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On Eisenstein series on quaternion unitary groups of degree 2

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Abstract. In this paper we give an explicit Fourier expansion of the Eisenstein series on certain quaternion unitary groups of degree 2 by means of Shimura's method. Moreover using an explicit formula of the Fourier coefficients of holomorphic Eisenstein series and Oda's lifted cusp forms, we give some numerical examples.

0. Introduction.

In this paper, we study Eisenstein series on a quaternion unitary group of degree 2 and give some examples. Automorphic forms of our type were studied by Arakawa [**1**]. We shall state an outline of this paper in the simplest case.

Though we treat totally real algebraic number fields as the basic field in this paper, to give a brief account of this paper, we state only the case of rational number field. Let B be an indefinite division quaternion algebra over $\boldsymbol{Q}$ and D the discriminant of B over $\boldsymbol{Q}$. Denote by $\alpha \mapsto \bar{\alpha}$ $(\alpha \in B)$ the canonical involution and put $\mathrm{Tr}_{B/\boldsymbol{Q}}(\alpha) = \alpha + \bar{\alpha}$. We denote by $\mathfrak{O}$ a maximal order of B. We define an algebraic group G over $\boldsymbol{Q}$ by

$$G_{\boldsymbol{Q}} := \left\{ g \in GL_2(B) \,\middle|\, g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

where $g^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(B)$. Then the group G_∞ of $\boldsymbol{R}$-rational point, which is isomorphic to $Sp(2, \boldsymbol{R})$, acts transitively on complex domain $\mathfrak{H}$ which is isomorphic to the Siegel upper half-plane of degree 2. We define a discrete subgroup Γ of G_∞ by

$$\Gamma := G_{\boldsymbol{Q}} \cap GL_2(\mathfrak{O})$$

(in this paper we treat a more general discrete subgroup defined in (1.2)). We denote by $M_l(\Gamma)$ [resp. $S_l(\Gamma)$], for a positive integer l, the space of holomorphic automorphic forms [resp. cusp forms] on $\mathfrak{H}$ of weight l. It is known that $S_l(\Gamma)$ is a finite dimensional vector space over $\boldsymbol{C}$, and an explicit formula for the dimension of $S_l(\Gamma)$ was obtained by Hashimoto [**4**]. For an even positive integer l, let

$$(0.1) \qquad E_l(Z, s) := \det(\operatorname{Im} Z)^{(2s-2l+3)/4} \sum_{\{C, D\}} \det(CZ + D)^{-l} |\det(CZ + D)|^{-s+l-3/2}$$

be the real analytic Eisenstein series for Γ. Here Z is a variable on $\mathfrak{H}$, s is a com-

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plex variable, and $\begin{pmatrix} * & * \\ C & D \end{pmatrix}$ runs over a complete system of representatives of $\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma \right\} \Big\backslash \Gamma$. The right-hand side of (0.1) converges absolutely and locally uniformly on

$$\{(Z,s) \mid Z \in \mathfrak{H}, \operatorname{Re} s > 3/2\}.$$

Especially for $l \geqq 4$

$$E_l(Z) := E_l(Z, l - 3/2)$$

belongs to $M_l(\Gamma)$.

We have an explicit formula for the Fourier expansion by an application of the Shimura's method (cf. [**17**]).

Theorem 0.1 (Theorem 3.4). *For* $\operatorname{Re} s > 3/2$ *and* $Z = X + iY \in \mathfrak{H}$, *the Eisenstein series has the following expansion*:

$$E_l(Z,s) = \sum_{\eta} a_l(\eta, Y, s) e[\operatorname{Tr}_{B/\boldsymbol{Q}}(\eta X)],$$

$$a_l(\eta, Y, s) = \det(Y)^{(2s-2l+3)/4} \delta(\eta = 0)$$
$$+ D^{-1} \det(Y)^{(2s-2l+3)/4} \xi_2 \left(J^{-1} Y, \eta J; \frac{2s+2l+3}{4}, \frac{2s-2l+3}{4} \right) \alpha_{\mathfrak{f}}(\eta, s),$$

where η *runs over the lattice defined in* (1.3), $\delta((*))$ *means* 1 *or* 0 *according as the condition* $(*)$ *is satisfied or not, and put* $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. *The function* ξ_2, *which is defined in* (2.8), *can be written by the confluent hypergeometric functions defined in Shimura* [**16**]. $\alpha_{\mathfrak{f}}$ *is the singular series and its explicit form is given in Proposition* 3.3.

From Theorem 0.1 we prove the analytic continuation and a functional equation without using Langlands' theory [**9**].

Theorem 0.2 (Theorem 3.7). *The Eisenstein series* $E_l(Z,s)$ *has a meromorphic continuation to the whole s-plane and*

$$\xi\left(s + \frac{3}{2}\right) \xi(2s+1) \prod_{j=0}^{l/2-1} \left(\frac{2s+3}{4} + j \right) \left(\frac{2s+1}{4} + j \right) \prod_{p \mid D} (p^{s+1/2} - p^{-s-1/2}) E_l(Z,s)$$

is invariant under $s \mapsto -s$. *Here* $\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) = \xi(1-s)$.

The right-hand side of (0.1) is no guarantee of the absolute convergence if $l = 2$ and $s = 1/2$. However, as in Shimura [**17**], we obtain the following result. Since $E_2(Z,s)$ is regular at $s = 1/2$, we can define $E_2(Z) := E_2(Z,s)|_{s=1/2}$.

Theorem 0.3 (Theorem 3.8).

$$E_2(Z) \in M_2(\Gamma)$$

We remark that in the case of Siegel modular forms of degree 2 the holomorphic Eisenstein series of weight 2 can not be constructed in this way (see [**8**], [**12**], [**17**]).

Moreover we give an explicit formula for the Fourier coefficients of the holomorphic Eisenstein series $E_l(Z)$ to compute numerical examples (cf. Theorem 3.10). By this formula, we verify that Fourier coefficients of $E_l(Z)$ are the rational number whose denominators are bounded (cf. Corollary 3.11) and satisfy the Maass relation (cf. Corollary 3.12).

In the last section (§4), we consider the case of $D = 6$. We introduce Oda's lifting (cf. [**13**], [**19**]), and give some numerical examples of Fourier coefficients of the holomorphic Eisenstein series and lifted cusp forms. Moreover we construct cusp forms which are not the lifted cusp form. For $l \geqq 5$ we can know the value of $\dim S_l(\Gamma)$ by Hashimoto's dimension formula (cf. [**4**]). As an application we prove the following result.

Theorem 0.4 (Theorem 4.4). *If* $D = 6$, *we have*

$$\dim_{\boldsymbol{C}} S_2(\Gamma) = 0, \quad \dim_{\boldsymbol{C}} S_4(\Gamma) = 2.$$

The existence of the holomorphic Eisenstein series of weight 2 (Theorem 0.3) and the structure of $S_6(\Gamma)$ (cf. (4.4)) play a basic role in proving the above theorem.

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Notation. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively, the ring of integers, the rational number field, the real number field, and the complex number field. For an associative ring R with an identity element, $R^\times$ denotes the group of all invertible elements and $M_m(R)$ the ring of all matrices size m with coefficients in R. We put $GL_m(R) = M_m(R)^\times$. If $X \in M_m(R)$, ${}^tX$ and $\mathrm{Tr}(X)$ stands for its transpose and trace. If R is commutative, $\det(X)$ stands for its determinant, and we denote by $SL_m(R)$ the special linear group of degree m. For real symmetric matrices X and Y, we write $X > Y$ to indicate that $X - Y$ is positive definite. If $X > 0$, we denote by $X^{1/2}$ its positive definite square root. Let k be a number field and $\mathfrak{o}$ the ring of integers. For each place v of k, we denote by k_v the v-completion of k, and by $|x|_v$ the module of x for an $x \in k_v^\times$. k_A [resp. $k_A^\times$] means the adele ring of k [resp. the idele group of k] and for $x = (x_v) \in k_A^\times$, put $|x|_A = \prod_v |x_v|_v$. For an algebraic group G defined over k, we denote by G_k the group of k-rational points of G. We abbreviate G_{k_v} to G_v. We let ∞ and $\mathbf{f}$ denote the sets of archimedean primes and non-archimedean primes of k, respectively. We denote by G_A, G_∞, and $G_\mathbf{f}$ the adelized group of G, the infinite part of G_A, and the finite part of G_A, respectively. Similar notation are used for an algebra or a vector space. Each prime ideal $\mathfrak{p}$ of k is identified with the corresponding finite place, and we denote by $\mathfrak{o}_\mathfrak{p}$ the ring of integer of $k_\mathfrak{p}$. If there is no fear of confusion, the maximal ideal $\mathfrak{p}\mathfrak{o}_\mathfrak{p}$ of $\mathfrak{o}_\mathfrak{p}$ is written as $\mathfrak{p}$. We denote by $\pi_\mathfrak{p}$ a prime element of $k_\mathfrak{p}$ and put $q_\mathfrak{p} = |\pi_\mathfrak{p}|_\mathfrak{p}$. When L is an $\mathfrak{o}$-module, put $L_v = L \otimes_v \mathfrak{o}_v$. For $z \in \boldsymbol{C}$, put $e[z] = \exp(2\pi i z)$. For a quaternion algebra B over k, we denote by $x \mapsto \bar{x}$ $(x \in B)$ the canonical involution of B over k, and put $\mathrm{Tr}_{B/k}(x) = x + \bar{x}$ and $N_{B/k}(x) = x\bar{x}$. We denote by B^- the set of pure quaternions, and any subset S of B we put $S^- = B^- \cap S$. The disjoint union of sets $Z_1, \dots, Z_s$ is denoted by $\coprod_{i=1}^s Z_i$. For $a \in \boldsymbol{R}$, the symbol $[a]$ denotes the integer not greater than a.

1. Definition of Eisenstein series.

1.1. Preliminaries.

Let k be a totally real algebraic number field of degree n over $\boldsymbol{Q}$, and let B be a division quaternion algebra over k; and denote by $\mathfrak{D}$ the product of prime ideals of k such that $B_{\mathfrak{p}}$ is division, and we call this the discriminant ideal of B over k. We assume that B is unramified at any infinite place of k. We denote by $\infty_1, \ldots, \infty_n$ all infinite places of k. Then by the above assumption on B, $B_{\infty_j} = B \otimes_k k_{\infty_j}$ is isomorphic to $M_2(\boldsymbol{R})$. So we identify B_{∞_j} with $M_2(\boldsymbol{R})$.

Let G be a linear algebraic group over k defined by

$$G_k := \left\{ g \in GL_2(B) \,\middle|\, g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \tag{1.1}$$

where $g^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$ for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(B)$. Then G_{∞_j} is isomorphic to

$$Sp(2, \boldsymbol{R}) := \left\{ g \in GL_4(\boldsymbol{R}) \,\middle|\, {}^t g \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\}.$$

Put

$$\mathfrak{H}_j := \{ Z_j \in B_{\infty_j} \otimes_{\boldsymbol{R}} \boldsymbol{C} \mid \mathrm{Tr}_{B/k}(Z_j) = 0, \ (\mathrm{Im}\, Z_j) J^{-1} \text{ is positive definite} \},$$

where $\mathrm{Im}\, Z_j$ means the imaginary part of Z_j, and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The domain $\mathfrak{H}_j$ is isomorphic to the Siegel upper half-plane of degree 2, and G_{∞_j} acts on $\mathfrak{H}_j$ transitively as a group of holomorphic automorphisms via the mapping

$$Z_j \mapsto g_j \langle Z_j \rangle := (A_j Z_j + B_j)(C_j Z_j + D_j)^{-1}, \quad g_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \in G_{\infty_j}.$$

Put

$$Z_{j,0} := iJ, \quad C_{\infty_j} := \{ g_j \in G_{\infty_j} \mid g_j \langle Z_{j,0} \rangle = Z_{j,0} \}.$$

The group C_{∞_j}, which is a maximal compact subgroup of G_{∞_j}, is isomorphic to the unitary group of degree 2, and $\mathfrak{H}_j$ is isomorphic to $G_{\infty_j}/C_{\infty_j}$. For $g_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}$ $\in G_{\infty_j}$ and $Z_j \in \mathfrak{H}_j$, we define a $\boldsymbol{C}$-valued holomrophic automorphic factor $J_j(g_j, Z_j)$ on $G_{\infty_j} \times \mathfrak{H}_j$ by

$$J_j(g_j, Z_j) := \det(C_j Z_j + D_j) \quad (1 \leqq j \leqq n).$$

Let $\mathfrak{H}$ be the direct product of $\mathfrak{H}_1 \times \cdots \times \mathfrak{H}_n$. We put $Z_0 = (Z_{1,0}, \ldots, Z_{n,0})$ and $C_\infty = C_{\infty_1} \times \cdots \times C_{\infty_n}$. The action of G_∞ on $\mathfrak{H}$ is given componentwise, namely,

$$g\langle Z \rangle := (g_1 \langle Z_1 \rangle, \ldots, g_n \langle Z_n \rangle),$$

and the automorphic factor on $G_\infty \times \mathfrak{H}$ is given by

$$J(g,Z) := \prod_{j=1}^{n} J_j(g_j, Z_j) \in \boldsymbol{C},$$

where $g = (g_1, \dots, g_n) \in G_\infty$, $Z = (Z_1, \dots, Z_n) \in \mathfrak{H}$.

Now, we fix a maximal order $\mathfrak{O}$ of B and a maximal two-sided ideal $\mathfrak{A}$. Then it is well-known that $\mathfrak{A}$ is uniquely written as

$$\mathfrak{A} := \prod_{\mathfrak{p} | \mathfrak{D}} \mathfrak{B}_{\mathfrak{p}}^{e_{\mathfrak{p}}},$$

where $\mathfrak{B}_\mathfrak{p}$ is a maximal ideal of $\mathfrak{O}_\mathfrak{p}$, $e_\mathfrak{p} = 0$ or 1. We denote $\mathfrak{D}_0$ [resp. $\mathfrak{D}_1$] the product of all primes such that $\mathfrak{p}|\mathfrak{D}$ and $e_\mathfrak{p} = 0$ [resp. $e_\mathfrak{p} = 1$].

For each prime ideal $\mathfrak{p}$, put

$$C_\mathfrak{p} := \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_\mathfrak{p} \,\middle|\, \alpha, \delta \in \mathfrak{O}_\mathfrak{p},\ \beta \in \mathfrak{A}_\mathfrak{p},\ \gamma \in \mathfrak{A}_\mathfrak{p}^{-1} \right\},$$

where $\mathfrak{O}_\mathfrak{p} = \mathfrak{O} \otimes_\mathfrak{o} \mathfrak{o}_\mathfrak{p}$ and $\mathfrak{A}_\mathfrak{p} = \mathfrak{A} \otimes_\mathfrak{o} \mathfrak{o}_\mathfrak{p}$. Then $C_\mathfrak{p}$ is a maximal compact subgroup of $G_\mathfrak{p}$ and $G_\mathfrak{p} = P_\mathfrak{p} C_\mathfrak{p}$, where P is a parabolic subgroup of G defined by

$$P_k := \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_k \,\middle|\, \gamma = 0 \right\}.$$

We abbreviate $\prod_\mathfrak{p} C_\mathfrak{p}$ to $C_\mathbf{f}$ and $C_\infty C_\mathbf{f}$ to C_A.

Put

$$(1.2) \qquad \Gamma_\mathfrak{A} := G_k \cap G_\infty C_\mathbf{f} = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_k \,\middle|\, \alpha, \delta \in \mathfrak{O},\ \beta \in \mathfrak{A},\ \gamma \in \mathfrak{A}^{-1} \right\}.$$

Then $\Gamma_\mathfrak{A}$ is a discrete subgroup of G_∞ such that the volume of $\Gamma_\mathfrak{A} \backslash G_\infty$ is finite. We denote by h the class number of k. Then the following lemma is easily seen (cf. [**4**], [**17**]).

Lemma 1.1. *There exist h elements $\beta_1, \dots, \beta_h$ of G_k such that*

$$G_k = \coprod_{i=1}^{h} P_k \beta_i \Gamma_\mathfrak{A}.$$

For a positive integer l, let $M_l(\Gamma_\mathfrak{A})$ denote the space of automorphic forms of weight l with respect to $\Gamma_\mathfrak{A}$;

$$M_l(\Gamma_\mathfrak{A}) := \left\{ f(Z) \,\middle|\, \begin{array}{l} f(Z) \text{ is holomorphic on } \mathfrak{H} \\ f(\gamma\langle Z\rangle) = J(g,Z)^l f(Z) \text{ for any } \gamma \in \Gamma_\mathfrak{A} \end{array} \right\},$$

and let $S_l(\Gamma_\mathfrak{A})$ denote the space of cusp forms of weight l with respect to $\Gamma_\mathfrak{A}$;

$$S_l(\Gamma_\mathfrak{A}) := \left\{ f(Z) \in M_l(\Gamma_\mathfrak{A}) \,\middle|\, |f(Z)| \prod_{v \in \infty} (\det(\operatorname{Im} Z_v))^{l/2} \text{ is bounded on } \mathfrak{H} \right\}.$$

The following fact is well-known ([**2**], [**14**]).

Proposition 1.2. *Let $\Gamma_{\mathfrak{A}}$ be the discrete subgroup of G_∞ given in* (1.2) *and let l be a positive integer. Then $f(Z) \in M_l(\Gamma_{\mathfrak{A}})$ has the following Fourier expansion*

$$f(Z) = a(0) + \sum_{\substack{\eta \in (\mathfrak{A}^-)^* \\ \eta J > 0}} a(\xi) e[\sigma(\eta Z)],$$

where $\sigma = \mathrm{Tr}_{k/\boldsymbol{Q}} \circ \mathrm{Tr}_{B/k}$ and $\eta J > 0$ means that $\eta_v J$ is positive definite for all $v \in \infty$. $(\mathfrak{A}^-)^*$ is a lattice defined by

$$(1.3) \qquad (\mathfrak{A}^-)^* := \{x \in B^- \mid \mathrm{Tr}_{B/k}(\eta x) \in \mathfrak{d}^{-1} \text{ for all } \eta \in \mathfrak{A}^-\},$$

where $\mathfrak{d}$ is the different of k over $\boldsymbol{Q}$. In particular, $f(Z) \in S_l(\Gamma_{\mathfrak{A}})$ is equivalent to $a(0) = 0$.

1.2. Eisenstein series.

For an even integer l, and a complex number s we define a function on G_A by

$$f_l(g,s) := \prod_v f_{l,v}(g,s), f_{l,v}(g,s) := \begin{cases} A_{\mathfrak{p}}(g)^s & \text{if } v = \mathfrak{p} \in \mathbf{f}, \\ A_{\infty_j}(g)^s J_j(w_{\infty_j}, Z_{0,j})^{-l} & \text{if } v = \infty_j \in \infty. \end{cases}$$

where $A_v(g) = |N_{B/k}(\alpha)|_v$ for $g = \begin{pmatrix} \alpha & * \\ 0 & \bar{\alpha}^{-1} \end{pmatrix} w \in P_A C$, $w = w_\infty w_{\mathbf{f}} \in C_\infty C_{\mathbf{f}}$. It can be easily seen that this is well-defined because l is an even integer. Now our Eisenstein series (as a function on G_A) is defined by

$$(1.4) \qquad E_l^{gr}(g,s) := \sum_{\gamma \in P_k \backslash G_k} f_l(\gamma g, s + 3/2).$$

The right-hand side of (1.4) converges absolutely and locally uniformly on

$$\{(g,s) \in G_A \times \boldsymbol{C} \mid g \in G_A, \mathrm{Re}\, s > 3/2\}.$$

Then we easily see that

$$(1.5) \qquad E_l^{gr}(\gamma g w, s) = E_l^{gr}(g,s) J(w_\infty, Z_0)^{-l} \quad \text{for } \gamma \in G_k,\ g \in G_A,\ w = w_\infty w_{\mathbf{f}} \in C_\infty C_{\mathbf{f}}.$$

Notice that

$$G_A = G_k G_\infty C$$

by virtue of the strong approximation theorem for G. Therefore we can define our Eisenstein series on $\mathfrak{H}$ by

$$(1.6) \qquad E_l(Z,s) := E_l^{gr}(g_\infty, s) J(g_\infty, Z_0)^l \quad \text{for } Z = g_\infty \langle Z_0 \rangle.$$

By Lemma 1.1

$$\begin{aligned} E_l^{gr}(g_\infty, s) &= \sum_{\gamma \in P_k \backslash G_k} f_l(\gamma g_\infty, s + 3/2) = \sum_{i=1}^h \sum_{\gamma \in P_k \backslash P_k \beta_i \Gamma_{\mathfrak{A}}} f_l(\gamma g_\infty, s + 3/2) \\ &= \sum_{i=1}^h \sum_{\gamma \in T_i} f_l(\gamma g_\infty, s + 3/2), \end{aligned}$$

where $T_i = (P_k \cap \beta_i \Gamma_{\mathfrak{A}} \beta_i^{-1}) \backslash \beta_i \Gamma_{\mathfrak{A}}$. Therefore we obtain

$$E_l(Z,s) = \left(\prod_{v \in \infty} \det(\mathrm{Im}\, Z_v)^{(2s-2l+3)/4} \right) \cdot \sum_{i=1}^{h} \prod_{v \in \mathbf{f}} A_v(\beta_i)^{s+3/2} \sum_{\gamma \in T_i} J(\gamma, Z)^{-l} |J(\gamma, Z)|^{-s+l-3/2}. \tag{1.7}$$

For $l > 3$ we define the holomorphic Eisenstein series by

$$E_l(Z) := E_l(Z, l - 3/2).$$

Then we can easily see $E_l(Z) \in M_l(\Gamma_{\mathfrak{A}})$. For a rational prime p, let $\boldsymbol{Q}_p$ denote the field of p-adic numbers. We define a character ψ of the adele ring k_A by $\psi = \prod_v \psi_v$, where

$$\psi_v(x) = \begin{cases} e\,[\text{the fractional part of } -\mathrm{Tr}_{k_v/\boldsymbol{Q}_p}(x)] & \text{for } x \in k_v \text{ if } v|p, \\ e[x] & \text{for } x \in k_v = \boldsymbol{R} \text{ if } v \in \infty. \end{cases} \tag{1.8}$$

We notice that ψ is trivial on k. Now we fix a Haar measure $d\mu_A(x) = \prod_v d\mu_v(x)$ on B_A^- such that

$$\int_{\mathfrak{A}_\mathfrak{p}^-} d\mu_\mathfrak{p}(x) = 1, \quad \int_{B_A^-/B^-} d\mu_A(x) = 1. \tag{1.9}$$

We put

$$d\mu_\infty := \prod_{v \in \infty} d\mu_v.$$

By the Fourier expansion of $E_l^{gr}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, s\right)$ as a function of $x \in B_A^-$, we have

$$\begin{aligned} E_l^{gr}(g,s) &= \sum_{\eta \in B^-} E_{l,\eta}^{gr}(g,s), \\ E_{l,\eta}^{gr}(g,s) &= \int_{B_A^-/B^-} E_l^{gr}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, s\right) \psi(-\mathrm{Tr}_{B/k}(\eta x))\, d\mu_A(x). \end{aligned} \tag{1.10}$$

By (1.5), we have

$$E_l^{gr}\left(\begin{pmatrix} 1 & x+u \\ 0 & 1 \end{pmatrix} g, s\right) = E_l^{gr}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, s\right),$$

for $u \in \mathfrak{A}_\mathfrak{p}^-$, $g \in G_\infty$. Thus $E_{l,\eta}^{gr}(g,s) \neq 0$ only when $\eta \in (\mathfrak{A}^-)^*$. Therefore we have

$$E_l^{gr}(g,s) = \sum_{\eta \in (\mathfrak{A}^-)^*} E_{l,\eta}^{gr}(g,s), \quad g \in G_\infty. \tag{1.11}$$

Since B is division, the following Bruhat decomposition is easily verified.

Lemma 1.3. *We have*

$$G_k = P_k \coprod P_k w N_k, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $N_k = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G_k \right\}$.

By (1.4), (1.10), and Lemma 1.3, we have

$$(1.12) \quad E_{l,\eta}^{gr}(g,s)$$
$$= \int_{B_A^-/B^-} \sum_{\gamma \in P_k \backslash G_k} f_l\left(\gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, s + \frac{3}{2}\right) \psi(-\mathrm{Tr}_{B/k}(\eta x))\, d\mu_A(x)$$
$$= \int_{B_A^-/B^-} \left\{ f_l\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, s + \frac{3}{2}\right) \psi(-\mathrm{Tr}_{B/k}(\eta x)) \right.$$
$$\left. + \sum_{u \in B^-} f_l\left(w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, s + \frac{3}{2}\right) \psi(-\mathrm{Tr}_{B/k}(\eta x)) \right\} d\mu_A(x)$$
$$= f_l\left(g, s + \frac{3}{2}\right) \delta(\eta = 0) + \int_{B_A^-} f_l\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} g, s + \frac{3}{2}\right) \psi(-\mathrm{Tr}_{B/k}(\eta x))\, d\mu_A(x),$$

where $\delta((*)) = 0$ or 1 according as the condition $(*)$ is satisfied or not. Therefore we obtain the following proposition.

Proposition 1.4. *Let l be an even integer and s a complex number such that* $\mathrm{Re}\, s > 3/2$. *If $g \in G_\infty$ we have*

$$E_l^{gr}(g,s) = \sum_{\eta \in (\mathfrak{A}^-)^*} E_{l,\eta}^{gr}(g,s),$$

$$E_{l,\eta}^{gr}(g,s) = f_l(g, s + 3/2)\delta(\eta = 0) + \alpha_{l,\infty}(\eta, g, s)\alpha_{\mathbf{f}}(\eta, s),$$

where

$$\alpha_{l,\infty}(\eta, g_\infty, s) = \prod_{v \in \infty} \alpha_{l,v}(\eta, g_v, s), \quad \alpha_{\mathbf{f}}(\eta, s) = \prod_{v \in \mathbf{f}} \alpha_v(\eta, s),$$

$$\alpha_{l,v}(\eta, g_v, s) = \int_{B_v^-} f_{l,v}\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} g_v, s + \frac{3}{2}\right) \psi_v(-\mathrm{Tr}_{B/k}(\eta x))\, d\mu_v(x),$$

$$\alpha_v(\eta, s) = \int_{B_v^-} A_{\mathfrak{p}}\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\right)^{s+3/2} \psi_v(-\mathrm{Tr}_{B/k}(\eta x))\, d\mu_v(x).$$

2. Local part.

For $0 \neq \eta \in B^-$, we put $K = k(\eta)$ and $K_{\mathfrak{p}} = K \otimes_k k_{\mathfrak{p}}$ for each prime $\mathfrak{p}$, and denote by p the rational prime divided by $\mathfrak{p}$. We denote by $\chi_\eta(\mathfrak{p})$ the Legendre symbol, i.e. it equals -1, 0, or 1 according as $\mathfrak{p}$ remains prime in K, ramifies in K, or splits in K. We denote by $(\mathfrak{A}_{\mathfrak{p}}^-)^*$ the dual lattice of $\mathfrak{A}_{\mathfrak{p}}^-$:

$$(\mathfrak{A}_{\mathfrak{p}}^-)^* = \{x \in B_{\mathfrak{p}}^- \mid \mathrm{Tr}_{B/k}(xy) \in \mathfrak{p}^{-\delta} \text{ for any } y \in \mathfrak{A}_{\mathfrak{p}}^-\},$$

where $\mathfrak{p}^{\delta}$ is the different of $k_{\mathfrak{p}}$ over $\boldsymbol{Q}_p$. An element $\eta \in \mathfrak{A}_{\mathfrak{p}}^{-}$ is said to be primitive if $\pi_{\mathfrak{p}}^{-1}\eta$ is not in $\mathfrak{A}_{\mathfrak{p}}^{-}$. We denote by $(\mathfrak{A}_{\mathfrak{p}}^{-})_{prim}^{*}$ the set of all primitive elements of $\mathfrak{A}_{\mathfrak{p}}^{-}$. For $0 \neq \eta \in (\mathfrak{A}_{\mathfrak{p}}^{-})^{*}$, we define integers $a_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ by the condition:

$$\eta = \pi_{\mathfrak{p}}^{a_{\mathfrak{p}}}\eta_0, \quad (2\pi_{\mathfrak{p}}^{\delta}\eta_0)^2 = d_{\mathfrak{p}}\pi_{\mathfrak{p}}^{2f_{\mathfrak{p}}}, \tag{2.1}$$

where $\eta_0 \in (\mathfrak{A}_{\mathfrak{p}}^{-})_{prim}^{*}$ and $d_{\mathfrak{p}}$ is a generator of the discriminant of $K_{\mathfrak{p}}/k_{\mathfrak{p}}$.

2.1. Ramified primes.

In this subsection $\mathfrak{p}$ denotes a prime ideal of k dividing $\mathfrak{D}$. Let K_0 be the unique unramified quadratic extension field of $k_{\mathfrak{p}}$. We realize $B_{\mathfrak{p}}$ as a cyclic algebra $(K_0, \pi_{\mathfrak{p}})$ i.e. $B_{\mathfrak{p}} = K_0 + K_0\Pi_{\mathfrak{p}}$, $\Pi_{\mathfrak{p}}^2 = \pi_{\mathfrak{p}}$, $\overline{\Pi}_{\mathfrak{p}} = -\Pi_{\mathfrak{p}}$, and $\Pi_{\mathfrak{p}}X\Pi_{\mathfrak{p}}^{-1} = \overline{X}$ for any $X \in K_0$ ($\Pi_{\mathfrak{p}}$ is a prime element of the division quaternion algebra $B_{\mathfrak{p}}$). We denote by $\mathfrak{O}_0$ the maximal order of K_0: so $\mathfrak{O}_{\mathfrak{p}} = \mathfrak{O}_0 + \mathfrak{O}_0\Pi_{\mathfrak{p}}$ is the maximal order of $B_{\mathfrak{p}}$ and $\mathfrak{B}_{\mathfrak{p}} = \pi_{\mathfrak{p}}\mathfrak{O}_0 + \mathfrak{O}_0\Pi_{\mathfrak{p}}$ is the maximal two-sided ideal of $\mathfrak{O}_{\mathfrak{p}}$. Take an element ι of $\mathfrak{O}_0^{\times}$ such that $\bar{\iota} = -\iota$.

Put

$$\mathfrak{A}_{\mathfrak{p}}(v) := B_{\mathfrak{p}}^{-} \cap \Pi_{\mathfrak{p}}^{-v}\mathfrak{A}_{\mathfrak{p}} \quad \text{for } v \in \boldsymbol{Z}.$$

For $\mathfrak{A}_{\mathfrak{p}}(v)$ we define the lattice $\mathfrak{A}_{\mathfrak{p}}^{*}(v)$ by

$$\mathfrak{A}_{\mathfrak{p}}^{*}(v) := \{x \in B_{\mathfrak{p}}^{-} \mid \mathrm{Tr}_{B/k}(xy) \in \mathfrak{p}^{-\delta} \text{ for any } y \in \mathfrak{A}_{\mathfrak{p}}(v)\}.$$

Then we easily see that

$$\mathfrak{A}_{\mathfrak{p}}^{*}(v) = \begin{cases} (\iota/2)\pi_{\mathfrak{p}}^{[v/2]-\delta}\mathfrak{o}_{\mathfrak{p}} + \pi_{\mathfrak{p}}^{[(v+1)/2]-\delta}\Pi_{\mathfrak{p}}^{-1}\mathfrak{O}_0 & \text{if } e_{\mathfrak{p}} = 0, \\ (\iota/2)\pi_{\mathfrak{p}}^{[(v+1)/2]-1-\delta}\mathfrak{o}_{\mathfrak{p}} + \pi_{\mathfrak{p}}^{[v/2]-\delta}\Pi_{\mathfrak{p}}^{-1}\mathfrak{O}_0 & \text{if } e_{\mathfrak{p}} = 1. \end{cases} \tag{2.2}$$

Put

$$L_{\mathfrak{A}_{\mathfrak{p}}}(v) := \mathfrak{A}_{\mathfrak{p}}^{*}(v) - \mathfrak{A}_{\mathfrak{p}}^{*}(v+1) \quad \text{for } v \in \boldsymbol{Z}.$$

From (2.2) we obtain the following lemma. (cf. [**19**])

Lemma 2.1. *Let η be an element of $L_{\mathfrak{A}_{\mathfrak{p}}}(v)$ ($v \in \boldsymbol{Z}$).*

(1) *When $e_{\mathfrak{p}} = 0$, η is written as*

$$\eta = (\iota/2)\pi_{\mathfrak{p}}^{[v/2]-\delta}x + \pi_{\mathfrak{p}}^{[(v+1)/2]-\delta}\Pi_{\mathfrak{p}}^{-1}X \quad (x \in \mathfrak{o}_{\mathfrak{p}}, X \in \mathfrak{O}_0).$$

If v is even, then $X \in \mathfrak{O}_0^{\times}$, $\chi_{\eta}(\mathfrak{p}) = 0$, and $f_{\mathfrak{p}} = -1$. If v is an odd integer, then $x \in \mathfrak{o}_{\mathfrak{p}}^{\times}$, $\chi_{\eta}(\mathfrak{p}) = -1$, and $f_{\mathfrak{p}} = 0$. Here, $f_{\mathfrak{p}}$ is defined in (2.1).

(2) *When $e_{\mathfrak{p}} = 1$, η is written as*

$$\eta = (\iota/2)\pi_{\mathfrak{p}}^{[(v+1)/2]-1-\delta}x + \pi_{\mathfrak{p}}^{[v/2]-\delta}\Pi_{\mathfrak{p}}^{-1}X \quad (x \in \mathfrak{o}_{\mathfrak{p}}, X \in \mathfrak{O}_0).$$

If v is an even integer, then $x \in \mathfrak{o}_{\mathfrak{p}}^{\times}$, $\chi_{\eta}(\mathfrak{p}) = -1$, and $f_{\mathfrak{p}} = -1$. If v is an odd integer, then $X \in \mathfrak{O}_0^{\times}$, $\chi_{\eta}(\mathfrak{p}) = 0$, and $f_{\mathfrak{p}} = -1$.

For $\eta \in B^{-}$ and $t \in \boldsymbol{Z}$ we put

$$V_{\mathfrak{p}}(\eta, t) := \int_{\mathfrak{A}_{\mathfrak{p}}(t)} \psi_{\mathfrak{p}}(\mathrm{Tr}_{B/k}(\eta x))\, d\mu_{\mathfrak{p}}(x) = \delta(\eta \in \mathfrak{A}_{\mathfrak{p}}^{*}(t)) \int_{\mathfrak{A}_{\mathfrak{p}}(t)} d\mu_{\mathfrak{p}}(x). \tag{2.3}$$

The value of $V_{\mathfrak{p}}(\eta, t)$ is given as follows:

Lemma 2.2. *For $\eta \in L_{\mathfrak{A}_{\mathfrak{p}}}(\nu)$ $(\nu \in \boldsymbol{Z})$, we have the following.*
(1) *Let $e_{\mathfrak{p}} = 0$, then*

$$V_{\mathfrak{p}}(\eta, 2t) = \begin{cases} q_{\mathfrak{p}}^{3t} & \text{if } t \leqq [\nu/2], \\ 0 & \text{if } [\nu/2]+1 \leqq t, \end{cases}$$

$$V_{\mathfrak{p}}(\eta, 2t+1) = \begin{cases} q_{\mathfrak{p}}^{3t+2} & \text{if } t \leqq [(\nu+1)/2]-1, \\ 0 & \text{if } [(\nu+1)/2] \leqq t. \end{cases}$$

(2) *Let $e_{\mathfrak{p}} = 1$, then*

$$V_{\mathfrak{p}}(\eta, 2t) = \begin{cases} q_{\mathfrak{p}}^{3t} & \text{if } t \leqq [\nu/2], \\ 0 & \text{if } [\nu/2]+1 \leqq t, \end{cases}$$

$$V_{\mathfrak{p}}(\eta, 2t+1) = \begin{cases} q_{\mathfrak{p}}^{3t+1} & \text{if } t \leqq [(\nu+1)/2]-1, \\ 0 & \text{if } [(\nu+1)/2] \leqq t. \end{cases}$$

Now we are ready to prove the following theorem:

Theorem 2.3. *Let the notation be as above. For $\mathfrak{p} | \mathfrak{D}$ and $\operatorname{Re} s > 3/2$, we have the following.*
(1) *If $\eta = 0$, then*

$$\alpha_{\mathfrak{p}}(0, s) = \begin{cases} (1 - q_{\mathfrak{p}}^{-s-3/2})(1 + q_{\mathfrak{p}}^{-s+1/2})(1 - q_{\mathfrak{p}}^{-2s})^{-1} & \textit{if } e_{\mathfrak{p}} = 0, \\ q_{\mathfrak{p}}^{s+3/2}(1 - q_{\mathfrak{p}}^{-s-3/2})(1 + q_{\mathfrak{p}}^{-s-1/2})(1 - q_{\mathfrak{p}}^{-2s})^{-1} & \textit{if } e_{\mathfrak{p}} = 1. \end{cases}$$

(2) *If $\eta \in (\mathfrak{A}^-)^*$ satisfies $\pi_{\mathfrak{p}}^{-a_{\mathfrak{p}}}\eta \in (\mathfrak{A}_{\mathfrak{p}}^-)^*_{prim}$, then*

$$\alpha_{\mathfrak{p}}(\eta, s) = \begin{cases} (1 - q_{\mathfrak{p}}^{-s-3/2})(1 - \chi_{\eta}(\mathfrak{p}) q_{\mathfrak{p}}^{-s+1/2}) q_{\mathfrak{p}}^{-a_{\mathfrak{p}} s} \hat{\alpha}_{\mathfrak{p}}(\eta, s) & \textit{if } e_{\mathfrak{p}} = 0, \\ q_{\mathfrak{p}}^{s+3/2}(1 - q_{\mathfrak{p}}^{-s-3/2})(1 + q_{\mathfrak{p}}^{-s-1/2})(1 - \chi_{\eta}(\mathfrak{p}) q_{\mathfrak{p}}^{-s-1/2})^{-1} q_{\mathfrak{p}}^{-a_{\mathfrak{p}} s} \hat{\alpha}_{\mathfrak{p}}(\eta, s) & \textit{if } e_{\mathfrak{p}} = 1, \end{cases}$$

where if $e_{\mathfrak{p}} = 0$

$$\hat{\alpha}_{\mathfrak{p}}(\eta, s) := \sum_{t=0}^{a_{\mathfrak{p}}} q_{\mathfrak{p}}^{(2t-a_{\mathfrak{p}})s} + (1 + \chi_{\eta}(\mathfrak{p})) \sum_{t=0}^{a_{\mathfrak{p}}-1} q_{\mathfrak{p}}^{(2t+1-a_{\mathfrak{p}})s+1/2}, \tag{2.4}$$

and if $e_{\mathfrak{p}} = 1$

$$\hat{\alpha}_{\mathfrak{p}}(\eta, s) := \sum_{t=0}^{a_{\mathfrak{p}}} q_{\mathfrak{p}}^{(2t-a_{\mathfrak{p}})s} - \chi_{\eta}(\mathfrak{p}) \sum_{t=0}^{a_{\mathfrak{p}}-1} q_{\mathfrak{p}}^{(2t+1-a_{\mathfrak{p}})s-1/2}. \tag{2.5}$$

Proof. In this proof we refer to the case $e_{\mathfrak{p}} = 0$ [resp. $e_{\mathfrak{p}} = 1$] as Case 0 [resp. Case 1]. Put

$$n_{\mathfrak{p}}(t) := \begin{cases} A_{\mathfrak{p}}\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\right)^{s+3/2} & \text{if } x \in \mathfrak{A}_{\mathfrak{p}}(t), t = 0, \\ A_{\mathfrak{p}}\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\right)^{s+3/2} & \text{if } x \in \mathfrak{A}_{\mathfrak{p}}(t) - \mathfrak{A}_{\mathfrak{p}}(t-1), t \geqq 1. \end{cases}$$

Then by

(Case 0) $$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \in P_\mathfrak{p} C_\mathfrak{p} & \text{if } x \in \mathfrak{A}_\mathfrak{p}(0), \\ \begin{pmatrix} x^{-1} & -1 \\ 0 & \bar{x} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ \bar{x}^{-1} & -1 \end{pmatrix} \in P_\mathfrak{p} C_\mathfrak{p} & \text{if } x \in \mathfrak{A}_\mathfrak{p}(t), t \geqq 1, \end{cases}$$

(Case 1) $$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} = \begin{cases} \begin{pmatrix} \Pi_\mathfrak{p}^{-1} & 0 \\ 0 & \overline{\Pi}_\mathfrak{p} \end{pmatrix}\begin{pmatrix} 0 & \Pi_\mathfrak{p} \\ \overline{\Pi}_\mathfrak{p}^{-1} & \overline{\Pi}_\mathfrak{p}^{-1} x \end{pmatrix} \in P_\mathfrak{p} C_\mathfrak{p} & \text{if } x \in \mathfrak{A}_\mathfrak{p}(0), \\ \begin{pmatrix} x^{-1} & -1 \\ 0 & \bar{x} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ \bar{x}^{-1} & -1 \end{pmatrix} \in P_\mathfrak{p} C_\mathfrak{p} & \text{if } x \in \mathfrak{A}_\mathfrak{p}(t), t \geqq 1, \end{cases}$$

we obtain

(Case 0) $\qquad n_\mathfrak{p}(t) = q_\mathfrak{p}^{-t(s+3/2)} \qquad$ for $t \geqq 0$,

(Case 1) $\qquad n_\mathfrak{p}(t) = q_\mathfrak{p}^{(-t+1)(s+3/2)} \qquad$ for $t \geqq 0$.

Hence, by (2.3) and Lemma 2.2, for $\eta \in L_{\mathfrak{A}_\mathfrak{p}}(v)$ $(0 \leqq v \in \boldsymbol{Z})$ we have

$$\begin{aligned} \alpha_\mathfrak{p}(\eta, s) &= n_\mathfrak{p}(0) V_\mathfrak{p}(\eta, 0) + \sum_{t=0}^{\infty} \{ n_\mathfrak{p}(2t+2) V_\mathfrak{p}(\eta, 2t+2) - n_\mathfrak{p}(2t+2) V_\mathfrak{p}(\eta, 2t+1) \\ &\quad + n_\mathfrak{p}(2t+1) V_\mathfrak{p}(\eta, 2t+1) - n_\mathfrak{p}(2t+1) V_\mathfrak{p}(\eta, 2t) \} \\ &= \sum_{t=0}^{[v/2]} \{ n_\mathfrak{p}(2t) V_\mathfrak{p}(\eta, 2t) - n_\mathfrak{p}(2t+1) V_\mathfrak{p}(\eta, 2t) \} \\ &\quad + \sum_{t=0}^{[(v+1)/2]-1} \{ n_\mathfrak{p}(2t+1) V_\mathfrak{p}(\eta, 2t+1) - n_\mathfrak{p}(2t+2) V_\mathfrak{p}(\eta, 2t+1) \}. \end{aligned}$$

Therefore by Lemma 2.1 and Lemma 2.2 we can obtain the following

(Case 0) $$\alpha_\mathfrak{p}(\eta, s) = (1 - q_\mathfrak{p}^{-s-3/2}) \left\{ (1 - q_\mathfrak{p}^{-s+1/2}) \sum_{t=0}^{[v/2]} q_\mathfrak{p}^{-2st} - (1 + \chi_\eta(\mathfrak{p})) q_\mathfrak{p}^{-2[v/2]s-s+1/2} \right\},$$

(Case 1) $$\alpha_\mathfrak{p}(\eta, s) = q_\mathfrak{p}^{s+3/2}(1 - q_\mathfrak{p}^{-s-3/2}) \left\{ (1 - q_\mathfrak{p}^{-s-1/2}) \sum_{t=0}^{[v/2]} q_\mathfrak{p}^{-2st} + \chi_\eta(\mathfrak{p}) q_\mathfrak{p}^{-2[v/2]s-s-1/2} \right\}.$$

Notice that $\eta \in L_{\mathfrak{A}_\mathfrak{p}}(2a_\mathfrak{p}) \coprod L_{\mathfrak{A}_\mathfrak{p}}(2a_\mathfrak{p}+1)$ if and only if $\pi_\mathfrak{p}^{-a_\mathfrak{p}} \eta \in (\mathfrak{A}_\mathfrak{p}^-)^*_{prim}$. So we see the second part of our theorem. The remaining part of our theorem can be proved by the same way. $\square$

Corollary 2.4. *Let the notation be the same as in the above theorem.*

(1) *For each $\eta \in (\mathfrak{A}_\mathfrak{p}^-)^*$, $\alpha_\mathfrak{p}(\eta, s)$ can be continued as a meromorphic function to the whole s-plane.*

(2) *For each $0 \neq \eta \in (\mathfrak{A}_\mathfrak{p}^-)^*$, $\hat{\alpha}_\mathfrak{p}(\eta, s)$ is an entire function on the whole s-plane, and it is invariant under $s \mapsto -s$.*

2.2. Unramified primes.

In this subsection we assume that $\mathfrak{p}$ is a prime ideal of k not dividing $\mathfrak{D}$: so $B_{\mathfrak{p}} \cong M_2(k_{\mathfrak{p}})$, $\mathfrak{A}_{\mathfrak{p}} \cong M_2(\mathfrak{o}_{\mathfrak{p}})$. We denote $\eta \in (\mathfrak{A}_{\mathfrak{p}}^-)^*$ which satisfies (2.1) by $\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}}$. Then we notice that

$$f_{\mathfrak{p}} \geq 0 \tag{2.6}$$

(cf. [**19**]). Shimura [**17**] treated $\alpha_{\mathfrak{p}}(\eta, s)$ in a more general situation. We calculate the explicit form of $\alpha_{\mathfrak{p}}(\eta, s)$ (cf. [**6**], [**7**]).

Theorem 2.5. *For $\mathfrak{p} \nmid \mathfrak{D}$ and $\mathrm{Re}\, s > 3/2$, we have the following.*

(1) *If $\eta = 0$, then*

$$\alpha_{\mathfrak{p}}(0, s) = (1 - q_{\mathfrak{p}}^{-s-3/2})(1 - q_{\mathfrak{p}}^{-2s-1})(1 - q_{\mathfrak{p}}^{-2s})^{-1}(1 - q_{\mathfrak{p}}^{-s+1/2})^{-1}.$$

(2) *If $\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}} \in (\mathfrak{A}^-)^*$, then*

$$\alpha_{\mathfrak{p}}(\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}}, s) = (1 - q_{\mathfrak{p}}^{-s-3/2})(1 - q_{\mathfrak{p}}^{-2s-1})(1 - \chi_\eta(\mathfrak{p}) q_{\mathfrak{p}}^{-s-1/2})^{-1} q_{\mathfrak{p}}^{(-a_{\mathfrak{p}}-f_{\mathfrak{p}})s} \hat{\alpha}_{\mathfrak{p}}(\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}}, s),$$

where

$$\hat{\alpha}_{\mathfrak{p}}(\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}}, s) := \sum_{t=0}^{a_{\mathfrak{p}}} \left\{ \sum_{k=0}^{a_{\mathfrak{p}}+f_{\mathfrak{p}}-t} q_{\mathfrak{p}}^{(t+2k)s+t/2-(a_{\mathfrak{p}}+f_{\mathfrak{p}})s} - \chi_\eta(\mathfrak{p}) \sum_{k=0}^{a_{\mathfrak{p}}+f_{\mathfrak{p}}-t-1} q_{\mathfrak{p}}^{(t+2k+1)s+t/2-1/2-(a_{\mathfrak{p}}+f_{\mathfrak{p}})s} \right\}. \tag{2.7}$$

As a corollary we get

Corollary 2.6. *The notations being the same as in the above theorem.*

(1) *For each $\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}} \in (\mathfrak{A}_{\mathfrak{p}}^-)^*$, $\alpha_{\mathfrak{p}}(\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}}, s)$ is continued as a meromorphic function to the whole s-plane, and saisfies the local Maass relation*

$$\alpha_{\mathfrak{p}}(\xi_{a_{\mathfrak{p}},f_{\mathfrak{p}}}, s) = \sum_{t=0}^{a_{\mathfrak{p}}} q_{\mathfrak{p}}^{(-s+1/2)t} \alpha_{\mathfrak{p}}(\eta_{0,\alpha_{\mathfrak{p}}+f_{\mathfrak{p}}-t}, s)$$

(2) *For each $\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}} \in (\mathfrak{A}_{\mathfrak{p}}^-)^*$, $\hat{\alpha}_{\mathfrak{p}}(\eta_{a_{\mathfrak{p}},f_{\mathfrak{p}}}, s)$ is an entire function on the whole s-plane and it is invariant under $s \mapsto -s$.*

2.3. Archimedean part.

Let W_m be the set of real symmetric matrices of size m and $d\mu(x)$ the ordinary Lebesgue measure on W_m. Set for $0 < g \in W_m$, $h \in W_m$

$$\xi_m(g, h; \alpha, \beta) := \int_{W_m} e[-\mathrm{Tr}(hx)] \det(x + ig)^{-\alpha} \det(x - ig)^{-\beta}\, d\mu(x), \tag{2.8}$$

which is convergent for $\mathrm{Re}(\alpha + \beta) > m$. This integral was studied by Shimura [**16**] and it is known that this can be expressed by generalized hypergeometric functions.

By the uniqueness of the Haar measure on B_∞^-, the following relation between $d\mu_\infty$ and $d\mu$:

$$d\mu_\infty(x) = \frac{1}{(\sqrt{|d_k|})^3 D_0 D_1^2} \prod_{v \in \infty} d\mu(x)_v. \tag{2.9}$$

Here, d_k denotes the discriminant $k/\boldsymbol{Q}$, and

$$D_i := \prod_{\mathfrak{p}|\mathfrak{D}_i} |\mathfrak{o}_\mathfrak{p}/\mathfrak{p}| \quad (i = 0, 1).$$

By (2.8) and (2.9), the following proposition is obtained.

Proposition 2.7. *Let l be an even integer and s a complex number. Then for $Z = g\langle Z_0\rangle = X + iY \in \mathfrak{H}$ $(g \in G_\infty)$,*

$$\alpha_{l,\infty}(\eta, g, s)J(g, Z_0)^l = \frac{1}{(\sqrt{|d_k|})^3 D_0 D_1^2} \cdot \prod_{v \in \infty}\left\{\det(Y_v)^{(2s-2l+3)/4}\xi_2\left(J^{-1}Y_v, \eta_v J; \frac{2s+2l+3}{4}, \frac{2s-2l+3}{4}\right)\right\}e[\sigma(\eta X)],$$

where $\sigma = \mathrm{Tr}_{k/\boldsymbol{Q}} \circ \mathrm{Tr}_{B/k}$.

Proof. Put $g_v = \begin{pmatrix} 1 & X_v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_v & 0 \\ 0 & \bar{y}_v^{-1} \end{pmatrix} \in P_v$.

(2.10) $\alpha_{l,v}(\eta, g_v, s)$

$$= \int_{B_v^-} f_{l,v}\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} g_v, s + \frac{3}{2}\right)\psi_v(-\mathrm{Tr}_{B/k}(\eta_v x))\, d\mu_v(x)$$

$$= e[\mathrm{Tr}_{B/k}(\eta_v X_v)]\int_{B_v^-} f_{l,v}\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\begin{pmatrix} y_v & 0 \\ 0 & \bar{y}_v^{-1} \end{pmatrix}, s + \frac{3}{2}\right)e[-\mathrm{Tr}_{B/k}(\eta_v x)]\, d\mu_v(x).$$

To obtain an explicit description of (2.10), we take a decomposition

$$\begin{pmatrix} 0 & 1 \\ 1 & x_v \end{pmatrix}\begin{pmatrix} y_v & 0 \\ 0 & \bar{y}_v^{-1} \end{pmatrix} = \begin{pmatrix} 1 & x_{1,v} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_{1,v} & 0 \\ 0 & \bar{y}_{1,v}^{-1} \end{pmatrix}\omega_v \in P_v C_v. \tag{2.11}$$

Comparing the action of both sides of (2.11) for Z_0, and automorphic factors, we have

$$f_{l,v}\left(\begin{pmatrix} 0 & 1 \\ 1 & x_v \end{pmatrix}\begin{pmatrix} y_v & 0 \\ 0 & \bar{y}_v^{-1} \end{pmatrix}, s + \frac{3}{2}\right)$$

$$= |\det(x_v\bar{y}_v^{-1}J^{-1}y_v^{-1}x_v + y_v J\bar{y}_v)|^{(-2s+2l-3)/4}\det(iy_v J + x_v\bar{y}_v^{-1})^{-l}$$

$$= (\det(y_v))^{s+3/2}\det(J^{-1}x_v + i{}^t\bar{y}_v\bar{y}_v)^{(-2s-2l-3)/4}\det(J^{-1}x_v - i{}^t\bar{y}_v\bar{y}_v)^{(-2s+2l-3)/4}$$

Hence, by (2.8) and (2.9), for $g = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & \bar{y}^{-1} \end{pmatrix} w \in P_\infty C_\infty = G_\infty$,

$$\alpha_{l,\infty}(\eta, g, s) = \frac{1}{(\sqrt{|d_k|})^3 D_0 D_1^2}J(w, Z_0)^{-l} \cdot \prod_{v \in \infty}\left\{\det(y_v)^{s+3/2}\xi_2\left({}^t\bar{y}_v\bar{y}_v, \eta_v J; \frac{2s+2l+3}{4}, \frac{2s-2l+3}{4}\right)\right\}e[\sigma(\eta X)]$$

On the other hand

$$g_v\langle Z_0\rangle = X_v + iJ\,{}^t\bar{y}_v\bar{y}_v.$$

Therefore we obtain this proposition. $\square$

3. Global part.

3.1. Hypergeometric functions.

In this section, we summarize some properties of hypergeometric functions which appear in the Fourier coefficients of the Eisenstein series. Put

$$\Gamma_m(s) := \pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma\left(s-\frac{j}{2}\right) \quad \text{for } s \in \boldsymbol{C},$$

where $\Gamma_1(s) = \Gamma(s)$ is the ordinary gamma function and we understand as $\Gamma_0(s) = 1$. We put

$$\kappa(m) := (m+1)/2 \quad \text{for } 0 \leqq m \in \boldsymbol{Z},$$
$$W_m := \{x \in M_m(\boldsymbol{R}) \mid {}^t x = x\}, \quad W_m^+ := \{x \in W_m \mid x > 0\}.$$

For non-negative integers p, q with $p+q=m$, we denote by $W_m(p,q)$ the subset of W_m consisting of the non-singular elements with p positive and q negative eigenvalues. We put for $x \in M_m(\boldsymbol{R})$

$$\begin{cases} \delta_+(x) := \text{the product of all positive real eigenvalues of } x, \\ \delta_-(x) := \delta_+(-x), \\ \tau(x) := \text{the sum of all absolute values of real eigenvalues of } x, \end{cases}$$

and denote by $\mu(x)$ [resp. $\lambda(x)$] the smallest [resp. largest] absolute value of non-zero eigen-values of x. We note that hg has only real eigenvalues, if $g \in W_m^+$ and $h \in W_m$. For $g \in W_m^+$, $h \in W_m$, and $(\alpha,\beta) \in \boldsymbol{C}^2$, put

$$\begin{aligned}(3.1)\quad &\eta_m^*(g,h;\alpha,\beta) \\ &:= \det(g)^{\alpha+\beta-\kappa(m)}\int_{\substack{W_m \\ x\pm h>0}} \exp(-\mathrm{Tr}(gx))\det(x+h)^{\alpha-\kappa(m)}\det(x-h)^{\beta-\kappa(m)}\,d\mu(x),\end{aligned}$$

which is convergent for $\mathrm{Re}(\alpha) > \kappa(m)-1$, $\mathrm{Re}(\beta) > m$. By [[**16**], (1.29), (4.3)]

$$\begin{aligned}(3.2)\qquad \xi_m(g,h;\alpha,\beta) = {}&\exp(\pi i m(\beta-\alpha)/2)2^m\pi^{m\kappa(m)}\Gamma_m(\alpha)^{-1}\Gamma_m(\beta)^{-1} \\ &\cdot \det(2g)^{-\alpha-\beta+\kappa(m)}\eta_m^*(2g,\pi h;\alpha,\beta)\end{aligned}$$

for $\mathrm{Re}(\alpha) > \kappa(m)-1$, $\mathrm{Re}(\beta) > m$, and

$$(3.3)\qquad \eta_m^*(g,0;\alpha,\beta) = \Gamma_m(\alpha+\beta-\kappa(m)).$$

We also use the following functions introduced by Shimura [[**16**], (4.6K)]: For $g \in W_m^+$,

$h \in W_m(p,q)$, and $(\alpha,\beta) \in \boldsymbol{C}^2$, put

$$(3.4) \qquad \omega_m(g,h;\alpha,\beta) := 2^{-p\alpha-q\beta}\Gamma_p(\beta-q/2)^{-1}\Gamma_q(\alpha-p/2)^{-1} \cdot \delta_+(hg)^{\kappa(m)-\alpha-q/4}\delta_-(hg)^{\kappa(m)-\beta-p/4}\eta_m^*(g,h;\alpha,\beta),$$

and put

$$(3.5) \qquad \omega_m(g,0;\alpha,\beta) := 1$$

The following theorem is one of the main results in [**16**].

THEOREM 3.1 (Shimura). (1) $\omega_m(g,h;\alpha,0) = \omega_m(g,h;\kappa(m),\beta) = 2^{-m\kappa(m)}\exp(-\mathrm{Tr}(gh))$ *if* $h \in W_m^+$.

(2) $\omega_m(g,h;\alpha,\beta)$ *extends to a holomorphic function in* (α,β) *to the whole* $\boldsymbol{C}^2$, *and is real analytic in* $(g,\alpha,\beta) \in W_m^+ \times \boldsymbol{C}^2$.

(3) $\omega_m(g,h;\alpha,\beta) = \omega_m(g,h;\kappa(m)-\beta,\kappa(m)-\alpha)$ *if* $\det(h) \neq 0$.

The following proposition, which is obtained from [**16**], will be used later (cf. [**12**] Proposition 3.4).

PROPOSITION 3.2. (1) *For* $g \in W_m(p,q)$ *with* $p+q=m$, *the function*

$$\Gamma_q(\alpha-p/2)^{-1}\Gamma_p(\beta-q/2)^{-1}\eta_m^*(g,h;\alpha,\beta)$$

extends to a holomorphic function in $(\alpha,\beta) \in \boldsymbol{C}^2$ *which is real analytic in* $(g,\alpha,\beta) \in W_m^+ \times \boldsymbol{C}^2$.

(2) *Let* $g \in W_m^+$, *and* $h \in W_m(p,q)(p+q=m)$. *Then, given a constant* $\rho > 0$ *and a compact subset* T *of* $\boldsymbol{C}^2$, *there exist constants* $C_1, C_2, C_3 > 0$ *depending only on* m, ρ, *and* T *such that*

$$|\Gamma_q(\alpha-p/2)^{-1}\Gamma_p(\beta-q/2)^{-1}\eta_m^*(2g,\pi h;\alpha,\beta)| \leqq C_1\exp(-\tau(hg))\tau(g)^{C_2}(\lambda(hg)^{C_3}+\mu(hg)^{-C_3})$$

for $(\alpha,\beta) \in T$, $\mu(g) \geqq \rho$.

3.2. Singular series.

Here we write an explicit formula for the singular series $\alpha_{\mathbf{f}}$ defined in Proposition 1.4. We denote by $\zeta_k(s)$ the Dedekind zeta-function, and denote by $L_k(s,\chi_\eta)$ the L-function for a quadratic character χ_η of $k(\eta)/k$. For $\mathrm{Re}\, s > 3/2$ and $0 \neq \eta \in (\mathfrak{A}_{\mathfrak{p}}^-)^*$, we put

$$\hat{\alpha}_{\mathbf{f}}(\eta,s) := \prod_{\mathfrak{p}\in\mathbf{f}} \hat{\alpha}_{\mathfrak{p}}(\eta,s),$$

where $\hat{\alpha}_{\mathfrak{p}}(\eta,s)$ is defined by (2.4), (2.5), and (2.7). The function $\hat{\alpha}_{\mathbf{f}}(\eta,s)$ is an entire function in s and satisfies

$$(3.6) \qquad \hat{\alpha}_{\mathbf{f}}(\eta,s) = \hat{\alpha}_{\mathbf{f}}(\eta,-s).$$

Putting our results of Theorem 2.3 and Theorem 2.5 together, we have the following proposition.

Proposition 3.3. *For* $\operatorname{Re} s > 3/2$ *we have*

$$\alpha_{\mathbf{f}}(0,s) = \frac{\zeta_k(2s)\zeta_k(s-\frac{1}{2})}{\zeta_k(s+\frac{3}{2})\zeta_k(2s+1)} \prod_{\mathfrak{p}|\mathfrak{D}_0} \frac{1-q_{\mathfrak{p}}^{-2s+1}}{1-q_{\mathfrak{p}}^{-2s-1}} \prod_{\mathfrak{p}|\mathfrak{D}_1} \frac{q_{\mathfrak{p}}^{s+3/2}(1-q_{\mathfrak{p}}^{-s+1/2})}{1-q_{\mathfrak{p}}^{-s-1/2}}$$

if $\eta = 0$, *and*

$$\alpha_{\mathbf{f}}(\eta,s) = \frac{L_k(s+\frac{1}{2},\chi_\eta)}{\zeta_k(s+\frac{3}{2})\zeta_k(2s+1)} \prod_{\mathfrak{p}|\mathfrak{D}_0} \frac{(1-\chi_\eta(\mathfrak{p})q_{\mathfrak{p}}^{-s-1/2})(1-\chi_\eta(\mathfrak{p})q_{\mathfrak{p}}^{-s+1/2})}{1-q_{\mathfrak{p}}^{-2s-1}}$$
$$\cdot \prod_{\mathfrak{p}|\mathfrak{D}_1} \frac{q_{\mathfrak{p}}^{s+3/2}}{1-q_{\mathfrak{p}}^{-s-1/2}} \left(2^n |d_k|^2 \, |d_K|^{-1/2} \sqrt{|N_{k/\boldsymbol{Q}}(\eta^2)|} \prod_{\substack{\mathfrak{p}|\mathfrak{D}_0 \\ \chi_\eta(\mathfrak{p})=0}} q_{\mathfrak{p}} \prod_{\mathfrak{p}|\mathfrak{D}_1} q_{\mathfrak{p}} \right)^{-s} \hat{\alpha}_{\mathbf{f}}(\eta,s)$$

if $0 \neq \eta \in (\mathfrak{A}^-)^*$. *Here* $K := k(\eta)$ *is a quadratic number field over* k, d_k [*resp.* d_K] *is the discriminant of* k [*resp.* K] *over* $\boldsymbol{Q}$.

From above proposition the singular series $\alpha_{\mathbf{f}}(\eta,s)$ has a meromorphic continuation to the whole s-plane and we obtain the following facts. For an arbitrary $s_0 \in \boldsymbol{C}$, there exist constants $\delta > 0$ and $0 \leqq t \in \boldsymbol{Z}$ depending only on s_0 such that the function

$$(s-s_0)^t \alpha_{\mathbf{f}}(\eta,s)$$

are holomorphic on $U_\delta(s_0)$ for all $\eta \in (\mathfrak{A}^-)^*$. Here $U_\delta(s_0) := \{s \in \boldsymbol{C} \,|\, |s-s_0| < \delta\}$. Moreover let s_0, δ, and t be as above. There exist positive constants C_1, C_2 depending only on s_0, δ, and t such that

$$|(s-s_0)^t \alpha_{\mathbf{f}}(\eta,s)| \leqq C_1 |N_{k/\boldsymbol{Q}}(\eta^2)|^{C_2} \tag{3.7}$$

for all $s \in U_\delta(s_0)$ and $\eta \in (\mathfrak{A}^-)^*$.

3.3. Continuation and functional equation of $E_l(Z,s)$

Hereafter we assume l be an even non-negative integer.

Theorem 3.4. (1) *For* $\operatorname{Re} s > 3/2$ *and* $Z = X + iY \in \mathfrak{H}$, *the Eisenstein series* $E_l(Z,s)$ *has the following expansion*;

$$E_l(Z,s) = \sum_{\eta \in (\mathfrak{A}^-)^*} a_l(\eta, Y, s) e[\sigma(\eta X)], \tag{3.8}$$

where

$$a_l(0,Y,s) = \prod_{v \in \infty} (\det(Y_v))^{(2s-2l+3)/4} + \frac{1}{(\sqrt{|d_k|})^3 D_0 D_1^2} \prod_{v \in \infty} \left\{ (\det(Y_v))^{(-2s-2l+3)/4} \right.$$
$$\left. \cdot\, 2^{-2s+2} \pi^3 \Gamma_2\left(\frac{2s+2l+3}{4}\right)^{-1} \Gamma_2\left(\frac{2s-2l+3}{4}\right)^{-1} \Gamma_2(s) \right\} \alpha_{\mathbf{f}}(0,s) \tag{3.9}$$

if $\eta = 0$, *and*

$$(3.10)\qquad a_l(\eta, Y, s) = \frac{1}{(\sqrt{|d_k|})^3 D_0 D_1^2} \prod_{v \in \infty} \left\{ (\det(Y_v))^{(-2s-2l+3)/4} 2^{-2s+2} \pi^3 \Gamma_2\left(\frac{2s+2l+3}{4}\right)^{-1} \right.$$
$$\left. \cdot \Gamma_2\left(\frac{2s-2l+3}{4}\right)^{-1} \eta_2^*\left(2J^{-1}Y_v, \pi\eta_v J, \frac{2s+2l+3}{4}, \frac{2s-2l+3}{4}\right) \right\} \alpha_{\mathbf{f}}(\eta, s)$$

if $\eta \neq 0$.

(2) *For any* $s_0 \in \boldsymbol{C}$, *there exist constants* $\delta > 0$ *and* $0 \leqq t \in \boldsymbol{Z}$ *depending only on* l *and* s_0 *such that*

$$(s - s_0)^t a_l(\eta, Y, s)$$

is holomorphic in s *on* $U_\delta(s_0)$ *for every* $\eta \in (\mathfrak{A}^-)^*$, *and the series*

$$(3.11)\qquad \sum_{\eta \in (\mathfrak{A}^-)^*} (s - s_0)^t a_l(\eta, Y, s) e[\mathrm{Tr}_{B/k}(\eta X)],$$

converges absolutely and locally uniformly in $(Z, s) \in \mathfrak{H} \times U_\delta(s_0)$. *Thus* (3.8)–(3.10) *give the analytic continuation of* $E_l(Z, s)$ *to the whole* s*-plane.*

Proof. The first assertion follows from Proposition 1.4, Proposition 2.7, (3.2), and (3.3). We shall prove the second assertion. By the assertion (1), Proposition 3.2, and Proposition 3.3, there exist positive constants $\delta > 0$ and $0 < t \in \boldsymbol{Z}$ depending only on l and s_0 such that

$$(s - s_0)^t a_l(\eta, Y, s)$$

is holomorphic in s on $U_\delta(s_0)$ and real analytic in (Y, s) on

$$\left(\prod_{v \in \infty} W_2^+\right) \times U_\delta(s_0).$$

By Proposition 3.2 and 3.7, for a given $\rho > 0$, there exist positive constants $C_1, \ldots, C_6$ depending only on ρ, l, s_0, and t such that

$$(3.12)\qquad |(s - s_0)^t a_l(0, Y, s)| \leqq C_1 \prod_{v \in \infty} (\mathrm{Tr}_{B/k}(J^{-1}Y_v))^{C_2}$$

$$(3.13)\qquad |(s - s_0)^t a_l(\eta, Y, s)| \leqq C_3 \exp\left(-\sum_{v \in \infty} \tau(\eta_v Y_v)\right) \left(\prod_{v \in \infty} \mathrm{Tr}_{B/k}(J^{-1}Y_v)\right)^{C_4}$$
$$\cdot |N_{k/\boldsymbol{Q}}(\eta^2)|^{C_5} \left(\prod_{v \in \infty} \lambda(\eta_v J)^{C_6} + \prod_{v \in \infty} \mu(\eta_v J)^{-C_6}\right)$$

for $\mu(J^{-1}Y_v) \geqq \rho$ $(v \in \infty)$ and $s \in U_\delta(s_0)$. From (3.12) and (3.13), there exist positive constants $C_7, \ldots, C_{10}$ depending only on δ, ρ, l, s_0, and t such that

$$(3.14)\quad \sum_{\eta\in(\mathfrak{A}^-)^*} |(s-s_0)^t a_l(\eta, Y, s) e[\sigma(\eta X)]|$$
$$\leqq C_7 \prod_{v\in\infty} (\mathrm{Tr}_{B/k}(J^{-1}Y_v))^{C_8} \Big\{ 1 + \sum_{\substack{\eta\in(\mathfrak{A}^-)^* \\ \eta\neq 0}} \exp\Big(-\sum_{v\in\infty} \tau(\eta_v Y_v)\Big) |N_{k/\mathbf{Q}}(\eta^2)|^{C_9}$$
$$\cdot \Big(\prod_{v\in\infty} \lambda(\eta_v J)^{C_{10}} + \prod_{v\in\infty} \mu(\eta_v J)^{-C_{10}} \Big) \Big\}$$

for $\mu(J^{-1}Y_v) \geqq \rho$ $(v\in\infty)$ and $s\in U_\delta(s_0)$. If $\mu(J^{-1}Y_v)\geqq \rho > 0$, then

$$(3.15)\qquad \tau(\eta_v Y_v) \geqq 2\rho |N_{B/k}(\eta_v)|^{1/2}$$

$$(3.16)\qquad \tau(\eta_v Y_v) \geqq \rho\lambda(\eta_v J)$$

Hence (3.14), (3.15), (3.16), and the Schwarz' inequality give

$$\sum_{\eta\in(\mathfrak{A}^-)^*} |(s-s_0)^t a_l(\eta, Y, s) e[\sigma(\eta X)]| \leqq C_{11} \Big(\prod_{v\in\infty} \mathrm{Tr}_{B/k}(J^{-1}Y_v) \Big)^{C_{12}}$$

for $\mu(J^{-1}Y_v)\geqq\rho$ $(v\in\infty)$, $s_0\in U_\delta(s_0)$. Here the constants C_{11}, C_{12} depend only on l, t, s_0, ρ and δ. This completes the proof of the assertion (2) of our theorem. □

In the rest of this section, we present a proof of a functional equation of $E_l(Z,s)$ by means of investigations of explicit Fourier coefficients given in Theorem 3.4. Put

$$\xi_k(s) := |d_k|^{s/2} \pi^{-sn/2} \Gamma(s/2)^n \zeta_k(s).$$

Then $\xi_k(s)$ is continued as a meromorphic function in s on the whole complex plane with only simple poles at $s=0,1$ and satisfies the functional equation $\xi_k(s)=\xi_k(1-s)$. For $0\neq\eta\in(\mathfrak{A}^-)^*$, $v\in\infty$, we assume that $\eta_v J\in W_2(p_v,q_v)$ $(p_v+q_v=2)$ and put

$$\Lambda_k(s,\chi_\eta) := |d_{k(\eta)}/d_k|^{s/2}\pi^{-sn/2} \prod_{v\in\infty} \Gamma((s+1-p_vq_v)/2) L_k(s,\chi_\eta).$$

Since χ_η is non-trivial (cf. Lemma 2.1), $\Lambda(s,\chi_\eta)$ is continued as an entire function and is invariant under $s\mapsto 1-s$. The following lemma is elementary.

Lemma 3.5. *Let l be an even non-negative integer and let p,q be non-negative integers with $p+q=2$. Then*

$$\Gamma_p((2s+2l+3)/4)^{-1}\Gamma_q((2s-2l+3)/4)^{-1}$$
$$= (-1)^{lpq/2} 2^{s-1/2} \pi^{-1+pq/2} \varepsilon_l^{(p,q)}(s) \frac{\Gamma((2s+3-2pq)/4)}{\Gamma((2s+3)/4)\Gamma((2s+1)/2)}.$$

Here

$$\varepsilon_l^{(p,q)}(s) := \begin{cases} \prod_{j=0}^{l/2-1}((2s+3)/4+j)^{-1}((2s+1)/4+j)^{-1} & \text{if } p=2, q=0, \\ \prod_{j=0}^{l/2-1}((-2s+3)/4+j)((-2s+1)/4+j) & \text{if } p=0, q=2, \\ \prod_{j=0}^{l/2-1}((-2s+1)/4+j)((2s+3)/4+j)^{-1} & \text{if } p=1, q=1. \end{cases}$$

Put

$$(3.17)\qquad Q_l(s) := \prod_{j=0}^{l/2-1}((2s+3)/4+j)((2s+1)/4+j).$$

Then we note that $Q_l(s)\varepsilon_l^{(p,q)}(s)$ is invariant under the substitution $s \mapsto -s$.

We now introduce the normalized Eisenstein series

$$(3.18)\qquad E_l^*(Z,s) := \xi_k\left(s+\frac{3}{2}\right)\xi_k(2s+1)Q_l(s)^n \prod_{\mathfrak{p}|\mathfrak{D}_0}(q_\mathfrak{p}^{s+1/2}-q_\mathfrak{p}^{-s-1/2}) \cdot \prod_{\mathfrak{p}|\mathfrak{D}_1}(q_\mathfrak{p}^2-q_\mathfrak{p}^{-s+3/2})E_l(Z,s).$$

Proposition 3.6. *Put*

$$E_l^*(Z,s) := \sum_{\eta\in(\mathfrak{A}^-)^*} a_l^*(\eta,Y,s)e[\sigma(\eta X)],$$

for $Z = X+iY \in \mathfrak{H}$ and $s \in \boldsymbol{C}$. Then $a_l^*(\eta,Y,s)$ extends to a meromorphic function in s to the whole complex plane, and is invariant under the substitution $s \mapsto -s$. Precisely, if $\eta \neq 0$ or $l \geqq 2$, then $a_l^*(\eta,Y,s)$ is holomorphic in s on the whole complex plane. On the other hand $a_0^*(0,Y,s)$ has only simple poles at $s = \pm 3/2$ in the case of $l = 0$.

Proof. First we assume $\eta \neq 0$. By (3.4), (3.10), and Proposition 3.3,

$$(3.19)\qquad a_l^*(\eta,Y,s) = \sqrt{\frac{D_1^3}{D_0}}|N_{k/\boldsymbol{Q}}(\eta^2)|^{-3/4}|d_K|^{-1/4}\prod_{v\in\infty}\Bigg\{\det(Y_v)^{-l/2}\delta_+(\eta_vY_v)^{(2l+q_v)/4} \cdot \delta_-(\eta_vY_v)^{(-2l+p_v)/4}\omega_2\left(2J^{-1}Y_v,\pi\eta_vJ;\frac{2s+2l+3}{4},\frac{2s-2l+3}{4}\right)\Bigg\} \cdot Q_{l,\eta}(s)\Lambda_k\left(s+\frac{1}{2},\chi_\eta\right)\prod_{\substack{\mathfrak{p}|\mathfrak{D}_0\\ \chi_\eta(\mathfrak{p})=-1}}\left\{q_\mathfrak{p}^{-1/2}(1+q_\mathfrak{p}^{s+1/2})(1+q_\mathfrak{p}^{-s+1/2})\right\} \cdot \hat{\alpha}_\mathfrak{f}(\eta,s),$$

where $\eta_vJ \in W_2(p_v,q_v)$, and we put

$$Q_{\eta,l}(s) := \prod_{v\in\infty}\{(-1)^{lp_vq_v/2}2^{(2l(p_v-q_v)+p_vq_v+3)/2}\pi^{(l(p_v-q_v)-1)/2}Q_l(s)\varepsilon_l^{(p_v,q_v)}(s)\}.$$

By Theorem 3.1(3), (3.6), and Lemma 3.5, we obtain results in this case. Secondarily let $\eta = 0$. From (3.9), Proposition 3.3, and (3.3) we obtain the following:

$$(3.20)\qquad a_l^*(0,Y,s) = \left(\prod_{v\in\infty}\det(Y_v)\right)^{(2s-2l+3)/2}\xi_k\left(s+\frac{3}{2}\right)\xi_k(2s+1)Q_l(s)^n \cdot \prod_{\mathfrak{p}|\mathfrak{D}_0}(q_\mathfrak{p}^{s+1/2}-q_\mathfrak{p}^{-s-1/2})\prod_{\mathfrak{p}|\mathfrak{D}_1}(q_\mathfrak{p}^2-q_\mathfrak{p}^{-s+3/2})$$

$$+\left(\prod_{v\in\infty}\det(Y_v)\right)^{(-2s-2l+3)/2}\xi_k\left(s-\frac{1}{2}\right)\xi_k(2s)Q_l(-s)^n$$
$$\cdot\prod_{\mathfrak{p}|\mathfrak{D}_0}(q_{\mathfrak{p}}^{s-1/2}-q_{\mathfrak{p}}^{-s+1/2})\prod_{\mathfrak{p}|\mathfrak{D}_1}(q_{\mathfrak{p}}^{s+3/2}-q_{\mathfrak{p}}^2).$$

Notice that the number of prime ideals dividing $\mathfrak{D}$ is even, since B is totally indefinite. So we get the functional equation. The possible poles are simple poles at $s=0$, $\pm 3/2$ and double poles at $s=\pm 1/2$. By virtue of the functional equation, $a_l^*(0,Y,s)$ is regular at $s=0$. Since $\mathfrak{D}_0\mathfrak{D}_1$ $(=\mathfrak{D})$ has at least two prime divisors, $a_l^*(0,Y,s)$ is regular at $s=\pm 1/2$. Furthermore if $l\geqq 2$, $Q_l(-3/2)=0$. So $a_l^*(0,Y,s)$ is regular at $s=\pm 3/2$. □

From Theorem 3.4(2) and Proposition 3.6, we obtain the following theorem.

Theorem 3.7. *For any $Z\in\mathfrak{H}$, the normalized Eisenstein series $E_l^*(Z,s)$ is continued as a meromorphic function in s on the whole complex plane and is invariant under $s\mapsto -s$. Precisely, if $l\geqq 2$, then $E_l^*(Z,s)$ is the entire function on the whole complex plane. On the other hand $E_0^*(Z,s)$ has only simple poles at $s=\pm 3/2$ in the case of $l=0$.*

The convergence of (1.7) is not guaranteed if $l=2$ and $s=3/2$. However, as in Shimura [**17**], we can construct the holomorphic Eisenstein series of weight 2 as follows:

Theorem 3.8. *We define*

$$E_2(Z):=E_2(Z,1/2).$$

Then $E_2(Z)$ is a holomorphic function in Z on $\mathfrak{H}$.

Proof. Since the second term of (3.20) vanishes at $s=1/2$, the constant term of $E_2^*(Z,1/2)$ does not depend on Z. By Theorem 3.1(1),

$$\omega_2(2J^{-1}Y_v,\pi\eta_vJ;2,0)=2^{-3}e[\mathrm{Tr}(iY_v\eta_v)]\quad\text{if } \eta_v\in W_2(2,0),$$

and

$$Q_2(1/2)\varepsilon_2^{(p_v,q_v)}(1/2)=0\quad\text{if } (p_v,q_v)\neq(2,0).$$

Therefore by (3.19) non-constant terms of $E_2^*(Z,1/2)$ are holomorphic in Z on $\mathfrak{H}$. We notice that the normalizing factor has neither zero nor pole at $s=1/2$. Therefore the Eisenstein series $E_2(Z,1/2)$ is holomorphic in Z on $\mathfrak{H}$. □

Remark. In the case of the Siegel modular form of degree 2, we can not construct the Eisenstein series of weight 2 in this way ([**8**], [**12**], [**17**]).

By Theorem 3.1(1), Theorem 3.4(1), and Theorem 3.8, we obtain the Fourier expansion of the holomorphic Eisenstein series of weight $l\geqq 2$.

Corollary 3.9. *Let l be an even positive integer with $l\geqq 2$.*

$$E_l(Z) = 1 + \sum_{\substack{\eta \in (\mathfrak{A}^-)^* \\ \eta J > 0}} a_l(\eta) e[\mathrm{Tr}_{B/k}(\eta Z)],$$

$$a_l(\eta) = |d_k|^{-4l+9/2} |d_K|^{l-3/2} \left\{ \frac{2^{2l}\pi^{2l-1}}{(2l-2)!} \right\}^n \prod_{\mathfrak{p} | \mathfrak{D}_0} \frac{(1 - \chi_\eta(\mathfrak{p}) q_\mathfrak{p}^{l-1})(1 - \chi_\eta(\mathfrak{p}) q_\mathfrak{p}^{l-2})}{q_\mathfrak{p}^{2l-2} - 1}$$

$$\cdot \prod_{\mathfrak{p} | \mathfrak{D}_1} \frac{1}{q_\mathfrak{p}^{l-1} - 1} \frac{L_k(l-1, \chi_\eta)}{\zeta_k(l) \zeta_k(2l-2)} F(\eta, l),$$

where $K := k(\eta)$ is a quadratic number field over k, d_k [resp. d_K] is the discriminant of k [resp. K] over $\boldsymbol{Q}$, and $F(\eta, l) := \prod_{\mathfrak{p} \in \mathbf{f}} F_\mathfrak{p}(\eta, l)$ is a product of integers. Here

$$F_\mathfrak{p}(\eta, l) := \begin{cases} q_\mathfrak{p}^{a_\mathfrak{p}(l-3/2)} \hat{\alpha}_\mathfrak{p}(\eta, l-3/2) & \text{if } \mathfrak{p} | \mathfrak{D}, \\ q_\mathfrak{p}^{(a_\mathfrak{p} + f_\mathfrak{p})(l-3/2)} \hat{\alpha}_\mathfrak{p}(\eta, l-3/2) & \text{if } \mathfrak{p} \nmid \mathfrak{D}. \end{cases}$$

3.4. Fourier coefficient of $E_l(Z)$.

In this section, we assume k is the rational number field $\boldsymbol{Q}$. To calculate numerical examples by the formula in Corollary 3.9, we shall rewrite this formula in an easier form.

THEOREM 3.10. *Let l be an even positive integer.*

$$E_l(Z) = 1 + \sum_{\substack{\eta \in (\mathfrak{A}^-)^* \\ \eta J > 0}} a_l(\eta) e[\mathrm{Tr}_{B/k}(\eta Z)],$$

$$a_l(\eta) = -\frac{4l B_{l-1,\chi_\eta}}{B_l B_{2l-2}} \prod_{p | D_0} \frac{(1 - \chi_\eta(p) p^{l-1})(1 - \chi_\eta(p) p^{l-2})}{p^{2l-2} - 1} \prod_{p | D_1} \frac{1}{p^{l-1} - 1} \prod_{p < \infty} F_p(\eta, l),$$

where χ_η is the Dirichlet character of $\boldsymbol{Q}(\eta)$ over $\boldsymbol{Q}$, B_m [resp. B_{m,χ_η}] is the m-th Bernoulli [resp. the generalized Bernoulli] number. We take the definition from [**11**] p. 89 [*resp.* [**11**] p. 94]. *We define even positive integers a, f as follows*:

$$a^{-1}\eta \in (\mathfrak{A}^-)^*_{prim}, \quad (2a^{-1}\eta)^2 = d_\eta f^2 \quad (d_\eta \text{ is discriminant of } \boldsymbol{Q}(\eta)/\boldsymbol{Q}).$$

Then we have

$$F_p(\eta, l) = \sum_{t=0}^{a_p} p^{(2l-3)t} + (1 + \chi_\eta(p)) \sum_{t=0}^{a_p - 1} p^{(2l-3)t + l - 1}$$

if $p | D_0$,

$$F_p(\eta, l) = \sum_{t=0}^{a_p} p^{(2l-3)t} - \chi_\eta(p) \sum_{t=0}^{a_p - 1} p^{(2l-3)t + l - 1}$$

if $p | D_1$, and

$$F_p(\eta, l) = \sum_{t=0}^{a_p} \left\{ \sum_{k=0}^{a_p + f_p - t} p^{(2l-3)k + (l-1)t} - \chi_\eta(p) \sum_{k=0}^{a_p + f_p - t - 1} p^{(2l-3)k + (l-1)t + l - 2} \right\}$$

if $p \nmid D$, where we put $a_p = \mathrm{ord}_p(a)$, $f_p = \mathrm{ord}_p(f)$.

Since $\prod_{p<\infty} F_p(\eta, l)$ is the finite product of integers, the following corollary easily follows from this theorem.

COROLLARY 3.11. *We keep the above notation. Then $a_l(\eta) \in \boldsymbol{Q}$, precisely there exists a constant $C \in \boldsymbol{Z}$ depending only on D and l such that $Ca_l(\eta) \in \boldsymbol{Z}$ for all η.*

Put

$$\mathscr{P} := \left\{ (d, f) \,\middle|\, \begin{array}{l} d \text{ is the discriminant of an imaginary quadratic number field} \\ f \text{ is a positive rational number satisfying (3.21)} \end{array} \right\}$$

$$\left(\frac{d}{p}\right) \neq 1 \quad \text{for } p|D,$$

$$\operatorname{ord}_p f = \begin{cases} 0 & \text{if } p|D_0, p \nmid d, \\ -1 & \text{if } p|D_0, p|d, \\ -1 & \text{if } p|D_1, \end{cases} \tag{3.21}$$

$$\operatorname{ord}_p f \geqq 0 \quad \text{if } p \nmid D.$$

Then there exists an $\eta \in (\mathfrak{A}^-)^*_{prim}$ such that $\eta J > 0$ and $(2\eta)^2 = df^2$ for one $(d, f) \in \mathscr{P}$. Moreover the converse is also valid (cf. Lemma 2.1, (2.6)). We say that a function

$$F(Z) = \sum_{\eta \in (\mathfrak{A}^-)^*} c(\eta) e[\operatorname{Tr}_{B/k}(\eta x)] \in M_l(\Gamma_{\mathfrak{A}})$$

satisfies the Maass relation, if there exists a function f_0 on $\boldsymbol{Z}_{>0} \times \mathscr{P}$ satisfying the following condition.

1. If $0 < t \in \boldsymbol{Z}$ and $\eta \in (\mathfrak{A}^-)^*_{prim}$ then

$$c(t\eta) = f_0(t; (d, f)),$$

where $(2\eta)^2 = df^2$.

2. The function f_0 satisfies the following recurrence formula:

$$f_0(t_1 t_2; (d, f)) = \sum_{r|t_2} r^{l-1} f_0\left(t_1; \left(d, \frac{t_2}{r} f\right)\right),$$

for positive integers t_1, t_2, where all the prime factor of t_1 divides D and t_2 is mutually prime with D.

We denote by $M_l^*(\Gamma_{\mathfrak{A}})$ the space of such functions. The subset $M_l^*(\Gamma_{\mathfrak{A}})$ of $M_l(\Gamma_{\mathfrak{A}})$ is the analogy of the Maass space in the case of the Siegel modular form of degree 2 (cf. [**10**]). Put

$$S_l^*(\Gamma_{\mathfrak{A}}) := S_l(\Gamma_{\mathfrak{A}}) \cap M_l^*(\Gamma_{\mathfrak{A}}).$$

Then we have the following corollary from Corollary 2.6(1) and Theorem 3.10.

COROLLARY 3.12. *Let l be a positive even integer. Then For $2 \leqq l \in \boldsymbol{Z}$*

$$E_l(Z) \in M_l^*(\Gamma_{\mathfrak{A}}).$$

4. Example of the case $D_0 = 6$, $D_1 = 1$.

In this section we give some examples of cusp forms on a quaternion unitary group of degree 2 over $\boldsymbol{Q}$ by using the Eisenstein series and Oda's lifting [**13**].

4.1. Examples by Oda lifting.

Let N be an odd squarefree positive integer and κ an odd positive integer. We put $M = 4N$. For a positive divisor Δ of N, we define a Dirichlet character (modulo M) by

$$\chi_\Delta(m) := \left(\frac{\Delta}{m}\right).$$

We denote by χ_0 the trivial character. Put

$$\Gamma_0(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \ (\mathrm{mod.}\, M) \right\}$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$, we put $\chi_\Delta(\gamma) = \chi_\Delta(d)$. We denote by $\mathfrak{M}_\kappa(M, \chi_\Delta)$ [resp. $\mathfrak{S}_\kappa(M, \chi_\Delta)$] the space of holomorphic modular [resp. cusp] forms on the complex upper half-plane $\mathfrak{h}$ of weight $\kappa/2$, with respect to $\Gamma_0(M)$ and with character χ_Δ (we use the same definitions as in [**18**]). For a rational prime p, the Hecke operator $T^M_{\kappa,\chi_\Delta}(p^2)$ acting on $\mathfrak{M}_\kappa(M, \chi_\Delta)$ is defined in [**18**].

Let B be an indefinite division quaternion algebra over $\boldsymbol{Q}$, D its discriminant and $\mathfrak{O}$ a maximal order of B. Let G have the same meaning as in (1.1). Let Γ be the intersection G and $GL_2(\mathfrak{O})$ i.e. $\Gamma := \Gamma_{\mathfrak{O}}$ in the notation of (1.2). $M_l(\Gamma)$ and $S_l(\Gamma)$, for a positive integer l, have the same meaning as in §1 respectively.

For each positive integer m we define Hecke operator $T_l(m)$ acting on $M_l(\Gamma)$ by

$$(T_l(m)f)(Z) := m^{2l-3} \sum_{\gamma \in \Gamma \backslash S_m} J(\gamma, Z)^{-l} f(\gamma\langle Z\rangle),$$

$$S_m := \left\{ g \in M_2(\mathfrak{O}) \,\middle|\, g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

where for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_m$

$$\gamma\langle Z\rangle := (AZ + B)(CZ + D)^{-1}, \quad J(\gamma, Z) := N_{B/\boldsymbol{Q}}(CZ + D).$$

For any prime number p there exists an element $\Pi_p \in \mathfrak{O}$ such that $N_{B/\boldsymbol{Q}}(\Pi_p) = p$, since B is an indefinite quaternion algebra over $\boldsymbol{Q}$. Let f be an element $M_l(\Gamma)$ such that

$$f(\Pi_p Z \Pi_p^{-1}) = \Lambda_p(f) f(Z) \quad (\Lambda_p(f) \in \boldsymbol{C}). \tag{4.1}$$

Then we notice that $\Lambda_p(f) = -1$ or 1, and particularly $\Lambda_p(f) = 1$ for $f \in M_l^*(\Gamma)$. For $p|D$ and $f \in M_l(\Gamma)$ satisfied (4.1), by the detail computation of $T_l(p)$, the relation between the Hecke operator $T_l(p)$ and Fourier coefficients of f is given as follows:

$$f(Z)=\sum_{\substack{\eta\in(\mathfrak{O}^-)^*\\ \eta J>0}} a(\eta)e[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)],$$

$$(T_l(p))f(Z)=\sum_{\substack{\eta\in(\mathfrak{O}^-)^*\\ \eta J>0}} b(\eta)e[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)],$$

where

$$b(\eta)=p^{2l-3}a(p^{-1}\eta)+p^{l-1}\delta_p(f,\eta)a(\eta)+a(p\eta), \tag{4.2}$$

$$\delta_p(f,\eta)=\begin{cases}0 & \text{if } \eta \text{ is } p\text{-primitive and } \chi_\eta(p)=0,\\ \Lambda_p(f) & \text{otherwise.}\end{cases}$$

If η is p-primitive, we understand that $a(p^{-1}\eta)=0$.

In the rest of this section we assume D is an even positive integer. Put

$$D':=D/2.$$

For $\eta\in(\mathfrak{O}^-)^*_{prim}$ we put

$$m_\eta:=-D'(2\eta)^2.$$

An explicit form of Oda's lifting $\mathfrak{S}_{2D}(2l-1,\chi_{D'})\mapsto S_l(\Gamma)$ is given as follows (cf. [**19**] §4):

Proposition 4.1. *The notation being as above, let l be an even integer ($l\geqq 6$), f an element of $\mathfrak{S}_{2l-1}(2D,\chi_{D'})$ and $(T^{2D}_{2l-1,\chi_{D'}}(p^2))f=\omega_p f$ for all prime p. Let the Fourier expansion of f at $i\infty$ be*

$$f(z):=\sum_{n=1}^{\infty}a(n)e[nz]$$

and put

$$J(f)(Z):=\sum_{\substack{\eta\in(\mathfrak{O}^-)^*\\ \eta J>0}} C_f(\eta)e[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)].$$

Here for $\eta\in(\mathfrak{O}^-)^*_{prim}$

$$C_f(\eta):=\tilde{C}_f(m_\eta)=\prod_{p|D}(1+\phi_p(m_\eta)p^{l-1}\omega_p^{-1})a(m_\eta),$$

where

$$\phi_p(m_\eta):=\begin{cases}1 & \text{if } p\neq 2 \text{ and } p|m_\eta, \text{ or } p=2 \text{ and } m_\eta\equiv -D' \ (\mathrm{mod.}\,4),\\ 0 & \text{otherwise.}\end{cases}$$

For general $\eta\in(\mathfrak{O}^-)^*$, we define $C_f(\eta)$ by the following recurrence formula (1) (2).

(1) *For $p|D$*

$$C_f(p\eta)=(\omega_p+p^{l-1}(1-\delta_p(\eta))+p^{2l-3}\omega_p^{-1})C_f(\eta)-p^{2l-3}C_f(p^{-1}\eta),$$

where

$$\delta_p(\eta) = \begin{cases} 0 & \text{if } \eta \text{ is } p\text{-primitive and } \chi_\eta(p) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

(2) *For* $\eta_0 \in (\mathfrak{O}^-)^*_{prim}$ *and a positive integer* t *which is mutually prime with* D

$$C_f(t\eta_0) = \sum_{r|t} r^{l-1} \tilde{C}_f((t/r)^2 m_{\eta_0}).$$

Then $J(f) \in S_l^*(\Gamma)$. *Moreover, put* $F := J(f)$ *then* F *satisfies the following facts.*

$$T_l(p)F = (\omega_p + p^{l-1} + p^{2l-3}\omega_p^{-1})F \quad \text{if } p|D,$$
$$T_l(p)F = (\omega_p + p^{l-1} + p^{l-2})F \qquad \text{if } p \nmid D.$$

By [**19**], we know that $\omega_p \neq 0$ if $p|2D$. Therefore we obtain the following (see [**15**]).

Proposition 4.2. *If* Δ *and* p *divide* $D/2$, *the map*

$$T^{2D}_{\kappa,\chi_p}(p) : \mathfrak{S}_\kappa(2D, \chi_\Delta) \to \mathfrak{S}_\kappa(2D, \chi_\Delta\chi_p) \quad \text{i.e.} \quad \left(\sum_{n=1}^{\infty} a(n)e[nz] \mapsto \sum_{n=1}^{\infty} a(pn)e[nz]\right)$$

is an isomorphism and commutative with all Hecke operators.

To obtain Fourier coefficients of the lifted cusp form numerically by Proposition 4.1 and Proposition 4.2, we introduce some examples of modular forms with respect to $\Gamma_0(4)$. Notice that $\Gamma_0(4)$ has three inequivalent cusps $i\infty$, 0, and $1/2$. Let $m \geqq 4$ be an even integer, we put

$$E^{(m)}(z) := -\sum_{n=1}^{\infty} (-1)^n 2^{(m-1)k} \sum_{\substack{d|n' \\ d>0}} d^{m-1} e[nz],$$

where for n we define the pair of integers (n', k) by $n = 2^k n'$ (n' is an odd positive integer). This is the Eisenstein series of weight m with respect to $\Gamma_0(4)$, which only vanishes at cusps $i\infty$ and 0. On the other hand the classical theta-function $\theta(z) := \sum_{n=-\infty}^{\infty} e[n^2 z] \in \mathfrak{S}_1(4, \chi_0)$ only vanishes at the cusp $1/2$.

The rest of this subsection will be devoted examples of lifted cusp forms in the case of $D = 6$.

Example 1 ($l = 6$) (cf. [**20**]).

Put $D = 6$. By [**3**], we obtain $\dim \mathfrak{S}_{11}(12, \chi_0) = 7$. Basis of $\mathfrak{S}_{11}(12, \chi_0)$ are given as follows:

$$f_1(z) := E^{(4)}(z)\theta(z)^3, \quad f_2(z) := \frac{1}{108}\left\{25 f_1(z) + \frac{1}{3} T^{12}_{11,\chi_0}(3^2) f_1(z)\right\},$$

$$f_3(z) := E^{(4)}(3z)\theta(z)\theta(3z)^2, \quad f_4(z) := \frac{1}{2} T^{12}_{11,\chi_0}(2^2) f_3(z), \quad f_5(z) := E^{(4)}(3z)\theta(z)^3,$$

$$f_6(z) := \frac{1}{2} T^{12}_{11,\chi_0}(2^2) f_5(z), \quad f_7(z) := E^{(4)}(z)\theta(z)\theta(3z)^2.$$

Using these basis, eigenbasis of $\mathfrak{S}_{11}(12,\chi_0)$ are given as follows:

$$\alpha : \alpha^2 + 156\alpha + 3^9 = 0,$$
$$\beta : \beta^2 - 9\beta + 128 = 0,$$
$$\gamma : \gamma^2 + 18\gamma + 128 = 0,$$

$$f_\alpha := -42 f_1 + (\alpha + 75) f_2, \quad T^{12}_{11,\chi_0}(2^2) f_\alpha = 16 f_\alpha, \quad T^{12}_{11,\chi_0}(3^2) f_\alpha = \alpha f_\alpha,$$
$$f_\beta := -84 f_1 + 192 f_2 - 12(2\beta + 79) f_3 - 3(\beta - 28) f_4 + 260 f_5 + 20 f_6 - 60 f_7,$$
$$T^{12}_{11,\chi_0}(2^2) f_\beta = 2\beta f_\beta, \quad T^{12}_{11,\chi_0}(3^2) f_\beta = 81 f_\beta,$$
$$f_\gamma := 168 f_1 - 384 f_2 + 24(\gamma + 125) f_3 + 3(\gamma - 64) f_4 + 8(\gamma - 57) f_5 - (\gamma + 48) f_6 + 168 f_7,$$
$$T^{12}_{11,\chi_0}(2^2) f_\gamma = 2\gamma f_\gamma, \quad T^{12}_{11,\chi_0}(3^2) f_\gamma = -81 f_\gamma,$$
$$f := -21 f_1 + 48 f_2 - 387 f_3 + 27 f_4 + 65 f_5 + 5 f_6 - 15 f_7,$$
$$T^{12}_{11,\chi_0}(2^2) f = -16 f, \quad T^{12}_{11,\chi_0}(3^2) f = 81 f.$$

By Proposition 4.1 and Proposition 4.2, we obtain the lifted cusp forms which belong to $S_6^*(\Gamma)$:

$$F_\alpha : = J\left(\frac{T^{12}_{11,\chi_0}(3) f_\alpha}{3(\alpha + 243)}\right), \quad F_\beta := J\left(\frac{-T^{12}_{11,\chi_0}(3) f_\beta}{9\beta}\right),$$
$$F_\gamma : = J\left(\frac{-T^{12}_{11,\chi_0}(3) f_\gamma}{36\gamma}\right), \quad F := J\left(\frac{T^{12}_{11,\chi_0}(3) f}{324}\right). \tag{4.3}$$

Their Fourier coefficients and eigenvalues with respect to $T_6(2)$ and $T_6(3)$ are given in Table I. From Table I eigenvalues of $F_\alpha, F_\beta, F_\gamma$, and F are distinct. By Hashimoto's dimension formula [**4**], we know $\dim S_6(\Gamma) = 4$. Therefore F_α is independent on the value of α and so are F_β and F_γ. So we obtain the following:

$$S_6(\Gamma) = J(\mathfrak{S}_{11}(12,\chi_3)) = F_\alpha \oplus F_\beta \oplus F_\gamma \oplus F. \tag{4.4}$$

Example 2 ($l = 8$) (cf. [**20**]).

Put $D = 6$. By [**3**], we obtain $\dim \mathfrak{S}_{15}(12,\chi_0) = 11$. Basis of $\mathfrak{S}_{15}(12,\chi_0)$ are given as follows:

$$g_1(z) := E^{(6)}(z)\theta(z)^3, \quad g_2(z) := E^{(4)}(z)\theta(z)^7,$$
$$g_3(z) := \frac{1}{972}\left\{\frac{1}{3} T^{12}_{15,\chi_0}(3^2) g_1(z) - 655 g_1(z)\right\},$$
$$g_4(z) := \frac{-1}{324}\left\{\frac{1}{27} T^{12}_{15,\chi_0}(3^2) g_2(z) + 41 g_2(z)\right\}, \quad g_5(z) := E^{(4)}(3z)\theta(z)\theta(3z)^6,$$
$$g_6(z) := E^{(4)}(3z)\theta(z)^3\theta(3z)^4, \quad g_7(z) := E^{(4)}(3z)\theta(z)^5\theta(3z)^2,$$
$$g_8(z) := E^{(4)}(3z)\theta(z)^7, \quad g_9(z) := E^{(6)}(3z)\theta(z)\theta(3z)^2,$$
$$g_{10}(z) := E^{(6)}(3z)\theta(z)^3, \quad g_{11}(z) := E^{(4)}(z)\theta(z)\theta(3z)^6.$$

TABLE I (Fourier coefficients of lifted cusp forms)

m_η	$J(\mathfrak{S}_7(12,\chi_3))$		$J(\mathfrak{S}_{11}(12,\chi_3))$				$J(\mathfrak{S}_{15}(12,\chi_3))$	
	H_α	H	F_α	F_β	F_γ	F	G_α	G_β
1	1	1	2	11	1	0	16	0
2	2	−1	3	−26	5	1	37	7
3	−12	0	−14	108	0	6	346	30
7	−4	−4	−14	28	140	−18	−1246	54
10	20	2	−170	940	−106	18	−1390	2214
11	4	16	114	−1588	−272	−10	−8998	1694
14	−44	−14	294	812	−98	50	−49462	−1906
17	−42	30	−1332	3234	510	0	−71328	0
19	20	−16	274	4012	−1648	−90	107978	−2322
22	−56	16	−388	−3304	−320	−252	79948	−8316
23	56	8	−1788	−4424	1640	28	38452	−4676
25	31	−41	2990	−9955	−689	0	−669200	0
26	52	−26	1014	11372	2698	50	−136798	−16042
30	−48	0	−3080	−41040	0	−696	76120	840
31	−100	44	754	5212	4892	270	−244318	100278
34	68	14	−1946	−6068	−2422	450	12218	17118
$T_l(2)$	2	20	80	50	−4	−16	320	−64
$T_l(3)$	63	−9	87	567	−81	567	3423	351

m_η	$J(\mathfrak{S}_{15}(12,\chi_3))$		
	G_γ	G_θ	G
1	5	$11a+1320$	2
2	17	$-238a-25320$	−17
3	0	$1620a+159408$	0
7	1204	$-16532a-1824432$	1012
10	−4706	$-7340a-589200$	994
11	5104	$81796a+8717808$	−1168
14	24094	$9364a+1145712$	2
17	22230	$-157806a-13462416$	−17220
19	−16112	$-26828a-4342992$	−18032
22	3344	$-567160a-58735776$	−20368
23	60824	442520a+44608416	17176
25	−72925	$188837a+28134744$	140558
26	−126902	174772a+23078640	−28042
30	0	$5947344a+648036288$	0
31	40708	$-160052a-17771568$	−29372
34	376498	$-1629212a-170719440$	−30002
$T_l(2)$	116	$-a+128$	320
$T_l(3)$	−729	5103	−729

Using these basis, eigenbasis of $\mathfrak{S}_{15}(12,\chi_0)$ are given as follows:

$$\alpha : \alpha^2 - 1236\alpha + 3^{13} = 0,$$
$$\beta : \beta^2 + 1836\beta + 3^{13} = 0,$$
$$\gamma : \gamma^2 + 12\gamma + 2^{13} = 0,$$
$$\lambda(a) : \lambda(a)^2 + a\lambda(a) + 2^{13} = 0 \quad \text{for } a^2 - 54a - 16992 = 0.$$

$$g_\alpha := 1038g_1 - (\alpha - 1965)g_3,$$
$$T^{12}_{15,\chi_0}(2^2)g_\alpha = 64g_\alpha, \quad T^{12}_{15,\chi_0}(3^2)g_\alpha = \alpha g_\alpha,$$
$$g_\beta = 90g_2 + (\beta + 1107)g_4,$$
$$T^{12}_{15,\chi_0}(2^2)g_\beta = -64g_\beta, \quad T^{12}_{15,\chi_0}(3^2)g_\beta = \beta g_\beta,$$
$$g_\gamma := -58g_1 - 80g_3 - 4\tau g_4 + 756g_5 + 75\tau g_6 + 162g_7 - 15\tau g_8 - 120\tau g_9 + 58g_{10} + 58g_{11} \quad \text{here } \tau := (\gamma + 38)/3,$$
$$T^{12}_{15,\chi_0}(2^2)g_\gamma = \gamma g_\gamma, \quad T^{12}_{15,\chi_0}(3^2)g_\gamma = -729g_\gamma,$$

$$g_{\lambda(a)} := \sum_{i=1}^{11} c_i g_i \quad \text{here:}$$
$$c_1 := 99(23a + 2976), \quad c_2 := 10(5a\lambda(a) + 541a + 492\lambda(a) + 58224),$$
$$c_3 := 3960(a + 120), \quad c_4 := -60(5a\lambda(a) + 541a + 492\lambda(a) + 58224),$$
$$c_5 := -594(311a + 34512), \quad c_6 := -31185(a + 2\lambda(a) + 64), \quad c_7 := 297(a + 192),$$
$$c_8 := 4455(a + 2\lambda(a) + 64), \quad c_9 := 990(a\lambda(a) + 146a + 174\lambda(a) + 14064),$$
$$c_{10} := -297(a + 192), \quad c_{11} := -1287(a + 192),$$
$$T^{12}_{15,\chi_0}(2^2)g_{\lambda(a)} = \lambda(a)g_{\lambda(a)}, \quad T^{12}_{15,\chi_0}(3^2)g_{\lambda(a)} = 729g_{\lambda(a)},$$
$$g := 173g_1 + 180g_3 - 9861g_5 + 1038g_7 - 173g_{10} - 173g_{11},$$
$$T^{12}_{15,\chi_0}(2^2)g = 64g, \quad T^{12}_{15,\chi_0}(3^2)g = -729g.$$

By Proposition 4.1 and Proposition 4.2, we obtain the lifted cusp forms which belong to $S_8^*(\Gamma)$:

$$G_\alpha := J\left(\frac{T^{12}_{15,\chi_0}(3)g_\alpha}{3(\alpha + 2187)}\right), \quad G_\beta := J\left(\frac{T^{12}_{15,\chi_0}(3)g_\beta}{3(\beta + 2187)}\right),$$
$$G_\gamma := J\left(\frac{-T^{12}_{15,\chi_0}(3)g_\gamma}{4(\gamma + 128)}\right), \quad G_{\lambda(a)} := J\left(\frac{T^{12}_{15,\chi_0}(3)g_{\lambda(a)}}{90(\lambda(a) + 128)}\right), \tag{4.5}$$
$$G := J\left(\frac{-T^{12}_{15,\chi_0}(3)g}{4320}\right).$$

Their Fourier coefficients and eigenvalues with respect to $T_8(2)$ and $T_8(3)$ are given in Table I. By the dimension formula [**4**], we know $\dim S_8(\Gamma) = 8$. From Table I we have

$$\dim J(\mathfrak{S}_{15}(12, \chi_3)) \geqq 6. \tag{4.6}$$

Example 3 ($l = 4$).
Put $D = 6$. Put

$$\begin{aligned} j_1(z) &:= \frac{\theta(z)^5 - \theta(z)^3\theta(3z)^2}{4}, \quad j_2(z) := \frac{\theta(z)^5 - \theta(z)\theta(3z)^4}{8}, \\ j_3(z) &:= T^{12}_{5,\chi_0}(2^2) j_1(z), \quad j_4(z) := T^{12}_{5,\chi_0}(2^2) j_2(z), \end{aligned}$$

which belong to $\mathfrak{M}_5(12, \chi_0)$. From [**3**], we know that $\dim \mathfrak{S}_5(12, \chi_0) = 1$ and $\dim \mathfrak{S}_7(12, \chi_0) = 3$. Therefore we can easily check that

$$j(z) := \frac{-8j_1(z) + 24j_2(z) + j_3(z) - 3j_4(z)}{4} \in \mathfrak{S}_5(12, \chi_0). \tag{4.7}$$

So basis of $\mathfrak{S}_7(12, \chi_0)$ can be given as follows:

$$h_1(z) := \theta(z)^2 j(z), \quad h_2(z) := \theta(3z)^2 j(z), \quad h_3(z) := T^{12}_{7,\chi_0}(2^2) h_1(z).$$

Using this, eigenbasis of $\mathfrak{S}_7(12, \chi_0)$ is given as follows:

$$\alpha : \alpha^2 + 6\alpha + 32 = 0$$

$$\begin{aligned} h_\alpha &:= \frac{(-8 + 4\alpha)h_1 - (108 + 18\alpha)h_2 + (2 - \alpha)h_3}{72} \\ & \quad T^{12}_{7,\chi_0}(2^2) h_\alpha = \alpha h_\alpha, \quad T^{12}_{7,\chi_0}(3^2) h_\alpha = 9h_\alpha \\ h &:= \frac{5h_1 + 9h_2 + h_3}{36} \\ & \quad T^{12}_{7,\chi_0}(2^2) h = 4h, \quad T^{12}_{7,\chi_0}(3^2) h = -9h \end{aligned}$$

By Proposition 4.2 we have

$$T^{12}_{7,\chi_0}(3) h_\alpha, \quad T^{12}_{7,\chi_0}(3) h \in \mathfrak{S}_7(12, \chi_3). \tag{4.8}$$

Proposition 4.1 does not guarantee of the justification if $l = 4$. However we formally apply this formula to 4.8, put

$$H_\alpha := J\left(\frac{T^{12}_{7,\chi_0}(3) h_\alpha}{6}\right), \quad H := J\left(\frac{T^{12}_{7,\chi_0}(3) h}{3}\right),$$

and calculate Fourier coefficients of H_α and H. Their Fourier coefficients and eigenvalues with respect to $T_4(2)$ and $T_4(3)$ are given in Table I.

4.2. Examples by Eisenstein series.

From Ibukiyama [**5**] we obtain the following lemma.

Lemma 4.3. *Put*

$$B := \boldsymbol{Q} + \boldsymbol{Q}a + \boldsymbol{Q}b + \boldsymbol{Q}ab, \quad a^2 = 6, \quad b^2 = 5, \quad ab = -ba.$$

Then B is an indefinite division quaternion algebra over $\boldsymbol{Q}$ with discriminant $D = 6$. Moreover

$$\mathfrak{O} = \boldsymbol{Z} + \boldsymbol{Z}\frac{1+b}{2} + \boldsymbol{Z}\frac{a(1+b)}{2} + \boldsymbol{Z}\frac{(1+a)b}{5}$$

is a maximal order of B.

The dual lattice $(\mathfrak{O}^-)^*$ of $\mathfrak{O}^-$ defined in (1.3) is given as follows:

$$(\mathfrak{O}^-)^* = \boldsymbol{Z}\frac{(1+a)b}{10} + \boldsymbol{Z}\frac{-5a-6b-ab}{60} + \boldsymbol{Z}\frac{5a-6b-ab}{60}.$$

For $\eta = x(1+a)b/10 + y(-5a-6b-ab)/60 + z(5a-6b-ab)/60 \in (\mathfrak{O}^-)^*$, we write $\eta = (x, y, z)$, and identify $x > 0$ and $3x^2 - y^2 - z^2 > 0$ with $\eta J > 0$, where we note that

$$-D'(2\eta)^2 = 3x^2 - y^2 - z^2 \quad (\text{in this case } D' = 3).$$

In the formula of Theorem 3.10, $\boldsymbol{Q}(\eta)$, a, and f is given as follows:

$$\boldsymbol{Q}(\eta) = \boldsymbol{Q}\left(\sqrt{-3(3x^2 - y^2 - z^2)}\right), \quad a = \gcd(x, y, z), \quad f = \sqrt{\frac{-3x^2 + y^2 + z^2}{3a^2 d_\eta}}.$$

By virtue of this we can calculate the value of Fourier coefficients of E_l. Put

$$\rho_l := -\frac{B_l B_{2l-2}(2^{2l-2} - 1)(3^{2l-2} - 1)}{4l}.$$

A few numerical examples of Fourier coefficients of $\rho_l E_l$ are given in Table II.

4.3. Application.

In this section, using the Fourier expansion of Eisenstein series and lifted cusp forms in Example I–III, we construct some examples of cusp forms which can not be obtained by Oda's lifting. Put

$$E_{2,2} := \frac{13104(\rho_2 E_2)^2 - 40\rho_4 E_4}{288} \in M_4(\Gamma).$$

From Table II and Proposition 1.2, we know that

$$E_{2,2} \in S_4(\Gamma). \tag{4.9}$$

We write the main result in this section.

Theorem 4.4. (1)

(2)
$$\dim S_2(\Gamma) = 0$$
$$\dim S_4(\Gamma) = 2$$

Table II (Fourier coefficients of holomorphic Eisenstein series)

m_η	$\rho_2 E_2$	$\rho_4 E_4$	$\rho_6 E_6$	$\rho_8 E_8$
0	$-1/12$	$91/40$	$-1144055/252$	$22854814717/240$
1	-2	30	-1870	273910
2	-2	138	-39850	24410722
3	-4	420	-250100	341010740
7	-4	3156	-11186900	83950326404
10	-4	7620	-55668020	852862571540
11	-4	9636	-85463540	1584626701364
14	-4	17556	-252966740	7598121102404
17	-12	36180	-644308500	27260636135460
19	-4	37572	-999666740	55305953899028
22	-8	55416	-1935065800	143427712564024
23	-8	61896	-2363559400	191476975767784
25	-14	94530	-3653514370	334367073086410
26	-4	82212	-4100529140	424815354499508
30	-16	132720	-7907361680	1078359569005040
31	-4	127572	-9048707540	1332675604695428
34	-4	160692	-13712411540	2429329353708068
35	-8	175080	-15629222440	2933066230165960
38	-8	215832	-22634197000	5005851318884248
39	-16	253680	-25741692560	5934577730222960
41	-12	321300	-33831452820	8331092065760100
43	-12	296172	-39482459100	11179712346051228
46	-8	344712	-53447365480	17330075038944808
47	-8	366888	-58908370600	19930611848994952
49	-14	502770	-75456757090	26538836615617690
50	-10	427938	-77807164850	29797863235785722
55	-8	539976	-119463364840	55365154158736744
57	-48	831600	-150969363600	71023313863647600
58	-12	624492	-151780110300	78192944865271068
59	-4	637188	-163791868340	87379893572413652
62	-12	735228	-204883755900	120623130915841932
65	-24	1031400	-269212522920	166552173291995400
66	-16	939120	-274625206160	181344498987181040
67	-12	895212	-290485652700	199697131315230108
70	-8	987816	-353642390440	265470814423644424
71	-8	1018632	-376854027880	291107744410228648
73	-24	1381320	-453967146600	354173917179449640
$T_l(2)$	5	41	545	8321
$T_l(3)$	7	271	19927	1596511

TABLE III

η	$S_4(\Gamma)$		$S_8(\Gamma)$
	H_1	H_2	G_1
(1, 1, 1)	1	1	1
(1, 1, 0)	2	−1	−2
(1, 0, 0)	−12	0	0
(2, 2, 1)	−4	−4	−40
(2, 1, 1)	20	2	1316
(2, 1, 0)	4	16	−1312
(3, 3, 2)	−44	−14	1028
(3, 3, 1)	−42	30	126
(3, 2, 2)	20	−16	15008
(3, 2, 1)	−56	16	−4256
(3, 2, 0)	56	8	−33584
(3, 1, 1)	31	−41	7879
(3, 1, 0)	52	−26	18284
$T_l(2)$	2	20	8
$T_l(3)$	63	−9	−729

Eigenbasis of $S_4(\Gamma)$ *is given as follows*:

$$H_1 := \frac{20E_{2,2} - T_4(2)E_{2,2}}{42},$$

$$T_4(2)H_1 = 2H_1, \quad T_4(3)H_1 = 63H_1,$$

$$H_2 := \frac{-2E_{2,2} + T_4(2)E_{2,2}}{156},$$

$$T_4(2)H_2 = 20H_2, \quad T_4(3)H_2 = -9H_2,$$

which the Fourier coefficients are given in Table III. (*Note*: $\dim S_4(\Gamma) = 2$ *is conjectured by Hashimoto* [**4**].)

The existence of the Eisenstein series of weight 2 (Theorem 3.8) and the structure of $S_6(\Gamma)$ (cf. (4.4)) play a basic role in proving the above theorem.

By (4.4), we can choose the following basis of $S_6(\Gamma)$:

$$e_1 := F_\gamma, \quad e_2 := F, \quad e_3 := \frac{F_\alpha - 2F_\gamma + 7F}{28}, \quad e_4 := \frac{-297F_\alpha + 14F_\beta + 440F_\gamma - 945F}{83160}.$$

Their Fourier coefficients are given as the following table.

m_η	e_1	e_2	e_3	e_4
1	1	0	0	0
2	0	1	0	0
3	5	6	1	0
7	140	-18	-15	1

Put

$$E_{2,2} = \sum_{\substack{\eta \in (\mathfrak{O}^-)^* \\ \eta J > 0}} c(\eta) e\,[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)],$$

$$T_4(2)E_{2,2} = \sum_{\substack{\eta \in (\mathfrak{O}^-)^* \\ \eta J > 0}} c^{(2)}(\eta) e\,[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)].$$

Then

$$(4.10)\qquad \begin{cases} c((1,1,1)) = 11, \quad c((1,1,0)) = -4, \quad c((1,0,0)) = -28, \\ c((2,2,1)) = -44, \quad c((2,1,1)) = 64, \quad c((2,1,0)) = 148, \end{cases}$$

$$(4.11)\qquad \begin{cases} c^{(2)}((1,1,1)) = 178, \quad c^{(2)}((1,1,0)) = -164, \quad c^{(2)}((1,0,0)) = -56, \\ c^{(2)}((2,2,1)) = -712, \quad c^{(2)}((2,1,1)) = 440, \quad c^{(2)}((2,1,0)) = 2792. \end{cases}$$

From (4.10) (4.11) we know that $E_{2,2}$ and $T_4(2)E_{2,2}$ is linearly independent. Therefore by virtue of $\dim S_6(\Gamma) = 4$ and the existence of E_2, we have

$$(4.12)\qquad 2 \leqq \dim S_4(\Gamma) \leqq 4$$

For $f \in S_4(\Gamma)$ we put

$$f(Z) := \sum_{\substack{\eta \in (\mathfrak{O}^-)^* \\ \eta J > 0}} c_f(\eta) e\,[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)]$$

Lemma 4.5. *Let* $f \in S_4(\Gamma)$. *If* $c_f((1,1,1)) = c_f((1,1,0)) = c_f((1,0,0)) = 0$, *then* $f = 0$.

Proof. For $f \in S_4(\Gamma)$ we assume $f \neq 0$ and

$$c_f((1,1,1)) = c_f((1,1,0)) = c_f((1,0,0)) = 0.$$

Since $E_2 f \in S_6(\Gamma)$ and the Fourier coefficients which correspond to

$$\eta \in (\mathfrak{O}^-)^*_{prim}, \quad m_\eta = -3(2\eta)^2 = 1, 2, 3,$$

are zero, we may put

$$\rho_2 E_2 f = e_4.$$

Therefore we can find Fourier coefficients of f by the method of undetermined coefficients. For each prime number $p|D$ we have $\Lambda_p(f) = 1$ since $\Lambda_p(E_2) = \Lambda_p(e_4) = 1$. So we put

$$T_4(p)f(Z) = \sum_{\substack{\eta \in (\mathfrak{O}^-)^* \\ \eta J > 0}} c_f^{(p)}(\eta) e\,[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)],$$

we obtain

$$(4.13) \qquad \begin{cases} c_f((1,1,1)) = 0, \quad c_f((1,1,0)) = 0, \quad c_f((1,0,0)) = 0, \\ c_f((2,2,1)) = -12, \quad c_f((2,1,1)) = 0, \quad c_f((2,1,0)) = 24, \end{cases}$$

$$(4.14) \qquad \begin{cases} c_f^{(2)}((1,1,1)) = 6, \quad c_f^{(2)}((1,1,0)) = 12, \quad c_f^{(2)}((1,0,0)) = -72, \\ c_f^{(2)}((2,2,1)) = 5352, \quad c_f^{(2)}((2,1,1)) = -18312, \quad c_f^{(2)}((2,1,0)) = 26136, \end{cases}$$

$$(4.15) \qquad \begin{cases} c_f^{(3)}((1,1,1)) = -144, \quad c_f^{(3)}((1,1,0)) = 432, \quad c_f^{(3)}((1,0,0)) = 1440, \\ c_f^{(3)}((2,2,1)) = 9449388, \quad c_f^{(3)}((2,1,1)) = -133419168, \\ c_f^{(3)}((2,1,0)) = 248756616. \end{cases}$$

By (4.10), (4.11), (4.13), (4.14), and (4.15), we know that $E_{2,2}$, $T_4(2)E_{2,2}$, f, $T_4(2)f$, and $T_4(3)f$ are linearly independent. This contradicts (4.12). $\square$

If $f \in S_2(\Gamma)$, then $fE_2 \in S_4(\Gamma)$ satisfies the condition of Lemma 4.5. Therefore we obtain the first assertion of Theorem 4.4. Moreover by (4.12) and Lemma 4.5 we know that

$$(4.16) \qquad 2 \leqq \dim S_4(\Gamma) \leqq 3.$$

Lemma 4.6. *Let $f(Z) \in S_4(\Gamma)$. If $c_f((1,1,1)) = c_f((1,1,0)) = 0$, then $f = 0$*

Proof. We assume $f \neq 0$ and

$$c_f((1,1,1)) = c_f((1,1,0)) = 0.$$

By the same trick as we used in the proof of Lemma 4.5, we may put

$$E_2 f = e_3 + te_4, \quad t \in \boldsymbol{C}.$$

Therefore we can find Fourier coefficients of f by the method of undetermined coefficients. So we put

$$T_4(2)f(Z) = \sum_{\substack{\eta \in (\mathfrak{O}^-)^* \\ \eta J > 0}} c_f^{(2)}(\eta) e\,[\mathrm{Tr}_{B/\boldsymbol{Q}}(\eta Z)],$$

we obtain

$$(4.17) \qquad \begin{cases} c_f((1,1,1)) = 0, \quad c_f((1,1,0)) = 0, \quad c_f((1,0,0)) = -12, \\ c_f((2,2,1)) = 180 - 12t, \quad c_f((2,1,1)) = 216, \quad c_f((2,1,0)) = 36 + 24t, \end{cases}$$

$$(4.18)\qquad \begin{cases} c_f^{(2)}((1,1,1)) = -72 + 6t, \quad c_f^{(2)}((1,1,0)) = -72 + 12t, \\ c_f^{(2)}((1,0,0)) = 1344 - 72t, \quad c_f^{(2)}((2,2,1)) = -6624 + 5352t, \\ c_f^{(2)}((2,1,1)) = 2326032 - 18312t, \quad c_f((2,1,0)) = 4817088 + 26136t. \end{cases}$$

By (4.10), (4.11), (4.17), and (4.18), we know that $E_{2,2}$, $T_4(2)E_{2,2}$, f, and $T_4(2)f$ are linearly independent. This contradicts (4.16). $\square$

(4.16) and Lemma 4.6 mean that $\dim S_4(\Gamma) = 2$. Hence we have proved the second assertion of Theorem 4.4.

Remark 1. Comparing the Fourier coefficients we conjecture that

$$H_1 = H_\alpha, \quad H_2 = H,$$

and H_1, $H_2 \in S_4^*(\Gamma)$ (cf. Table I and III).

Remark 2. From Table I, we conjecture

$$\dim J(\mathfrak{S}_{15}(12, \chi_3)) = 6.$$

We can obtain another cusp form of $S_8(\Gamma)$ ($\dim S_8(\Gamma) = 8$ by Hashimoto [**4**]):

$$G_1 := -\frac{11791}{48440} G_\alpha + \frac{13}{88} G_\beta - \frac{169}{153} G_\gamma + \frac{45851}{27720} G_\lambda^{(1)} - \frac{1027}{2520} G_\lambda^{(2)} - \frac{1188}{2941} G - \frac{13}{2} E_2 F_\alpha,$$

which Fourier coefficients are given in Table III. Here we put

$$G_\lambda^{(1)} := \frac{G_{\lambda(a)} - G_{\lambda(\bar{a})}}{2\sqrt{17721}}, \quad G_\lambda^{(2)} := \frac{G_{\lambda(a)} + G_{\lambda(\bar{a})}}{2},$$

where $G_{\lambda(a)}$ [*resp.* $G_{\lambda(\bar{a})}$] corresponds to $a = 27 + \sqrt{17721}$ [*resp.* $\bar{a} = 27 - \sqrt{17721}$]. It seems that the cusp form G_1 is an eigenfunction for all $T_8(p)$ and G_1 does not satisfy the Maass relation. But we could not obtain the last cusp form in $S_8(\Gamma)$ from lifted cusp forms and the Eisenstein series.

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