepsilon, (\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n + (4/2)\log p/\log\varepsilon]$ if $\log p/\log\varepsilon < 4/9$. As an example, $p = 2389$ gives that $\log p/\log\varepsilon < 4/9$ [**13**].

Remark 4. Let $\varepsilon = (t+u\sqrt{p})/2 > 1$ be the fundamental unit of $\boldsymbol{Q}(\sqrt{p})$ ($p \equiv 5 \pmod 8$ a prime). Then the integers $t \geq 1$ and $u \geq 1$. If $u > p$, then $\varepsilon = (t+u\sqrt{p})/2 > p^{3/2}/2 + p\sqrt{p}/2 = p^{3/2}$ which gives $2/3 > \log p/\log\varepsilon$. If prime $p \equiv 5 \pmod 8$ and $u > p$, then, by (1) of Theorem 3.5, the class number $h$ of $\boldsymbol{Q}(\sqrt{p})$ is an odd integer lying in the closed interval $[(\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n - (5/2)\log p/\log\varepsilon, (\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n + (4/2)\log p/\log\varepsilon]$ which contains at most three integers. If prime $p \equiv 5 \pmod{24}$ and $u > p$, then, by (2) of Theorem 3.5, the class number $h$ of $\boldsymbol{Q}(\sqrt{p})$ is an odd integer lying in the closed interval $[V, W] = [(\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/12]}\chi(n)/n - (11/2)\log p/\log\varepsilon, (\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/12]}\chi(n)/n + (10/2)\log p/\log\varepsilon]$ with $W - V = (21/2)\log p/\log\varepsilon < (21/2)(2/3) = 7$. Since $W - V < 8$, we can use Corollary 1 (iii) of [**12**] (cf. [**12**, page 388 and page 390]) to determine exactly value of $h$. To be more precise, we quote Corollary 1 (iii) of [**12**] as follows: If prime $p \equiv 5 \pmod 8$, then $h(-p) \equiv 3gTUh(p) + (p-5) \pmod{16}$, where $h(n)$ is the class number of the quadratic field $\boldsymbol{Q}(\sqrt{n})$, $\varepsilon^g = ((t+u\sqrt{p})/2)^g = T + U\sqrt{p}$, $g = 3$ if $t \equiv u \equiv 1 \pmod 2$, and $g = 1$ otherwise.

For prime $p \equiv 5 \pmod 8$ and $p > e^{34}$, by a slight improvement in Pólya's inequality, we can obtain better estimates for $L(1,\chi)$ than Theorem 3.4 does. Therefore, we can obtain better estimates for the class number $h$ of $\boldsymbol{Q}(\sqrt{p})$ than Theorem 3.5 does.

Proposition 3.6. *If* $\chi_C$ *is any primitive character modulo* $C$ *and* $C > e^{34}$, *then*

$$\left|\sum_{n=1}^{m}\chi_C(n)\right| < \frac{2}{3}\sqrt{C}\log C$$

*for any positive integer m.*

Proof. See, for example, [**1**, Chapter 8, Exercise 14]. □

If $r \leq [k/2]$ is a positive number such that $\sum_{n=1}^{[r]}\chi(n) = 0$, then, by (3.2) and Proposition 3.6, we have, for $k > e^{34}$,

$$\sum_{n=1}^{[r]}\frac{\chi(n)}{n}-\frac{2}{3}\frac{(k-r)}{rk}\sqrt{k}\log k < L(1,\chi) < \sum_{n=1}^{[r]}\frac{\chi(n)}{n}+\frac{2}{3}\frac{(k-2r)}{rk}\sqrt{k}\log k \tag{3.6}$$

for the real even primitive character $\chi$ modulo $k$. Hence, for $k > e^{34}$,

$$\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]}\frac{\chi(n)}{n} - \frac{(k-r)}{3r}\frac{\log k}{\log\varepsilon} < h \tag{3.7}$$

$$< \frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]}\frac{\chi(n)}{n} + \frac{(k-2r)}{3r}\frac{\log k}{\log\varepsilon},$$

where $h$ is the class number, and $\varepsilon\ (>1)$ is the fundamental unit of $\boldsymbol{Q}(\sqrt{k})$.

To sum up, we have proved the following:

Theorem 3.7. *For a real quadratic field $\boldsymbol{Q}(\sqrt{k})$ with fundamental unit $\varepsilon\ (>1)$, if $k > e^{34}$ and there exists positive number $r \le [k/2]$ such that $A(r) = 0$ and $((2k-3r)/3r)\log k/\log\varepsilon < 1$, then the class number $h$ of $\boldsymbol{Q}(\sqrt{k})$ is*

$$h = \left[\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]}\frac{\chi(n)}{n} + \frac{(k-2r)}{3r}\frac{\log k}{\log\varepsilon}\right].$$

Remark 5. If the norm $\varepsilon\bar{\varepsilon} = -1$, then, by Lemma 3.2 and the genus theory of quadratic number field (cf. Corollary 3 of [**8**]), the condition $((2k-3r)/3r)\log k/\log\varepsilon < 1$ in Theorem 3.7 can be replaced by $((2k-3r)/3r)\log k/\log\varepsilon < 2$ and the conclusion

$$h = \left[\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]}\frac{\chi(n)}{n} + \frac{(k-2r)}{3r}\frac{\log k}{\log\varepsilon}\right]$$

should be replaced by

$$h = \left[\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[r]}\frac{\chi(n)}{n} + \frac{(k-2r)}{3r}\frac{\log k}{\log\varepsilon}\right] - i,$$

where $i = 0$ or $i = 1$ depends on whether $[(\sqrt{k}/2\log\varepsilon)\sum_{n=1}^{[r]}\chi(n)/n + ((k-2r)/3r)\log k/\log\varepsilon] + E(k)$ is an even integer or an odd integer, where $E(k)$ is 1 or 0 depending on whether $k$ is a prime or not a prime. For the case $\varepsilon\bar{\varepsilon} = -1$, the similar replacement is also applied to Theorem 3.1.

By Lemma 3.3, inequalities (3.6) and (3.7), we have immediately the following theorems.

Theorem 3.8.

(1) *If prime $p \equiv 5 \pmod 8$ and $p > e^{34}$, then*

$$\sum_{n=1}^{[p/6]}\frac{\chi(n)}{n} - \frac{10}{3}\frac{\log p}{\sqrt{p}} < L(1,\chi) < \sum_{n=1}^{[p/6]}\frac{\chi(n)}{n} + \frac{8}{3}\frac{\log p}{\sqrt{p}}.$$

(2) *If prime* $p \equiv 5 \pmod{24}$ *and* $p > e^{34}$, *then*

$$\sum_{n=1}^{[p/12]} \frac{\chi(n)}{n} - \frac{22}{3}\frac{\log p}{\sqrt{p}} < L(1,\chi) < \sum_{n=1}^{[p/12]} \frac{\chi(n)}{n} + \frac{20}{3}\frac{\log p}{\sqrt{p}}.$$

*In both cases,* $\chi$ *is the real even primitive character modulo* $p$.

THEOREM 3.9.

(1) *If prime* $p \equiv 5 \pmod 8$ *and* $p > e^{34}$, *then*

$$\frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/6]} \frac{\chi(n)}{n} - \frac{10}{6}\frac{\log p}{\log\varepsilon} < h < \frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/6]} \frac{\chi(n)}{n} + \frac{8}{6}\frac{\log p}{\log\varepsilon}.$$

(2) *If prime* $p \equiv 5 \pmod{24}$ *and* $p > e^{34}$, *then*

$$\frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/12]} \frac{\chi(n)}{n} - \frac{22}{6}\frac{\log p}{\log\varepsilon} < h < \frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/12]} \frac{\chi(n)}{n} + \frac{20}{6}\frac{\log p}{\log\varepsilon}.$$

*In both cases,* $h$ *is the class number, and* $\varepsilon\ (>1)$ *is the fundamental unit of* $\boldsymbol{Q}(\sqrt{p})$.

COROLLARY 3.10. *If prime* $p \equiv 5 \pmod 8$, $p > e^{34}$ *and* $u > p$, *then*

$$h = \left[\frac{\sqrt{p}}{2\log\varepsilon}\sum_{n=1}^{[p/6]} \frac{\chi(n)}{n} + \frac{8}{6}\frac{\log p}{\log\varepsilon}\right] - i,$$

*where* $\varepsilon = (t + u\sqrt{p})/2 > 1$ *is the fundamental unit of* $\boldsymbol{Q}(\sqrt{p})$ *and*

$$i = \begin{cases} 0, & \text{if } \left[\dfrac{\sqrt{p}}{2\log\varepsilon}\displaystyle\sum_{n=1}^{[p/6]} \frac{\chi(n)}{n} + \frac{8}{6}\frac{\log p}{\log\varepsilon}\right] \text{ is odd;} \\ 1, & \text{otherwise.} \end{cases}$$

PROOF. By Remark 4, we have $(2/3) > \log p/\log\varepsilon$. Since $(18/6)\log p/\log\varepsilon < 2$, there are at most 2 positive integers lying in the closed interval $[(\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n - (10/6)\log p/\log\varepsilon, (\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n + (8/6)\log p/\log\varepsilon]$. By Lemma 3.2, the class number $h$ of $\boldsymbol{Q}(\sqrt{p})$ is odd. Hence the corollary is proved. □

REMARK 6. As before, let $\varepsilon = (t + u\sqrt{p})/2 > 1$ be the fundamental unit of $\boldsymbol{Q}(\sqrt{p})$ ($p \equiv 5 \pmod 8$ a prime). Then it is well-known that the norm $N(\varepsilon) = \varepsilon\bar{\varepsilon} = -1$. If $u > 1$, then $2 > \log p/\log\varepsilon$. If $u = 1$, then $p = t^2 + 4$. Since the function $g(x) = (\sqrt{x-4} + \sqrt{x})/2 - x^{10/21}$ is positive and strictly increasing for $x \geq 29$, we have $2.1 > \log p/\log\varepsilon$ for prime $p = t^2 + 4 \geq 29$. Therefore we have that $2.1 > \log p/\log\varepsilon$ for prime $p \equiv 5 \pmod 8$ and $p \geq 29$. Now, for prime $p \equiv 5 \pmod 8$ and $p > e^{34}$, by (1) of Theorem 3.9, the class number $h$ of $\boldsymbol{Q}(\sqrt{p})$ is an odd integer lying in the closed interval $[V, W] = [(\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n - (10/6)\log p/\log\varepsilon, (\sqrt{p}/2\log\varepsilon)\sum_{n=1}^{[p/6]}\chi(n)/n + (8/6)\log p/\log\varepsilon]$ with $W - V = (18/6)\log p/\log\varepsilon < (18/6)(2.1) = 6.3$. Again, since $W - V < 8$, we can use Corollary 1 of [**12**] to determine exactly value of $h$.

## 4. Chowla's conjecture and Yokoi's conjecture.

For $x > 0$, set

$$B(x) = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} + \frac{1}{x+5} - \frac{1}{x+6} - \frac{1}{x+7} + \frac{1}{x+8} + \frac{1}{x+9} - \frac{1}{x+10} - \frac{1}{x+11} - \frac{1}{x+12} + \frac{1}{x+13} + \frac{1}{x+14} + \frac{1}{x+15} - \frac{1}{x+16} - \frac{1}{x+17} - \frac{1}{x+18} - \frac{1}{x+19}.$$

Then we have the following lemma.

Lemma 4.1.
(1) $B(x) < 0$ *for* $x > 2$.
(2) $|B(x)| > |B(t+x)|$ *for* $x > 12$ *and* $t > 0$.

Proof. For $x > 2$, it is easy to verify the following inequalities:

$$\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+4} < 0, \quad -\frac{1}{x+2} + \frac{1}{x+3} < 0,$$

$$-\frac{1}{x+7} + \frac{1}{x+8} + \frac{1}{x+9} - \frac{1}{x+10} < 0,$$

$$-\frac{1}{x+11} - \frac{1}{x+12} + \frac{1}{x+13} + \frac{1}{x+14} + \frac{1}{x+15} - \frac{1}{x+16} < 0$$

and

$$\frac{1}{x+5} - \frac{1}{x+6} - \frac{1}{x+17} - \frac{1}{x+18} - \frac{1}{x+19} < 0.$$

Thus, the statement (1) is proved.

Using the proof of statement (1) and the easy exercise:

$$\left| -\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} \right| = \left| \frac{-10-4x}{(x+1)(x+2)(x+3)(x+4)} \right| > \left| -\frac{1}{x+t+1} + \frac{1}{x+t+2} + \frac{1}{x+t+3} - \frac{1}{x+t+4} \right|$$

for $x > 0$ and $t > 0$, we can derive statement (2) without difficulty. □

Note. $B(1) \approx 0.0585302$.

Using Davenport's result $T(v, 0, \chi) > 0$ [**4**] and Lemma 4.1, we have the following proposition.

Proposition 4.2. *Let* $d \equiv 0$ *or* 1 (mod 4) *be a fundamental discriminant such that* $\chi_d(q) = -1$ *for prime* $q \leq 19$, *where* $\chi_d$ *is the real even primitive character modulo* $d$.

*Then*

$$T(v, 20, \chi_d) > 0$$

*for integer* $v \geq 1$.

Proof. The integer $d$ such that $\chi_d(q) = -1$ for prime $q \leq 19$ is greater than 9172 [**7**]. For integer $v \geq 1$,

$$\begin{aligned} T(v, 20, \chi_d) &= \sum_{n=21}^{20+d} \frac{\chi_d(vd+n)}{vd+n} \\ &= -\sum_{n=1}^{20} \frac{\chi_d(vd+n)}{vd+n} + \sum_{n=1}^{20} \frac{\chi_d(vd+n)}{vd+n} \\ &\quad + \sum_{n=21}^{d} \frac{\chi_d(vd+n)}{vd+n} + \sum_{n=1}^{20} \frac{\chi_d(vd+d+n)}{vd+d+n} \\ &= T(v, 0, \chi_d) - \sum_{n=1}^{20} \frac{\chi_d(n)}{vd+n} + \sum_{n=1}^{20} \frac{\chi_d(n)}{vd+d+n} \\ &= T(v, 0, \chi_d) - B(vd+1) + B(vd+d+1). \end{aligned}$$

Since $T(v, 0, \chi_d) > 0$ for $v \geq 0$ [**4**] and, by Lemma 4.1, $-B(vd+1) + B(vd+d+1) > 0$ for integer $v \geq 1$, therefore we have $T(v, 20, \chi_d) > 0$ for integer $v \geq 1$. □

Problem 1. Let $\chi$ be a real even primitive character modulo $k$ such that $\chi(q) = -1$ for prime $q \leq 19$. Is $T(0, 20, \chi) > 0$ always true?

Before proposing the next problem, we recall the following conjectures on the class number of real quadratic fields:

$(C_1)$ (S. Chowla): Let $D$ be a square-free rational integer of the form $D = (2n)^2 + 1$ for natural number $n$. Then, there exist exactly 6 real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ of class number one, that is $(D, n) = (5, 1), (17, 2), (37, 3), (101, 5), (197, 7), (677, 13)$.

$(C_2)$ (H. Yokoi): Let $D$ be a square-free rational integer of the form $D = n^2 + 4$ for natural number $n$. Then, there exist exactly 6 real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ of class number one, that is $(D, n) = (5, 1), (13, 3), (29, 5), (53, 7), (173, 13), (293, 17)$. In [**6**], H. K. Kim. M.-G. Leu and T. Ono proved that at least one of the two conjectures $(C_1)$, $(C_2)$ is true and that for the other case there are at most 7 quadratic fields $\boldsymbol{Q}(\sqrt{D})$ of class number one.

In relation to these two conjectures, we propose the following problem:

Problem 2. Is $T(0, 20, \chi_D) > 0$ true for any square-free integer $D$ ($> 8844444$) of the form $D = (2n)^2 + 1$ or $n^2 + 4$ ($n \in \boldsymbol{N}$) with real even primitive character $\chi_D$ having $\chi_D(q) = -1$ for prime $q \leq 19$?

Remark 7. If Problem 2 is true, then $L(1, \chi) > B(1)$. Hence, by applying Dirichlet's class number formula and following the easy procedure used in [**6**], one can prove the conjectures $(C_1)$ and $(C_2)$ without condition.

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